THÈSE

présentée à

L'U.F.R des Sciences et Techniques de L'université de Franche-Comté

pour obtenir le

GRADE DE DOCTEUR DE L'UNIVERSITÉ DE FRANCHE-COMTÉ

Spécialité: Mathématiques et Applications

MARCHÉS AVEC COÛTS DE TRANSACTION: APPROXIMATIONS DE LELAND ET ARBITRAGE

EMMANUEL DENIS

Table des matières

Remerciements

Je tiens d'abord à remercier Youri Kabanov, mon directeur de thèse, pour sa générosité. Je remercie tous les membres du jury pour le temps qu'ils m'ont consacré.
Je suis reconnaissant à l'ensemble du personnel du Centre de Télé-enseignement de mathématiques à Besançon sans qui cette thèse n'aurait pas vu le jour. J'ai une pensée pour le secrétariat de mathématiques, souriant et disponible.
Merci à Dimitri de Vallière pour sa collaboration fructueuse.
J'exprime ma gratitude à Bruno Saussereau et Christophe Stricker qui ont cru en moi.
Enfin, je remercie ma famille et mon épouse, Agnès, pour son soutien durant ces deux années passionnantes.

Résumé

Cette thèse s'est déroulée de Septembre 2006 à Mai 2008 sous la direction de Youri Kabanov. Elle aborde plusieurs problèmes qui se posent pour les marchés financiers soumis à des coûts de transaction.
Nous revisitons d'abord la méthode d'approximation des portefeuilles de couverture des options Européennes suggérée par Leland pour le call Européen. On met en évidence la convergence en probabilité des portefeuilles discrétisés vers le pay-off lorsque ce dernier est bien plus général. Dans le même esprit, on mesure la vitesse de convergence en estimant la moyenne de l'erreur quadratique. Cela nous conduit à formuler un théorème de convergence en loi de l'erreur d'approximation du type « central-limite ». Toutefois, le modèle de Black et Scholes utilisé est critiquable dans la pratique puisque la volatilité est supposée constante. C'est pourquoi, nous proposons d'établir un théorème de convergence en probabilité analogue au précédent lorsque la volatilité ne dépend pas seulement du temps mais aussi de l'actif risqué sous-jacent.

Enfin, on s'intéresse à des marchés continus plus abstraits décrits par des cônes générés par les coûts de transactions. Nous formulons quelques notions d'arbitrage mais surtout on propose une description duale des prix de couverture des options américaines comme cela a déjà été fait pour les marchés discrétisés.

Abstract

This thesis has been supervised by Youri Kabanov between September 2006 and May 2008. It deals with different problems on financial markets under transaction costs. The first part is devoted to the method of approximation suggered by Leland in order to hedge European options. We show that we can prove the convergence in probability, conjectured by Leland and proved by Lott with the European call, for a more general pay-off. We estimate the rate of convergence by computing the mean square error which leads us to establish a functional limit theorem, that is a kind of "central-limit theorem ". The second part is about arbitrage and hedging of American options for models in continuous time. The main theorem extends the hedging theorem for the American options in discrete time.

Chapitre 1

Introduction

Les mathématiques financières constituent un domaine des mathématiques appliquées ayant pour objectif la modélisation des phénomènes régissant les marchés financiers. Elles fournissent ainsi aux traders des outils pour spéculer. Louis Bachelier (1870-1946) est considéré comme le fondateur de la finance mathématique. Sa thèse (1900) intitulée « Théorie de la Spéculation »contient des idées novatrices pour analyser les marchés financiers, introduisant l'utilisation du mouvement brownien, l'une des découvertes les plus importantes du vingtième siècle. L'essor des mathématiques financières a depuis été spectaculaire notamment grâce au développement du calcul stochastique.

Dans le souci de décrire le plus vraisemblablement possible la réalité des marchés financiers, une théorie des marchés avec coûts de transaction, prenant en compte les différents frais inhérents à la spéculation, est en pleine expansion. De nombreux articles paraissent sur le sujet et beaucoup de problèmes restent ouverts.

Notre sujet principal (première partie) s'appuie sur le fameux article de Heyne Leland [21] qui, dans le cadre du modèle de Black et Scholes, propose une méthode pour couvrir le call Européen lorsqu'on introduit des coûts de transaction proportionnels au mouvement du portefeuille, c'est à dire proportionnels au volume d'actions achetées ou vendues. Lorsque le taux de transaction n'est pas constant, on suppose que ce dernier est d'autant plus faible que le trader spécule un grand nombre de fois, ce qui traduit l'idée d'une offre commerciale de la part de l'intermédiaire (une banque par exemple) à qui s'adresse le propriétaire du portefeuille pour vendre ou acheter des actions. Précisément, il est supposé que le taux est défini par $k_n = k_0 n^{-\alpha}$ où n est le nombre de révisions et $\alpha \in [0, 1/2]$ est un paramètre constant ainsi que k_0. Leland propose une procédure efficace et simple à mettre en oeuvre puisque sa stratégie est discrétisée ; on modifie la valeur du portefeuille à des dates de révision fixées à l'avance et on maintient son portefeuille jusqu'à la prochaine date. Sans coût de transaction, la stratégie de réplication à suivre est bien connue et des formules explicites sont données. Le portefeuille correspondant est continu (dans la pratique, on discrétise) et réplique exactement le pay-off $(S_1 - K)^+$ du call Européen . Leland propose de suivre cette dernière stratégie en substituant la volatilité du modèle considéré par une volatilité modifiée afin de compenser les coûts de transaction.

Il a été démontré par Lott [24] puis Kabanov et Safarian [18] que cette approche est efficace pour un grand nombre de révisions puisqu'on obtient une convergence en probabilité du portefeuille discrétisé de Leland vers le call Européen lorsque le nombre de révisions n converge vers $+\infty$. Malheureusement, c'est faux lorsque le taux de transaction est constant ($\alpha = 0$) puisqu'apparaît une erreur systématique qui cependant est fournie explicitement dans [18] et donne ainsi aux traders une information précieuse. D'ailleurs, Pergamenshchikov [26] s'est intéressé à ce cas. Il a évalué la vitesse de convergence (d'ordre $n^{1/4}$) et a formulé un théorème de convergence en loi de l'erreur d'approximation. On peut citer aussi le travail récent de Sekine et Yano qui proposent de diminuer l'erreur de couverture lorsque la valeur terminale de l'action (à la date d'échéance) est proche du strike K. Enfin, Kabanov et Gamys [12] estiment la vitesse de convergence (d'ordre $n^{1/2}$) dans le cas où $\alpha = 1/2$.

Ici, commence notre travail. Il est tout naturel de se demander si l'approche de Leland est encore valable pour des options Européennes différentes du call Européen définies par une fonction de pay-off h autre que $h(x) = (x - K)^+$. Le chapitre 2 apporte une réponse positive à cette question sous la condition que h soit assez régulière et convexe. Il s'avère qu'une erreur d'approximation systématique apparaît si h n'est pas convexe. Le problème de couverture approximative reste donc ouvert pour les fonctions non convexes, pour lesquelles une bonne connaissance des EDP non linéaires semble nécessaire. Notons que contrairement au travail initié par Leland, il n'est pas nécessaire de choisir des dates de révision uniformes comme cela est suggéré dans l'article [14].

Afin de préciser la vitesse de convergence, un travail similaire à celui de [12] est proposé dans le chapitre 3. La principale difficulté par rapport au cas initial du call Européen est d'estimer les dérivées successives de la fonction $\widehat{C}(t, x)$ générant le portefeuille de Leland $\widehat{C}(t, S_t)$. En effet, ces dernières sont nécessaires car la démonstration de la convergence en probabilité mais aussi celle de la convergence de la moyenne quadratique de l'erreur d'approximation reposent sur une utilisation intensive du calcul analytique (approximations de Taylor) et du calcul stochastique. Il ressort que concentrer les dates de révision autour de l'échéance semble améliorer la vitesse de convergence.

Grâce à l'étude de la moyenne quadratique du chapitre 3, l'erreur d'approximation apparaît dans le cas $\alpha = 1/2$ comme composée essentiellement d'une martingale. Dans l'esprit du travail initié par Pergamenshchikov pour $\alpha = 0$ [26], nous nous intéressons donc à la convergence en loi de l'erreur d'approximation amplifiée par la vitesse de convergence $n^{1/2}$ ($n^{1/4}$ lorsque $\alpha = 0$ [26]). Nous énonçons ainsi dans le chapitre 4 un théorème du type « central-limite », utile dans la pratique puisqu'il fournit des intervalles de confiance. La preuve du théorème ainsi proposé s'appuie sur la théorie développée par Jacod et Shiryaev [16].

Le chapitre 5 est consacré à l'étude de la convergence en probabilité de l'erreur d'approximation tout comme dans le chapitre 2 sauf qu'ici la volatilité du modèle décrivant l'actif risqué est une fonction dépendant du temps mais aussi de la valeur de l'action. La méthode pour prouver la convergence est sensiblement la même. La grande difficulté est d'estimer les dérivées successives de $\widehat{C}$ en tant que solution d'une EDP. En effet, des estimations existent dans la littérature mais malheureusement elles ne nous satisfont pas lorsque $\alpha < 1/2$ car alors l'EDP dépend de n. C'est pourquoi, nous avons dû reprendre les estimations faites par Friedman [11] afin de préciser l'influence de la variable n sur les inégalités vérifiées par les dérivées. Nous avons alors réussi à prouver que la convergence en probabilité du portefeuille de couverture selon Leland vers le pay-off est toujours vérifiée tout au moins pour $\alpha \in]1/4, 1/2]$.

La deuxième partie est consacrée à la théorie de l'arbitrage. Pour un marché donné, on veut savoir s' il est possible de faire des profits (gains positifs non-nuls sur un ensemble nonnégligeable) en partant d'un capital initial nul. Un portefeuille le permettant s'appelle un arbitrage. On s'intéresse aussi aux prix des options ; à quel prix dois-je vendre à mon client une option Européenne (respectivement Américaine) afin d'être en mesure de démarrer un portefeuille dont la valeur à la date d'échéance (respectivement à tout instant) sera au moins égale au revenu promis (on parle de sur-réplication) ? C'est un sujet de première importance en finance.

Pour les modèles sans coût de transaction, la théorie est déjà très développée. L'absence d'arbitrage est équivalente à l'existence d'une probabilité sous-laquelle le processus $(S_t)_{t\in[0,T]}$ décrivant l'actif risqué est une martingale. Pour les modèles discrets, il s'agit du fameux théorème de Dalang-Morton-Willinger tandis que pour les modèles à temps continu, Delbaen et Schachermayer ont introduit la condition « No Free Lunch ». Dans les deux cas, la théorie développée s'appuie sur le théorème de séparation de Hahn-Banach qui a donné naissance au fameux théorème de Kreps-Yann. De nombreux articles traitent le sujet. Notons que des théorèmes de sur-réplication existent utilisant le théorème de décomposition optionnelle (voir [19]).

Pour les modèles avec coûts de transaction, la théorie de l'arbitrage est bien développée dans le cas discret. La modélisation mathématique s'appuie sur la notion de processus générant des cônes. En particulier, le cône de solvabilité $\widehat{K}$ qu'on aura l'occasion d'introduire a un rôle essentiel. Si un portefeuille exprimé en quantité d'actions détenues est dans ce dernier, on peut, moyennant des transactions, se ramener à un portefeuille dont toutes les positions sont positives. L'absence d'arbitrage (cas discret) est équivalente à l'existence d'une martingale évoluant dans le dual positif $\widehat{K}^*$ du cône $\widehat{K}$. Lorsqu'il n'y a pas d'opportunité d'arbitrage, on sait alors définir les prix de sur-réplication aussi bien pour les options Européennes qu'Américaines [1], [19].

Pour les modèles avec coûts de transaction et en temps continu, la théorie est moins développée. Elle a été initiée par Kabanov avec son modèle $\mathcal{X}$ pour lequel il fournit un théorème de sur-réplication des options Européennes mais aussi par Campi et Schachermayer qui permettent l'extension de ce dernier grâce au modèle $\mathcal{Y}$ plus général. Ici commence notre travail de la partie 2. Dans le cas discret, on peut observer que les théorèmes de sur-réplication sont énoncés sous des conditions de non-arbitrage. On est alors naturellement amené à se demander si les conditions utilisées pour les théorèmes de sur-réplication des modèles $\mathcal{X}$ ou $\mathcal{Y}$ sont équivalentes à l'absence d'arbitrage. Dans le chapitre 6, on propose une notion d'arbitrage pour un modèle très proche du modèle $\mathcal{X}$. Le théorème de sur-réplication des options Europènnes est toujours valable. Pour les options Américaines, le résultat de Bouchard et Temam [1] dans le cas discret conduit à une idée de démonstration dans la cas continu qui au premier abord s'avère inefficace pour le modèle $\mathcal{X}$. Mais elle se révèle fructueuse pour le modèle $\mathcal{Y}$ (voir travail commun avec De Vallière et Kabanov [6]) permettant ainsi de conclure aussi pour le modèle $\mathcal{X}$. Le chapitre 7 montre que les conditions utilisées pour les théorèmes de sur-réplication des options Européennes et Américaines sont équivalentes à une condition de non-arbitrage dans le cas où le marché est défini par des processus de prix et de coûts de transaction qui sont « en escalier ». Enfin, dans l'esprit du travail initié par Kabanov et Kramkov pour des marchés sans coût de transaction [17], sont proposées dans le chapitre 8, différentes notions d'arbitrage pour des marchés dont l'horizon converge vers ∞.

Première partie

Leland's Approximations

Chapter 2

Approximate Hedging for the Leland–Lott Hedging Strategy for General Pay-offs

In 1985 Leland suggested an approach to pricing contingent claims under proportional transaction costs. Its main idea is to use the classical Black–Scholes formula with a suitably enlarged volatility for a periodically revised portfolio whose terminal value approximates the pay-off $h(S_T) = (S_T - K)^+$ of the call-option. In subsequent studies, Lott, Kabanov and Safarian, Gamys and Kabanov provided a rigorous mathematical analysis and established that the hedging portfolio approximates this pay-off in the case where the transaction costs decrease to zero as the number of revisions tends to infinity. The arguments used heavily the explicit expressions given by the Black–Scholes formula leaving open the problem whether the Leland approach holds for more general options and other types of price processes. In this paper we show that for a large class of the pay-off functions Leland's method can be successfully applied. On the other hand, if the pay-off function $h(x)$ is not convex, then this method does not work.

2.1 Introduction

In his famous paper [21] Leland suggested, in the framework of a two-asset model of financial market with proportional transaction costs, a modification of the Black–Scholes approach to pricing contingent claims. The idea is very simple: one can use the Black–Scholes formula but not with a true volatility parameter σ but with an artificially enlarged one, $\widehat{\sigma}$. A theoretical justification of this approach is based on the replication principle: the terminal value of a "real-world" self-financing portfolio, revised at sufficiently large number n of dates t_k, should approximate the terminal pay-off. Leland gave an explicit formula for enlarged volatility $\widehat{\sigma}$ which may depend on n. His pricing methodology is of great practical importance, in particular, due to an easy implementation.

However, a mathematical validation of this "approximate replication principle" happened to be quite delicate. The first rigorous result was obtained by Lott [24] who shown that the convergence in probability, as it was conjectured by Leland, holds when the transaction costs coefficient $k_n = k_0 n^{-\alpha}$ decreases to zero for $\alpha = 1/2$ (in this case, $\widehat{\sigma}$ does not depend on n). On the other hand, for the constant k_0 the replication principle fails to be true. This was observed by Kabanov and Safarian [18] who calculated the limiting approximation error. They also proved that the replication error tends to zero when $\alpha \in]0, 1/2[$. Interesting limit theorems for the case $\alpha = 0$ (i.e. constant k) were obtained by Granditz and Schachinger [13] and Pergamenshchikov [26]. Results on the first-order asymptotics of the L^2-norm of the approximation error can be found in [12]. All mentioned papers deal with the call option, i.e. with the particular pay-off function $h(x) = (x - K)^+$. Even in this case the arguments need a lot of estimates. The explicit expressions given by the Black–Scholes formula simplify calculations which are quite involved.

The limits of applicability of the Leland approach remains an open problem. In this paper we address this issue and establish convergence results for more general pay-off functions and non-uniform revision intervals following the methodology of [12]. In particular, we show, for the case $\alpha \in]0, 1/2]$, that the approximation error converges to zero for convex pay-off functions of "moderate" growth. For non-convex pay-off functions we calculate the systematic error depending on the value of the stock price at maturity. We find this limiting error also for $\alpha = 0$ (Theorem 2.5.1).

2.2 Main Results

We consider the standard two-asset model with the time horizon $T = 1$ assuming that it is specified under the martingale measure, the non-risky asset is the *numéraire*, and the price of risky asset is given by the formula

$$S_t = S_0 \exp\left\{\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right\}$$

where W is a Wiener process. So, $dS_t = \sigma_t S_t dW_t$. We assume that σ_t is a strictly positive and continuous function on $[0, 1]$ verifying the Lipschitz condition

$$|\sigma_t - \sigma_u| \leqslant L|t - u|$$

where $L > 0$ is a constant. In particular, we have $\sigma_t \in [\underline{\sigma}, \overline{\sigma}]$ where $\underline{\sigma} > 0$. Note that

$$S_t \sim S_0 \exp\{\alpha_t \xi - \alpha_t^2/2\}$$

where $\alpha_t^2 = \int_0^t \sigma_s^2 ds$ and $\xi \sim \mathcal{N}(0, 1)$.

Recall that, according to Black and Scholes, the price of the contingent claim $h(S_1)$ is the initial value of the replicating portfolio

$$V_t = V_0 + \int_0^t H_r dS_r = E(h(S_1)|\mathcal{F}_t) = C(t, S_t),$$

where

$$\begin{aligned} C(t, x) &= E\,h(x \exp\{\overline{\rho}_t \xi - \overline{\rho}_t^2/2\}), \\ \overline{\rho}_t^2 &= \int_t^1 \sigma_s^2 ds, \end{aligned}$$

and the replication strategy is $H_r = C_x(r, S_r)$.

In the model with proportional transaction costs and a finite number of revisions the current value of the portfolio process at time t is described as

$$V_t^n = V_0^n + \int_0^t H_u^n dS_u - \sum_{t_i \leqslant t} k_n S_{t_i} |H_{i+1}^n - H_i^n| \tag{2.2.1}$$

where H^n is a piecewise-constant process with $H^n = H_i^n$ on the interval $]t_{i-1}, t_i]$, $t_i = t_i^n$, $i \leqslant n$, are the revision dates, and H_i^n are $\mathcal{F}_{t_{i-1}}$-measurable random variables. Of course, V_0^n is the initial endowment. We assume that the transaction costs coefficient verifies

$$k = k_n = k_0 n^{-\alpha}, \qquad \alpha \in [0, 1/2], \tag{2.2.2}$$

and the dates t_i are defined by a strictly increasing function $g \in C^1[0,1]$ with $g(0) = 0$, $g(1) = 1$, so that $t_i = g(i/n)$. Let denote by f the inverse of g. The "enlarged volatility", in general, depending on n, is given by the formula

$$\widehat{\sigma}_t^2 = \sigma_t^2 + \sigma_t k_n n^{\frac{1}{2}} \sqrt{8/\pi} \sqrt{f'(t)} = \sigma_t^2 + \sigma_t \gamma_n(t). \tag{2.2.3}$$

We call the Leland strategy the process H^n with

$$H_i^n = \widehat{C}_x(t_{i-1}, S_{t_i - 1})$$

where the function $\widehat{C}(t,x)$ is the solution of the Cauchy problem:

$$\widehat{C}_t(t,x) + \frac{1}{2}\widehat{\sigma}_t^2 x^2 \widehat{C}_{xx}(t,x) = 0, \qquad \widehat{C}(1,x) = h(x). \tag{2.2.4}$$

Its solution can be written as

$$\widehat{C}(t,x) = \int_{-\infty}^{\infty} h(xe^{\rho_t y - \rho_t^2/2}) \varphi(y) dy \tag{2.2.5}$$

where φ is the Gaussian density and $\rho_t^2 = (\rho_t^n)^2 = \int_t^1 \widehat{\sigma}_s^2 ds$; to simplify formulae we shall omit frequently the subscript t at ρ.

Note that $\widehat{\sigma}_s^2 \geqslant \underline{\sigma}^2 + cn^{\frac{1}{2}-\alpha}$ for a constant $c > 0$ and, therefore,

$$\rho_t^2 \geqslant \left(\underline{\sigma}^2 + cn^{\frac{1}{2}-\alpha}\right)(1-t).$$

We use the abbreviations $\widehat{H}_t = \widehat{C}_x(t, S_t)$ and $\widehat{h}_t = \widehat{C}_{xx}(t, S_t)$. We define $V_0^n := \widehat{C}(0, S_0)$.

We shall use the following hypothesis on the "cadence " of revisions:

Assumption (G1): $g' > 0$, $g'' \in C[0,1[$ and there exists a constant $\lambda \in [0,1[$ such that $g''(t)(1-t)^\lambda$ is bounded on $[0,1[$.

Assumption (G2): the function g is concave, $g'' \in C[0,1[$ and there exists a constant $\lambda \in [0,1[$ such that $g''(t)(1-t)^\lambda$ is bounded on $[0,1[$. Moreover, we have some constants $k_1 > k_2$ in $]0, 1/2+\alpha[$, $c_1, c_2 > 0$ and $h_0 > 0$ near to 0 such that, for n large enough:

(i) $c_1 (1-g(u))^{k_1} \leqslant g'(u) \leqslant c_2 (1-g(u))^{k_2}$ for u near to 1,

(ii) $g'\left(1 - \frac{1}{n}\right) \geqslant \left(\frac{1}{n}\right)^\mu, \quad \mu \in [0, k_1]$,

(iii) $\lim_{h\to 0}\sup_{t\leqslant 1-h}\frac{1-g(t-h)}{1-g(t)}\leqslant c_2$,

(iv) $\sup_{t<1}\left|\frac{f'(t-h)}{f'(t)}-1\right|\leqslant \frac{c_2 h}{1-t}, \quad 0\leqslant h\leqslant h_0$,

(v) $\frac{|g''(u)|}{g'(u)^2}\leqslant \frac{c_2}{(1-g(u))^{3/2}}, \quad \forall u<1$,

(vi) $\frac{f'(t_2)}{f'(t_1)}\leqslant c_2\frac{1-t_1}{1-t_2} \quad$ if $t_1-h\leqslant t_2\leqslant t_1<1, \quad 0\leqslant h\leqslant h_0$.

(vii) $\frac{g'(x)^{3/4}}{\sqrt{1-g(x)}}\leqslant \frac{c}{\sqrt{1-x}}$, for x near to 1

It is easy to see that, in this two cases, the following properties hold:

Lemma 2.2.1. *Assume that $(G1)$ or $(G2)$ hold. Then there exists a constant $\gamma>1$ such that for $i=0,..,n-1$ and n large enough:*

$$\Delta t_i = t_i - t_{i-1} = g'(i-1/n)n^{-1} + n^{-\gamma}o_n(1), \tag{2.2.6}$$

and, moreover, for some constants $d_1, d_2>0$

$$\Delta t_i \leqslant d_1 n^{-1}, \tag{2.2.7}$$

$$\frac{1-t_{i-1}}{1-t_i}\leqslant d_1, \quad \frac{f'(t_{i-1})}{f'(t_i)}\leqslant d_1, \tag{2.2.8}$$

$$\sup_{u\in[t_{i-1},t_i[} f'(u)(t_i-u)n\leqslant d_1, \tag{2.2.9}$$

$$1-t_{n-1}\geqslant d_2 n^{-(\mu+1)}. \tag{2.2.10}$$

Note that the assumption (G2) is verified by the functions

$$g_\mu(t) = 1-(1-t)^\mu,\ \mu>1.$$

Our hypothesis on the pay-off function is as follows:

Assumption (H): h is a continuous function on $[0,\infty[$ which is twice differentiable except the points $K_1<\cdots<K_p$ where h' and h'' admit right and left limits; h' is bounded and $|h''(x)|\leqslant Mx^{-\beta}$ for $x\geqslant K_p$ where $\beta\geqslant 3/2$.

Let $K_0=0$ and $K_{p+1}=\infty$. Then h'' is bounded while h verifies the inequality $|h(x)|\leqslant M_1(1+x)$ with some constant M_1. The function $\widehat{C}(t,x)$ is continuous on $[0,1]\times\mathbb{R}$.

Put

$$\begin{aligned}
\theta_1(x,S_1) &:= \frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} h'(S_1 e^{\sqrt{x}y+x/2})y\varphi(y)dy,\\
\varepsilon_\alpha &:= \frac{S_1}{2}\int_0^\infty [\theta_1(x,S_1)-|\theta_1(x,S_1)|]dx, \quad \alpha\in]0,1/2[,\\
\varepsilon_{1/2} &:= \frac{1}{2}k_0\sqrt{\frac{8}{\pi}}\int_0^1 \sigma_t\sqrt{f'(t)}\left(\widehat{C}_{xx}(t,S_t)-|\widehat{C}_{xx}(t,S_t)|\right)dt.
\end{aligned}$$

Theorem 2.2.2. *Let $\alpha \in]0, 1/2[$. Suppose that $(G1)$ holds. Then*

$$P\text{-}\lim_n V_1^n = h(S_1) + \varepsilon_\alpha. \tag{2.2.11}$$

Let $\alpha = 1/2$. Suppose that $(G1)$ or $(G2)$ is fulfilled. Then

$$P\text{-}\lim_n V_1^n = h(S_1) + \varepsilon_{1/2}. \tag{2.2.12}$$

If h is a convex function, then $\varepsilon_\alpha = 0$ (for $\alpha \in [0, 1/2]$).

2.3 Estimates

In the following subsections we establish some properties of the solution of the Cauchy problem (2.2.4) needed for the proof of Theorem 2.2.2.

2.3.1 Explicit Formulae

Lemma 2.3.1. *Let $\widehat{C}(t, x)$ is given by (2.2.5). Then*

$$\frac{\partial^{k+1}\widehat{C}(t,x)}{\partial x^{k+1}} = \frac{1}{\rho^k x^k}\int_{-\infty}^{\infty} h'(xe^{\rho y+\rho^2/2})P_k(y)\varphi(y)dy, \qquad k \geqslant 0,$$

where $P_k(y) = y^k + a_{k-1}(\rho)y^{k-1} + \cdots + a_0(\rho)$ is a polynomial of degree k whose coefficients $a_i(\rho)$ are polynomials in ρ of degree $k-1$.

Proof. By the change of variable $z = z(y, x, \rho) = xe^{\rho y - \rho^2/2}$ with the inverse

$$y = y(z, x, \rho) = -\frac{1}{\rho}\left(\ln\frac{x}{z} - \frac{\rho^2}{2}\right)$$

we transform (2.2.5) to the form more convenient to differentiate:

$$\widehat{C}(t,x) = \frac{1}{\rho}\int_0^\infty \frac{h(z)}{z}\varphi(y(z,x,\rho))dz.$$

It follows that

$$\widehat{C}_x(t,x) = \frac{1}{\rho^2}\int_0^\infty \frac{h(z)}{xz}y(z,x,\rho)\varphi(y(z,x,\rho))dz$$

because we can differentiate under the sign of the integral. Indeed, it is easily seen that for every $x_0 > 0$ the integrand in the last formula, for x varying in a sufficiently small interval $]x_0 - \alpha, x_0 + \alpha[$, can be dominated by an L^1-function of variable z which does not depend of x.

Turning back to the integration with respect to y, we have the formula

$$\widehat{C}_x(t,x) = \frac{1}{\rho x}\int_{-\infty}^\infty h(z(x,y,\rho))y\varphi(y)dy.$$

Splitting the integral and integrating by parts on each interval $]\widetilde{\delta}_j, \widetilde{\delta}_{j+1}[$ with

$$\widetilde{\delta}_j = \frac{1}{\rho}\ln\frac{K_j}{x} + \frac{\rho}{2},$$

$j = 0, .., p+1$ and $K_0 = 0$, $K_{p+1} = \infty$, we deduce from here, after the change of variable $y' = y - \rho$, that

$$\widehat{C}_x(t,x) = \int_{-\infty}^{\infty} h'(\tilde{z}(y,x,\rho))\varphi(y)dy \tag{2.3.13}$$

where $\tilde{z} = \tilde{z}(y,x,\rho) = xe^{\rho y + \rho^2/2}$.

In particular, $|\widehat{C}_x(t,x)| \leqslant ||h'||_\infty$.

Similar calculations give the formulae:

$$\widehat{C}_{xx}(t,x) = \frac{1}{\rho x}\int_{-\infty}^{\infty} h'(\tilde{z}(y,x,\rho))y\varphi(y)dy, \tag{2.3.14}$$

$$\widehat{C}_{xxx}(t,x) = \frac{1}{\rho^2 x^2}\int_{-\infty}^{\infty} h'(\tilde{z}(y,x,\rho))P_2(y)\varphi(y)dy, \tag{2.3.15}$$

$$\widehat{C}_{xxxx}(t,x) = \frac{1}{\rho^3 x^3}\int_{-\infty}^{\infty} h'(\tilde{z}(y,x,\rho))P_3(y)\varphi(y)dy \tag{2.3.16}$$

with

$$\begin{aligned} P_2(y) &= y^2 - \rho y - 1, \\ P_3(y) &= y^3 - 3\rho y^2 + (2\rho^2 - 3)y + 3\rho. \end{aligned}$$

The general formula for the derivatives in x follows by induction. Indeed, assume that

$$\frac{\partial^k \widehat{C}(t,x)}{\partial x^k} = \frac{1}{\rho^{k-1}x^{k-1}}\int_{-\infty}^{\infty} h'(xe^{\rho y + \rho^2/2})P_{k-1}(y)\varphi(y)dy,$$

where $P_{k-1}(y) = y^{k-1} + a_{k-2}(\rho)y^{k-2} + \cdots + a_0(\rho)$ is a polynomial of degree $k-1$ whose coefficients $a_i(\rho)$ are polynomials in ρ of degree $k-2$. With the change of variable $y = y' - \rho$, we obtain

$$\frac{\partial^k \widehat{C}(t,x)}{\partial x^k} = \frac{1}{\rho^{k-1}x^{k-1}}\int_{-\infty}^{\infty} h'(xe^{\rho y - \rho^2/2})P_{k-1}(y-\rho)\varphi(y-\rho)dy.$$

By the change of variable $z = z(y,x,\rho) = xe^{\rho y - \rho^2/2}$, we write

$$\frac{\partial^k \widehat{C}(t,x)}{\partial x^k} = \frac{1}{\rho^{k-1}x^{k-1}}\int_{-\infty}^{\infty} \frac{h'(z)}{\rho z}P_{k-1}\left(y(z,x,\rho) - \rho\right)\varphi\left(y(z,x,\rho) - \rho\right)dz$$

and we deduce that

$$\begin{aligned} \frac{\partial^{k+1} \widehat{C}(t,x)}{\partial x^{k+1}} &= \frac{1-k}{\rho^{k-1}x^k}\int_{-\infty}^{\infty} \frac{h'(z)}{\rho z}P_{k-1}\left(y(z,x,\rho) - \rho\right)\varphi\left(y(z,x,\rho) - \rho\right)dz, \\ &+ \frac{1}{\rho^{k-1}x^{k-1}}\int_{-\infty}^{\infty} \frac{h'(z)}{\rho z}P_{k-1}\left(y(z,x,\rho) - \rho\right)\frac{y(z,x,\rho) - \rho}{\rho x}\varphi\left(y(z,x,\rho) - \rho\right)dz, \\ &+ \frac{1}{\rho^{k-1}x^{k-1}}\int_{-\infty}^{\infty} \frac{h'(z)}{\rho z}\frac{1}{\rho x}P'_{k-1}\left(y(z,x,\rho) - \rho\right)\varphi\left(y(z,x,\rho) - \rho\right)dz. \end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial^{k+1}\widehat{C}(t,x)}{\partial x^{k+1}} &= \frac{1}{\rho^k x^k}\int_{-\infty}^{\infty}\frac{h'(z)}{\rho z}P_k\left(y(z,x,\rho)-\rho\right)\varphi(y(z,x,\rho)-\rho)dz,\\ &= \frac{1}{\rho^k x^k}\int_{-\infty}^{\infty}h'(xe^{\rho y-\rho^2/2})P_k\left(y-\rho\right)\varphi(y-\rho)dy,\end{aligned}$$

where $P_k(y) = (1-k)\rho P_{k-1}(y) + yP_{k-1}(y) + P'_{k-1}(y)$ is a polynomial of degree k because of the degree of P_{k-1}. The coefficient of the main term x^k is clearly equal to unit whereas the other coefficients are polynomials in ρ of degree $k-1$ by induction. Then, we can conclude using the change of variable $y = y' + \rho$.

By similar reasoning we obtain, using the previous lemma and the PDE 2.2.4:

Lemma 2.3.2. *Let $\widehat{C}(t,x)$ is given by (2.2.5). Then*

$$\widehat{C}_t(t,x) = \frac{-\widehat{\sigma}_t^2 x}{2\rho}\int_{-\infty}^{\infty}h'(xe^{\rho y+\rho^2/2})y\varphi(y)dy, \tag{2.3.17}$$

$$\widehat{C}_{tx}(t,x) = \frac{\widehat{\sigma}_t^2}{2\rho^2}\int_{-\infty}^{\infty}h'(xe^{\rho y+\rho^2/2})Q_2(y)\varphi(y)dy, \tag{2.3.18}$$

$$\widehat{C}_{xxt}(t,x) = \frac{\widehat{\sigma}_t^2}{2\rho^3 x}\int_{-\infty}^{\infty}h'(xe^{\rho y+\rho^2/2})Q_3(y)\varphi(y)dy, \tag{2.3.19}$$

with

$$\begin{aligned}Q_2(y) &= -y^2-\rho y+1,\\ Q_3(y) &= -y^3-\rho y^2+3y+\rho.\end{aligned}$$

2.3.2 Inequalities

Lemma 2.3.3. *There is a constant $c > 0$ such that*

$$|\widehat{C}_{xx}(t,x)| \leqslant c\frac{1}{\rho x^{3/2}}e^{-\rho^2/8}\sum_{j=1}^{p}\exp\left\{-\frac{1}{2}\frac{\ln^2(K_j/x)}{\rho^2}\right\} + c\frac{1}{x^{3/2}}e^{-\rho^2/8}. \tag{2.3.20}$$

Proof. We integrate by parts the integral of the formula (2.3.14) on each interval $]\delta_j,\delta_{j+1}[$ with

$$\delta_j = \frac{1}{\rho}\ln\frac{K_j}{x} - \frac{\rho}{2}.$$

As

$$\{y:\ K_j < \tilde{z}(y,x,\rho) < K_{j+1}\} =]\delta_j,\delta_{j+1}[,$$

we have:

$$\widehat{C}_{xx}(t,x) = \sum_{j=0}^{p}\frac{1}{\rho x}\left[-h'(\tilde{z})\varphi(y)\right]_{\delta_j}^{\delta_{j+1}} + \sum_{j=0}^{p}\frac{1}{\rho x}\int_{\delta_j}^{\delta_{j+1}}\rho\tilde{z}h''(\tilde{z})\varphi(y)dy.$$

Notice that $\widehat{C}_{xx}(t,x) \geqslant 0$ if h is convex.

Using the change of variable $u = y + \rho/2$ and the boundedness of h'', we have:

$$\left|\frac{1}{\rho x}\int_{-\infty}^{\delta_p} \rho\tilde{z}h''(\tilde{z})\varphi(y)dy\right| = e^{-\rho^2/8}\left|\int_{-\infty}^{\ln(K_p/x)/\rho} e^{\frac{3}{2}\rho y}h''(xe^{\rho y})\varphi(y)dy\right|$$

$$\leqslant c\, e^{-\rho^2/8}e^{\frac{3}{2}\ln(K_p/x)} \leqslant c\frac{1}{x^{3/2}}e^{-\rho^2/8},$$

where c is a constant.

Using the assumption on the growth of h'', we get in a similar way that

$$\left|\frac{1}{\rho x}\int_{\delta_p}^{\infty} \rho\tilde{z}h''(\tilde{z})\varphi(y)dy\right| = e^{-\rho^2/8}\left|\int_{\ln(K_p/x)/\rho}^{\infty} e^{\frac{3}{2}\rho y}h''(xe^{\rho y})\varphi(y)dy\right|$$

$$\leqslant e^{-\rho^2/8}\int_{\ln(K_p/x)/\rho}^{\infty} e^{\frac{3}{2}\rho y-\beta\rho y}x^{-\beta}\varphi(y)dy \leqslant c\frac{1}{x^{3/2}}e^{-\rho^2/8}.$$

Noting that $-h'(\tilde{z})\varphi(y) = 0$ for $y = \delta_0$ and $y = \delta_{p+1}$, we dominate the first sum by the estimate

$$\frac{1}{\sqrt{2\pi}\rho x}\sum_{j=1}^{p} 2||h'||_\infty e^{-\delta_j^2/2}.$$

The desired inequality follows from the above bounds.

Recall the following identity (see [12]):

Lemma 2.3.4. *Let $\eta \sim \mathcal{N}(0,1)$. Then for any real numbers $a \neq 0$, b and c*

$$Ee^{c\eta}e^{-(a\eta+b)^2} = \frac{1}{\sqrt{2a^2+1}}\exp\left\{-\frac{\tilde{b}^2}{2a^2+1}+\tilde{b}^2-b^2\right\}$$

where $\tilde{b} = b - c/(2a)$.

It will serve to get the following:

Corollary 2.3.5. *There exists a constant c such that for $t \in [0,1[$*

$$ES_t^4\widehat{C}_{xx}^2(t,S_t) \leqslant c\frac{1}{\rho}e^{-\rho^2/4}.$$

Proof. By (2.3.20)

$$S_t^m\widehat{C}_{xx}^2(t,S_t) \leqslant \frac{K}{\rho^2}\sum_{j=1}^{p} e^{c_t\eta}e^{-(a_t\eta+b_t^j)^2} + Ke^{-\rho^2/4}S_t^{m-3}$$

with $c_t = (m-2)\alpha_t$, $a_t = \alpha_t/\rho$, and

$$b_t^j = \frac{1}{\rho}\left(\ln\frac{S_0}{K_j} - \frac{1}{2}\alpha_t^2\right) + \frac{\rho}{2}.$$

Then

$$ES_t^m\widehat{C}_{xx}^2(t,S_t) \leqslant K\sum_{j=1}^{p}\frac{\exp\{-B_t^j\}}{\rho\sqrt{2\alpha_t^2+\rho^2}} + Ke^{-\rho^2/4}$$

for some constant K and

$$B_t^j = \frac{\left(\ln \frac{S_0}{K_j} - \frac{1}{2}\alpha_t^2 - \frac{1}{2}(m-3)\rho^2\right)^2}{2\alpha_t^2 + \rho^2} - \frac{(m-2)(m-4)}{4}\rho^2.$$

We conclude by taking $m = 4$.

Similarly, we can deduce the following bounds:

Corollary 2.3.6. *There exists a constant c such that for $t \in [0, 1[$*

$$ES_t^2 \widehat{C}_{xx}^2(t, S_t) \leqslant c \left(\sum_{j=1}^{p} \frac{1}{\rho^2 \sqrt{2u^2 + 1}} \exp\left\{ -\frac{v_j^2}{2u^2 + 1} \right\} + e^{-\rho^2/4} \right)$$

where c is a constant, $u = \alpha_t/\rho$ and

$$v_j = \frac{\ln(S_0/K_j) - \alpha_t^2/2}{\rho} + \frac{1}{2}\rho.$$

Corollary 2.3.7. *There exists a constant c such that for $t \in [\frac{1}{2}, 1[$,*

$$ES_t^2 \widehat{C}_{xx}^2(t, S_t) \leqslant c \left(\frac{1}{\rho} + e^{-\rho^2/4} \right).$$

With the same technique we can prove the following estimates:

Lemma 2.3.8. *There exists a constant c such that*

$$\begin{aligned}
|\widehat{C}_{xxx}(t, x)| &\leqslant \frac{ce^{-\rho^2/8}}{\rho^2 x^{5/2}} \left(L(x, \rho) + \rho \right), \\
|\widehat{C}_{xxxx}(t, x)| &\leqslant ce^{-\rho^2/8} x^{-7/2} P_3(\rho^{-1}), \\
|\widehat{C}_{tx}(t, x)| &\leqslant \frac{c\widehat{\sigma}^2 e^{-\frac{\rho^2}{8}}}{x^{1/2}\rho^2} \left(L(x, \rho) + \rho + \rho^2 \right), \\
|\widehat{C}_{xxt}(t, x)| &\leqslant c\widehat{\sigma}^2 e^{-\rho^2/8} x^{-3/2} (\rho^{-1} + \rho^{-3}),
\end{aligned}$$

where P_3 is a polynomial of the third order and

$$L(x, \rho) = \sum_{j=1}^{p} \frac{|\ln(x/K_j)|}{\rho} \exp\left\{ -\frac{\ln^2(x/K_j)}{2\rho^2} \right\}.$$

Lemma 2.3.9. *There exists a constant c and a polynomial Q of third order such that*

$$ES_t^m \widehat{C}_{tx}^2(t, S_t) \leqslant c\widehat{\sigma}_t^4 Q(\rho^{-1}) e^{-\rho^2/4}.$$

Proof. It suffices to use Lemma 2.3.8 and observe that

$$ES_t^m \ln^2 \frac{S_t}{K} \exp\left\{ -\frac{\ln^2(S_t/K)}{\rho^2} \right\} \leqslant c(\rho^5 + \rho^3).$$

2.4 Proof of Theorem 2.2.2

By the Ito formula we get that

$$\widehat{C}_x(t,S_t)=\widehat{C}_x(0,S_0)+M_t^n+A_t^n \tag{2.4.21}$$

where

$$\begin{aligned} M_t^n &:= \int_0^t \sigma_u S_u \widehat{C}_{xx}(u,S_u)dW_u,\\ A_t^n &:= \int_0^t \left[\widehat{C}_{xt}(u,S_u)+\frac{1}{2}\sigma_u^2 S_u^2 \widehat{C}_{xxx}(u,S_u)\right]du. \end{aligned}$$

The process M^n is a square integrable martingale on $[0,1]$ in virtue of Corollaries 2.3.6 and 2.3.7.

Following [19] we represent the difference $V_1^n - h(S_1)$ in a convenient form.

Lemma 2.4.1. *We have $V_1^n - h(S_1) = F_1^n + F_2^n + F_3^n$ where*

$$\begin{aligned} F_1^n &:= \int_0^1 (H_t^n - \widehat{H}_t)dS_t - k_n|H_{t_n}^n - H_{t_{n-1}}^n|S_{t_n}, \\ F_2^n &:= \frac{1}{2}\int_0^1 \sigma_t\gamma_n(t)S_t^2|\widehat{C}_{xx}(t,S_t)|dt - k_n\sum_{i=1}^{n-1}|H_{t_i}^n - H_{t_{i-1}}^n|S_{t_i},\\ F_3^n &= \frac{1}{2}\int_0^1 \sigma_t\gamma_n(t)S_t^2\left(\widehat{C}_{xx}(t,S_t)-|\widehat{C}_{xx}(t,S_t)|\right)dt. \end{aligned} \tag{2.4.22}$$

Note that $F_3^n = 0$ if h is a convex function.

Put $D_i^a := \{(x,y):\ x\in[1/a,a], xe^y \in]K_i,K_{i+1}[\}$, $a>0$, $i=0,...,p$.

Lemma 2.4.2. *The mapping $(x,y)\mapsto h'(xe^y)$ is a Lipschitz function on each set D_i^a, i.e. there exists a constant L_a such that*

$$|h'(xe^y)-h'(ze^u)| \leqslant L_a(|x-z|+|y-u|)$$

for all $x,z\in[1/a,a]$ and y, u such that xe^y, $ze^u \in]K_i,K_{i+1}[$, $i=0,...,p$.

Proof. Let us consider the representation

$$h'(xe^y)-h'(ze^u)=\int_{ze^u}^{xe^y} h''(s)ds.$$

Since h'' is bounded, the assertion for $i<p$ is obvious. For $i=p$, we use the assumption that $|h''(s)|\leqslant Ms^{-\beta}$ for $s\geqslant K_p$ where $\beta\geqslant 3/2$.

Lemma 2.4.3. *For any $\alpha\in[0,1/2]$,*

$$P\text{-}\lim_n F_1^n = 0. \tag{2.4.23}$$

Proof. Because the processes H^n and $\widehat{H}^n$ are bounded, we obtain the convergence to zero in L^2 of the stochastic integral by checking that the difference $H_u^n - \widehat{H}_u^n$ tends to zero. To this end, we note that this difference for $u\in[t_{i-1},t_i[$ can be expressed as

$$\sum_{j=0}^{p}\left[\int_{I_j(u)} h'(\tilde{z}(y,S_u,\rho_u))\varphi(y)dy - \int_{I_j(t_{i-1})} h'(\tilde{z}(y,S_{t_{i-1}},\rho_{t_{i-1}}))\varphi(y)dy\right]$$

with $\delta_j(u) = \frac{1}{\rho_u}\ln(K_j/S_u) - \frac{1}{2}\rho_u$ and $I_j(u) = [\delta_j(u), \delta_{j+1}(u)]$. Note that $|\rho_u^n - \rho_{t_{i-1}}^n| \leqslant c_u\sqrt{k_n}n^{-1/4}$ for some constant c_u. The measure of the symmetric difference of the intervals $I_j(u)$ and $I_j(t_{i-1})$ tends to zero as $n \to \infty$. At last,

$$\int_{I_j(u)\cap I_j(t_{i-1})} |h'(\tilde{z}(y, S_u, \rho_u))\varphi(y) - h'(\tilde{z}(y, S_{t_{i-1}}, \rho_{t_{i-1}}))|\varphi(y)dy \to 0$$

in virtue of the previous lemma.

For the second term in (2.4.22) we note that

$$\widehat{C}_x(t_{n-1}, S_{t_{n-1}}) = \int_{-\infty}^{\infty} h'(S_{t_{n-1}}e^{\rho y+\rho^2/2})\varphi(y)dy$$

with $\rho = \rho_{t_{n-1}} \to 0$ as $n \to \infty$. We conclude using the Lebesgue theorem.

We write $F_2^n = \sum_{i=1}^5 L_i^n$ where

$$\begin{aligned}
L_1^n &:= \frac{1}{2}\int_0^1 \sigma_t\gamma_n(t)S_t^2|\widehat{h}_t|dt - \frac{1}{2}\int_0^1 \sum_{i=1}^{n-1}\sigma_{t_{i-1}}\gamma_n(t_{i-1})S_{t_{i-1}}^2|\widehat{h}_{t_{i-1}}|I_{]t_{i-1},t_i]}(t)dt\\
L_2^n &:= \sum_{i=1}^{n-1}|\widehat{h}_{t_{i-1}}|S_{t_{i-1}}^2\left(\frac{1}{2}\sigma_{t_{i-1}}\gamma_n(t_{i-1})\Delta t_i - k_n\sigma_{t_{i-1}}\, n^{1/2}\sqrt{\Delta t_i f'(t_{i-1})}|\Delta W_{t_i}|\right),\\
L_3^n &:= k_n\sum_{i=1}^{n-1}\sigma_{t_{i-1}}S_{t_{i-1}}^2|\widehat{h}_{t_{i-1}}|n^{1/2}\sqrt{\Delta t_i f'(t_{i-1}}|\Delta W_{t_i}| - k_n\sum_{i=1}^{n-1}S_{t_{i-1}}|\Delta M_{t_i}|,\\
L_4^n &:= k_n\sum_{i=1}^{n-1}S_{t_{i-1}}|\Delta M_{t_i}| - k_n\sum_{i=1}^{n-1}S_{t_{i-1}}|\Delta\widehat{H}_{t_i}|,\\
L_5^n &:= -k_n\sum_{i=1}^{n-1}\Delta S_{t_i}|\Delta\widehat{H}_{t_i}|
\end{aligned}$$

where we use the abbreviations $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$ etc.

Lemma 2.4.4. *For any $\alpha \in [0, 1/2]$ both terms whose difference defines L_1^n converge almost surely, as $n \to \infty$, to the random variable J_α given by the formula*

$$J_\alpha = \frac{1}{2}S_1\int_0^\infty |\theta_1(x, S_1)|\,dx, \quad \alpha \in [0, 1/2[, \tag{2.4.24}$$

$$J_{1/2} = \frac{1}{2}\int_0^1 \sigma_t\gamma_n(t)S_t^2|\widehat{h}_t|dt. \tag{2.4.25}$$

Therefore, $L_1^n \to 0$ a.s.

Proof. Let us consider first the case $\alpha < 1/2$. We shall argue for ω outside the null set $\cup_i\{S_1 = K_i\}$. Recalling the definition $\widehat{h}_t = \widehat{C}_{xx}(t, S_t)$ we make the substitution $x = (\rho_t^n)^2$ in integral in the representation of L_1^n and transform this integral to the form

$$\frac{1}{2}\int_0^{\rho_0^2}\sigma_t\frac{\gamma_n(t)}{\widehat{\sigma}_t^2}S_t^2|\widehat{C}_{xx}(t, S_t)|dx.$$

There is an abuse of notation here: we should write $t(x)$ or even $t^n(x)$ instead of t. The function $x \mapsto t(x)$ is the inverse of the function $t \mapsto (\rho_t^n)^2$, so it depends also on n. It converges to unit as $n \to \infty$ when $\alpha \in [0, 1/2[$. This follows from the bound $x \geqslant c(1 + n^{1/2-\alpha})(1-t)$ with some constant c. With the same abuse of notation, we infer from the formula (2.3.14) that

$$S_t^2 \widehat{C}_{xx}(t, S_t) = S_t \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} h'(S_t e^{\sqrt{x}y + x/2}) y\varphi(y) dy.$$

Since h' is bounded and continuous except the points K_i, we get from here $S_t^2 \widehat{C}_{xx}(t, S_t) \to S_1^2 \widehat{C}_{xx}(1, S_1)$.

The bound (2.3.20) implies that

$$\left| \frac{\sigma_t \gamma_n(t)}{\widehat{\sigma}_t^2} S_t^2 C_{xx}(t, S_t) \right| \leqslant c e^{-x/8}(x^{-1/2} + 1)$$

and we obtain required convergence of the integral by the Lebesgue theorem.

In a similar way, we rewrite the second term:

$$\frac{1}{2} \int_0^{\rho_0^2} \sum_{i=1}^{n-1} \sigma_{t_{i-1}} \frac{\gamma_n(t_{i-1})}{\widehat{\sigma}_t^2} S_{t_{i-1}}^2 |\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})| 1_{[x_i, x_{i-1}[}(x) dx$$

where $x_i := (\rho_{t_i}^n)^2$.

Making use the hypothesis $(G1)$ we get that $\gamma_n(t) \to \infty$ and

$$\frac{\sigma_{t_{i-1}} \gamma_n(t_{i-1}) - \sigma_t \gamma_n(t)}{\widehat{\sigma}_t^2} \to 0$$

when $t \in]t_{i-1}, t_i]$, i.e. $x \in [x_i, x_{i-1}[$. Thus, $\sigma_{t_{i-1}} \gamma_n(t_{i-1}) / \widehat{\sigma}_t^2 \to 1$. The end of the reasoning is the same as for the integral term.

In the case $\alpha = 1/2$ the convergence is obvious for the first term. Moreover, the function $\gamma_n(t)$ does not depend of n and $L_1^n \to 0$ because of convergence of the Riemann sums to the integral.

Lemma 2.4.5. *For any $\alpha \in [0, 1/2]$, we have P-$\lim_n L_2^n = 0$.*

Proof. Taking into account the independence of increments of the Wiener process and the equalities

$$E|\Delta W_{t_i}| = \sqrt{2/\pi} \sqrt{\Delta t_i},$$

$$E\left(\frac{1}{2} \gamma_n(t_{i-1}) \Delta t_i - k_n \sqrt{n \Delta t_i f'(t_{i-1})} |\Delta W_{t_i}| \right)^2 = (1 - 2/\pi) k_n^2 n f'(t_{i-1}) (\Delta t_i)^2,$$

we obtain that

$$E(L_2^n)^2 = (1 - 2/\pi) k_n^2 \sum_{i=1}^{n-1} \sigma_{t_{i-1}}^2 E S_{t_{i-1}}^4 \widehat{C}_{xx}^2(t_{i-1}, S_{t_{i-1}}) f'(t_{i-1}) \, n (\Delta t_i)^2$$

where $f'(t_{i-1}) \Delta t_i \, n$ is bounded. In virtue of Lemma 2.3.5, we have:

$$E(L_2^n)^2 \leqslant c \, n^{\alpha/2 - 1/4} k_n^2 \sum_{i=1}^{n-1} (1 - t_{i-1})^{-1/2} \Delta t_i,$$

and so $E(L_2^n)^2 \to 0$ as $n \to \infty$.

Lemma 2.4.6. *For any $\alpha \in [0, 1/2]$, we have P-$\lim_n L_3^n = 0$.*

Proof. In the case of the assumption $(G1)$, we can prove that

$$n^{1/2}\sqrt{f'(t_{i-1})\Delta t_i} = 1 + \varepsilon_n$$

where $\varepsilon_n = o_n(n^{-\tau})$ and $\tau > 0$. We deduce that $L_3^n = C^n + D^n$ with

$$C^n := k_n\varepsilon_n \sum_{i=1}^{n-1} \sigma_{t_{i-1}} S_{t_{i-1}}^2 |\widehat{h}_{t_{i-1}}| \, |\Delta W_{t_i}|$$

and

$$D^n := k_n \sum_{i=1}^{n-1} \left[\sigma_{t_{i-1}} S_{t_{i-1}}^2 |\widehat{h}_{t_{i-1}}| \, |\Delta W_{t_i}| - S_{t_{i-1}} |\Delta M_{t_i}|\right].$$

From Corollary 2.3.5, it follows that:

$$||C^n||_2 \leqslant k_n\varepsilon_n \, n^{\alpha/4-1/8} \sum_{i=1}^{n-1} \frac{\exp\left\{-n^{1/2-\alpha}(1-t_{i-1})/8\right\}}{(1-t_{i-1})^{1/4}} \sqrt{\Delta t_i}.$$

Since $|x|e^{-|x|}$ is bounded, we deduce that

$$||C^n||_2 \leqslant \widetilde{c}\,\varepsilon_n \int_0^{1-\frac{c}{n}} \frac{dt}{1-t} \leqslant \widetilde{c}\,\varepsilon_n \ln n$$

where c, $\widetilde{c}$ are some constants. Then, $||C^n||_2 \to 0$ as $n \to \infty$.

In the case of the assumption $(G2)$ and $\alpha = 1/2$, we can establish that

$$n^{1/2}\sqrt{f'(t_{i-1})\Delta t_i} = 1 + \varepsilon_i$$

where

$$|\varepsilon_i| \leqslant c\frac{\Delta t_i}{1-t_i}$$

and c is a constant. Then, we deduce that

$$||C^n||_2 \leqslant cn^{-1/2} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^{3/2}}{(1-t_i)^{5/4}} \leqslant c\frac{\ln n}{n^{3/4}}.$$

The remaining part is similar to the proof in [19]:

$$|D^n| \leqslant D_1^n + D_2^n$$

where

$$\begin{aligned}
D_1^n &:= k_n \sum_{i=1}^{n-1} S_{t_{i-1}} \left| \int_{t_{i-1}}^{t_i} (S_{t_{i-1}}\widehat{h}_{t_{i-1}} - S_u\widehat{h}_u)\sigma_u dW_u \right|, \\
D_2^n &:= k_n \sum_{i=1}^{n-1} \left| \int_{t_{i-1}}^{t_i} (\sigma_{t_{i-1}} - \sigma_u) S_{t_{i-1}}^2 \widehat{h}_{t_{i-1}} dW_u \right|.
\end{aligned}$$

We have $\|D_2^n\|_2 \to 0$ because of the assumption on σ whereas

$$E|D_1^n| \leqslant ck_n \sum_{i=1}^{n-1} \left(\int_{t_{i-1}}^{t_i} E(S_{t_{i-1}}\widehat{h}_{t_{i-1}} - S_u\widehat{h}_u)^2 du \right)^{1/2}.$$

By the Ito formula, we obtain that

$$d[S_t\widehat{h}_t] = d[S_t\widehat{C}_{xx}(t,S_t)] = f_t dW_t + g_t dt$$

where

$$\begin{aligned} f_t &:= \sigma_t S_t \widehat{C}_{xx}(t,S_t) + \sigma_t S_t^2 \widehat{C}_{xxx}(t,S_t) = \frac{\sigma_t}{\rho^2}\int_{-\infty}^{\infty} h'(\tilde{z})(y^2-1)\varphi(y)dy, \\ g_t &:= S_t\widehat{C}_{xxt}(t,S_t) + \frac{1}{2}\sigma_t^2 S_t^3 \widehat{C}_{xxxx}(t,S_t) + \sigma_t^2 S_t^2 \widehat{C}_{xxx}(t,S_t). \end{aligned}$$

Then,

$$E(S_{t_{i-1}}\widehat{h}_{t_{i-1}} - S_t\widehat{h}_t)^2 \leqslant 2\int_{t_{i-1}}^{t_i} Ef_u^2 du + 2\Delta t_i \int_{t_{i-1}}^{t_i} Eg_u^2 du.$$

From Lemma 2.3.8, it follows that

$$\begin{aligned} E|D_1^n| &\leqslant cn^{-1/2}\sum_{i=1}^{n-1}\frac{\Delta t_i}{1-t_i} + c\frac{k_n}{n^{1/2(1/2-\alpha)}}\sum_{i=1}^{n-1}\frac{(\Delta t_i)^{3/2}}{(1-t_i)^{3/2}} \\ &\leqslant cn^{-1/2}\ln n + c\frac{n^{-\alpha}\ln n}{n^{1/2(1/2-\alpha)}} \end{aligned}$$

where c is a constant. Then, $E|D_1^n|$ converge to zero.

Lemma 2.4.7. *For any $\alpha \in]0,1/2]$, we have P-$\lim_n L_4^n = 0$. For $\alpha = 0$, the sequence L_n^4 is bounded in probability.*

Proof. Using again the inequality $||a_1| - |a_2|| \leqslant |a_1 - a_2|$ we get that

$$\begin{aligned} |L_n^4| &\leqslant ck_n \sum_{i=1}^{n-1} S_{t_{i-1}}|\Delta A_{t_i}| \\ &\leqslant ck_n \int_0^1 |\widehat{C}_{xt}(u,S_u)|du + k_n\int_0^1 \sigma_u^2 S_u^2 |\widehat{C}_{xxx}(u,S_u)|du \end{aligned}$$

where c is a constant. Moreover,

$$\int_0^1 |\widehat{C}_{xt}(u,S_u)|du = \int_0^{\rho_0^2} |\widehat{C}_{xt}(u,S_u)|\widehat{\sigma}_u^{-2}dx,$$

where $u(x)$ is defined by $x = \rho_u^2$. Thus, by Lemma 2.3.8,

$$\int_0^1 |\widehat{C}_{xt}(u,S_u)|du \leqslant c\int_0^{\rho_0^2} G_1(x)dx$$

where

$$G_1(x) := \frac{1}{x}e^{-x/8}\left(\sum_{j=1}^{p}\frac{|\ln(S_u/K_j)|}{\sqrt{x}}\exp\left\{-\frac{\ln^2(S_u/K_j)}{2x}\right\} + \sqrt{x} + x\right).$$

Since $0 \leqslant 1-u \leqslant c\,x\,n^{\alpha-1/2}$, it follows that $u \to 1$ as $n \to \infty$ for $\alpha \in [0,1/2[$. We can apply the Lebesgue theorem by dominating the function whether $x \leqslant 1$ or not because $x \leqslant 1$ implies that u is sufficiently near from 1 independently of x for $n \geqslant n_0$. Indeed, outside of the null-set $\cup_i \{S_1 = K_i\}$, we have $0 < a \leqslant |\ln(S_u/K_j)| \leqslant b$ for some constants a, b (depending on ω) provided that u is sufficiently near unit.

For $\alpha = 1/2$, the majorant is independent of n but $k_n \to 0$. Thus,

$$k_n \int_0^1 |\widehat{C}_{xt}(u, S_u)| du \to 0 \text{ for } \alpha \in]0, 1/2].$$

The reasoning is similar to analyze the second term using Lemma 2.3.8:

$$\int_0^1 \sigma_u^2 S_u^2 |\widehat{C}_{xxx}(u, S_u)| du \leqslant c n^{\alpha-1/2} \int_0^{\rho_0^2} G_2(x) dx$$

where

$$G_2(x) := \frac{1}{x^{3/2}} \sum_{j=1}^{p} \exp\left\{-\frac{\ln^2(S_u/K_j)}{2x}\right\} + \frac{1}{\sqrt{x}} e^{-x/8}.$$

Thus,

$$k_n \int_0^1 \sigma_u^2 S_u^2 |\widehat{C}_{xxx}(u, S_u)| du \to 0.$$

Lemma 2.4.8. *For any $\alpha \in [0, 1/2]$ we have P-$\lim_n L_5^n = 0$.*

Proof. Since $\max_i |\Delta S_{t_i}| \to 0$ as $n \to \infty$, it suffices to verify that the sequence $k_n \sum_{i=1}^n |\Delta \widehat{H}_{t_i}|$ is bounded in probability. But this follows from the preceding lemmas.

Lemma 2.4.9. *$F_3^n \to \varepsilon_\alpha$ a.s. as $n \to \infty$.*

Proof. Only the case $\alpha \in [0, 1/2[$ needs to be considered. Needed arguments are based on the change of variables $x = \rho_t^2$ and the observation that $n^{1/2-\alpha}/\widehat{\sigma}_t^2$ converges to $\left(\sigma_1 k_0 \sqrt{8/\pi}\sqrt{f'(1)}\right)^{-1}$ for a fixed x.

Inspecting the formulations of above lemmas, we observe that all terms $L_j^n \to 0$ in probability when $\alpha \in]0, 1/2]$ and, hence, Theorem 2.2.2 is proven.

2.5 Constant Coefficient: Discrepancy

An inspection of the proof of Theorem 2.2.2 reveals that almost arguments hold also for $\alpha = 0$, i.e. when the transaction costs coefficient does not depend on the number of portfolio revisions, but in Lemma 2.4.7, in this case, we have non-trivial limits. This observation leads to the following result.

Theorem 2.5.1. *Let $k = k_0 \geqslant 0$ (i.e. $\alpha = 0$). Suppose that h is convex or concave and the assumptions (H) and (G1) hold. Then*

$$P\text{-}\lim_n V_1^n = h(S_1) + J_1 - J_2(k_0) + \varepsilon_0 \tag{2.5.26}$$

where J_1 is defined (as Lemma (2.4.4)) by the formula

$$\begin{aligned} J_1 &:= \frac{1}{2} S_1 \int_0^\infty |\theta_1(S_1, x)|\, dx, \\ \theta_1(S, x) &:= \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} h'(S e^{\sqrt{x}y + x/2}) y \varphi(y) dy \end{aligned}$$

and J_2 is defined by

$$J_2(k_0) := \frac{1}{2} S_1 \int_0^{\infty} j_2(S_1, x) dx$$

where

$$\begin{aligned} j_2(S,x) &:= |\theta_1(S,x)| \exp\left\{-\theta^2(S,x)/2\right\} + k_0 \left[2\Phi\left(\theta(S,x)\right) - 1\right] \theta_2(S,x), \\ \theta_2(S,x) &:= \frac{1}{x} \int_{-\infty}^{\infty} h'(Se^{\sqrt{x}y + x/2})(-y^2 - \sqrt{x}y + 1)\varphi(y) dy, \\ \theta(S,x) &:= k_0 \sqrt{\frac{2}{\pi}} \frac{\theta_2(S,x)}{\theta_1(S,x)}, \quad \Phi(x) = \int_{-\infty}^{x} \varphi(t) dt. \end{aligned}$$

Proof. In virtue of Lemmas 2.4.4 – 2.4.6 for $\alpha = 0$, the "chained" terms L_1^n, L_2^n, and L_3^n are differences of sequences of random variables converging to the common limit J_1. Thus, in our representation of L_4^n, the first component also converges to J_1 and it remains to check the convergence property for the second component, i.e.

$$k_0 \sum_{i=1}^{n-1} S_{t_{i-1}} |\widehat{H}_{t_i} - \widehat{H}_{t_{i-1}}| \to J_2(k_0). \tag{2.5.27}$$

We put

$$Z_i^n = \left|\sigma_{t_{i-1}} \lambda_i S_{t_{i-1}}^2 \widehat{h}_{t_{i-1}} \Delta W_{t_i} + S_{t_{i-1}} \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}}) \Delta t_i\right|$$

where

$$\lambda_i = \lambda_i^n = n^{1/2} \sqrt{\Delta t_i f'(t_{i-1})} = 1 + o_n(1),$$

and we represent the left-hand side of (2.5.27) as the sum $I_1^n + I_2^n + I_3^n$ with

$$\begin{aligned} I_1^n &:= k_0 \sum_{i=1}^{n-1} S_{t_{i-1}} |\widehat{H}_{t_i} - \widehat{H}_{t_{i-1}}| - k_0 \sum_{i=1}^{n-1} Z_i^n, \\ I_2^n &:= k_0 \sum_{i=1}^{n-1} [Z_i^n - E(Z_i^n | \mathcal{F}_{t_{i-1}})], \\ I_3^n &:= k_0 \sum_{i=1}^{n-1} E(Z_i^n | \mathcal{F}_{t_{i-1}}). \end{aligned}$$

Using the inequality $||a_1| - |a_2|| \leqslant |a_1 - a_2|$ and regrouping terms, we estimate I_1^n as follows:

$$\begin{aligned} |I_1^n| \leqslant\ & k_0 \sum_{i=1}^{n-1} S_{t_{i-1}} |\Delta M_{t_i} - \sigma_{t_{i-1}} \lambda_i S_{t_{i-1}} \widehat{h}_{t_{i-1}} \Delta W_{t_i}| \\ & + k_0 \sum_{i=1}^{n} S_{t_{i-1}} \left|\Delta A_{t_i} - \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}}) \Delta t_i\right|. \end{aligned}$$

The first sum above coincides with the majorant for $|L_3^n|$ which, as it was established in the proof of Lemma 2.4.6, converges to zero in probability. The second sum is dominated, up to a random but fixed multiplier, by

$$\int_0^1 |\widehat{C}_{xxx}(t, S_t)| dt + \int_0^1 \sum_{i=1}^{n-1} |\widehat{C}_{xt}(t, S_t) - \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}})| 1_{[t_{i-1}, t_i[}(t) dt.$$

As we have already shown in Lemma 2.4.7, the first integral converges to zero. The convergence of the second term to zero (of course, outside the null-set where S_1 takes one of the values K_i) follows by our usual arguments based on the change of variables $x = \rho_u^2$ and dominated convergence. Using the same consideration as in the proof of Lemma 2.4.5, we can show that $I_2^n \to 0$ in probability. Indeed, define the sequence

$$M_j^n = \sum_{i=1}^{j-1} [Z_i^n - E(Z_i^n | \mathcal{F}_{t_{i-1}})].$$

It is a martingale and for its quadratic characteristics we have the bound

$$\langle M^n, M^n \rangle_n \leqslant \sum_{i=1}^{n-1} \sigma_{t_{i-1}}^2 \lambda_i^2 S_{t_{i-1}}^4 \widehat{h}_{t_i}^2 \Delta t_i + \sum_{i=1}^{n-1} S_{t_{i-1}}^2 \widehat{C}_{xt}^2(t_{i-1}, S_{t_{i-1}})(\Delta t_i)^2.$$

The first sum in the right-hand side converges to zero in virtue of Lemma 2.4.5 while the second one converges to zero in L^1 in virtue of Lemma 2.3.9. By the Lenglart inequality I_2^n converges to zero in probability.

For $\xi \sim \mathcal{N}(0,1)$ and constants $\alpha > 0$, $\beta \in \mathbb{R}$, we have the formula

$$E|\alpha\xi + \beta| = \sqrt{\frac{2}{\pi}} \alpha e^{-\frac{\beta^2}{2\alpha^2}} + \beta[2\Phi(\beta/\alpha) - 1]$$

implying, due to the independence of increments of the Wiener process, the representation

$$I_3^n = k_0 \int_0^1 f_n(t)dt + k_0 \int_0^1 g_n(t)dt$$

where

$$\begin{aligned} f_n(t) &:= \sum_{i=1}^{n-1} f_n(t_{i-1}) I_{]t_{i-1}, t_i]}(t), \\ g_n(t) &:= \sum_{i=1}^{n-1} g_n(t_{i-1}) I_{]t_{i-1}, t_i]}(t) \end{aligned}$$

with

$$\begin{aligned} f_n(t_{i-1}) &:= \sqrt{\frac{2}{\pi}} \frac{\sigma_{t_{i-1}} \lambda_i S_{t_{i-1}}^2}{\sqrt{\Delta t_i}} |\widehat{h}_{t_{i-1}}| \exp\left\{ -\frac{\widehat{C}_{xt}^2(t_{i-1}, S_{t_{i-1}})\Delta t_i}{2\sigma_{t_{i-1}}^2 \lambda_i^2 S_{t_{i-1}}^2 \widehat{h}_{t_{i-1}}^2} \right\}, \\ g_n(t_{i-1}) &:= S_{t_{i-1}} \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}}) \left[2\Phi\left(\frac{\widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}})\sqrt{\Delta t_i}}{\sigma_{t_{i-1}} \lambda_i S_{t_{i-1}} \widehat{h}_{t_{i-1}}} \right) - 1 \right]. \end{aligned}$$

Using the change of variables $x = \rho_t^2$ and putting $x_i = \rho_{t_i}^2$, we have :

$$I_3^n = k_0 \int_0^{\rho_0^2} f_n(x)dx + k_0 \int_0^{\rho_0^2} g_n(x)dx$$

where

$$
\begin{aligned}
f_n(x) &= \sum_{i=1}^{n-1} f_n(x_{i-1}) I_{]x_{i-1},x_i]}(x), \\
g_n(x) &= \sum_{i=1}^{n-1} g_n(x_{i-1}) I_{]x_{i-1},x_i]}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
f_n(x_{i-1}) &= \sqrt{\frac{2}{\pi}} \frac{\sigma_{t_{i-1}} \lambda_i S_{t_{i-1}}}{\widehat{\sigma}_t^2 \sqrt{\Delta t_i}} \left| \theta_1(S_{t_{i-1}}, x_{i-1}) \right| \exp\left\{ -\widetilde{\theta}_i^2(S_{t_{i-1}}, x_{i-1})/2 \right\}, \\
g_n(x_{i-1}) &= \frac{\widehat{\sigma}_{t_{i-1}}^2 S_{t_{i-1}}}{2\widehat{\sigma}_t^2} \theta_2(S_{t_{i-1}}, x_{i-1}) \left[2\Phi\left(\widetilde{\theta}_i(S_{t_{i-1}}, x_{i-1}) \right) - 1 \right], \\
\widetilde{\theta}_i(S,x) &= \frac{\widehat{\sigma}_{t_{i-1}}^2 \theta_2(S,x) \sqrt{\Delta t_i}}{2\sigma_{t_{i-1}} \lambda_i \theta_1(S,x)}.
\end{aligned}
$$

Of course, there is an abuse of notation here since t_{i-1} and t are functions depending, respectively, on x_{i-1} and x but also on n. Note that $x \in [x_i, x_{i-1}]$ if and only if $t \in [t_{i-1}, t_i]$ where $|t_i - t_{i-1}| \leqslant an^{-1}$ for some constant a. Hence, we have also $|x_{i-1} - x_i| \leqslant cn^{-1/2}$. Moreover, the equality $x = \rho_t^2$ implies that $1 - t \leqslant cn^{-1/2}$ where c is a constant (recall that ρ depends on n). Then, for each fixed x , t_{i-1} and t converge to unit as $n \to \infty$.
Moreover, using the Taylor approximation, we can easily establish that

$$
\sqrt{\Delta t_i} \widehat{\sigma}_t^2 = \sigma_t^2 \sqrt{\Delta t_i} + \sigma_t k_0 \sqrt{\frac{8}{\pi}} \sqrt{\frac{g'(f(t_{i-1}))}{g'(f(t))}} + o_n(1).
$$

Then, for x fixed, $\sqrt{\Delta t_i} \widehat{\sigma}_t^2 \to \sigma_1 k_0 \sqrt{8/\pi}$ due to the uniform continuity of $g' \circ f$. Thus,

$$
f_n(x) \to \frac{1}{2k_0} S_1 \left| \theta_1(S_1, x) \right| \exp\left\{ -\frac{k_0^2}{\pi} \frac{\theta_2^2(S_1, x)}{\theta_1^2(S_1, x)} \right\}
$$

as $n \to \infty$. Since

$$
\frac{\widehat{\sigma}_{t_{i-1}}^2}{\widehat{\sigma}_t^2} = \frac{\widehat{\sigma}_{t_{i-1}}^2 (\Delta t_i)^{1/2}}{\widehat{\sigma}_t^2 (\Delta t_i)^{1/2}} \to 1,
$$

$$
g_n(x) \to \frac{1}{2} S_1 \theta_2(S_1, x) \left[2\Phi\left(k_0 \sqrt{\frac{2}{\pi}} \frac{\theta_2(S_1, x)}{\theta_1(S_1, x)} \right) - 1 \right].
$$

The most delicate point is to justify the domination of f_n and g_n to use the Lebesgue theorem. In particular, we have to add the convergence of the last functions because of the term $\theta_1(S_1, x)$. But we can assume that $\theta_1(S_1, x) \neq 0$. Indeed, suppose that $\theta_1(S_1, x) = 0$. Then,

$$
\int_{-\infty}^{\infty} h'(S_1 e^{\sqrt{x} y + x/2}) y \varphi(y) dy = 0,
$$

$$
\sum_{j=0}^{p} \left[-h'(S_1 e^{\sqrt{x} y + x/2}) \varphi(y) \right]_{\delta_j}^{\delta_{j+1}} + \int_{-\infty}^{\infty} S_1 \sqrt{x} e^{\sqrt{x} y + x/2} h''(S_1 e^{\sqrt{x} y + x/2}) dy = 0
$$

where

$$\delta_j = \frac{1}{\sqrt{x}} \ln \frac{K_j}{S_1} - \frac{\sqrt{x}}{2}$$

and, therefore, $K_j < S_1 e^{\sqrt{x}y + x/2} < K_{j+1}$ if and only if $y \in]\delta_j, \delta_{j+1}[$. Thus, if h is convex or concave then either h' is creasing and $h'' \geqslant 0$, or h' is decreasing and $h'' \leqslant 0$. So, each term above is either positive or negative. Then $\theta_1(S_1, x) = 0$ leads to $h'(K_{j-}) = h'(K_{j+})$ and $h'' = 0$ on $]K_j, K_{j+1}[$ (this means that $h(x) = ax + b$ for some constants a, b and $Z_i^n = 0$).

Justification for f_n. We have $\sqrt{\Delta t_i} \geqslant a n^{-1/2}$ and $\widehat{\sigma}_t^2 \geqslant b n^{1/2}$ for some constants a, b while $S_u(\omega)$ is bounded on $[0, 1]$. Otherwise, observing $f_n(t)$ and using (2.3.20), we can deduce that $|f_n(x)| \leqslant c\, e^{-x/8}/\sqrt{x}$.

Justification for g_n. The bounds $\widehat{\sigma}_t^2 \leqslant a\sqrt{n}$ and $\widehat{\sigma}_{t_{i-1}}^2 \geqslant b\sqrt{n}$ imply that the quotient of the two last terms is bounded. Inspecting $g_n(t)$ and using Lemma 2.3.8, we can write

$$|g_n(x)| \leqslant \frac{c}{x_{i-1}^{3/2}} e^{-x_{i-1}/8} \left(\sum_{j=1}^{p} \exp\left\{ -\frac{\ln^2(S_{t_{i-1}}/K_j)}{2x_{i-1}} \right\} + x_{i-1} + x_{i-1}^{3/2} \right)$$

for $x \in]x_i, x_{i-1}]$. Hence, $|g_n(x)| \leqslant c(x^{-3/2} e^{-x/8} + x^{-1/2} + 1)$ for $x \geqslant 1$.

For $x \leqslant 1$, the relation $x = \rho_t^2$ implies that $0 \leqslant 1 - t \leqslant c\, n^{-1/2}$ and $\ln^2(S_{t_{i-1}}/K_j) \geqslant \epsilon > 0$ outside of the null-set $\cup_i \{S_1 = K_i\}$ provided that $n \geqslant n_0$ and knowing that $|t - t_{i-1}| \leqslant |\Delta t_i| \leqslant c n^{-1}$. Thus,

$$|g_n(x)| \leqslant \frac{c}{x_{i-1}^{3/2}} e^{-x_{i-1}/8} \left(\sum_{j=1}^{p} \exp\left\{ -\frac{\epsilon}{2x_{i-1}} \right\} + x_{i-1} + x_{i-1}^{3/2} \right).$$

Using the fact that the function $y e^{-y}$ is bounded on $[0, \infty[$, we infer that

$$|g_n(x)| \leqslant c(x^{-1/2} + 1) \quad \text{for } x \leqslant 1.$$

Theorem 2.5.1 is proven.

Chapter 3

Mean Square Error for the Leland–Lott Hedging Strategy for General Pay-offs

In the previous chapter, we have seen that the Leland strategy produces a portfolio whose the terminal value converges in probability to the pay-off $h(S_1)$ if h is a convex function. For the case $\alpha = 1/2$, it was shown in [12] that it converges also in L^2 if the pay-off is $h(S_1) = (S_1 - K)^+$ and for non-uniform revision intervals. In this chapter, we show that this is always true for a more general contingent claim $h(S_1)$ and for $\alpha \in]0, 1/2]$.

3.1 Theorems

We assume that the model is the classical Black–Scholes model with transaction costs of Chapter 2. Although, we suppose that the volatility is constant: the risky asset is defined by the equation

$$dS_t = \sigma S_t dW_t$$

where W is a Wiener process.

Our objective is to extend the result, that we can find in [12], giving the rate of convergence of the mean square replication error. For this, we shall assume that the pay-off $h(S_1)$ is defined by the function h verifying the same conditions (H) as in Chapter 2. Furthermore, we shall obtain an interesting representation for the error process $n^{1/2}(V_t^n - \widehat{V}_t)$ (recall that $\widehat{V}_t = \widehat{C}(t, S_t)$) as a sum of a martingale and a residual term which uniformly tends to 0.

We note $\Lambda_t = E\widehat{C}^2_{xx}(t, S_t)S_t^4$ and

$$\Lambda(x) = \frac{1}{x}\int_{-\infty}^{\infty} e^{2\sigma z - \sigma^2} \left(\int_{-\infty}^{\infty} h'\left(e^{\sigma z - \frac{\sigma^2}{2} + \sqrt{x}y + \frac{x}{2}}\right) y\varphi(y)dy\right)^2 \varphi(z)dz.$$

Theorem 3.1.1. $\alpha = 1/2$. *Let $h(S_1)$ be the contingent claim where h is a convex function verifying the condition (H). Suppose that the assumptions (G1) or (G2) hold. Then, the mean square approximation error of the Leland-Lott strategy is such that:*

$$E(V_t^n - \widehat{V}_t)^2 = E(1/2)_t n^{-1} + o(n^{-1}), \qquad n \to \infty \tag{3.1.1}$$

with the coefficient

$$E(1/2)_t = \sigma^2 \int_0^t \left[\frac{2k_0\sigma\sqrt{2/\pi}\sqrt{f'(u)}+\sigma^2}{2f'(u)} + k_0^2\left(1-\frac{2}{\pi}\right)\right] \Lambda_u du.$$

Note that the convergence is uniform on $[0,1]$.

Corollary 3.1.2. *We have the following approximation for* $\alpha = 1/2$*:*

$$n^{1/2}(V_t^n - \widehat{V}_t) = M_t^n + \varepsilon_t^n$$

where $M_t^n = \sum_{t_i^n \leqslant t} Y_i^n + Z_i^n$ *is a martingale with*

$$\begin{aligned} Y_i^n &= \frac{\sigma^2}{2} n^{1/2}\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})S_{t_{i-1}}^2\left[\Delta t_i - (W_{t_{i-1}} - W_{t_i})^2\right], \\ Z_i^n &= k_0\sigma\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})S_{t_{i-1}}^2\left[\sqrt{\frac{2}{\pi}}\sqrt{\Delta t_i} - |W_{t_{i-1}} - W_{t_i}|\right], \end{aligned}$$

and $E(\sup_t \varepsilon_t^n)^2 \to 0$.

Let $p(\alpha)$ be such that $0 \leqslant p(\alpha) < \alpha$ for $\alpha < 1/2$. Then we have:

Theorem 3.1.3. $\alpha \in]0, 1/2[$. *Let* $h(S_1)$ *be the contingent claim where* h *is a convex function verifying the condition (H). Then, the mean square approximation error of the Leland-Lott strategy is such that:*

In the case where $g = g_\mu$, $\mu > 1$,

$$n^{1/2+\alpha}E(V_1^n - \widehat{V}_1)^2 \to 0. \tag{3.1.2}$$

Under the assumption (G_1),

$$n^{p(\alpha)}E(V_1^n - \widehat{V}_1)^2 \to 0. \tag{3.1.3}$$

3.2 Proof of Theorem 3.1.3

We recall the representation of the hedging error that we can find in [19]:

Lemma 3.2.1. *We have the equality* $V_t^n - \widehat{C}(t, S_t) = F_{1t}^n + F_{2t}^n$ *where*

$$\begin{aligned} F_{1t}^n &= \sigma \sum_{i=1}^n \int_{t_{i-1}\wedge t}^{t_i \wedge t} \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(u, S_u)\right) S_u dW_u, \\ F_{2t}^n &= \sigma k_0 \sqrt{\frac{2}{\pi}} n^{1/2-\alpha} \int_0^t \sqrt{f'(u)} S_u^2 \widehat{C}_{xx}(u, S_u) du \\ &\quad - k_0 n^{-\alpha} \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \left|\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})\right| S_{t_i} \end{aligned}$$

where $\hat{t}_{n-1}(t) = \max_{i \leqslant n-1}\{t_i : t_i \leqslant t\}$.

Inspired by the fact that

$$\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) \simeq \widehat{C}_x(u, S_u) + \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})(S_{t_{i-1}} - S_u),$$

we note:

$$\begin{aligned}
P^n_{1t} &= \sum_{i=1}^{n-1} \sigma \widehat{h}_{t_{i-1}} S^2_{t_{i-1}} \int_{t_{i-1}\wedge t}^{t_i \wedge t} \left(1 - \frac{S_u}{S_{t_{i-1}}}\right) \frac{S_u}{S_{t_{i-1}}} dW_u, \\
P^n_{2t} &= k_0 n^{-\alpha} \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \widehat{h}_{t_{i-1}} S^2_{t_{i-1}} \left[\sigma \sqrt{\frac{2}{\pi}} n^{1/2} \sqrt{f'(t_{i-1})} \Delta t_i - \left|\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right|\right].
\end{aligned}$$

Let $R^n_i = F^n_i - P^n_i$ for $i = 1, 2$. We have

$$V^n_t - \widehat{V}_t = P^n_{1t} + P^n_{2t} + R^n_{1t} + R^n_{2t}.$$

So, our objective is to show that

$$n^{1/2+\alpha} E(P^n_{1t} + P^n_{2t})^2 \to E(\alpha)_t$$

where $E(\alpha)_t$ is a coefficient depending on t and

$$nE(R^n_{it})^2 \to 0 \text{ as } n \to \infty.$$

From the Taylor formula, we can deduce that $\widehat{C}_x(t_{i-1}, S_{t_{i-1}})$ is equal to the following sum:

$$\begin{aligned}
&\widehat{H}_u + \widehat{h}_{t_{i-1}}(S_{t_{i-1}} - S_u) + \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}})(t_{i-1} - u) - \frac{1}{2}\widehat{C}_{xxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})(S_{t_{i-1}} - S_u)^2 \\
&- \widehat{C}_{xxt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})(S_{t_{i-1}} - S_u)(t_{i-1} - u) - \frac{1}{2}\widehat{C}_{xtt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})(t_{i-1} - u)^2
\end{aligned} \tag{3.2.4}$$

if $u \in [t_{i-1}, t_i[$, where $(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})$ is a random variable $\mathcal{F}_{t_i}$-measurable.

It follows that we have

$$R^n_1 = \sigma\left(R^n_{10} - R^n_{11} - R^n_{12} - R^n_{13} + 2R^n_{14}\right)$$

where:

$$\begin{aligned}
R^n_{10}(t) &= \int_{t_{n-1}\wedge t}^{t} \left(\widehat{C}_x(t_{n-1}, S_{t_{n-1}}) - \widehat{C}_x(u, S_u)\right) S_u dW_u \\
R^n_{11}(t) &= \sum_{i=1}^{n-1} \int_{t_{i-1}\wedge t}^{t_i\wedge t} \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}})(u - t_{i-1}) S_u dW_u, \\
R^n_{12}(t) &= \frac{1}{2}\sum_{i=1}^{n-1} S^3_{t_{i-1}} \int_{t_{i-1}\wedge t}^{t_i\wedge t} \widehat{C}_{xxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \left(1 - \frac{S_u}{S_{t_{i-1}}}\right)^2 \frac{S_u}{S_{t_{i-1}}} dW_u, \\
R^n_{13}(t) &= \frac{1}{2}\sum_{i=1}^{n-1} S_{t_{i-1}} \int_{t_{i-1}\wedge t}^{t_i\wedge t} \widehat{C}_{xtt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})(u - t_{i-1})^2 \frac{S_u}{S_{t_{i-1}}} dW_u, \\
R^n_{14}(t) &= \frac{1}{2}\sum_{i=1}^{n-1} S^2_{t_{i-1}} \int_{t_{i-1}\wedge t}^{t_i\wedge t} \widehat{C}_{xxt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \left(1 - \frac{S_u}{S_{t_{i-1}}}\right)(u - t_{i-1}) \frac{S_u}{S_{t_{i-1}}} dW_u.
\end{aligned}$$

In the same way, we write:

$$R_2^n = R_{20}^n + \cdots + R_{24}^n$$

where

$$\begin{aligned}
R_{20}^n(t) &= \sigma k_0 \sqrt{\frac{2}{\pi}} n^{1/2-\alpha} \int_{\hat{t}_{n-1}}^t S_u^2 \widehat{h}_u \sqrt{f'(u)} du, \\
R_{21}^n(t) &= \sigma k_0 n^{1/2-\alpha} \sqrt{\frac{2}{\pi}} \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \int_{t_{i-1}}^{t_i} S_u^2 \widehat{h}_u \sqrt{f'(u)} - S_{t_{i-1}}^2 \widehat{h}_{t_{i-1}} \sqrt{f'(t_{i-1})} du, \\
R_{22}^n(t) &= k_n \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \widehat{h}_{t_{i-1}} |S_{t_i} - S_{t_{i-1}}| (S_{t_{i-1}} - S_{t_i}), \\
R_{23}^n(t) &= k_n \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \Theta_i (S_{t_i} - S_{t_{i-1}}), \\
R_{24}^n(t) &= k_n \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \Theta_i S_{t_{i-1}}, \\
\Theta_i &= \widehat{h}_{t_{i-1}} |S_{t_i} - S_{t_{i-1}}| - |\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})|. \qquad (3.2.5)
\end{aligned}$$

3.2.1 Analyze of the Main Terms

Lemma 3.2.2. *We have the following uniform convergences :*

If $\alpha \in]0, 1/2[$, then

$$n^{1/2+\alpha} \sup_t E(P_{1t}^n)^2 \to 0.$$

If $\alpha = 1/2$, then

$$nE(P_{1t}^n)^2 \to \frac{\sigma^4}{2} \int_0^t \frac{\Lambda_u}{f'(u)} du,$$

where $\Lambda_t = E\widehat{C}_{xx}^2(t, S_t) S_t^4$ doesn't depend on n.

Proof. By the independence of the increments of the Wiener process, we have only:

$$E(P_{1t}^n)^2 = \sigma^2 \sum_{i=1}^{n-1} \Lambda_{t_{i-1}} \int_{t_{i-1} \wedge t}^{t_i \wedge t} E\left(1 - \frac{S_u}{S_{t_{i-1}}}\right)^2 \frac{S_u^2}{S_{t_{i-1}}^2} du$$

where:

$$E\left(1 - \frac{S_u}{S_{t_{i-1}}}\right)^2 \frac{S_u^2}{S_{t_{i-1}}^2} = \sigma^2 (u - t_{i-1}) + (u - t_{i-1}) O(n^{-1}).$$

It follows that:

$$E(P_{1t}^n)^2 = \frac{\sigma^4}{2} \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \Lambda_{t_{i-1}} (\Delta t_i)^2 (1 + O(n^{-1})) + \frac{\sigma^4}{2} \Lambda_{\hat{t}_{n-1}} (t - \hat{t}_{n-1})^2 (1 + O(n^{-1}))$$

where $\Delta t_i = g'(\theta_i)/n$ with $\theta_i \in [(i-1)/n, i/n]$. We deduce that

$$E(P_{1t}^n)^2 \sim \frac{\sigma^4}{2} \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \frac{\Lambda_{t_{i-1}}}{n f'(g(\theta_i))} \Delta t_i$$

and we can write:

$$nE(P_{1t}^n)^2 = \frac{\sigma^4}{2}\int_0^t f_n(u)du + o(1)$$

where

$$f_n(t) = \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \frac{\Lambda_{t_{i-1}}}{f'(g(\theta_i))} 1_{[t_{i-1},t_i[}(t)$$

In the case where $\alpha = 1/2$, the function Λ doesn't depend on n and verifies:

$$\Lambda_t \leqslant \frac{c}{\sqrt{1-t}}$$

where $c > 0$ is a constant. Moreover, f' is bounded from below. So, there exists a constant $\widetilde{M}$ such that

$$|f_n(t)| \leqslant \frac{\widetilde{M}}{\sqrt{1-t}}.$$

We can conclude, applying the Lebesgue theorem, that uniformly in t

$$nE(P_{1t}^n)^2 \to \frac{\sigma^4}{2}\int_0^t \frac{\Lambda_u}{f'(u)}du.$$

In the case where $\alpha < 1/2$,

$$n^{3/2-\alpha}E(P_{1t}^n)^2 = \frac{\sigma^4(1+o(n^{-1}))}{2}\sum_{t_i \leqslant \hat{t}_{n-1}(t)} \Lambda_{t_{i-1}}(\Delta t_i n)\frac{\Delta t_i n^{1/2-\alpha}}{x_{i-1}-x_i}(x_{i-1}-x_i) + o(1)$$

where $x_i = \rho_{t_i}^2$. So, we have:

$$n^{1/2+\alpha}\sup_t E(P_{1t}^n)^2 \leqslant n^{2\alpha-1}\frac{\sigma^4(1+o(n^{-1}))}{2}\int_0^\infty f_n(x)dx + o(1)$$

where

$$f_n(x) = \sum_{i=1}^{n-1} \Lambda_{t_{i-1}}(\Delta t_i n)\frac{\Delta t_i n^{1/2-\alpha}}{x_{i-1}-x_i}1_{]x_i,x_{i-1}]}(x).$$

Recall that

$$0 \leqslant \Lambda_{t_{i-1}} \leqslant \frac{c}{\sqrt{x_{i-1}}}e^{-x_{i-1}/4} \leqslant \frac{c}{\sqrt{x}}e^{-x/4}$$

where c is a constant and $x \in]x_i, x_{i-1}]$. For each fixed $x \in]x_i, x_{i-1}]$, $x = \rho_t^2 \geqslant cn^{1/2-\alpha}(1-t)$ where $t \in [t_{i-1}, t_i[$. It follows that not only $t \to 1$ but also $t_i, t_{i-1} \to 1$.

We have $\Delta t_i = g'(\theta_i)n^{-1}$ where $\theta_i \in [(i-1)/n, i/n]$. It follows that $g(\theta_i) \to 1$ and $\theta_i \to 1$ since f is continuous. So, $\Delta t_i n \to g'(1)$. Furthermore, we have

$$\frac{\Delta t_i n^{1/2-\alpha}}{x_{i-1}-x_i} = \left(\frac{\sigma^2}{n^{1/2-\alpha}} + \sigma k_0\sqrt{\frac{8}{\pi}}\frac{1}{\Delta t_i}\int_{t_{i-1}}^{t_i}\sqrt{f'(s)}ds\right)^{-1}$$

which converges to

$$\left(\sigma k_0\sqrt{\frac{8}{\pi}}\sqrt{f'(1)}\right)^{-1}.$$

Note that

$$\Lambda_{t_{i-1}} = \frac{1}{x_{i-1}} \int_{-\infty}^{\infty} e^{2\sigma\sqrt{t_{i-1}}z-\sigma^2 t_{i-1}} \Upsilon_i(z)\varphi(z)dz$$

where

$$\Upsilon_i(z) = \left(\int_{-\infty}^{\infty} h'\left(e^{\sigma\sqrt{t_{i-1}}z - \frac{\sigma^2 t_{i-1}}{2} + \sqrt{x_{i-1}}y + \frac{x_{i-1}}{2}} \right) y\varphi(y)dy \right)^2.$$

Applying the Lebesgue theorem, we deduce that $\Lambda_{t_{i-1}}$ converges to

$$\Lambda(x) = \frac{1}{x} \int_{-\infty}^{\infty} e^{2\sigma z - \sigma^2} \left(\int_{-\infty}^{\infty} h'\left(e^{\sigma z - \frac{\sigma^2}{2} + \sqrt{x}y + \frac{x}{2}} \right) y\varphi(y)dy \right)^2 \varphi(z)dz.$$

From now on, we can apply once again the Lebesgue theorem to conclude that

$$\frac{\sigma^4(1+O(n^{-1}))}{2} \int_0^{\infty} f_n(x)dx \to \frac{\sigma^3 g'(1)}{4k_0\sqrt{f'(1)}} \sqrt{\frac{\pi}{2}} \int_0^{\infty} \Lambda(x)dx$$

where

$$\Lambda(x) \leqslant \frac{c}{\sqrt{x}} e^{-x/4}.$$

It follows that

$$n^{1/2+\alpha} \sup_t E(P_{1t}^n)^2 \to 0.$$

Lemma 3.2.3. *For $\alpha \in]0, 1/2[$, we have the following convergences:*

$$n^{1/2+\alpha} E(P_{21}^n)^2 \to \frac{k_0\sigma\left(1-\frac{2}{\pi}\right)}{2\sqrt{f'(1)}} \sqrt{\frac{\pi}{2}} \int_0^{\infty} \Lambda(x)dx$$

$$n^{1/2+\alpha} E(P_{2t}^n)^2 \to 0, \; \forall t \in [0, 1[.$$

For $\alpha = 1/2$, the following convergence is uniform on $[0,1]$:

$$E(P_{2t}^n)^2 \to k_0^2\sigma^2 \left(1 - \frac{2}{\pi}\right) \int_0^t \Lambda_u du.$$

Proof. We write $P_2^n = A^n + B^n$ where

$$A_t^n = k_n \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \widehat{h}_{t_{i-1}} S_{t_{i-1}}^2 \left[\sigma\sqrt{\frac{2}{\pi}} n^{1/2} \sqrt{f'(t_{i-1})} \Delta t_i - G\left(\frac{\sigma\sqrt{\Delta t_i}}{2} \right) \right],$$

$$B_t^n = k_n \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \widehat{h}_{t_{i-1}} S_{t_{i-1}}^2 \left[G\left(\frac{\sigma\sqrt{\Delta t_i}}{2} \right) - \left| \frac{S_{t_i}}{S_{t_{i-1}}} - 1 \right| \right]$$

where $G(x) = 4\Phi(x) - 2$, i.e.

$$G\left(\frac{\sigma\sqrt{\Delta t_i}}{2} \right) = E\left| \frac{S_{t_i}}{S_{t_{i-1}}} - 1 \right|.$$

Moreover, we have

$$\begin{aligned} G\left(\frac{\sigma\sqrt{\Delta t_i}}{2}\right) &= \sigma\sqrt{\frac{2}{\pi}}\sqrt{\Delta t_i} + (\Delta t_i)o_n(1), \\ \sigma\sqrt{\frac{2}{\pi}}n^{1/2}\sqrt{f'(t_{i-1})}\Delta t_i &= \sigma\sqrt{\frac{2}{\pi}}\sqrt{\Delta t_i}\,\varepsilon_i \end{aligned}$$

where $\varepsilon_i = n^{1/2}\sqrt{\Delta t_i}\sqrt{f'(t_{i-1})}$ verifies

$$|\varepsilon_i - 1| \leqslant \frac{c\Delta t_i}{1-t_i}$$

because of $(G2)$ or $(G1)$ according to Lemma 3.3.3. So, we can write for some constant c:

$$\begin{aligned} \sup_t |A_t^n| &\leqslant ck_n \sum_{i=1}^{n-1} \widehat{h}_{t_{i-1}} S_{t_{i-1}}^2 \frac{(\Delta t_i)^{3/2}}{1-t_i}, \\ n^{1/2(1/2+\alpha)}\|\sup_t A_t^n\|_2 &\leqslant cn^{1/4(1/2-\alpha)} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^{3/2}}{(1-t_i)^{5/4}} \leqslant \frac{cn^{1/4(1/2-\alpha)}}{n^{1/4}}\ln n \to 0. \end{aligned}$$

We first analyze $B^n = B$ in the case $\alpha = 1/2$.

$$nEB_t^2 = k_0^2 \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \Lambda_{t_{i-1}} E\left[G\left(\frac{\sigma\Delta t_i}{2}\right) - \left|\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right|\right]^2$$

because of independence. But, it is easy to obtain that:

$$E\left[G\left(\frac{\sigma\sqrt{\Delta t_i}}{2}\right) - \left|\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right|\right]^2 = \left(1-\frac{2}{\pi}\right)\sigma^2\Delta t_i(1+o_n(1)).$$

It follows that, uniformly on $[0,1]$,

$$nEB_t^2 \to k_0^2\sigma^2\left(1-\frac{2}{\pi}\right)\int_0^t \Lambda_t dt.$$

If $\alpha < 1/2$, we use the change of variable $x = \rho_t^2$ and we obtain that

$$n^{1/2+\alpha}E(B_t^2) = (1+o_n(1))\int_{\rho_t^2}^{\rho_0^2} f_n(x)dx$$

where

$$f_n(x) = k_0^2\sigma^2\left(1-\frac{2}{\pi}\right)\sum_{i=1}^n \Lambda_{t_{i-1}} \frac{\Delta t_i n^{1/2-\alpha}}{x_{i-1}-x_i} 1_{]x_{i-1},x_i]}(x)$$

and $x_i = \rho_{t_i}^2$ with $t_i \leqslant t$. We have already shown that $\Lambda_{t_{i-1}} \to \Lambda(x)$ and:

$$\frac{\Delta t_i n^{1/2-\alpha}}{x_{i-1}-x_i} \to \left(\sigma k_0\sqrt{\frac{8}{\pi}}\sqrt{f'(1)}\right)^{-1}.$$

So, we can conclude. Indeed, if $t<1$, $\rho_t^2 \geqslant cn^{1/2-\alpha}(1-t)$ which implies that $\rho_t^2 \to \infty$, otherwise $\rho_1^2 = 0$.

Lemma 3.2.4. *For $\alpha = 1/2$, we have the following uniform convergence on $[0,1]$:*

$$nE(P_{1t}^n P_{2t}^n) \to \frac{\sigma^3 k_0}{2}\sqrt{\frac{2}{\pi}}\int_0^t \frac{\Lambda_u}{\sqrt{f'(u)}}du.$$

Proof. In virtue of the previous lemma, it suffices to analyze the convergence of $nEP_{1t}^n B_t^n$. For this, we note $D_t^n = P_{1t}^n B_t^n$.

$$ED_t^n = \sigma k_n \sum_{t_i \leqslant \hat{t}_{n-1}} \Lambda_{t_{i-1}} E\left(\int_{t_{i-1}}^{t_i}\left(1 - \frac{S_u}{S_{t_{i-1}}}\right)\frac{S_u}{S_{t_{i-1}}}dW_u\left[G\left(\frac{\sigma\Delta t_i}{2}\right) - \left|\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right|\right]\right).$$

Moreover, we can easily obtain that

$$E\left(\int_{t_{i-1}}^{t_i}\left(\frac{S_u}{S_{t_{i-1}}} - 1\right)\frac{S_u}{S_{t_{i-1}}}dW_u - \int_{t_{i-1}}^{t_i}\sigma(W_u - W_{t_{i-1}})dW_u\right)^2 = (\Delta t_i)^2 o_n(1),$$

$$E\left(\left|\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right| - \sigma|W_{t_i} - W_{t_{i-1}}|\right)^2 = e^{\sigma^2\Delta t_i} - 1 - \sigma^2\Delta t_i \leqslant c(\Delta t_i)^2.$$

Then, we deduce that

$$\begin{aligned} ED_t^n &= \frac{\sigma^3 k_n}{2}\sum_{t_i \leqslant \hat{t}_{n-1}(t)} \Lambda_{t_{i-1}} E\left(|\Delta W_{t_i}|^3 - |\Delta W_{t_i}|\Delta t_i\right)(1 + o_n(n^{-1})), \\ ED_t^n &= \frac{\sigma^3 k_n}{2}\sqrt{\frac{2}{\pi}}\sum_{t_i \leqslant \hat{t}_{n-1}(t)} \Lambda_{t_{i-1}}(\Delta t_i)^{3/2}(1 + o_n(n^{-1})) \end{aligned}$$

where

$$(\Delta t_i)^{1/2} = \sqrt{g'\left((i-1)/n\right)}n^{-1/2}(1 + o_n(1)).$$

Using the Lebesgue theorem, we can conclude that

$$nED_t^n \to \frac{\sigma^3 k_0}{2}\sqrt{\frac{2}{\pi}}\int_0^t \frac{\Lambda_u}{\sqrt{f'(u)}}du$$

uniformly on $[0,1]$.

Corollary 3.2.5. *For $\alpha < 1/2$,*

$$\begin{aligned} &n^{1/2+\alpha}E(P_{11}^n + P_{21}^n)^2 \to \frac{k_0\sigma\left(1 - 2/\pi\right)}{2\sqrt{f'(1)}}\sqrt{\frac{\pi}{2}}\int_0^\infty \Lambda(x)dx, \\ &n^{1/2+\alpha}E(P_{1t}^n + P_{2t}^n)^2 \to 0, \quad \forall t \in [0,1[\,. \end{aligned}$$

Corollary 3.2.6. *For $\alpha = 1/2$,*

$$nE(P_{1t}^n + P_{2t}^n)^2 \to \sigma^2\int_0^t\left[\frac{2k_0\sigma\sqrt{2/\pi}\sqrt{f'(u)} + \sigma^2}{2f'(u)} + k_0^2\left(1 - \frac{2}{\pi}\right)\right]\Lambda_u du$$

uniformly on $[0,1]$.

3.2.2 Analyse of the Residual Terms

Lemma 3.2.7. $n^{1/2+\alpha}E(\sup_t R^n_{10}(t))^2 \to 0$.

Proof. If $t < t_{n-1}$, $R^n_{10}(t) = 0$. Then,

$$\sup_t |R^n_{10}(t)| = \sup_{t\geqslant t_{n-1}} \left| \int_{t_{n-1}}^{t} \left(\widehat{C}_x(t_{n-1}, S_{t_{n-1}}) - \widehat{C}_x(u, S_u) \right) S_u dW_u \right|$$

and the Doob inequality leads to $E\sup_t (R^n_{10}(t))^2 \leqslant 4E(R^n_{10}(1))^2$. Then

$$n^{1/2+\alpha}E(\sup_t R^n_{10}(t))^2 \leqslant 4n^{1/2+\alpha} \int_{t_{n-1}}^{1} E\left(\widehat{C}_x(t_{n-1}, S_{t_{n-1}}) - \widehat{C}_x(t, S_t) \right)^2 S_t^2 dt.$$

Moreover, $|\widehat{C}_x(t_{n-1}, S_{t_{n-1}}) - \widehat{C}_x(t, S_t)|S_t$ is equal to

$$\left| \int_{-\infty}^{\infty} \left(h'(S_{t_{n-1}} e^{\rho_{t_{n-1}} y + \rho^2_{t_{n-1}}/2}) - h'(S_t e^{\rho_t y + \rho_t^2/2}) \right) \varphi(y) dy \right| S_t$$

which is dominated by

$$\kappa_n = \int_{-\infty}^{\infty} \sup_{t_{n-1}\leqslant t\leqslant 1} S_t \left| h'(S_{t_{n-1}} e^{\rho_{t_{n-1}} y + \rho^2_{t_{n-1}}/2}) - h'(S_t e^{\rho_t y + \rho_t^2/2}) \right| \varphi(y) dy.$$

The random variable κ_n converges almost surely to 0 out of the null-set $S_1 \in \{K_1, \cdots, K_p\}$ because of the continuity of h' and is bounded from above by $\widetilde{k}\sup_t S_t$ where $\widetilde{k}$ is a constant. Applying the Lebesgue theorem, we can conclude from the inequality

$$n^{1/2+\alpha}E(\sup_t R^n_{10}(t))^2 \leqslant const\, n^{\alpha-1/2} E\kappa_n^2.$$

Lemma 3.2.8. $n^{1/2+\alpha}E(\sup_t R^n_{11}(t))^2 \to 0$.

Proof. Using the Doob inequality, we obtain that $E(\sup_t R^n_{11}(t))^2 \leqslant 4E(R^n_{11}(1))^2$ and by independence of the increments of the Wiener process, we deduce that

$$n^{1/2+\alpha}E(R^n_{11}(1))^2 = n^{1/2+\alpha} \sum_{i=1}^{n-1} E\widehat{C}^2_{xt}(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} \int_{t_{i-1}}^{t_i} (u - t_{i-1})^2 E\left(\frac{S_u}{S_{t_{i-1}}} \right)^2 du,$$

$$n^{1/2+\alpha}E(R^n_{11}(1))^2 \leqslant cn^{1/2+\alpha} \sum_{i=1}^{n-1} E\widehat{C}^2_{xt}(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} (\Delta t_i)^3 \leqslant cn^{-1/4} \ln n,$$

since Lemma 3.3.5 gives

$$E\widehat{C}^2_{xt}(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} \leqslant c \frac{n^{1/2(1/2-\alpha)} f'(t_{i-1})}{(1 - t_{i-1})^{3/2}}$$

and $nf'(t_{i-1})\Delta t_i$ is bounded.

Lemma 3.2.9. $n^{1/2+\alpha}E(\sup_t R^n_{12}(t))^2 \to 0$.

As previously, we have the Doob inequality $E(\sup_t R^n_{12}(t))^2 \leqslant 4E(R^n_{12}(1))^2$ and

$$4E(R^n_{12}(1))^2 = \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} E\left(\widehat{C}^2_{xxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) S^6_{t_{i-1}} \left(1 - \frac{S_t}{S_{t_{i-1}}}\right)^4 \frac{S^2_t}{S^2_{t_{i-1}}}\right) dt$$

where, from Lemma 3.3.15, we recall that there exists a constant c such that:

$$\begin{aligned} E\widehat{C}^4_{xxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) &\leqslant \frac{c}{\rho^8_{t_i}} \epsilon(a), \\ E\left(1 - \frac{S_t}{S_{t_{i-1}}}\right)^{16} &\leqslant c(t_i - t_{i-1})^8, \end{aligned}$$

and $\epsilon(a) \to 0$ as $a \to 1$. Using the Cauchy-Schwartz inequality, we deduce that

$$\begin{aligned} n^{1/2+\alpha} E(R^n_{12}(1))^2 &\leqslant cn^{1/2+\alpha} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^3}{n^{2(1/2-\alpha)}(1-t_i)^2} \epsilon(a) \\ &\leqslant c\frac{n^{1/2+\alpha}}{n^{2(1/2-\alpha)}} \frac{\ln n}{n} \end{aligned}$$

which proves the convergence to 0 in the case $\alpha < 1/2$. Otherwise, we split the sum whether $t_i \geqslant a$ or not and we use the convergence $\epsilon(a) \to 0$ as $a \to 1$. Then, the most difficult part is to analyse in the case of the assumption $(G2)$ the inequality:

$$n \sum_{t_i \geqslant a} \frac{(\Delta t_i)^3}{f'(t_i)(1-t_i)^2} \epsilon(a) \leqslant \epsilon(a) \sum_{t_i \geqslant a} \frac{\Delta t_i}{f'(t_i)(1-t_i)}$$

where $f'(t_i) = 1/g'(f(t_i))$ is such that

$$g'(f(t_i)) \leqslant c_2 \left(1 - g(f(t_i))\right)^{k_2} \leqslant c_2(1-t_i)^{k_2}.$$

It follows that

$$n \sum_{t_i \geqslant a} \frac{(\Delta t_i)^3}{f'(t_i)(1-t_i)^2} \epsilon(a) \leqslant \epsilon(a) \sum_{t_i \geqslant a} \frac{\Delta t_i}{(1-t_i)^{1-k_2}}$$

and we can easily conclude.

Lemma 3.2.10. $n^{1/2+\alpha} E(\sup_u R^n_{13}(u))^2 \to 0.$

Proof. We have the Doob inequality $E(\sup_u R^n_{13}(u))^2 \leqslant 4E(R^n_{13}(1))^2$ and

$$4E(R^n_{13}(1))^2 \leqslant \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} E\left(\widehat{C}^2_{xtt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})(t - t_{i-1})^4 S^2_t\right) dt.$$

Moreover, using Lemma 3.3.14 and the Cauchy-Schwartz inequality, we deduce that

$$E\left(\widehat{C}^2_{xtt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) S^2_t\right) \leqslant \frac{c}{(1-t_i)^4}.$$

Then, we obtain that

$$n^{1/2+\alpha} E(R^n_{13}(1))^2 \leqslant cn^{1/2+\alpha} \sum_{i=1}^{n-1} \frac{(\Delta t_i)^5}{(1-t_i)^4} \leqslant c\frac{n^{1/2+\alpha}}{n} \ln n.$$

We can easily conclude in the case where $\alpha \neq 1/2$.

If $\alpha = 1/2$, we first consider the sum with $t_{i-1} \leqslant a$ choosing a sufficiently near to 1: we can easily conclude. If $t_{i-1} \geqslant a$, we know that

$$E\left(\widehat{C}^2_{xtt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})S_t^2\right) \leqslant \frac{c}{(1-t_i)^4}\varepsilon_a$$

where $\varepsilon_a \to 0$ as $a \to 1$. So, we can conclude if the assumption (G_1) holds.

Under the assumption (G_2), the reasoning is the same if $a \leqslant t_{i-1} \leqslant 1 - 1/n$. Indeed, we have to estimate:

$$n\varepsilon_a \sum_{a \leqslant t_{i-1} \leqslant 1-\frac{1}{n}} \frac{(\Delta t_i)^5}{(1-t_i)^4} \leqslant \frac{n\varepsilon_a}{n^4}\int_0^{1-1/n} \frac{dt}{(1-t)^4} \leqslant c\,\varepsilon_a.$$

Otherwise, if $t_{i-1} \geqslant 1 - 1/n$, we can write $\Delta t_i = g'(\theta_i)n^{-1}$ where $g'(\theta_i) \leqslant n^{-k_2}$ because of (G_2). Thus, we analyse the following sum:

$$n \sum_{t_{i-1} \geqslant 1-\frac{1}{n}} \frac{(\Delta t_i)^5}{(1-t_i)^4} \leqslant c\frac{\ln n}{n^{k_2}} \to 0.$$

Lemma 3.2.11. $n^{1/2+\alpha}E(\sup_u R^n_{14}(u))^2 \to 0$.

Proof. We have the Doob inequality $E(\sup_u R^n_{14}(u))^2 \leqslant 4E(R^n_{14}(1))^2$ and

$$4E(R^n_{14}(1))^2 = \sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i} E\left(S^4_{t_{i-1}}\widehat{C}^2_{xxt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})\left(1 - \frac{S_t}{S_{t_{i-1}}}\right)^2 (t-t_{i-1})^2\frac{S_t^2}{S^2_{t_{i-1}}}\right)dt.$$

From Lemma 3.3.16, we deduce that

$$E\left(S^4_{t_{i-1}}\widehat{C}^2_{xxt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}})\left(1 - \frac{S_t}{S_{t_{i-1}}}\right)^2 \frac{S_t^2}{S^2_{t_{i-1}}}\right) \leqslant c\frac{t-t_{i-1}}{(1-t_i)^3}.$$

Then,

$$n^{1/2+\alpha}E(R^n_{14}(1))^2 \leqslant cn^{1/2+\alpha}\sum_{i=1}^{n-1}\frac{(t_i-t_{i-1})^4}{(1-t_i)^3} \leqslant c\frac{n^{1/2+\alpha}}{n}\ln n.$$

Thus, we can conclude easily if $\alpha \neq 1/2$. Otherwise, we consider the case $t_{i-1} \geqslant a$ with a near to unit and we conclude with the same reasoning as the one used in the previous lemma, using Lemma 3.3.16.

Lemma 3.2.12. $n^{1/2+\alpha}E(\sup_t R^n_{20}(t))^2 \to 0$.

First, we prove that $n^{1/2+\alpha}(\sup_t R^n_{20}(t))^2$ is bounded from above by an integrable random variable. Indeed, we have

$$0 \leqslant R^n_{20}(t) \leqslant \frac{cn^{1/2-\alpha}}{n^{1/2(1/2-\alpha)}}\int_{\hat{t}_{n-1}(t)}^t \frac{S_u^{1/2}f'(u)^{1/4}}{\sqrt{1-u}}du$$

where

$$\int_{\hat{t}_{n-1}(t)}^t \frac{f'(u)^{1/4}du}{\sqrt{1-u}} \leqslant cn^{-1/2}$$

for a constant c. Indeed, it is clear if f' is bounded. In the case of assumption (G_2), we choose a fixed a sufficiently near to unit. If $t \leqslant a$, it is obvious that

$$\int_{\hat{t}_{n-1}(t)}^{t} \frac{f'(u)^{1/4} du}{\sqrt{1-u}} \leqslant c_a n^{-1/2}.$$

If $t \geqslant a$, we note $\hat{t}_{n-1} = t_{i-1}$ where $t \in [t_{i-1}, t_i[$. Using the change of variable $x = f(u)$, we have

$$\int_{t_{i-1}}^{t} \frac{f'(u)^{1/4} du}{\sqrt{1-u}} \leqslant \int_{(i-1)/n}^{i/n} \frac{g'(x)^{3/4}}{\sqrt{1-g(x)}} dx.$$

Note that $\hat{t}_{n-1}(t) = t_{i-1} \geqslant a - c/n$ where c is a constant since $t \geqslant a$ and $\Delta t_i \leqslant c/n$. Then, $(i-1)/n \geqslant f(a - c/n)$ which implies that $(i-1)/n$ is also near to unit as $a \to 1$. Because of (G_2), we deduce that

$$\int_{(i-1)/n}^{i/n} \frac{g'(x)^{3/4}}{\sqrt{1-g(x)}} dx \leqslant c \int_{(i-1)/n}^{i/n} \frac{dx}{\sqrt{1-x}} \leqslant \frac{c}{\sqrt{n}}.$$

Then, in all cases, there exists a constant c such that

$$n^{1/2+\alpha} (\sup_t R_{20}^n(t))^2 \leqslant c \sup_u S_u.$$

From now on, it suffices to prove that $n^{1/2+\alpha}(\sup_t R_{20}^n(t))^2 \to 0$ almost surely and to apply the Lebesgue theorem. But we can prove easily that $\sup_u S_u^2 \widehat{C}_{xx}(u, S_u) < \infty$ out of the null set $\{S_1 = K_1, \cdots, K_p\}$. So, there exists a.s. a constant $c(w)$ such that

$$n^{1/2+\alpha} (\sup_t R_{20}^n(t))^2 \leqslant \frac{c(w)}{n^{1/2+\alpha}}$$

if f' is bounded. Thus, we can conclude in the case of the assumption (G_1). Otherwise, we use the property of (G_2), $g'(f(t)) \geqslant c(1-t)^{k_1}$ if t is near to 1. It follows that if $t \geqslant a$, with a fixed a closed to unit, then

$$\int_{\hat{t}_{n-1}(t)}^{t} \sqrt{f'(t)} dt \leqslant c \int_{\hat{t}_{n-1}(t)}^{t} \frac{dt}{(1-t)^{k_1/2}} \leqslant \frac{c}{n^{1-k_1/2}}$$

where we recall that $k_1 < 1/2 + \alpha$. Indeed, it is clear if $t \geqslant t_{n-1}$. If $t < t_{n-1}$, we use the inequality $f'(t) \leqslant f'(t_{n-1}) \leqslant n^{\mu} \leqslant n^{k_1}$. Then,

$$n^{1/2+\alpha} (\sup_t R_{20}^n(t))^2 \leqslant \frac{c(w)}{n^{1/2+\alpha-k_1}}$$

and we can conclude.

Lemma 3.2.13. $n^{1/2+\alpha} E(\sup_t R_{21}^n(t))^2 \to 0.$

Proof. Let be $\Psi(t,x) = x^2 \widehat{C}_{xx}(t,x)\sqrt{f'(t)}$. The Ito formula give us

$$\begin{aligned} \Psi(t, S_t) = \quad & \Psi(t_{i-1}, S_{t_{i-1}}) + \int_{t_{i-1}}^{t} \frac{\partial \Psi}{\partial x}(u, S_u) \sigma S_u dW_u + \int_{t_{i-1}}^{t} \frac{\partial \Psi}{\partial t}(u, S_u) du \\ & + \frac{1}{2} \int_{t_{i-1}}^{t} \frac{\partial^2 \Psi}{\partial x^2}(u, S_u) \sigma^2 S_u^2 du, \end{aligned}$$

where

$$\begin{aligned}
\frac{\partial \Psi}{\partial t}(t,x) &= x^2\left[\widehat{C}_{xxt}(t,x)\sqrt{f'(t)}+\widehat{C}_{xx}(t,x)\frac{f''(t)}{2\sqrt{f'(t)}}\right],\\
\frac{\partial \Psi}{\partial x}(t,x) &= \left[2x\widehat{C}_{xx}(t,x)+x^2\widehat{C}_{xxx}(t,x)\right]\sqrt{f'(t)},\\
\frac{\partial^2 \Psi}{\partial x^2}(t,x) &= \left[2\widehat{C}_{xx}(t,x)+4x\widehat{C}_{xxx}(t,x)+x^2\widehat{C}_{xxxx}(t,x)\right]\sqrt{f'(t)}.
\end{aligned}$$

If we note $X_t = S_t^2\widehat{C}_{xx}(t,x)\sqrt{f'(t)}$ then $dX_t = \mu_t dt + \beta_t dW_t$ where

$$\begin{aligned}
\mu_t &= \frac{\partial \Psi}{\partial t}(t,S_t)+\frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(t,S_t)\sigma^2 S_t^2,\\
\beta_t &= \frac{\partial \Psi}{\partial x}(t,S_t)\sigma S_t.
\end{aligned}$$

We write $n^{\frac{1}{2}(\frac{1}{2}+\alpha)}R_{21}^n(t) = A_t^n + B_t^n$ with

$$\begin{aligned}
A_t^n &= n^{\frac{1}{2}(\frac{1}{2}+\alpha)}\sigma k_0 n^{\frac{1}{2}-\alpha}\sqrt{\frac{2}{\pi}}\sum_{t_i\leqslant \hat{t}_{n-1}(t)}\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{t}\beta_u dW_u\right)dt,\\
B_t^n &= n^{\frac{1}{2}(\frac{1}{2}+\alpha)}\sigma k_0 n^{\frac{1}{2}-\alpha}\sqrt{\frac{2}{\pi}}\sum_{t_i\leqslant \hat{t}_{n-1}(t)}\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{t}\mu_u du\right)dt.
\end{aligned}$$

From Lemma 3.3.8, there exists a constant c such that:

$$E\beta_t^2 \leqslant c\left(ES_t^4\widehat{h}_t^2 + ES_t^6\widehat{C}_{xxx}^2(t,S_t)\right)f'(t) \leqslant \frac{cf'(t)}{n^{\frac{3}{2}(\frac{1}{2}-\alpha)}(1-t)^{\frac{3}{2}}}.$$

Using hypothesis (G_1) or (G_2) , we claim that there exists a constant $\widetilde{c}$ such that

$$\frac{|f''(t)|}{\sqrt{f'(t)}} \leqslant \widetilde{c}\,\frac{\sqrt{f'(t)}}{1-t}.$$

Thus, we obtain, for some constant c, the following inequality:

$$E\mu_t^2 \leqslant \frac{cf'(t)}{n^{\frac{1}{2}(\frac{1}{2}-\alpha)}(1-t)^{\frac{5}{2}}}. \tag{3.2.6}$$

By the stochastic Fubini Theorem, we obtain that

$$A_t^n = n^{\frac{1}{2}(\frac{1}{2}+\alpha)}\sigma k_0 n^{\frac{1}{2}-\alpha}\sqrt{\frac{2}{\pi}}\sum_{t_i\leqslant \hat{t}_{n-1}(t)}\int_{t_{i-1}}^{t_i}(t_i-u)\beta_u dW_u.$$

Moreover, we have the Doob inequality $E\left(\sup_t A_t^n\right)^2 \leqslant 4E\left(A_1^n\right)^2$ where, from the boundedness of $\sqrt{(t_i-u)/(1-u)}$ and $f'(u)(t_i-u)n$ if $u\in[t_{i-1},t_i[$, we deduce the following estimates:

$$\begin{aligned}
E\left(A_1^n\right)^2 &\leqslant cn^{3/2-\alpha}\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}(t_i-u)^2 E\beta_u^2 du,\\
E\left(A_1^n\right)^2 &\leqslant cn^{3/2-\alpha}\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}\frac{(t_i-u)^2 f'(u)}{n^{3/2(1/2-\alpha)}(1-u)^{3/2}}du,\\
E\left(A_1^n\right)^2 &\leqslant \frac{c}{n^{1/2(1/2-\alpha)}}\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}\frac{(t_i-u)}{(1-u)^{3/2}}du \leqslant c\frac{\ln n}{\sqrt{n}} \to 0.
\end{aligned}$$

Then, we can conclude that $E\left(\sup_t A_t^n\right)^2 \to 0$.

Secondly, we write:

$$\begin{aligned}
B_t^n &= cn^{3/4-\alpha/2}\sum_{t_i\leqslant \hat{t}_{n-1}(t)}\int_{t_{i-1}}^{t_i}\mu_u\int_{t_{i-1}}^{t_i} I_{t\geqslant u}dtdu,\\
B_t^n &= cn^{3/4-\alpha/2}\sum_{t_i\leqslant \hat{t}_{n-1}(t)}\int_{t_{i-1}}^{t_i}(t_i-u)\mu_u du.
\end{aligned}$$

Then,

$$\sup_t |B_t^n| \leqslant cn^{3/4-\alpha/2}\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}(t_i-u)|\mu_u|du.$$

It follows that there exists a constant c such that $E\sup_t |B_t^n|^2 \leqslant cn^{3/2-\alpha}\Upsilon^n$ where

$$\begin{aligned}
\Upsilon^n &= E\left(\int_0^1\sum_{i=1}^{n-1}(t_i-u)|\mu_u| I_{]t_{i-1},t_i]}(u)du\right)^2,\\
\Upsilon^n &= E\int_0^1\int_0^1\sum_{i,j=1}^{n-1}(t_i-u)(t_j-v)|\mu_u||\mu_v| I_{]t_{i-1},t_i]}(u) I_{]t_{j-1},t_j]}(v)dudv.
\end{aligned}$$

Using the Cauchy–Schwartz inequality and 3.2.6, we get that

$$\begin{aligned}
\Upsilon^n &\leqslant \int_0^1\int_0^1\sum_{i,j=1}^{n-1}(t_i-u)(t_j-v)\left(E\mu_u^2\right)^{1/2}\left(E\mu_v^2\right)^{1/2} I_{]t_{i-1},t_i]}(u) I_{]t_{j-1},t_j]}(v)dudv,\\
\Upsilon^n &\leqslant \left(\int_0^1\sum_{i=1}^{n-1}(t_i-u)\left(E\mu_u^2\right)^{1/2} I_{]t_{i-1},t_i]}(u)du\right)^2,\\
\Upsilon^n &\leqslant const\left(\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}\frac{\sqrt{(t_i-u)f'(u)n}\sqrt{(t_i-u)n^{-1/2}}}{n^{1/4(1/2-\alpha)}(1-u)^{5/4}}du\right)^2,\\
\Upsilon^n &\leqslant \frac{const}{n}\left(\sum_{i=1}^{n-1}\frac{(\Delta t_i)^{3/2}}{(1-t_i)^{5/4}}\right)^2 \leqslant \frac{const}{n^{3/2}}\left(\sum_{i=1}^{n-1}\frac{\Delta t_i}{1-t_i}\right)^2,\\
\Upsilon^n &\leqslant \frac{const}{n^{3/2}}\ln^2 n.
\end{aligned}$$

Then, we can deduce that $E\sup_t |B_t^n|^2 \to 0$ and the result follows.

Lemma 3.2.14. $n^{1/2+\alpha}E(\sup_t R_{22}^n(t))^2 \to 0$.

Proof. We write

$$-R_{22}^n(t) = k_n \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \widehat{h}_{t_{i-1}} S_{t_{i-1}}^2 \chi_i = M^n(t) + N^n(t)$$

where M^n is a martingale defined by

$$\begin{aligned} M^n(t) &= k_n \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \widehat{h}_{t_{i-1}} S_{t_{i-1}}^2 \left[\chi_i - E\chi_i\right], \\ \chi_i &= \left(\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right)^2 sign\left(\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right) \end{aligned}$$

and

$$N^n(t) = k_n \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \widehat{h}_{t_{i-1}} S_{t_{i-1}}^2 E\chi_i.$$

Note that there exists a constant $k \geqslant 0$ such that

$$E\chi_i = k(\Delta t_j)^{3/2}\left(1 + o(n^{-1/4})\right).$$

Indeed, $\left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1\right)^2 sign\left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1\right)$ has the same law than

$$\left(\exp\left\{\sigma\sqrt{\Delta t_j}\xi - \sigma^2\Delta t_j/2\right\} - 1\right)^2 \left(1_{\xi \geqslant \sigma\sqrt{\Delta t_j}/2} - 1_{\xi \leqslant \sigma\sqrt{\Delta t_j}/2}\right)$$

where ξ is the standard Gaussian variable so that ξ and $-\xi$ has the same law. It follows that

$$E\left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1\right)^2 sign\left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1\right)$$

is equal to

$$E\left[\left(e^{u\xi - u^2/2} - 1\right)^2 - \left(e^{-u\xi - u^2/2} - 1\right)^2\right] 1_{\xi \geqslant u/2} - E\left(e^{-u\xi - u/2} - 1\right)^2 1_{|\xi| \leqslant u/2}$$

where $u = \sigma\sqrt{\Delta t_j}$. Moreover,

$$E\left(e^{-u\xi - u^2/2} - 1\right)^2 1_{|\xi| \leqslant u/2} \leqslant u^4$$

whereas, from [19], we recall that

$$E\left[\left(e^{u\xi - u^2/2} - 1\right)^2 - \left(e^{-u\xi - u^2/2} - 1\right)^2\right] 1_{\xi \geqslant u/2} = \frac{2}{\sqrt{2\pi}}u^3 + O(u^4).$$

We can deduce that for n sufficiently large, we have $0 \leqslant E\chi_i \leqslant c(\Delta t_i)^{3/2}$. From the Doob inequality, we have $E\left(\sup_t M^n(t)\right)^2 \leqslant 4E\left(M^n(1)\right)^2$. Moreover, the independence of the increments of the Brownian motion implies that

$$E\left(M^n(1)\right)^2 = k_n^2 \sum_{i=1}^{n-1} E\widehat{C}_{xx}^2(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^4 E\left(\chi_i - E\chi_i\right)^2.$$

Then, there exists a constant c such that

$$n^{1/2+\alpha}E\left(M^n(1)\right)^2 \leqslant \frac{c}{n^{1/4}}.$$

At last, for n large enough , $E\chi_i \geqslant 0$. Hence, $0 \leqslant \sup_t N^n(t) \leqslant N^n(1)$. In order to prove that $n^{1/2+\alpha}EN^n(1)^2 \to 0$, we first analyse the following sum

$$n^{1/2+\alpha}k_n^2\sum_{i=1}^{n-1}E\widehat{C}^2_{xx}(t_{i-1},S_{t_{i-1}})S^4_{t_{i-1}}(E\chi_i)^2 \leqslant \frac{c}{n^{7/4}}$$

where c a constant. Using the Cauchy-Schwartz inequality, we also have

$$n^{1/2-\alpha}\sum_{t_i<t_j\leqslant t_{n-1}}E\widehat{h}_{t_{i-1}}S^2_{t_{i-1}}\widehat{h}_{t_{j-1}}S^2_{t_{j-1}}E\chi_iE\chi_j \leqslant \frac{c}{n^{1/2}}.$$

Then, we deduce that $n^{1/2+\alpha}EN^n(1)^2 \to 0$ and we conclude that

$$n^{1/2+\alpha}E(\sup_t R^n_{22}(t))^2 \to 0.$$

Lemma 3.2.15. $n^{1/2+\alpha}E(\sup_t R^n_{23}(t))^2 \to 0$.

Proof. We observe that $\sup_t |R^n_{23}(t)|$ is dominated by

$$k_n\sum_{i=1}^{n-1}\left|\widehat{C}_x(t_i,S_{t_i}) - \widehat{C}_x(t_{i-1},S_{t_{i-1}}) - \widehat{h}_{t_{i-1}}(S_{t_i}-S_{t_{i-1}})\right||S_{t_i}-S_{t_{i-1}}|.$$

Applying 3.2.4 with $t = t_i$, $i = 1,\cdots,n-1$, it is sufficient to estimate the following sums 3.2.7, $\cdots$,3.2.10. First, from Lemma 3.3.5, we have:

$$E\widehat{C}^2_{xt}(t_{i-1},S_{t_{i-1}})(\Delta t_i)^2(S_{t_i}-S_{t_{i-1}})^2 \leqslant c\frac{(\Delta t_i)^3 n^{\frac{1}{2}(\frac{1}{2}-\alpha)}f'(t_{i-1})^{1/4}}{(1-t_i)^{3/2}}$$

which leads to

$$n^{\frac{1}{2}(\frac{1}{2}+\alpha)}\left\|k_n\sum_{i=1}^{n-1}\widehat{C}_{xt}(t_{i-1},S_{t_{i-1}})(\Delta t_i)(S_{t_i}-S_{t_{i-1}})\right\|_2 \leqslant c\frac{1}{n^{1/8}} \to 0. \tag{3.2.7}$$

Secondly, from Lemma 3.3.15, we have

$$E\widehat{C}^2_{xxx}(\widetilde{t}_{i-1},\widetilde{S}_{t_{i-1}})(S_{t_i}-S_{t_{i-1}})^6 \leqslant \frac{c(\Delta t_i)^3}{n^{2(1/2-\alpha)}(1-t_{i-1})^2}$$

and we deduce that

$$n^{\frac{1}{2}(\frac{1}{2}+\alpha)}\left\|k_n\sum_{i=1}^{n-1}\widehat{C}_{xxx}(\widetilde{t}_{i-1},\widetilde{S}_{t_{i-1}})(S_{t_i}-S_{t_{i-1}})^3\right\|_2 \leqslant c\frac{\ln n}{n^{1/2}} \to 0. \tag{3.2.8}$$

Thirdly, from Lemma 3.3.16, we have

$$E\widehat{C}^2_{xxt}(\widetilde{t}_{i-1},\widetilde{S}_{t_{i-1}})(S_{t_i}-S_{t_{i-1}})^4(\Delta t_i)^2 \leqslant \frac{c(\Delta t_i)^4}{(1-t_i)^3}$$

and it follows that

$$(3.2.9) \qquad n^{\frac{1}{2}(\frac{1}{2}+\alpha)}\left\|k_n\sum_{i=1}^{n-1}\widehat{C}_{xxt}(\widetilde{t}_{i-1},\widetilde{S}_{t_{i-1}})(S_{t_i}-S_{t_{i-1}})^2\Delta t_i\right\|_2 \leqslant c\frac{\ln n}{n^{1/4}} \to 0.$$

Finally, from Lemma 3.3.14, we have

$$E\widehat{C}^2_{xtt}(\widetilde{t}_{i-1},\widetilde{S}_{t_{i-1}})(S_{t_i}-S_{t_{i-1}})^2(\Delta t_i)^4 \leqslant \frac{c(\Delta t_i)^5}{(1-t_i)^4}$$

and

$$(3.2.10) \qquad n^{\frac{1}{2}(\frac{1}{2}+\alpha)}\left\|k_n\sum_{i=1}^{n-1}\widehat{C}_{xtt}(\widetilde{t}_{i-1},\widetilde{S}_{t_{i-1}})(S_{t_i}-S_{t_{i-1}})(\Delta t_i)^2\right\|_2 \leqslant c\frac{\ln n}{n^{1/4}} \to 0.$$

Lemma 3.2.16. *Assume that* $\alpha \in]0,1/2[$.

If the revision function is $g_b(t) = 1-(1-t)^b$, $b>1$, *we have*

$$n^{1/2(1/2+\alpha)}E(\sup_t R^n_{24}(t))^2 \to 0.$$

If the assumption (G_1) *holds, we have only*

$$n^{p(\alpha)}E(\sup_t R^n_{24}(t))^2 \to 0$$

where $p(\alpha)<\alpha$.

Proof. We first suppose that the revision dates are defined by the functions g_b. We can claim that $\sup_t |R^n_{24}(t)|$ is bounded by the random variable

$$k_n\sum_{i=1}^{n-1}\left|\widehat{C}_x(t_i,S_{t_i})-\widehat{C}_x(t_{i-1},S_{t_{i-1}})-\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}})\left(S_{t_i}-S_{t_{i-1}}\right)\right|S_{t_{i-1}}.$$

Using the Ito formula for the increments $\widehat{C}_x(t_i,S_{t_i})-\widehat{C}_x(t_{i-1},S_{t_{i-1}})$, we obtain that

$$\begin{aligned}\sup_t|R^n_{24}(t)| \leqslant\ & k_n \sum_{i=1}^{n-1} S_{t_{i-1}}\left|\int_{t_{i-1}}^{t_i}\sigma S_u\left[\widehat{C}_{xx}(u,S_u)-\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}})\right]dW_u\right.\\ (3.2.11) \qquad & + \left.\int_{t_{i-1}}^{t_i}\left[\widehat{C}_{xt}(u,S_u)+\frac{1}{2}\sigma^2S_u^2\widehat{C}_{xxx}(u,S_u)\right]du\right|.\end{aligned}$$

We deduce that

$$n^{1/2(1/2+\alpha)}\|\sup_t R^n_{24}(t)\|_2 \leqslant T^1_n + T^2_n$$

where

$$T^1_n = \sigma k_0 n^{1/2(1/2-\alpha)}\sum_{i=1}^{n-1}\left(\int_{t_{i-1}}^{t_i}ES^2_{t_{i-1}}S^2_u\left(\widehat{h}_u-\widehat{h}_{t_{i-1}}\right)^2du\right)^{1/2}$$

and

$$T^2_n = k_0 n^{1/2(1/2-\alpha)}\sum_{i=1}^{n-1}(\Delta t_i)^{1/2}\left(\int_{t_{i-1}}^{t_i}ES^2_{t_{i-1}}\left(\widehat{C}_{xt}(u,S_u)+\frac{1}{2}\sigma^2S_u^2\widehat{C}_{xxx}(u,S_u)\right)^2du\right)^{1/2}.$$

We shall prove that $T_n^1 \to 0$. Using the Taylor Formula, we deduce that $\widehat{C}_{xx}(u, S_u) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})$ is equal to

$$\widehat{C}_{xx}(u, S_u) - \widehat{C}_{xx}(u, S_{t_{i-1}}) + \widehat{C}_{xx}(u, S_{t_{i-1}}) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})$$

which can be written as

$$\widehat{C}_{xxx}(u, S_{t_{i-1}})(S_u - S_{t_{i-1}}) + \frac{1}{2}\widehat{C}_{xxxx}(u, \widetilde{S}_{t_{i-1}})(S_u - S_{t_{i-1}})^2 + \widehat{C}_{xxt}(\widetilde{t}_{i-1}, S_{t_{i-1}})(u - t_{i-1}).$$

Using estimations from Appendix, we obtain that there exists a constant c such that

$$ES_{t_{i-1}}^2 S_u^2 \left(\widehat{C}_{xx}(u, S_u) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})\right)^2$$

is dominated by the sum

$$\frac{c\Delta t_i}{n^{\frac{7}{4}(1/2-\alpha)}(1-t_i)^{\frac{7}{4}}} + \frac{c(\Delta t_i)^2}{n^{3(1/2-\alpha)}(1-t_i)^3 f'(t_i)^{3/2}}\varepsilon_a + \frac{c(\Delta t_i)^2}{n^{3/2(1/2-\alpha)}(1-t_i)^{11/4}}$$

where $\varepsilon_a \to 0$ as $a \to 1$. The last estimate comes from Lemma 3.3.12. Indeed, the proof is the same because $\rho_{\widetilde{t}_{i-1}} \leqslant \rho_{t_{i-1}}$.

Then, we can easily deduce that $T_n^1 \to 0$ since we assume $\alpha < 1/2$.

We shall prove that $T_n^2 \to 0$. We have from Appendix the following inequalities:

$$\begin{aligned} ES_{t_{i-1}}^2 \widehat{C}_{xt}^2(u, S_u) &\leqslant \frac{c\,(f'(t_i))^{1/8}\, n^{1/4(1/2-\alpha)} e^{-c\rho_{t_i}^2}}{(1-t_i)^{7/4}}, \\ ES_{t_{i-1}}^2 S_u^4 \widehat{C}_{xxx}^2(u, S_u) &\leqslant \frac{c}{n^{7/4(1/2-\alpha)}(1-t_i)^{7/4}} \end{aligned}$$

where $c > 0$ is a constant. So, it follows that $T_n^2 \to 0$. Indeed, we have to examine the two following sums. First,

$$n^{1/2(1/2-\alpha)} \sum_{i=1}^{n-1} \frac{\Delta t_i}{n^{7/8(1/2-\alpha)}(1-t_i)^{7/8}} \leqslant \frac{const}{n^{3/8(1/2-\alpha)}} \to 0.$$

Secondly, we have to analyze the sum

$$n^{5/8(1/2-\alpha)} \sum_{i=1}^{n-1} \frac{\Delta t_i f'(t_i)^{1/16} e^{-cn^{1/2-\alpha} f'(t_i)^{1/2}(1-t_i)}}{(1-t_i)^{7/8}}.$$

We strike the latter on two parts. The first contains the terms verifying $t_i \leqslant a$ where a is chosen sufficiently near to 1. The convergence to 0 is easy to check. For the second part, since we assume that $g = g_b$, we deduce from $\Delta t_i = g_b'(\theta_i)/n$ with $\theta_i \in [(i-1)/n, i/n]$ that

$$\Delta t_i \leqslant (1 - \frac{i-1}{n})^{b-1}\frac{1}{n}.$$

Moreover,

$$\begin{aligned} 1 - t_i &= (1 - \frac{i}{n})^b, \\ f'(t_i) &= (1-t_i)^{1/b-1} = (1-i/n)^{1-b}. \end{aligned}$$

Then, it suffices to analyse

$$S_n = n^{5/8(1/2-\alpha)} \sum_{t_i \geqslant a} \frac{e^{-cn^{1/2-\alpha} f'(t_i)^{1/2}(1-\frac{i-1}{n})^b}}{(1-\frac{i-1}{n})^{15/16(1-b)}}$$

verifying

$$S_n \leqslant c\, \frac{n^{5/8(1/2-\alpha)}}{n^{15/16(1/2-\alpha)}} \sum_{t_i \geqslant a} \frac{1}{(1-\frac{i-1}{n})} \frac{1}{n}.$$

Indeed, we use the boundedness of $|X|e^{-|X|}$. Then, the convergence to 0 is guaranteed. However, in the case of the assumption G_1, the reasoning is the same but we can't use the deceleration of g.

Lemma 3.2.17. *For $\alpha = 1/2$, $nE(\sup_t R_{24}^n(t))^2 \to 0$.*

Proof. We write

$$-n^{1/2} R_{24}^n(t) = k_0 \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \gamma_i$$

where $\gamma_i = |\alpha_i + \beta_i| - |\alpha_i|$ and, using the Taylor Formula,

$$\begin{aligned}
\alpha_i &= \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \left(S_{t_i}/S_{t_{i-1}} - 1\right), \\
\beta_i &= S_{t_{i-1}} \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}}) \Delta t_i + \frac{1}{2} S_{t_{i-1}}^3 \widehat{C}_{xxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \left(S_{t_i}/S_{t_{i-1}} - 1\right)^2 + \\
&\quad S_{t_{i-1}}^2 \widehat{C}_{xxt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \left(S_{t_i}/S_{t_{i-1}} - 1\right) \Delta t_i + S_{t_{i-1}} \widehat{C}_{xtt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) (\Delta t_i)^2.
\end{aligned}$$

Then, we get that $-n^{1/2} R_{24}^n(t) = A^n(t) + B^n(t)$ with

$$\begin{aligned}
A^n(t) &= \sum_{t_i \leqslant \hat{t}_{n-1}(t)} \gamma_i - E\left(\gamma_i | \mathcal{F}_{t_{i-1}}\right), \\
B^n(t) &= \sum_{t_i \leqslant \hat{t}_{n-1}(t)} E\left(\gamma_i | \mathcal{F}_{t_{i-1}}\right).
\end{aligned}$$

First, we prove that $E \sup_t \left(A^n(t)\right)^2 \to 0$. By the Doob inequality, we have

$$E \sup_t \left(A^n(t)\right)^2 \leqslant 4E\left(A^n(1)\right)^2 \leqslant 4 \sum_{i=1}^{n-1} E\gamma_i^2.$$

Recall that

$$\alpha_i + \beta_i = \left(\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})\right) S_{t_{i-1}}.$$

Then, using the inequality $||a| - |b|| \leqslant |a - b|$ and the Ito formula for the last increment, we deduce that $E \sup_t \left(A^n(t)\right)^2 \leqslant const\,(E_1^n + E_2^n)$ where

$$\begin{aligned}
E_1^n &= \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} E S_{t_{i-1}}^2 S_u^2 \left(\widehat{C}_{xx}(u, S_u) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})\right)^2 du, \\
E_2^n &= \sum_{i=1}^{n-1} \Delta t_i \int_{t_{i-1}}^{t_i} E S_{t_{i-1}}^2 \left(\widehat{C}_{xt}(u, S_u) + \frac{1}{2}\sigma^2 S_u^2 \widehat{C}_{xxx}(u, S_u)\right)^2 du.
\end{aligned}$$

In order to prove that $E_1^n \to 0$, we apply the Taylor Formula to $\widehat{C}_{xx}(u, S_u) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})$ as in the previous Lemma. Using estimations from Appendix, we deduce that

$$ES_{t_{i-1}}^2 S_u^2 \left(\widehat{C}_{xx}(u, S_u) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})\right)^2$$

is dominated by:

$$\frac{c\Delta t_i}{(1-t_i)^{\frac{7}{4}}} + \frac{const\,(\Delta t_i)^2}{(1-t_i)^3 f'(t_i)^{3/2}}\varepsilon_a + \frac{const\,(\Delta t_i)^2}{(1-t_i)^{11/4}}$$

where $\varepsilon_a \to 0$ as $a \to 1$. The last estimate comes from Lemma 3.3.12. Indeed, the proof is the same because $\rho_{\tilde{t}_{i-1}} \leqslant \rho_{t_{i-1}}$. Then, we have to analyse the following sums where c is a constant: first,

$$\sum_{i=1}^{n-1} \frac{(\Delta t_i)^2}{(1-t_i)^{\frac{7}{4}}} \leqslant \frac{c\ln n}{n^{1/4}} \to 0.$$

Secondly, we examine the sum

$$\sum_{i=1}^{n-1} \frac{(\Delta t_i)^3}{(1-t_i)^3 f'(t_i)^{3/2}}\varepsilon_a.$$

We first deal with the terms verifying $t_i \leqslant a$ for a fixed a chosen sufficiently near to the unit. Thus, the convergences to 0 is ensured. For $t_i \geqslant a$ and $\alpha = 1/2$, we have,

$$\sum_{t_i \geqslant a} \frac{(\Delta t_i)^3}{(1-t_i)^3 f'(t_i)^{3/2}}\varepsilon_a \leqslant c\,\varepsilon_a$$

in the case of (G_1), which converges to 0 as $a \to 1$. In the case of the assumption (G_2), we have, with a near to 1

$$f'(t_i) = \frac{1}{g'(f(t_i))} \geqslant \frac{c}{((1-g(f(t_i)))^{k_2}} \geqslant \frac{c}{(1-t_i)^{k_2}}$$

where $0 < k_2 < 1$. It follows that

$$\sum_{t_i \geqslant a} \frac{(\Delta t_i)^3}{(1-t_i)^3 f'(t_i)^{3/2}}\varepsilon_a \leqslant \frac{c\ln n}{n^{3/2k_2}} \to 0.$$

Thirdly,

$$\sum_{i=1}^{n-1} \frac{c\,(\Delta t_i)^3}{(1-t_i)^{11/4}} \leqslant \frac{c\ln n}{n^{1/4}} \to 0.$$

At last, $E_2^n \to 0$. Indeed, we have

$$E_2^n \leqslant c\sum_{i=1}^{n-1} \frac{(\Delta t_i)^2 f'(t_i)^{1/8}}{(1-t_i)^{7/4}} \leqslant \frac{c}{n^{1/4}} \to 0.$$

From now on, we shall prove that $E\sup_t (B^n(t))^2 \to 0$. For this, we note that

$$\sup_t (B^n(t))^2 \leqslant \mathcal{X}_1^n + \mathcal{X}_2^n$$

where

$$\mathcal{X}_1^n = \sum_{i=1}^{n-1} \left(E(\gamma_i|\mathcal{F}_{t_{i-1}})\right)^2,$$
$$\mathcal{X}_2^n = 2 \sum_{1\leqslant i<j\leqslant n-1} \left|E(\gamma_i|\mathcal{F}_{t_{i-1}})\right| \left|E(\gamma_j|\mathcal{F}_{t_{j-1}})\right|.$$

Since $E\mathcal{X}_1^n \leqslant \sum_{i=1}^{n-1} E\gamma_i^2$, it suffices to use the previous estimations in order to prove that $E\mathcal{X}_1^n \to 0$. Moreover, $\mathcal{X}_2^n = T_1^n + T_2^n$ where, for a fixed a near to 1,

$$T_1^n = 2 \sum_{1\leqslant i<j\leqslant n-1; t_j>a} \left|E(\gamma_i|\mathcal{F}_{t_{i-1}})\right| \left|E(\gamma_j|\mathcal{F}_{t_{j-1}})\right|,$$
$$T_2^n = 2 \sum_{1\leqslant i<j\leqslant n-1; t_j\leqslant a} \left|E(\gamma_i|\mathcal{F}_{t_{i-1}})\right| \left|E(\gamma_j|\mathcal{F}_{t_{j-1}})\right|.$$

We shall prove that $ET_1^n \to 0$. Using the Cauchy–Schwarz Inequality, we obtain that

$$ET_1^n \leqslant c \sum_{1\leqslant i<j\leqslant n-1; t_j>a} \sqrt{E\gamma_i^2}\sqrt{E\gamma_j^2} \leqslant c \left(\sum_{i=1}^{n-1} \sqrt{E\gamma_i^2}\right) \left(\sum_{t_j>a} \sqrt{E\gamma_j^2}\right).$$

Moreover, $\sqrt{E\gamma_i^2}$ is dominated by three terms that we can deduce from the previous analyse of E_1^n and E_2^n:

$$\sqrt{E\gamma_i^2} \leqslant \frac{\Delta t_i f'(t_i)^{1/16}}{(1-t_i)^{\frac{7}{8}}} + \frac{(\Delta t_i)^{3/2}\varepsilon_a}{(1-t_i)^{3/2} f'(t_i)^{3/4}} + \frac{(\Delta t_i)^{3/2}}{(1-t_i)^{11/8}}.$$

Then, it suffices to estimate the following sums.

First, the sums where the following term Σ_1, verifying

$$\Sigma_1 = \sum_{i=1}^{n-1} \frac{\Delta t_i f'(t_i)^{1/16}}{(1-t_i)^{7/8}} \leqslant const \sum_{i=1}^{n-1} \frac{\Delta t_i}{(1-t_i)^{7/8+k_1/16}} \leqslant const,$$

appears in the development of the product dominating ET_1^n. They correspond with:

$$\Sigma_1 \sum_{t_j\geqslant a} \frac{\Delta t_j f'(t_j)^{1/16}}{(1-t_j)^{7/8}} \leqslant \widetilde{\varepsilon}(a),$$
$$\Sigma_1 \sum_{t_j\geqslant a} \frac{(\Delta t_j)^{3/2}\varepsilon_a}{(1-t_j)^{3/2} f'(t_j)^{3/4}} \leqslant \widetilde{\varepsilon}(a),$$
$$\Sigma_1 \sum_{t_j\geqslant a} \frac{(\Delta t_j)^{3/2}}{(1-t_j)^{11/8}} \leqslant \Sigma_1 \sum_{t_j\geqslant a} \frac{\Delta t_j}{(1-t_j)^{7/8}} \leqslant \widetilde{\varepsilon}(a)$$

where $\widetilde{\varepsilon}(a)$ is a function verifying $\widetilde{\varepsilon}(a) \to 0$ as $a \to 1$.

Secondly, for the sums where the following term Σ_2, verifying

$$\Sigma_2 = \sum_{i=1}^{n-1} \frac{(\Delta t_i)^{3/2}}{(1-t_i)^{3/2} f'(t_i)^{3/4}} \leqslant const$$

appears, the reasoning is the same as the previous one since we obtain analogous inequalities. Thirdly, the conclusion is the same where

$$\Sigma_3 = \sum_{i=1}^{n-1} \frac{(\Delta t_i)^{3/2}}{(1-t_i)^{11/8}} \leqslant const$$

appears.

From now on, we shall prove that $ET_2^n \to 0$. Using the Cauchy–Schwarz inequality, we get that

$$ET_2^n \leqslant const \sum_{1\leqslant i<j\leqslant n-1, t_{j-1}<a} \sqrt{E\beta_i^2}\sqrt{EE(\gamma_j|\mathcal{F}_{t_{j-1}})^2}.$$

But, we have

$$|\alpha_j+\beta_j| - |\alpha_j| = |\beta_j| sgn(\alpha_j\beta_j) + 2\left(|\beta_j| - |\alpha_j|\right) 1_{\alpha_j\beta_j\leqslant 0, |\alpha_j|\leqslant|\beta_j|}$$

and

$$|E(\gamma_j|\mathcal{F}_{t_{j-1}})| \leqslant |E(\beta_j sgn\alpha_j|\mathcal{F}_{t_{j-1}})| + 2E(|\beta_j| 1_{|\beta_j|\geqslant|\alpha_j|}|\mathcal{F}_{t_{j-1}}).$$

Note that $sgn\alpha_j = sgn(S_{t_{i-1}}/S_{t_i} - 1)$ since $\widehat{C}_{xx}(t,x) \geqslant 0$. In virtue of Lemma 3.3.18, we obtain the following inequalities

$$\begin{aligned}
|E(\beta_j sgn\alpha_j|\mathcal{F}_{t_{j-1}})| &\leqslant \frac{cS_{t_{j-1}}^{1/2}}{(1-t_j)^2}(\Delta t_j)^{3/2}, \\
E\beta_j^2 &\leqslant \frac{c(\Delta t_j)^2}{(1-t_j)^2}, \\
E(\beta_j^2|\mathcal{F}_{t_{j-1}})^{1/2} &\leqslant \frac{cS_{t_{j-1}}^{1/2}\Delta t_j}{1-t_j}
\end{aligned}$$

where c is a constant. Then, using the Cauchy–Schwarz inequality, we deduce that

$$E(|\beta_j| 1_{|\beta_j|\geqslant|\alpha_j|}|\mathcal{F}_{t_{j-1}}) \leqslant E(\beta_j^2|\mathcal{F}_{t_{j-1}})^{1/2} P\left(|\beta_j| \geqslant |\alpha_j| \big| \mathcal{F}_{t_{j-1}}\right)^{1/2}.$$

Moreover, $|\beta_j| \leqslant \widetilde{\beta}_j$ where

$$\widetilde{\beta}_j = cS_{t_{j-1}}^{1/2}\left(\widetilde{\beta}_j^a + \widetilde{\beta}_j^b + \widetilde{\beta}_j^c\right)$$

is defined using Lemma 2.3.8 with

$$\begin{aligned}
\widetilde{\beta}_j^a &= \frac{\Delta t_j}{1-t_j} + \frac{1}{1-t_j}\left(1 + \frac{S_{t_{j-1}}^{5/2}}{S_{t_j}^{5/2}}\right)\left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1\right)^2, \\
\widetilde{\beta}_j^b &= \frac{1}{(1-t_j)^{3/2}}\left(1 + \frac{S_{t_{j-1}}^{3/2}}{S_{t_j}^{3/2}}\right)\left|\frac{S_{t_j}}{S_{t_{j-1}}} - 1\right|\Delta t_j \\
\widetilde{\beta}_j^c &= \left(1 + \frac{S_{t_{j-1}}}{S_{t_j}}\right)\left(\frac{\Delta t_j}{1-t_j}\right)^2.
\end{aligned}$$

Then,

$$P(|\beta_j| \geqslant |\alpha_j| | \mathcal{F}_{t_{j-1}}) \leqslant P\left(|\widetilde{\beta}_j| \geqslant |\alpha_j| | \mathcal{F}_{t_{j-1}}\right)^{1/2} \leqslant l(S_{t_{j-1}})$$

where

$$\begin{aligned} l(x) &= P\Big(c_1 u^2 + c_2(1+\eta_u^{-5/2})(\eta_u - 1)^2 + c_3(1+\eta_u^{-3/2})|\eta_u - 1|u^2 \\ &\quad + c_4(1+\eta_u^{-1/2})u^4 \geqslant \widehat{C}_{xx}(t_{j-1}, x)x^{3/2}|\eta_u - 1|\Big) \end{aligned}$$

and

$$\begin{aligned} c_1 &= c_2 = \frac{c}{1-t_j},\ c_3 = \frac{c}{(1-t_j)^{3/2}},\ c_4 = \frac{c}{(1-t_j)^2} \\ u &= \sigma\sqrt{\Delta t_j},\ \eta_u \sim e^{u\xi - u^2/2},\ \xi \sim \mathcal{N}(0,1) \end{aligned}$$

with c a constant. We note

$$C = 4(c_1 + \cdots + c_4), \qquad C_5(x) = \frac{C}{\widehat{C}_{xx}(t_{j-1}, x)x^{3/2}}.$$

Note that we can assume $S_t \in [1/m, m]$, $\forall t$ for m large enough because

$$P\left(\forall t\,, S_t < 1/m \text{ or } S_t > m\right) \to 0$$

as $m \to \infty$. Since we suppose that $t_{j-1} < a$, we can assume that there exists N_m such that $C_5 < N_m$. We can deduce from Lemma 3.3.19 that $l(x) \leqslant L(N_m)u$ for $x \in [1/m, m]$.

We note

$$A_m = \{\forall t\,, S_t \in [1/m, m]\}\,.$$

We have the following inequality:

$$ET_2^n \leqslant const\,(A_n + B_n + C_n)$$

where

$$\begin{aligned} A_n &= \sum_{t_i < t_j \leqslant a} \frac{\Delta t_i (\Delta t_j)^{3/2}}{(1-t_i)(1-t_j)^2} \leqslant \frac{const(a)}{n^{1/2}} \to 0, \\ B_n &= \sum_{t_i < t_j \leqslant a} \frac{\Delta t_i \Delta t_j}{(1-t_i)(1-t_j)} \sqrt{E\left(\sup_t S_t\, I_{A_m^c}\right)} \leqslant const(a) E\left(\sup_t S_t\, I_{A_m^c}\right), \\ C_n &= \sum_{t_i < t_j \leqslant a} \frac{\Delta t_i (\Delta t_j)^{3/2}}{(1-t_i)(1-t_j)} const(m) \leqslant \frac{const(a,m)}{n^{1/2}}. \end{aligned}$$

From now on, it suffices to fix m large enough to conclude that $ET_2^n \to 0$ and finally $n^{1/2+\alpha} \sup_t E(R_{24}^n(t))^2 \to 0$ for $\alpha = 1/2$.

3.2.3 Proof of Corollary 3.1.2

From the previous analyze, we deduce the following approximation for $\alpha = 1/2$:

$$n^{1/2}(V_t^n - \widehat{V}_t) = M_t^n + \varepsilon_t^n$$

where $M_t^n = \sum_{t_i^n \leqslant t} \widetilde{Y}_i^n + \widetilde{Z}_i^n$ is defined by

$$\begin{aligned}
\widetilde{Y}_i^n &= \sigma n^{1/2} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \int_{t_{i-1}}^{t_i} \left(1 - \frac{S_t}{S_{t_{i-1}}}\right) \frac{S_t}{S_{t_{i-1}}} dW_t, \\
\widetilde{Z}_i^n &= k_0 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \left[G\left(\frac{\sigma\sqrt{\Delta t_i}}{2}\right) - \left|\frac{S_{t_i}}{S_{t_{i-1}}} - 1\right|\right]
\end{aligned}$$

and $E(\sup_t \varepsilon_t^n)^2 \to 0$. Indeed, we note $\widetilde{P}_1^n(t) = \sum_{t_i^n \leqslant t} \widetilde{Y}_i^n$ and we recall that

$$P_1^n(t) = n^{1/2} \sum_{i=1}^{n-1} \sigma \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \int_{t_{i-1}\wedge t}^{t_i \wedge t} \left(1 - \frac{S_t}{S_{t_{i-1}}}\right) \frac{S_t}{S_{t_{i-1}}} dW_t.$$

We observe that

$$\sup_t (P_1^n(t) - \widetilde{P}_1^n(t))^2 \leqslant 2\sigma^2(\theta^n + \xi^n)$$

where

$$\begin{aligned}
\theta^n &= n \sup_i \sup_{t\in[t_{i-1},t_i[} \widehat{h}_{t_{i-1}}^2 S_{t_{i-1}}^4 R_i^2(t), \\
\xi^n &= \frac{n}{4} \sup_i \sup_{t\in[t_{i-1},t_i[} \widehat{h}_{t_{i-1}}^2 S_{t_{i-1}}^4 \left(t - t_{i-1} - (W_t - W_{t_{i-1}})^2\right)^2
\end{aligned}$$

and

$$R_i(t) = \int_{t_{i-1}}^t \left(\frac{S_t}{S_{t_{i-1}}} - 1\right) \frac{S_t}{S_{t_{i-1}}} dW_t - \int_{t_{i-1}}^t \sigma(W_t - W_{t_{i-1}}) dW_t.$$

In the proof of Lemma 3.2.4, we have shown that $ER_i^2(t_i) \leqslant (\Delta t_i)^2 o_n(1)$. Using the Doob inequality and the independence of the increments of the Wiener process, we deduce that

$$E\theta^n \leqslant n \sum_{i=1}^n \frac{1}{\sqrt{1 - t_{i-1}}} (\Delta t_i)^2 o_n(1)$$

so that $E\theta^n \to 0$.

We write $\xi^n \leqslant (\xi_a^n + \xi_b^n)/4$ with

$$\begin{aligned}
\xi_a^n &= n \sup_i \sup_{t\in[t_{i-1},t_i[} \widehat{h}_{t_{i-1}}^2 S_{t_{i-1}}^4 \left(\Delta^i t - (\Delta^i W_t)^2\right)^2 I_{\overline{\Delta}^i W > K\sqrt{\Delta t_i}}, \\
\xi_b^n &= n \sup_i \sup_{t\in[t_{i-1},t_i[} \widehat{h}_{t_{i-1}}^2 S_{t_{i-1}}^4 \left(\Delta^i t - (\Delta^i W_t)^2\right)^2 I_{\overline{\Delta}^i W \leqslant K\sqrt{\Delta t_i}},
\end{aligned}$$

where $K > 0$ is a constant, $\Delta^i t = t - t_{i-1}$, $\Delta^i W_t = W_t - W_{t_{i-1}}$ and

$$\overline{\Delta}^i W = \sup_{t\in[t_{i-1},t_i[} \left|\Delta^i W_t\right|.$$

We will show that we can make $E\xi_a^n$ arbitrary small provided that K is large enough. Indeed, using the independence, the Cauchy–Schwarz and Bienaymé–Tchebychev inequalities, we deduce that

$$E\xi_a^n \leqslant \frac{const}{K} n \sum_{i=1}^{n} \frac{1}{\sqrt{1-t_{i-1}}} (\Delta t_i)^2 \leqslant \frac{const}{K}$$

recalling that

$$E \sup_{t\in[t_{i-1},t_i[} \left(\int_{t_{i-1}}^{t} (W_t - W_{t_{i-1}}) dW_t \right)^4 \leqslant const (\Delta t_i)^4$$

because of the Burkholder–Davis–Gundy inequalities [20].

Then, with a fixed K large enough, we shall prove that $E\xi_b^n \to 0$. For this, we observe that a.s. $\xi_b^n \to 0$. Indeed, the Levy modulus [20] ensures that a.s.(ω),

$$\max_i |\Delta W_{t_i}| \leqslant const(\omega) \frac{\ln n}{n^{1/2}}$$

for n large enough. Moreover, the singularity generated by $\widehat{h}_{t_{i-1}}$ disappears in the neighborhood of the unit, out of the null-set $S_1 \in \{K_1, \cdots, K_p\}$. But, we also have the inequality $\xi_b^n \leqslant const(K) \sup_t S_t$. Then, we can conclude applying the Lebesgue theorem. Thus, we can replace P_1^n by $\widetilde{P}_1^n$. In a similar way, we can substitute

$$\int_{t_{i-1}}^{t_i} \left(\frac{S_t}{S_{t_{i-1}}} - 1 \right) \frac{S_t}{S_{t_{i-1}}} dW_t$$

for

$$\int_{t_{i-1}}^{t_i} \sigma (W_t - W_{t_{i-1}}) dW_t = \frac{\sigma}{2} (\Delta W_{t_i})^2 - \frac{\sigma \Delta t_i}{2}.$$

A last, we define

$$T_i = G\left(\frac{\sigma\sqrt{\Delta t_i}}{2} \right) - \left| \frac{S_{t_i}}{S_{t_{i-1}}} - 1 \right| - \sigma \sqrt{\frac{2}{\pi}} \sqrt{\Delta t_i} + \sigma |W_{t_{i-1}} - W_{t_i}|.$$

In virtue of the proofs of Lemmas 3.2.3 and 3.2.4, we have $ET_i^2 \leqslant c(\Delta t_i)^2$ for a constant c. Moreover,

$$\varrho^n(t) = \sum_{t_i^n \leqslant t} k_0 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 T_i$$

is a martingale. Thus, $E\left(\sup_t \varrho^n(t)\right)^2 \leqslant 4E\left(\varrho^n(1)\right)^2$ with $E\left(\varrho^n(1)\right)^2 \to 0$. It follows that we can replace $\widetilde{Z}_i^n$ by Z_i^n.

3.3 Appendix

We give here some necessary calculus and inequalities for the present work. Although, we use some results that we can find in Chapter 2. In particular, we can show the next one in a similar way.

Lemma 3.3.1. *We have:*

$$\begin{aligned}
\widehat{C}_{xxt}(t,x) &= \frac{\widehat{\sigma}_t^2}{2\rho_t^3 x}\int_{-\infty}^{\infty} h'(xe^{\rho_t y+\rho_t^2/2})P_1(\rho_t,y)\varphi(y)dy,\\
\widehat{C}_{xtt}(t,x) &= -\frac{\rho_t''}{\rho_t}\int_{-\infty}^{\infty} h'(xe^{\rho_t y+\rho_t^2/2})P_2(\rho_t,y)\varphi(y)dy\\
&\quad +\frac{\widehat{\sigma}_t^4}{2\rho_t^4}\int_{-\infty}^{\infty} h'(xe^{\rho_t y+\rho_t^2/2})P_3(\rho_t,y)\varphi(y)dy,\\
\widehat{C}_{xxxt}(t,x) &= \frac{\widehat{\sigma}_t^2}{2\rho_t^4 x^2}\int_{-\infty}^{\infty} h'(xe^{\rho_t y+\rho_t^2/2})P_4(\rho_t,y)\varphi(y)dy
\end{aligned}$$

where

$$\begin{aligned}
P_1(x,y) &= -y^3-xy^2+3y+x,\\
P_2(x,y) &= -y^2-xy+1,\\
P_3(x,y) &= y^4-(4+x^2)y^2+2xy+x^2+1,\\
P_4(x,y) &= -y^4+2xy^3+(6-x^2)y^2-8xy+x^2-3.
\end{aligned}$$

Moreover, we have the following inequalities:

Lemma 3.3.2.

$$\begin{aligned}
|\widehat{C}_{xxt}(t,x)| &\leqslant c\frac{e^{-\rho_t^2/8}}{x^{3/2}}\frac{\widehat{\sigma}_t^2}{\rho_t^3}\left(\sum_{j=1}^{p}\left(\varrho_j(x)^2+\rho_t^2/4+1\right)e^{-\varrho_j(x)^2/2}+\rho_t+\rho_t^3\right),\\
|\widehat{C}_{xtt}(t,x)| &\leqslant \mathcal{X}^1(t,x)+\mathcal{X}^2(t,x)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{X}^1(t,x) &= c\frac{e^{-\rho_t^2/8}}{\sqrt{x}}\frac{|\rho_t''|}{\rho_t}\left(\sum_{j=1}^{p}\varrho_j(x)e^{-\varrho_j(x)^2/2}+\rho_t+\rho_t^2\right),\\
\mathcal{X}^2(t,x) &= c\frac{e^{-\rho_t^2/8}}{\sqrt{x}}\frac{\widehat{\sigma}_t^4}{\rho_t^4}\left(\sum_{j=1}^{p}\left(\varrho_j(x)^3+\varrho_j(x)\right)e^{-\varrho_j(x)^2/2}+\sum_{j=1}^{4}\rho_t^j\right)
\end{aligned}$$

and $\varrho_j(x)=|\ln(K_j/x)|/\rho_t$.

Lemma 3.3.3. *Assume that the assumptions* $(G1)$ *or* $(G2)$ *hold, then there exists a constant* c *such that* $\varepsilon_i=n^{1/2}\sqrt{\Delta t_i}\sqrt{f'(t_{i-1})}$, $i\leqslant n-1$ *verifies* $|\varepsilon_i-1|\leqslant c\Delta t_i/(1-t_i)$ *for* n *large enough.*

Proof. First, we suppose that the assumption $(G1)$ holds.

We have

$$\Delta_i=g'(\frac{i-1}{n})\frac{1}{n}+\frac{1}{2}g''(\theta_i)\frac{1}{n^2},$$

where $\theta_i\in[(i-1)/n,i/n]$, which implies that

$$n\Delta t_i f'(t_{i-1})=1+\frac{g''(\theta_i)}{2n}f'(t_{i-1}).$$

We deduce that:

$$|\varepsilon_i - 1| \leqslant c\frac{|g''(\theta_i)|(1-\theta_i)^\lambda}{(1-\theta_i)^\lambda}\Delta t_i \leqslant \frac{c\Delta t_i}{(1-f(u_i))^\lambda}$$

where $u_i = g(\theta_i) \in [t_{i-1}, t_i]$. Using the fact that f' is bounded from below, we obtain:

$$|\varepsilon_i - 1| \leqslant c\frac{\Delta t_i}{(1-u_i)^\lambda} \leqslant c\frac{\Delta t_i}{1-t_i}.$$

Secondly, we suppose that the assumption $(G2)$ holds.

We have obviously

$$|\varepsilon_i - 1| \leqslant |n\Delta t_i f'(t_{i-1}) - 1|,$$

where $\Delta t_i = g'(\theta_i)n^{-1}$ and $\theta_i \in [(i-1)/n, i/n]$, which implies that $h_i = g(\theta_i) - t_{i-1}$ verifies $h_i \in [0, \Delta t_i]$. Then using $(G2)$, we obtain that:

$$|\varepsilon_i - 1| \leqslant \left|\frac{f'(g(\theta_i) - h_i)}{f'(g(\theta_i))} - 1\right| \leqslant const\,\frac{\Delta t_i}{1-g(\theta_i)} \leqslant const\,\frac{\Delta t_i}{1-t_i}.$$

The following lemma is of first importance in order to specify some expectations with t near to unit as we shall see further.

Lemma 3.3.4. *Suppose that $t \leqslant u < 1$, $m \in \mathbb{R}$, $q \in 2\mathbb{N}$ and $K > 0$. There exists a constant $c = c(m,q)$ such that*

$$ES_u^m \ln^q \frac{S_u}{K} \exp\left\{-\frac{\ln^2(S_u/K)}{\rho_t^2}\right\} \leqslant cP_q(\rho_t)$$

where

$$\begin{aligned} P_0(\rho_t) &= \rho_t, \\ P_2(\rho_t) &= \rho_t^3 + \rho_t^5, \\ P_4(\rho_t) &= \rho_t^5 + \rho_t^7 + \rho_t^9, \\ P_{2q}(\rho_t) &= \rho_t^{2q+1} + \rho_t^{2q+3} + \cdots + \rho_t^{4q+1}. \end{aligned}$$

Proof. We note $p = \ln \frac{S_0}{K} - \sigma^2 u/2$, $\alpha = \sigma\sqrt{u}$ and

$$A(q) = ES_u^m \ln^q \frac{S_u}{K} \exp\left\{-\frac{\ln^2(S_u/K)}{\rho_t^2}\right\}.$$

Then,

$$\begin{aligned} A(q) &= \frac{S_0^m}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(p+\alpha y)^q \exp\left\{\alpha m y - \alpha^2 m/2 - \frac{1}{\rho_t^2}(p+\alpha y)^2 - y^2/2\right\}dy, \\ A(q) &= \frac{S_0^m e^{A_1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(p+\alpha y)^q \exp\left\{-\frac{1}{2}\left(1+\frac{2\alpha^2}{\rho_t^2}\right)y^2 + \alpha\left(m - \frac{2p}{\rho_t^2}\right)y\right\}dy \end{aligned}$$

where

$$A_1 = -\frac{\alpha^2 m}{2} - \frac{p^2}{\rho_t^2}.$$

Let $y = z/A_2$ with $A_2 = \sqrt{1 + 2\alpha^2/\rho_t^2}$. Then

$$A(q) = \frac{S_0^m e^{A_4}}{\sqrt{2\pi}A_2} \int_{-\infty}^{\infty} (p + \frac{\alpha z}{A_2})^q \exp\left\{ -\frac{1}{2}\left[z^2 - 2(A_3/A_2)z + A_3^2/A_2^2\right]\right\} dz$$

where $A_3 = \alpha\,(m - 2p/\rho_t^2)$ and $A_4 = A_1 + A_3^2/(2A_2^2)$. After the change of variable $y = z - A_3/A_2$, we obtain that

$$A(2) = \frac{S_0^m \rho_t e^{A_4}}{\sqrt{\rho_t^2 + 2\alpha^2}} \left[\left(p + \frac{\alpha\rho_t^2 A_3}{\rho_t^2 + 2\alpha^2}\right)^2 + \frac{\alpha^2\rho_t^2}{\rho_t^2 + 2\alpha^2}\right].$$

Moreover, if $u \geqslant t$, then $\rho_t^2 \geqslant \sigma^2(1-t)$ implies that

$$\rho_t^2 + 2\alpha^2 \geqslant \sigma^2(1-t) + \sigma^2 u \geqslant \sigma^2.$$

We have

$$A_4 = -\frac{m\alpha^2}{2} - \frac{p^2}{\rho_t^2} + \frac{\alpha^2\rho_t^2}{2(\rho_t^2 + 2\alpha^2)}\left(m^2 + \frac{4p^2}{\rho_t^4} - \frac{4pm}{\rho_t^2}\right)$$

where p, α are bounded. But, the term

$$\frac{\alpha^2\rho_t^2}{2(\rho_t^2 + 2\alpha^2)} m^2$$

is obviously bounded whereas we can establish the following inequality

$$\frac{\alpha^2\rho_t^2}{2(\rho_t^2 + 2\alpha^2)}\frac{4p^2}{\rho_t^4} \leqslant \frac{p^2}{\rho_t^2}.$$

The following term

$$\left| \frac{\alpha^2\rho_t^2}{2(\rho_t^2 + 2\alpha^2)}\frac{4pm}{\rho_t^2} \right|$$

is also bounded. It follows that e^{A_4} is bounded and we can conclude easily for $q = 2$. In a similar way, we can conclude for any $q \in 2\mathbb{N}$ because we use in particular the property

$$\int_{-\infty}^{\infty} y^k \varphi(y) dy = 0$$

if $k \in 2\mathbb{N} + 1$.

From now on, we can deduce the following results.

Corollary 3.3.5. *If $m \in \mathbb{R}$ and $u \geqslant t$, then there exists a constant $c_m > 0$ such that*

$$ES_u^m \widehat{C}_{xt}^2(t, S_u) \leqslant \frac{c_m \widehat{\sigma}_t^4}{\rho_t^3} e^{-\rho_t^2/8}.$$

Proof. Indeed, it suffices to use Lemma 2.3.8 established in Chapter 2 and apply the previous lemma.

In a similar way, we have:

Corollary 3.3.6. *If $m \in \mathbb{R}$ and $u \geqslant t$, then there exists a constant $c_m > 0$ such that*

$$ES_u^m \widehat{C}_{xt}^4(t, S_u) \leqslant \frac{c_m \widehat{\sigma}_t^8}{\rho_t^7} e^{-\rho_t^2/8}.$$

Corollary 3.3.7. *If $m \in \mathbb{R}$ and $u \geqslant t$, then there exists a constant $c_m > 0$ such that*

$$ES_u^m \widehat{C}_{xx}^4(t, S_u) \leqslant \frac{c_m}{\rho_t^3} e^{-\rho_t^2/4}.$$

Corollary 3.3.8. *If $m \in \mathbb{R}$ and $u \geqslant t$, then there exists a constant $c_m > 0$ such that*

$$ES_u^m \widehat{C}_{xxx}^2(t, S_u) \leqslant \frac{c_m}{\rho_t^3} e^{-\rho_t^2/8}.$$

Corollary 3.3.9. *If $m \in \mathbb{R}$ and $u \geqslant t$, then there exists a constant $c_m > 0$ such that*

$$ES_u^m \widehat{C}_{xxt}^2(t, S_u) \leqslant \frac{c_m \widehat{\sigma}_t^4}{\rho_t^5} e^{-\rho_t^2/8}.$$

Corollary 3.3.10. *If $m \in \mathbb{R}$ and $u \geqslant t$, then there exists a constant $c_m > 0$ such that*

$$ES_u^m \widehat{C}_{xxx}^4(t, S_u) \leqslant \frac{c_m}{\rho_t^7} e^{-\rho_t^2/8}.$$

Corollary 3.3.11. *If $m \in \mathbb{R}$ and $u \geqslant t$, then there exists a constant $c_m > 0$ such that*

$$ES_u^m \widehat{C}_{xxxx}^2(t, S_u) \leqslant \frac{c_m}{\rho_t^5} e^{-\rho_t^2/8}.$$

Corollary 3.3.12. *If $m \in \mathbb{R}$ and $u \geqslant t$, then there exists a constant $c_m > 0$ such that*

$$ES_u^m \widehat{C}_{xxt}^4(t, S_u) \leqslant \frac{c_m \widehat{\sigma}_t^8}{\rho_u^{11}} e^{-\rho_t^2/8}.$$

Let $\widetilde{S}_{t_{i-1}} \in [S_{t_{i-1}}, S_{t_i}]$ and $\widetilde{t}_{i-1} \in [t_{i-1}, t_i]$ be random variables. We have the following inequalities:

Lemma 3.3.13. *There exists a constant c such that*

$$E\widehat{C}_{xt}^4(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{ce^{-\rho_{t_i}^2/4}}{(1-t_i)^4}.$$

Moreover, if $\alpha = 1/2$, there exists a bounded function $\epsilon(a)$ verifying $\epsilon(a) \to 0$ as $a \to 1$ such that

$$E\widehat{C}_{xt}^4(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{c}{(1-t_i)^4}\epsilon(a)$$

if $t_{i-1} \geqslant a$, $i \leqslant n-1$ and n is sufficiently large.

Proof. We have $\widetilde{S}_{t_{i-1}}^m \leqslant S_{t_{i-1}}^m + S_{t_i}^m$, and $\rho_{\widetilde{t}_{i-1}} \geqslant \rho_{t_i}$. Furthermore, in virtue of 2.3.8, recall that we have

$$|\widehat{C}_{xt}(t, x)| \leqslant c \frac{\widehat{\sigma}_t^2 e^{-\rho_t^2/8}}{x^{1/2}\rho_t^2}.$$

Then, the first result is obvious. Moreover,

$$|\widehat{C}_{tx}(t, x)| \leqslant \frac{c\widehat{\sigma}^2 e^{-\frac{\rho^2}{8}}}{x^{1/2}\rho^2}\left(L(x, \rho) + \rho + \rho^2\right),$$

where

$$L(x,\rho) = \sum_{j=1}^{p} \frac{|\ln(x/K_j)|}{\rho} \exp\left\{-\frac{\ln^2(x/K_j)}{2\rho^2}\right\} = \sum_{j=1}^{p} L^j(x,\rho).$$

On the sets $\Gamma^{i,j} = \{S_{t_{i-1}} \vee S_{t_i} \leqslant K_j/e^{\sqrt{\rho_{t_{i-1}}}}\} \cup \{S_{t_{i-1}} \wedge S_{t_i} \geqslant K_j e^{\sqrt{\rho_{t_{i-1}}}}\}$, we have:

$$\begin{aligned}\left|\frac{\ln(\widetilde{S}_{t_{i-1}}/K_j)}{\rho_{\widetilde{t}_{i-1}}}\right| \exp\left\{-\frac{\ln^2(\widetilde{S}_{t_{i-1}}/K_j)}{2\rho^2_{\widetilde{t}_{i-1}}}\right\} &\leqslant c\left|\frac{\ln(S_{t_{i-1}}/K_j)}{\rho_{t_{i-1}}}\right| \exp\left\{-\frac{\ln^2(S_{t_{i-1}}/K_j)}{2\rho^2_{t_{i-1}}}\right\} \\ &+ c\left|\frac{\ln(S_{t_i}/K_j)}{\rho_{t_{i-1}}}\right| \exp\left\{-\frac{\ln^2(S_{t_i}/K_j)}{2\rho^2_{t_{i-1}}}\right\}\end{aligned}$$

Indeed, $\rho^2_{t_{i-1}}/\rho^2_{t_i}$ is bounded because of the boundedness of $\Delta t_i/(1-t_i)$. Secondly, the mapping $x \to L^j(x,\rho_{t_{i-1}})$ is respectively increasing and decreasing on the intervals $]0, K_j/e^{\sqrt{\rho_{t_{i-1}}}}]$ and $[K_j e^{\sqrt{\rho_{t_{i-1}}}}, \infty[$. Then, we deduce that

$$E\left(\frac{1}{\widetilde{S}^{1/2}_{t_{i-1}}\rho^2_{\widetilde{t}_{i-1}}} e^{-\frac{\rho^2_{\widetilde{t}_{i-1}}}{8}} L^j(\widetilde{S}_{t_{i-1}}, \rho_{\widetilde{t}_{i-1}})\right)^4 1_{\Gamma^{i,j}} \leqslant \frac{c}{\rho^8_{t_i}} \frac{\rho^{9/2}_{t_{i-1}}}{\rho^4_{t_i}} \leqslant \frac{c}{\rho^8_{t_i}} \rho^{1/2}_{t_i}$$

if $i \leqslant n-1$ and $\alpha = 1/2$. Indeed, it suffices to use the Cauchy-Schwartz inequality and Lemma 3.3.4 with $q = 8$. Moreover, if $\alpha = 1/2$, ρ_t does not depend on n and it is easy to show that $\rho_{t_i} \to 0$ as $t_i \to 1$ even if the assumption $(G2)$ holds.

Finally, it suffices to note that $E\sup_t S^m_t < \infty$ and $P(\Omega\setminus\Gamma^{i,j})$ converges to 0 as $t_{i-1} \geqslant a$ converges to unit. Indeed, we have a.s. $S_1 < K_j$ or $S_1 > K_j$ and S_u is near to S_1 if $u \geqslant a$ whereas $\rho_{t_i} \to 0$ provided that a is sufficiently close to unit. It follows that we can apply the Lebesgue theorem in order to have for $t_{i-1} \geqslant a$

$$E\left(\frac{1}{\widetilde{S}^{1/2}_{t_{i-1}}\rho^2_{\widetilde{t}_{i-1}}} e^{-\frac{\rho^2_{\widetilde{t}_{i-1}}}{8}} L^j(\widetilde{S}_{t_{i-1}}, \rho_{\widetilde{t}_{i-1}})\right)^4 1_{\Omega\setminus\Gamma^{i,j}} \leqslant \frac{c}{\rho^8_{t_i}}\epsilon_a$$

where $\epsilon_a \to 0$ as $a \to 1$. So, we can conclude about the lemma because the difficult part is solved.

In the same way, we can prove the following results:

Lemma 3.3.14. *There exists a constant c such that*

$$E\widehat{C}^4_{xtt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{ce^{-\rho^2_{t_i}/4}}{(1-t_i)^8}.$$

Moreover, if $\alpha = 1/2$, there exists a bounded function $\epsilon(a)$ verifying $\epsilon(a) \to 0$ as $a \to 1$ such that

$$E\widehat{C}^4_{xtt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{c}{(1-t_i)^8}\epsilon(a)$$

if $t_{i-1} \geqslant a$, $i \leqslant n-1$ and n is sufficiently large.

Note that, in the case of the assumption (G_2), we also use the inequality:

$$\frac{g''(u)}{g'(u)^2} \leqslant \frac{c_4}{(1-g(u))^{3/2}}, \ \forall u < 1$$

in order to have

$$\frac{\rho_t''}{\rho_t} \leqslant \frac{c}{(1-t)^2}.$$

Lemma 3.3.15. *There exists a constant c such that*

$$E\widehat{C}^4_{xxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{ce^{-\rho^2_{t_i}/4}}{\rho^8_{t_i}}.$$

Moreover, if $\alpha = 1/2$, there exists a bounded function $\epsilon(a)$ verifying $\epsilon(a) \to 0$ as $a \to 1$ such that

$$E\widehat{C}^4_{xxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{c}{(1-t_i)^4}\epsilon(a)$$

if $t_{i-1} \geqslant a$, $i \leqslant n-1$ and n is sufficiently large.

Lemma 3.3.16. *There exists a constant c such that*

$$E\widehat{C}^4_{xxt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{ce^{-\rho^2_{t_i}/4}}{n^{2(1/2-\alpha)}(1-t_i)^6 f'(t_i)}.$$

Moreover, if $\alpha = 1/2$, there exists a bounded function $\epsilon(a)$ verifying $\epsilon(a) \to 0$ as $a \to 1$ such that

$$E\widehat{C}^4_{xxt}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{c}{(1-t_i)^6}\epsilon(a)$$

if $t_{i-1} \geqslant a$, $i \leqslant n-1$ and n is sufficiently large.

Lemma 3.3.17. *There exists a constant c such that*

$$E\widehat{C}^4_{xxxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{ce^{-\rho^2_{t_i}/4}}{\rho^{12}_{t_i}}.$$

Moreover, if $\alpha = 1/2$, there exists a bounded function $\epsilon(a)$ verifying $\epsilon(a) \to 0$ as $a \to 1$ such that

$$E\widehat{C}^4_{xxxx}(\widetilde{t}_{i-1}, \widetilde{S}_{t_{i-1}}) \leqslant \frac{c}{f'(t_i)^3(1-t_i)^6}\epsilon(a)$$

if $t_{i-1} \geqslant a$, $i \leqslant n-1$ and n is sufficiently large.

In order to conclude about the main theorem of this chapter, we add the two following lemmas, valid for $\alpha = 1/2$ and used in Lemma 3.2.17.

Lemma 3.3.18. *We have the following inequalities for $j \leqslant n-1$:*

$$\begin{aligned} |E(\beta_j sgn\alpha_j|\mathcal{F}_{t_{j-1}})| &\leqslant \frac{cS^{1/2}_{t_{j-1}}}{(1-t_j)^2}(\Delta t_j)^{3/2}, \\ E(\beta_j^2|\mathcal{F}_{t_{j-1}}) &\leqslant \frac{cS_{t_{j-1}}(\Delta t_j)^2}{(1-t_j)^2}. \end{aligned}$$

Proof. First, we prove that

$$|E(\beta_j sgn\alpha_j|\mathcal{F}_{t_{j-1}})| \leqslant \frac{cS^{1/2}_{t_{i-1}}}{(1-t_j)^2}(\Delta t_j)^{3/2}.$$

For this, we note that

$$|E(\beta_j sgn\alpha_j|\mathcal{F}_{t_{j-1}})| \leqslant \mathcal{X}_1 + \mathcal{X}_2$$

where

$$\begin{aligned}
\mathcal{X}_1 &= \left| E\left(S_{t_{j-1}} \widehat{C}_{xt}(S_{t_{j-1}}, t_{j-1}) \Delta t_j sgn(\alpha_j) | \mathcal{F}_{t_{j-1}} \right) \right|, \\
\mathcal{X}_2 &= \left| E\left(\widetilde{\mathcal{X}}^1 + \widetilde{\mathcal{X}}^2 + \widetilde{\mathcal{X}}^3 | \mathcal{F}_{t_{j-1}} \right) \right|, \\
\widetilde{\mathcal{X}}^1 &= \frac{1}{2} S^3_{t_{j-1}} \widehat{C}_{xxx}(\widetilde{S}_{t_{j-1}}, \widetilde{t}_{j-1}) \left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1 \right)^2 sgn(\alpha_j), \\
\widetilde{\mathcal{X}}^2 &= S^2_{t_{j-1}} \widehat{C}_{xxt}(\widetilde{S}_{t_{j-1}}, \widetilde{t}_{j-1}) \left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1 \right) \Delta t_j sgn(\alpha_j), \\
\widetilde{\mathcal{X}}^3 &= S_{t_{j-1}} \widehat{C}_{xtt}(\widetilde{S}_{t_{j-1}}, \widetilde{t}_{j-1}) (\Delta t_j)^2 sgn(\alpha_j).
\end{aligned}$$

By independence, we get that

$$\mathcal{X}_1 = S_{t_{j-1}} \left| \widehat{C}_{xt}(S_{t_{j-1}}, t_{j-1}) \right| \Delta t_j \left| E\, sgn\left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1 \right) \right|.$$

We recall that, from [12], we have

$$Esgn\left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1 \right) = -\frac{1}{\sqrt{2\pi}} \sqrt{\Delta t_j} + O(\Delta t_j).$$

Then, we deduce that

$$\mathcal{X}_1 \leqslant \frac{const\, S^{1/2}_{t_{j-1}} (\Delta t_j)^{3/2}}{1 - t_{j-1}}. \tag{3.3.12}$$

We write

$$\widetilde{\mathcal{X}}^1 = \widetilde{\mathcal{X}}^1_a + \widetilde{\mathcal{X}}^1_b + \widetilde{\mathcal{X}}^1_c$$

where, using the Taylor approximation, we get that

$$\begin{aligned}
\widetilde{\mathcal{X}}^1_a &= \frac{1}{2} S^3_{t_{j-1}} \widehat{C}_{xxx}(S_{t_{j-1}}, t_{j-1}) \left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1 \right)^2 sgn(\alpha_j), \\
\widetilde{\mathcal{X}}^1_b &= \frac{1}{2} S^3_{t_{j-1}} \widehat{C}_{xxxx}(S^*_{t_{j-1}}, t^*_{j-1}) \left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1 \right)^2 sgn(\alpha_j) \left(\widetilde{S}_{t_{j-1}} - S_{t_{j-1}} \right), \\
\widetilde{\mathcal{X}}^1_c &= \frac{1}{2} S^3_{t_{j-1}} \widehat{C}_{xxxt}(S^*_{t_{j-1}}, t^*_{j-1}) \left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1 \right)^2 sgn(\alpha_j) (\widetilde{t}_{j-1} - t_{j-1})
\end{aligned}$$

where $t^*_{j-1} \in [t_{j-1}, \widetilde{t}_{j-1}]$ and $S^*_{t_{j-1}} \in [S_{t_{j-1}}, \widetilde{S}_{t_{j-1}}]$ are random variables. Recall the following approximation from [12]:

$$E\left(\frac{S_{t_j}}{S_{t_{j-1}}} - 1 \right)^2 sgn(\alpha_j) = \frac{2}{\sqrt{2\pi}} (\Delta t_j)^{3/2} + O((\Delta t_j)^2).$$

Then, by independence, we easily deduce that

$$\left| E(\widetilde{\mathcal{X}}^1_a | \mathcal{F}_{t_{j-1}}) \right| \leqslant \frac{const\, S^{1/2}_{t_{j-1}} (\Delta t_j)^{3/2}}{1 - t_{j-1}}. \tag{3.3.13}$$

Since $S^*_{t_{j-1}} \in [S_{t_{j-1}}, S_{t_j}]$, we deduce, from Appendix, that

$$\begin{aligned} \left|\widehat{C}_{xxxx}(S^*_{t_{j-1}}, t^*_{j-1})\right| &\leqslant \frac{const}{(1-t_j)^{3/2}}\left(\frac{1}{S^{7/2}_{t_{j-1}}}+\frac{1}{S^{7/2}_{t_j}}\right), \\ \left|\widehat{C}_{xxxt}(S^*_{t_{j-1}}, t^*_{j-1})\right| &\leqslant \frac{const}{(1-t_j)^{2}}\left(\frac{1}{S^{5/2}_{t_{j-1}}}+\frac{1}{S^{5/2}_{t_j}}\right). \end{aligned}$$

Then, using the independence of $S_{t_j}/S_{t_{j-1}}$ relatively to $\mathcal{F}_{t_{j-1}}$, the Cauchy-Schwarz inequality and the property

$$E\left(S_{t_j}/S_{t_{j-1}}-1\right)^{2m} \leqslant const\,(\Delta t_j)^m,\ m\in\mathbb{N},$$

we deduce that

$$\left|E(\widetilde{\mathcal{X}}^1_b+\widetilde{\mathcal{X}}^1_c|\mathcal{F}_{t_{j-1}})\right| \leqslant \frac{const\, S^{1/2}_{t_{j-1}}(\Delta t_j)^{3/2}}{(1-t_{j-1})^2}. \tag{3.3.14}$$

In a similar way, knowing that

$$\left(S_{t_j}/S_{t_{j-1}}-1\right) sgn(\alpha_j) = \left|S_{t_j}/S_{t_{j-1}}-1\right|,$$

we have

$$\left|E(\widetilde{\mathcal{X}}^2|\mathcal{F}_{t_{j-1}})\right| \leqslant E\left(S^2_{t_{j-1}}\left|\widehat{C}_{xxt}(\widetilde{S}_{t_{j-1}},\widetilde{t}_{j-1})\right|\left(S_{t_j}/S_{t_{j-1}}-1\right)\Delta t_j\, sgn(\alpha_j)|\mathcal{F}_{t_{j-1}}\right)$$

where

$$\left|\widehat{C}_{xxt}(\widetilde{S}_{t_{j-1}},\widetilde{t}_{j-1})\right| \leqslant \frac{const}{(1-t_j)^{3/2}}\left(\frac{1}{S^{3/2}_{t_{j-1}}}+\frac{1}{S^{3/2}_{t_j}}\right).$$

It follows that,

$$\left|E(\widetilde{\mathcal{X}}^2|\mathcal{F}_{t_{j-1}})\right| \leqslant \frac{const\, S^{1/2}_{t_{j-1}}(\Delta t_j)^{3/2}}{(1-t_{j-1})^{3/2}}. \tag{3.3.15}$$

Finally, with the same argument, since we have

$$\left|\widehat{C}_{xtt}(\widetilde{S}_{t_{j-1}},\widetilde{t}_{j-1})\right| \leqslant \frac{const}{(1-t_j)^{2}}\left(\frac{1}{S^{1/2}_{t_{j-1}}}+\frac{1}{S^{1/2}_{t_j}}\right),$$

we deduce that

$$\left|E(\widetilde{\mathcal{X}}^3|\mathcal{F}_{t_{j-1}})\right| \leqslant \frac{const\, S^{1/2}_{t_{j-1}}(\Delta t_j)^{2}}{(1-t_{j-1})^{2}}. \tag{3.3.16}$$

Then, from the inequalities 3.3.12,$\cdots$,3.3.16 we can conclude about the first assertion of the lemma.

For the second assertion, we follow the same reasoning. We get that

$$E(\beta_j^2|\mathcal{F}_{t_{j-1}}) \leqslant const \sum_{i=1}^{6}\mathcal{X}_i$$

where:

$$
\begin{aligned}
\mathcal{X}_1 &= S^2_{t_{j-1}} \widehat{C}^2_{xt}(S_{t_{j-1}}, t_{j-1})(\Delta t_j)^2, \\
\mathcal{X}_2 &= S^6_{t_{j-1}} \widehat{C}^2_{xxx}(S_{t_{j-1}}, t_{j-1}) E(S_{t_j}/S_{t_{j-1}} - 1)^4, \\
\mathcal{X}_3 &= S^8_{t_{j-1}} \widehat{C}^2_{xxxx}(S^*_{t_{j-1}}, t^*_{j-1}) E(S_{t_j}/S_{t_{j-1}} - 1)^6, \\
\mathcal{X}_4 &= S^6_{t_{j-1}} \widehat{C}^2_{xxxt}(S^*_{t_{j-1}}, t^*_{j-1}) E(S_{t_j}/S_{t_{j-1}} - 1)^4 (\Delta t_j)^2, \\
\mathcal{X}_5 &= S^4_{t_{j-1}} \widehat{C}^2_{xxt}(\widetilde{S}_{t_{j-1}}, \widetilde{t}_{j-1}) E(S_{t_j}/S_{t_{j-1}} - 1)^2 (\Delta t_j)^2, \\
\mathcal{X}_6 &= S^2_{t_{j-1}} \widehat{C}^2_{xtt}(\widetilde{S}_{t_{j-1}}, \widetilde{t}_{j-1})(\Delta t_j)^4.
\end{aligned}
$$

From estimations of the successives derivatives of $\widehat{C}$, we obtain a constant c such that:

$$
\begin{aligned}
\mathcal{X}_1 + \mathcal{X}_2 &\leqslant \frac{c\, S_{t_{j-1}} (\Delta t_j)^2}{(1-t_j)^2}, \\
\mathcal{X}_3 + \mathcal{X}_5 &\leqslant \frac{c\, S_{t_{j-1}} (\Delta t_j)^3}{(1-t_j)^3}, \\
\mathcal{X}_4 + \mathcal{X}_6 &\leqslant \frac{c\, S_{t_{j-1}} (\Delta t_j)^4}{(1-t_j)^4}.
\end{aligned}
$$

Since we have for $j \leqslant n-1$,

$$
\frac{\Delta t_j}{1-t_j} \leqslant const \, \frac{\Delta t_j}{1-t_{j-1}} \leqslant const,
$$

we can easily conclude about the second assertion.

We consider

$$
\begin{aligned}
l(x) &= P\Big(c_1 u^2 + c_2 (1+\eta_u^{-5/2})(\eta_u - 1)^2 + c_3 (1+\eta_u^{-3/2}) |\eta_u - 1| u^2 \\
&\quad + c_4 (1+\eta_u^{-1/2}) u^4 \geqslant \widehat{C}_{xx}(t_{j-1}, x) x^{3/2} |\eta_u - 1| \Big),
\end{aligned}
$$

where $\eta_u = e^{u\xi - u^2/2}$, $\xi \sim \mathcal{N}(0,1)$ and

$$
\begin{aligned}
C &= 4(c_1 + \cdots + c_4), \\
C_5(x) &= \frac{C}{\widehat{C}_{xx}(t_{j-1}, x) x^{3/2}}.
\end{aligned}
$$

We have the following result :

Lemma 3.3.19. *There exists a continuous function F on $\mathbb{R}^+$ such that*

$$
l(x) \leqslant F(N) u 1_{C_5(x) \leqslant N} + 1_{C_5(x) > N}.
$$

Proof. We can easily establish that $l(x)$ is less than the probability

$$
P\Big(C/4 \left(u^2 + (1+\eta_u^{-5/2})(\eta_u - 1)^2 + (1+\eta_u^{-3/2})|\eta_u - 1| u^2 + (1+\eta_u^{-1/2}) u^4 \right) \geqslant C_5^{-1} C |\eta_u - 1| \Big).
$$

It follows that $l(x) \leqslant W + X + Y + Z$ where

$$\begin{aligned} W &= P\left(|\eta_u - 1| \leqslant C_5 u^2\right), \\ X &= P\left(|\eta_u - 1| \leqslant C_5(1+\eta_u^{-5/2})(\eta_u - 1)^2\right), \\ Y &= P\left(|\eta_u - 1| \leqslant C_5(1+\eta_u^{-3/2})|\eta_u - 1|u^2\right), \\ Z &= P\left(|\eta_u - 1| \leqslant C_5(1+\eta_u^{-1/2})u^4\right). \end{aligned}$$

We note that

$$|\eta_u - 1| = e^{u\xi - u^2/2} - 1 \Leftrightarrow \xi \geqslant u/2$$

and

$$e^{u\xi - u^2/2} - 1 \leqslant Nu^2 \Leftrightarrow \xi \leqslant \frac{1}{u}\ln(Nu^2+1) + u/2.$$

In a similar way,

$$1 - e^{u\xi - u^2/2} \leqslant Nu^2 \Leftrightarrow \xi \geqslant \frac{1}{u}\ln(-Nu^2+1) + u/2.$$

It suffices to analyse the case $C_5 \leqslant N$. Then,

$$\begin{aligned} W &\leqslant P(|\eta_u - 1| \leqslant Nu^2), \\ W &\leqslant P\left(\frac{u}{2} \leqslant \xi \leqslant \frac{1}{u}\ln(Nu^2+1) + u/2\right) + P\left(\frac{1}{u}\ln(-Nu^2+1) + u/2 \leqslant \xi \leqslant u/2\right), \\ W &\leqslant Nu + \frac{|\ln(-Nu^2+1)|}{u}. \end{aligned}$$

We note

$$\widetilde{K} = \max_{x\in[0,1/2]} \frac{|\ln(1-x)|}{x}.$$

In the case where $u^2 \geqslant 1/2N$, it is obvious that $W \leqslant \sqrt{2N}u$ whereas, if $Nu^2 \leqslant 1/2$, we have

$$\frac{|\ln(-Nu^2+1)|}{Nu^2} \leqslant \widetilde{K}.$$

So, $W \leqslant F_1(N)u$ where $F_1(N) = (1+\widetilde{K})N + \sqrt{2N}$.

Always for $C_5 \leqslant N$,

$$\begin{aligned} X &\leqslant P\left(1 \leqslant C_5(1+\eta_u^{-5/2})|\eta_u - 1|\right), \\ X &\leqslant P\left(|\eta_u - 1| \geqslant 1/2N\right) + P\left(|\eta_u - 1| \geqslant \eta_u^{5/2}/2N\right). \end{aligned}$$

Moreover, from the Bienaymé-Tchebychev inequality, we deduce a constant a such that

$$P\left(|\eta_u - 1| \geqslant 1/2N\right) \leqslant aNu$$

whereas

$$\begin{aligned} P\left(|\eta_u - 1| \geqslant \eta_u^{5/2}/2N\right) &= P\left(|\eta_u - 1| \geqslant \eta_u^{5/2}/2N, \eta_u \geqslant 1\right) + P\left(1 - \eta_u \geqslant \eta_u^{5/2}/2N, \eta_u \leqslant 1\right), \\ P\left(|\eta_u - 1| \geqslant \eta_u^{5/2}/2N\right) &\leqslant P\left(|\eta_u - 1| \geqslant 1/2N\right) + P\left(1 - \eta_u^{5/2} \geqslant \eta_u^{5/2}/2N\right) \end{aligned}$$

where $P\left(1 - \eta_u^{5/2} \geqslant \eta_u^{5/2}/2N\right) = P\left(\eta_u^{-5/2} - 1 \geqslant 1/2N\right)$.

But $\eta_u^{5/2} \sim e^{35u^2/8}\eta_{5u/2}$ because $\xi =^{law} -\xi$. So, we have

$$P\left(\eta_u^{-5/2} - 1 \geqslant 1/2N\right) \quad \leqslant \quad P\left(e^{35u^2/8}(\eta_{5u/2} - 1) \geqslant 1/4N\right) + P\left(e^{35u^2/8} - 1 \geqslant 1/2N\right) \leqslant a_1 N u$$

provided that u is bounded. Then, we can conclude that there exists a continuous function F_2 such that $X \leqslant F_2(N)u$ if $C_5 \leqslant N$.

It is easy to find a continuous function F_3 such that $Y \leqslant F_3(N)u$ if $C_5 \leqslant N$.

Finally,

$$\begin{aligned}
Z &\leqslant P\left(|\eta_u - 1| \leqslant 2Nu^4\right) + P\left(|\eta_u - 1| \leqslant 2Nu^4\eta_u^{-1/2}\right), \\
Z &\leqslant F_1(2Nu^2)u + P\left(|\eta_u - 1| \leqslant 2Nu^4, \eta_u \geqslant 1\right) + P\left(1 - \eta_u \leqslant 2Nu^4\eta_u^{-1}, \eta_u \leqslant 1\right), \\
Z &\leqslant 2F_1(2Nu^2)u + P\left(\eta_u^2 - \eta_u + 2Nu^4 \geqslant 0, \eta_u \leqslant 1\right).
\end{aligned}$$

In the case where $\sqrt{2\sqrt{2N}}u \geqslant 1$, we have

$$P\left(\eta_u^2 - \eta_u + 2Nu^4 \geqslant 0, \eta_u \leqslant 1\right) \leqslant \sqrt{2\sqrt{2N}}u.$$

Otherwise $x^2 - x + 2Nu^4 = 0$ holds if and only if

$$x = \frac{1 \pm \sqrt{1 - 8Nu^4}}{2}.$$

So, we estimate:

$$P\left(\eta_u \leqslant \frac{1 - \sqrt{1 - 8Nu^4}}{2}\right) \leqslant P\left(|\eta_u - 1| \geqslant 1/2\right) \leqslant a_2 u$$

where a_2 is a constant. We have also

$$\begin{aligned}
P\left(\eta_u \geqslant \frac{1 + \sqrt{1 - 8Nu^4}}{2}\right) &\leqslant P\left(\frac{1}{u}\ln\left(\frac{1 + \sqrt{1 - 8Nu^4}}{2}\right) + \frac{u}{2} \leqslant \xi \leqslant \frac{u}{2}\right), \\
P\left(\eta_u \geqslant \frac{1 + \sqrt{1 - 8Nu^4}}{2}\right) &\leqslant \left|\frac{1}{u}\ln\left(\frac{1 + \sqrt{1 - 8Nu^4}}{2}\right)\right| \leqslant K^*\sqrt{N}u
\end{aligned}$$

where

$$K^* = \max_{x \in [0, 1/\sqrt{8}]} \left|\frac{1}{x}\ln\left(\frac{1 + \sqrt{1 - 8x^2}}{2}\right)\right|.$$

Then, we have found a continuous function F_4 such that $Y \leqslant F_4(N)u$ if $C_5 \leqslant N$. We can conclude about the lemma considering $F = F_1 + \cdots + F_4$.

Chapter 4

Functional Limit Theorem for Leland–Lott Hedging Strategy

Leland's approach to the hedging of derivatives under proportional transaction costs is based on an approximate replication of the contingent claim using the classical Black–Scholes formulae with a suitably enlarged volatility. The formal mathematical framework is a scheme of series, i.e. a sequence of models with the transaction costs coefficients $k_n = k_0 n^{-\alpha}$ where $\alpha \in [0, 1/2]$ and n is the number of the revision intervals. The enlarged volatility $\widehat{\sigma}_n$, in general, depends on n except the case $\alpha = 1/2$. If the parameter is $\alpha = 0$, the approximation errors $V_T^n - V_T$ converge to a non-trivial random variable ξ. For the case of call option where $V_T = (S_T - K)^+$, it was shown by Pergamenshchikov that the sequence of random variables $n^{1/4}(V_T^n - V_T - \xi)$ converges in law to a mixture of Gaussian distributions. In this chapter, we treat the case $\alpha = 1/2$ with non-uniform revision intervals and a more general pay-off $h(S_T)$. We show that the sequence $n^{1/2}(V_T^n - V_T)$ converges in law and calculate the limit. Our main result is an application of the theory of diffusion approximation.

4.1 Introduction and Formulation of the Main Result.

We assume that the model is the classical Black–Scholes model under transaction costs defined in Chapter 3 where the volatility is constant. The study of convergence happens to be a mathematically interesting issue. The only limit theorem (in narrow sense, i.e. dealing with the convergence of distributions) is the Pergamenchtikov theorem: for $\alpha = 0$ and $h(x) = (x - K)^+$, the sequence $n^{1/4}(V_1^n - V_1 - \xi)$ converges in law to a mixture of Gaussian distributions, [26] (see also [13]). The known exact rate (Chapter 3) for the L^2-convergence if $\alpha = 1/2$ indicates that in this case the approximation errors multiplied by the amplifying factor growing as $n^{1/2}$ also should converge in law. The aim of this chapter is to show this property: for $\alpha = 1/2$ the sequence of random variables $X_1^n := n^{1/2}(V_1^n - V_1)$ converges in law. In fact, we prove a more general result on the diffusion approximation which claims that the whole process $X^n := n^{1/2}(V^n - \widehat{V})$ converges in law (in the Skorohod space), and calculate the limit. We do this for the model with non-uniform revision intervals and a general pay-off in the setting of Chapter 3 using heavily its results.

Put $\rho_t^2 = \int_t^1 \widehat{\sigma}_s^2 ds$ and $\widehat{V}_t = \widehat{C}(t, S_t)$.

Theorem 4.1.1. *Suppose that the conditions (G_1) or (G_2) and (H) hold. Then, the distribution of the process $X^n := n^{1/2}(V^n - \widehat{V})$ in the Shorohod space $\mathcal{D}[0,1]$ converges weakly to the distribution of the process*

$$X_t = \int_0^t F(t, S_t) dW'_t \tag{4.1.1}$$

where W' is a Wiener process and

$$F(t,x) = \left[\frac{\sigma^4}{2} \frac{1}{f'(t)} + k_0 \sqrt{\frac{2}{\pi}} \frac{\sigma^3}{\sqrt{f'(t)}} + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi}\right) \right]^{1/2} \widehat{C}_{xx}(t,x) x^2.$$

Note that the limiting process X is not a diffusion but only the second component of the diffusion process (S, X).

4.2 Proof of Theorem 4.1.1

4.2.1 Preliminaries

First of all, we recall Corollary 3.1.2 established in Chapter 3.

Corollary 4.2.1. *We have the following approximation for $\alpha = 1/2$:*

$$n^{1/2}(V_t^n - \widehat{V}_t) = M_t^n + \varepsilon_t^n$$

where $M_t^n = \sum_{t_i^n \leqslant t} Y_i^n + Z_i^n$ is a martingale with

$$\begin{aligned} Y_i^n &= \frac{\sigma^2}{2} n^{1/2} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \left[\Delta t_i - (W_{t_{i-1}} - W_{t_i})^2 \right], \\ Z_i^n &= k_0 \sigma \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 \left[\sqrt{\frac{2}{\pi}} \sqrt{\Delta t_i} - |W_{t_{i-1}} - W_{t_i}| \right] \end{aligned}$$

and $E(\sup_t \varepsilon_t^n)^2 \to 0$.

In virtue of Lemma 3.31 p 316 in [16], it is sufficient to establish the functional limit theorem for the process M^n.

4.2.2 Diffusion Approximation

For the reader convenience, we formulate a theorem on identification of the limit process which is deduced from Theorem 4.3.5.

Let $(\Omega, \mathcal{F}, F) = (D[0,T], \mathcal{D}, \mathbf{D} = (\mathcal{D}_t)_{t \leqslant T}, Q)$ be the natural stochastic basis constructed on the Shorohod space of d-dimensional càdlàg functions on $[0,T]$ and let $C[0,T]$ be its subspace formed by continuous functions. We suppose that $X = (X_t)_{t \in [0,T]}$ is the canonical process $X(\alpha) = \alpha$ defined on $\mathcal{D}[0,T]$. On this basis, we also consider:

- (i) $C = (C^{i,j})_{i,j \leqslant d}$ a continuous adapted process with $C_0 = 0$ and $C_t - C_s$ is a symmetric nonnegative matrix for all $s \leqslant t$,
- (ii) the stopping time $S_a(\alpha) = inf\{t > 0 : |\alpha(t)| > a$ or $|\alpha(t-)| > a\}$,
- (iii) $C(a) = C^{S_a}$.

Let $X^n = (X^n_t)_{t\leqslant T}$ be a d-dimensional semimartingale defined on a stochastic basis $(\Omega^n, \mathcal{F}^n, \mathbf{F}^n, P^n)$. Let μ^n and (B^n, C^n, ν^n) be the jump measure and the triplet of predictable characteristics of X^n.

H is a fixed continuous truncation function and we define

$$\tilde{C}^{n,i,j} = C^{n,i,j} + (H^i H^j) * \nu^n - \Sigma_{s\leqslant .}\Delta B^{n,i}_s \Delta B^{n,j}_s.$$

We consider $S^n_a = S_a \circ X^n$, $B(a)^n = (B^n)^{S^n_a}$, $\tilde{C}(a)^n = (\tilde{C}^n)^{S^n_a}$ and $\nu(a)^n = (\nu^n)^{S^n_a}$.

Finally, we say that an increasing càdlàg process B strong majorizes an increasing càdlàg process A , and we note $A \prec B$ if $A_t - A_s \leqslant B_t - B_s$, $\forall s \leqslant t$.

Theorem 4.2.2. *Suppose that the sequence $\mathcal{L}(X^n)$ weakly converges to a limit P, a probability measure on $\mathcal{B}(D[0,T])$ which only charges $C[0,T]$. Assume that for $t \in [0,1]$, $a > 0$ and $g \in C_1(\mathbb{R}^d)$*[1]*, we have:*

(*i*) *a*) $B(a)^n_t \to_P 0$,

b) $\tilde{C}(a)^n_t - C(a)_t \circ X^n \to_P 0$,

c) $g * \nu(a)^n_t \to_P 0$.

(*ii*) $P-a.s., \sum_{i,j\leqslant d} C(a)^{i,j} \prec F(a)$

where $s \mapsto F(a)_s$ is an increasing and continuous determinist function.

(*iii*) *the function $\alpha \mapsto C_t(\alpha)$ is P-a.s. Shorohod-continuous on $D[0,T]$.*

Then, X is a continuous P-local martingale and its quadratic characteristic is given by $\langle X, X\rangle = C$.

4.2.3 Reformulation of the Problem

To apply the above limit theorem we need to reformulate our problem in terms of semimartingales. To this end we consider the two-dimensional process $X^n = (X^{1n}, X^{2n})$ with

$$\begin{aligned} X^{1n}_t &= \sum_{i=0}^{n-1} S_{t_i} I_{[t_i, t_{i+1}[}(t), \quad X^{1n}_1 = X^{1n}_{t_{n-1}}, \\ X^{2n}_t &= \sum_{t^n_i \leqslant t} U^n_i, \quad i \leqslant n-1 \end{aligned}$$

where

$$U^n_i = Y^n_i + Z^n_i,$$

$$Y^n_i = \frac{\sigma^2}{2} n^{1/2} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} \left[\Delta t_i - (W_{t_i} - W_{t_{i-1}})^2\right]$$

and

$$Z^n_i = k_0 \sigma \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) S^2_{t_{i-1}} \left[\sqrt{\frac{2}{\pi}}\sqrt{\Delta t_i} - |W_{t_i} - W_{t_{i-1}}|\right].$$

[1] $C_1(\mathbb{R}^d)$ is a set of positive and bounded functions vanishing in a neighborhood of the origin.

We view the process X^n as defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}^n = (\mathcal{F}^n), P)$ with $\mathcal{F}^n_t = F_{t^n_{i-1}}$ for $t \in [t^n_{i-1}, t^n_i[$. We consider a fixed truncation function $H(x) = x\delta(x)$ where $\delta(x)$ is a continuous function verifying $0 \leqslant \delta(x) \leqslant 1$, $\delta(x) = 1$ if $|x| \leqslant 1$ and $\delta(x) = 0$ if $|x| > 2$. Then, H is clearly bounded and $|\delta(x) - 1| \leqslant I_{|x|>1}$. It is easily seen that the triplet of predictable characteristics of X^n associated with the truncation function H is $(B^n, 0, \nu^n)$ where

$$B^n_t = \sum_{t_i \leqslant t} E(H(\Delta S_{t_i}, U_i)|\mathcal{F}_{t_{i-1}}), \quad i \leqslant n-1$$

$$\nu^n([0,t] \times \Gamma) = \sum_{t_i \leqslant t} E(I_\Gamma(\Delta S_{t_i}, U_i)|\mathcal{F}_{t_{i-1}}), \quad i \leqslant n-1.$$

The components of the matrix-valued process $\tilde{C}^n_t$ are as follows:

$$\begin{aligned}
\tilde{C}^{n,1,1}_t &= \sum_{t_i \leqslant t} E\left((\Delta S_{t_i})^2 \delta^2(\Delta X^n_{t_i})|\mathcal{F}_{t_{i-1}}\right) - \left(E(\Delta S_{t_i}\delta(\Delta X^n_{t_i})|\mathcal{F}_{t_{i-1}})\right)^2, \\
\tilde{C}^{n,1,2}_t &= \tilde{C}^{n,2,1}_t = \sum_{t_i \leqslant t} E(\Delta S_{t_i} U_i \delta^2(\Delta X^n_{t_i})|\mathcal{F}_{t_{i-1}}) \\
&\qquad -E(\Delta S_{t_i}\delta(\Delta X^n_{t_i})|\mathcal{F}_{t_{i-1}}) E(U_i \delta(\Delta X^n_{t_i})|\mathcal{F}_{t_{i-1}}), \\
\tilde{C}^{n,2,2}_t &= \sum_{t_i \leqslant t} E(U_i^2 \delta^2(\Delta X^n_{t_i})|\mathcal{F}_{t_{i-1}}) - \left(E(U_i\delta(\Delta X^n_{t_i})|\mathcal{F}_{t_{i-1}})\right)^2, \quad i \leqslant n-1.
\end{aligned}$$

We define the matrix process

$$C = C(t, \alpha) = \int_0^t c(s, \alpha_s) ds, \quad \alpha = (\alpha_1, \alpha_2)$$

where:

$$\begin{aligned}
c^{11}(t,x) &= (\sigma x_1)^2, \\
c^{12}(t,x) &= c^{21}(t,x) = 0, \\
c^{22}(t,x) &= F^2(t, x_1).
\end{aligned}$$

We can observe that $c = c(t,x)$ is continuous in x for any $t < 1$.
For each $T \leqslant 1$, we note Y^T the process which is the restriction of Y on the interval $[0, T]$ and P^T the unique solution-measure of the following sde:

$$(S) \begin{cases} dX^1_t = & \sigma X^1_t dW_t, \\ dX^2_t = & F(t, X^1_t) dB_t, \\ X_0 = & (1,0),\ t \in [0,T]. \end{cases}$$

where (W, B) is a standard brownian motion.
We describe the steps which lead us to prove Theorem 4.1.1:

Step 1 : The sequence $X^{n,T}$ is C-tight for all $T \in [0,1]$.
Step 2 : The sequence $X^{n,T}$ converges weakly to P^T for all $T \in [0,1[$.
Step 3 : The sequence $X^{n,1}$ converges weakly to $P = P^1$.

4.2.4 Tightness

The process X^n is a locally square integrable locale martingale. In virtue of Theorem 4.13 p 322 in [16], it suffices to show that the sequence of processes $G^n = \langle X^{1n}, X^{1n}\rangle + \langle X^{2n}, X^{2n}\rangle$ defined on $[0,T]$ is C-tight to conclude that the sequence X^n is tight. But Lemma 4.3.1 claims that G^n converges in probability to

$$G_{\cdot} = \langle S, S\rangle_{\cdot} + \int_0^{\cdot} F^2(t, S_t)dt$$

uniformly on $[0,T]$. We can deduce that G^n, as a random variable from Ω^n to $\mathcal{D}[0,T]$, not only converges in probability to G according to the Shorohod topology but also converges weakly to G. So, the sequence G^n is tight.

Moreover, the continuity of G implies that the sequence G^n is C-tight. Finally, because of Lemma 4.3.2, we have $P(\sup_t |\Delta X_t^n| > \epsilon) \to 0$, $\forall \epsilon > 0$ which implies, according to Proposition 3.26 p 315 in [16], that the sequence X^n is C-tight.

4.2.5 Limit Measure

We choose the $\mathbb{R}^2$-norm defined by

$$|x| = |(x_1, x_2)| = Max(|x_1|, |x_2|).$$

For more convenience, we note $X^{n,T} = X^n$ where $T < 1$. We shall apply Theorem 4.2.2. From the previous step, we can assume that a subsequence of $\mathcal{L}(X^n)$ weakly converges to a limit P which only charges $C[0,T]$.

The condition $(i)a)$ is verified. Indeed, from Lemma 4.3.3, we have the convergence $P(\sup_t |B_t^n| \geqslant \epsilon) \to 0$.

In virtue of Lemma 4.3.4, we have:

$$P\left(\sup_t \left|\tilde{C}_t^n - \int_0^t c(s, X_s^n)ds\right| \geqslant \epsilon\right) \to 0,$$

and we deduce that $(i)b)$ is also verified.

Note that if $g \in C_1(\mathbb{R}^d)$ then g is bounded and there exists a constant r such that $g(x) = 0$ for $|x| \leqslant r$ (see definition p 354 in [16]). Moreover, from Chapter 3, there exists a constant c such that

$$E(\Delta S_{t_i})^4 + EU_i^4 \leqslant c\frac{(\Delta t_k)^2}{(1-t_{k-1})^{3/2}} \leqslant c_T(\Delta t_k)^2.$$

Using the Bienaymé–Tchebychev inequality, we deduce that $(i)c)$ is verified.

We have P- a.s., $S_a(\alpha) = \inf\{t : |\alpha(t)| \geqslant a\}$. It follows that, P- a.s., $|\alpha(s)| \leqslant a$ if $s \in [0, S_a(\alpha)]$. It suffices to consider

$$F(a)_s = \left(\sigma^2 a^2 + \frac{d}{1-T}a\right) s$$

where d is a constant to conclude that (ii) is verified.

Recall that a sequence $\alpha_n \to \alpha \in C[0,T]$ according to the Shorohod topology if and only if $\alpha_n \to \alpha$ locally uniformly. Then, we can easily conclude that $\alpha \mapsto C_t(\alpha)$ is P-a.s. continuous and (iii) holds.

From now on, we can conclude from Theorem 4.2.2 that X has for characteristic $(0, C, 0)$ and it is a continuous local martingale. We consider the two-dimensional process (W, B) defined as follows :

$$\begin{aligned} W_t &= \int_0^t \sigma^{-1}(X_u^1)^{-1} dX_u^1, \\ B_t &= \int_0^t F^{-1}(u, X_u^1) dX_u^2. \end{aligned}$$

It is easy to prove that (W, B) is a standard Brownian motion in virtue of the Levy characterization. Thus, X verifies the sde (S) and $P = P^T$.

4.2.6 Identification of the Limit

We consider the mapping $\Psi_T : \alpha \mapsto \alpha^T$ from $\mathcal{D}[0,1]$ to $\mathcal{D}[0,T]$ where α^T is the restriction of α on $\mathcal{D}[0,T]$. It is a continuous function according to the Shorohod topology (see 1.14 p 292 in [16]). We also define for any probability μ on $\mathcal{B}(\mathcal{D}[0,1])$, the probability $\mu_T(A) = \mu(\psi_T \in A)$ on $\mathcal{B}(\mathcal{D}[0,T])$. We note $A_T = \psi_T(A)$.

Then, it is easy to deduce that $P_T^1 = P^T$. Furthermore, we have the following lemma:

Lemma 4.2.3. *Assume that μ is a probability on $\mathcal{B}(\mathcal{D}[0,1])$ which only charges $C[0,1]$. Then, for any compact subset according to the Shorohod Topology which is included in $C([0,1)$, we have:*

$$\mu(A) = \lim_{T \nearrow 1} \mu_T(A_T).$$

Proof. We have $\mu_T(A_T) = E_\mu 1_{\alpha^T \in A_T}$ and $\mu(A) = E_\mu 1_{\alpha \in A}$.

If $\alpha \in A$ it is obvious that $\alpha^T \in A_T$. Hence $1_{\alpha^T \in A_T} \to 1_{\alpha \in A}$ as $T \nearrow 1$.

Suppose that $\alpha \notin A$ and $\alpha^T \in A_T$ for an infinite family of $T < 1$. Since μ only charges $C[0,1]$, we can assume that α is continuous. For each T, there exists $\widetilde{\alpha}_{(T)} \in A$ such that $\widetilde{\alpha}_{(T)}^T = \alpha^T$. Since A is a compact subset, we can assume that $\lim_{T \nearrow 1} \widetilde{\alpha}_{(T)} = \widetilde{\alpha} \in A$ where $\widetilde{\alpha}$ is continuous. It follows that $\widetilde{\alpha}_{(T)} \to \widetilde{\alpha}$ uniformly on $[0,1]$. We can deduce that $\widetilde{\alpha}(u) = \alpha(u)$ for any $u \in [0,1[$. We also have $|\widetilde{\alpha}_{(T)}(T) - \widetilde{\alpha}(T)| \to 0$ as $T \to 1$ whereas $\widetilde{\alpha}(T) \to \widetilde{\alpha}(1)$ and $\widetilde{\alpha}_{(T)}(T) = \alpha(T) \to \alpha(1)$ since $\alpha \in C[0,1]$. Finally, we have $\alpha = \widetilde{\alpha} \in A$ which leads to a contradiction. Then, $1_{\alpha^T \in A_T} \to 1_{\alpha \in A}$ μ -a.s. and $\mu_T(A_T) \to \mu(A)$.

We shall conclude about our main theorem. Assume that a subsequence of $\mathcal{L}(X^n)$ weakly converges to a limit Q, a probability measure on $\mathcal{B}(\mathcal{D}[0,1])$ which only charges $C[0,1]$ since X^n is C-tight. We deduce that $\mathcal{L}(X^{n,T})$ weakly converges to the limit Q_T, which is a probability measure on $\mathcal{B}(\mathcal{D}[0,T])$ for any $T < 1$. From the second step, it follows that $Q_T = P^T = P_T^1$ where Q and $P = P^1$ only charges $C[0,1]$. Lemma 4.2.3 implies that for any compact subset of $D[0,1]$, we have the following equalities:

$$\begin{aligned} Q(A) &= Q(A \cap C[0,1]) = \lim_{T \nearrow 1} Q_T\left(\{A \cap C[0,1]\}_T\right) \\ &= \lim_{T \nearrow 1} P_T^1\left(\{A \cap C[0,1]\}_T\right) = P^1(A \cap C[0,1]) = P^1(A). \end{aligned}$$

Recall that, for any σ-finite measure defined on the borels of a polish space, we have

$$\mu(B) = \sup_{K \in \mathcal{K}} \{\mu(K) : K \subset B\}$$

where $\mathcal{K}$ is the set of compacts. Then, we deduce that $Q = P^1$ and our main theorem is proved.

Note that we can follow an other method to establish the proof of Theorem 4.1.1 [7]. For this, it suffices to use Theorem IX.3.39 in [16] where local uniqueness property holds in virtue of Lemma IX.4.4.

4.3 Appendix

Lemma 4.3.1. *G^n converges in probability to the process*

$$G_{\cdot} = \langle S, S\rangle_{\cdot} + \int_0^{\cdot} F^2(t, S_t)dt$$

uniformly on $[0, T]$.

Proof. Note that:

$$\langle X^{1n}, X^{1n}\rangle_t = \sum_{t_i \leqslant t} E\left((\Delta S_{t_i})^2 | \mathcal{F}_{t_{i-1}}\right), \quad i \leqslant n-1.$$

Using the independence of the increments of the Wiener process we have

$$\langle X^{1n}, X^{1n}\rangle_t = \sum_{t_i \leqslant t} S^2_{t_{i-1}} E(S_{t_i}/S_{t_{i-1}} - 1)^2 = \sigma^2 \sum_{t_i \leqslant t} S^2_{t_{i-1}} \Delta t_i + o_n(1),$$

where $o_n(1)$ is a sequence of random variables converging to zero almost surely uniformly on $[0, 1]$. It follows that $\langle X^{1n}, X^{1n}\rangle \to \langle S, S\rangle$ in probability uniformly on [0,T].

We have

$$\langle X^{2n}, X^{2n}\rangle_t = \sum_{t_k \leqslant t} E(U_k^2 | \mathcal{F}_{t_{k-1}}), \quad k \leqslant n-1$$

where the independence of the increments of the Wiener process gives us:

$$E(U_k^2 | \mathcal{F}_{t_{k-1}}) = F^2(t_{k-1}, S_{t_{k-1}})\Delta t_k. \tag{4.3.2}$$

Here, there is an abuse of notation since the conditional expectation gives a similar expression to F whose the only difference is the multiplier where t_{k-1} is replaced by $t^*_{k-1} \in [t_{k-1}, t_k]$. We note

$$F_n^2(t, S) = \sum_{i=1}^{n-1} F^2(t_{i-1}, S_{t_{i-1}}) 1_{]t_{i-1}, t_i]}(t).$$

Then, we have

$$\sup_t \left| \langle X^{2n}, X^{2n}\rangle_t - \int_0^t F^2(s, S_s)ds \right| \leqslant \int_0^1 \left| F_n^2(t, X_t^{1n}) - F^2(t, S_t)\right| dt$$

which converges to 0, using the Lebesgue theorem. Indeed, it suffices to argue out of the null-set $S_1 \in \{K_1, \cdots, K_p\}$ where a.s., we have $\widehat{C}_{xx}(s, S_t)S_t^2 \leqslant const(\omega)$.

Lemma 4.3.2. $P\left(\sup_t |\Delta X_t^n| > \epsilon\right) \to 0, \quad \forall \epsilon > 0.$

Proof. The mapping $t \mapsto S_t$ is a.s. uniformly continuous in $[0,1]$ whereas, almost surely, there exists some constant $c(w)$ such that

$$\max_k |W_{t_k} - W_{t_{k-1}}| \leqslant c(w)\ln(n)n^{-1/2}$$

for n sufficiently large (see [20]). Moreover, recall that a.s. $\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})S_{t_{i-1}}^2$ is bounded, so the result follows.

Lemma 4.3.3. $P(\sup_t |B_t^n| \geqslant \epsilon) \to 0.$

Proof. Recall that $\overline{\delta}(x) := 1 - \delta(x)$ verifies $0 \leqslant \overline{\delta}(x) \leqslant I_{|x|>1}$. Moreover, we have

$$\begin{aligned} B_t^{n,1} &= -\sum_{t_k \leqslant t} E(\Delta S_{t_k}\overline{\delta}(\Delta X_{t_k}^n)|\mathcal{F}_{t_{k-1}}), \\ B_t^{n,2} &= -\sum_{t_k \leqslant t} E(U_k\overline{\delta}(\Delta X_{t_k}^n)|\mathcal{F}_{t_{k-1}}). \end{aligned}$$

Then, in order to prove convergence in L^1, it suffices to estimate the following sum:

$$\begin{aligned} \sup_t \sum_{t_k \leqslant t} E(|\Delta S_{t_k}|1_{\{|U_k|>1\}}) &\leqslant \sum_{t_k \leqslant 1} E(|\Delta S_{t_k}|\, U_k^2 1_{\{|U_k>1|\}}) \\ &\leqslant \sum_{k=1}^{n-1} \left(E(\Delta S_{t_k})^2\right)^{1/2} \left(EU_k^4\right)^{1/2}. \end{aligned}$$

Moreover, we know (see Chapter 3) that there exists a constant c such that

$$E(\Delta S_{t_k})^2 \leqslant c\Delta t_k, \quad EU_k^4 \leqslant \frac{(\Delta t_k)^2}{(1-t_{k-1})^{3/2}}.$$

So, we can conclude that

$$\sup_t \sum_{t_k \leqslant t} E(\Delta |S_{t_k}|1_{\{U_k>1\}}) \to 0.$$

The reasoning is the same for the other terms.

Lemma 4.3.4. $P\left(\sup_t \left|\tilde{C}_t^n - \int_0^t c(s, X_s^n)ds\right| \geqslant \epsilon\right) \to 0.$

Proof. If we note $\widetilde{\delta}(x) := 1 - \delta^2(x)$, we have also $0 \leqslant \widetilde{\delta}(x) \leqslant I_{|x|>1}$. Then, we have

$$\begin{aligned} \tilde{C}_t^{n,1,1} &= \sum_{t_i \leqslant t} E\left((\Delta S_{t_i})^2|\mathcal{F}_{t_{i-1}}\right) - E\left((\Delta S_{t_i})^2\widetilde{\delta}(\Delta X_{t_i}^n)|\mathcal{F}_{t_{i-1}}\right) \\ &\quad - \left(E(\Delta S_{t_i}\delta(\Delta X_{t_i}^n)|\mathcal{F}_{t_{i-1}})\right)^2, \quad i \leqslant n-1 \end{aligned}$$

where we have already proved that

$$\sum_{t_i \leqslant t} E\left((\Delta S_{t_i})^2|\mathcal{F}_{t_{i-1}}\right) \to [S,S]_t \text{ uniformly in probability }.$$

Furthermore, we can use the arguments of the previous lemma and the Jensen inequality to prove the uniform convergence in L^1 to 0 of the other terms. In a similar way, we have

$$\begin{aligned}\tilde{C}_t^{n,2,2} &= \sum_{t_i \leqslant t} E(U_i^2|\mathcal{F}_{t_{i-1}}) - E(U_i^2\widetilde{\delta}(\Delta X_{t_i}^n)|\mathcal{F}_{t_{i-1}}) \\ &- \left(E(U_i\delta(\Delta X_{t_i}^n)|\mathcal{F}_{t_{i-1}})\right)^2, \, i \leqslant n-1\end{aligned}$$

where

$$\sum_{t_i \leqslant t} E(U_i^2|\mathcal{F}_{t_{i-1}}) \to \int_0^t c^{22}(s, X_s^n)ds$$

uniformly on $[0,1]$ according to Lemma 4.3.1. The other terms converges uniformly to 0 in L^1 as previously.

4.3.1 Identification Theorem

We formulate a theorem on identification of the limit process suggested in [16] which is a little more general but adapted to our purposes.

Let $(\Omega, \mathcal{F}, F) = (D[0,T], \mathcal{D}, \mathbf{D} = (\mathcal{D}_t)_{t \leqslant T}, Q)$ be the natural stochastic basis constructed on the Shorohod space of d-dimensional càdlàg functions on $[0,T]$ and let $C[0,T]$ be its subspace formed by continuous functions. We suppose that $X = (X_t)_{t \in [0,T]}$ is the canonical process $X(\alpha) = \alpha$ defined on $\mathcal{D}[0,T]$. On this basis, we also consider:

- (i) $B = (B^i)_{i \leqslant d}$ a predictable process with finite variation, over finite intervals and $B_0 = 0$,
- (ii) $C = (C^{i,j})_{i,j \leqslant d}$ a continuous adapted process with $C_0 = 0$ and $C_t - C_s$ is a symmetric nonnegative matrix for all $s \leqslant t$,
- (iii) ν a predictable random measure on $\mathbb{R}^+ \times \mathbb{R}^d$ which charges neither $[0,T] \times 0$ nor $0 \times \mathbb{R}^d$, such that$(1 \wedge |x^2|) * \nu_t(w) < \infty$, $\int \nu(w, t \times dx)H(x) = \Delta B_t(w)$ and $\nu(w, t \times \mathbb{R}^d) \leqslant 1$ identically,
- (iv) the stopping time $S_a(\alpha) = inf\{t > 0 : |\alpha(t)| > a \text{ or } |\alpha(t-)| > a\}$,
- (vi) $B(a) = B^{S_a}$, $C(a) = C^{S_a}$ and $\nu(a) = \nu^{S_a}$.

H is a fixed continuous truncation function and we define

$$\tilde{C}^{i,j} = C^{i,j} + (H^i H^j) * \nu - \Sigma_{s \leqslant .} \Delta B_s^i \Delta B_s^j.$$

Let $X^n = (X_t^n)_{t \leqslant T}$ be a d-dimensional semimartingale defined on a stochastic basis $(\Omega^n, \mathcal{F}^n, \mathbf{F}^n, P^n)$ such that $X_0^n = X_0$ is a constant. Let μ^n and (B^n, C^n, ν^n) be the jump measure and the triplet of predictable characteristics of X^n.
We consider $S_a^n = S_a \circ X^n$, $B(a)^n = (B^n)^{S_a^n}$, $\tilde{C}(a)^n = (\tilde{C}^n)^{S_a^n}$ and $\nu(a)^n = (\nu^n)^{S_a^n}$.

Finally, we say that an increasing càdlàg process B strong majorizes an increasing càdlàg process A , and we note $A \prec B$ if $\forall s \leqslant t$, $A_t - A_s \leqslant B_t - B_s$.

Theorem 4.3.5. *Suppose that the sequence $\mathcal{L}(X^n)$ weakly converges to a limit P, a probability measure on $\mathcal{B}(D[0,T])$. Let D be a dense subset of $[0,T]$ which is contained in $J(X)^c$ where*

$$J(X) = \{t > 0 : P(\Delta X_t \neq 0) > 0\}.$$

Moreover, assume that for each $t \in D$, $a > 0$ and $g \in C_1(\mathbb{R}^d)$, we have:

$$\begin{array}{ll} (i) & a)B(a)^n_t - B(a)_t \circ X^n \to_P 0, \\ & b)\tilde{C}(a)^n_t - \tilde{C}(a)_t \circ X^n \to_P 0, \\ & c)g * \nu(a)^n_t - (g * \nu(a)_t) \circ X^n \to_P 0. \\ (ii) & P - a.s., \sum_{i,j \leqslant d} VarB(a)^i + \tilde{C}(a)^{i,j} + g * \nu(a) \prec F(a) \end{array}$$

where $s \mapsto F(a)_s$ is an increasing and continuous determinist function.

(iii) the function $\alpha \mapsto B_t(\alpha), \alpha \mapsto \tilde{C}_t(\alpha)$, and $\alpha \mapsto g * \nu_t(\alpha)$ are P-a.s. Shorohod-continuous on $D[0,T]$.

Then, X is a P-semimartingale with characteristics (B, C, ν).

Proof. According to [16], we introduce necessary (but sophisticated) notations in order to apply Theorem 2.21 page 80:

$$\begin{aligned} X'_t &= X_t - \sum_{s \leqslant t} \left[\Delta X_s - H(\Delta X_s)\right], \\ V_t &= X'_t - B_t - X_0, \\ X'^n_t &= X^n_t - \sum_{s \leqslant t} \left[\Delta X^n_s - H(\Delta X^n_s)\right], \\ V^n_t &= X'^n_t - B^n_t - X^n_0. \end{aligned}$$

Note that $X'^n = X' \circ X^n$. Recall that $C_2(\mathbb{R}^d)$ is defined page 354 in [16] as a subclass of all continuous bounded functions from $\mathbb{R}^d$ to $\mathbb{R}$ vanishing in a neighborhood of the origin and having a limit at infinity. Moreover, $C_1(\mathbb{R}^d)$ is defined as a subclass of $C_2(\mathbb{R}^d)$ having only nonnegative functions which contains all functions $g_a(x) = (a|x| - 1)^+ \wedge 1$ for all positive rationals a, and with the following property: let η_n, η be positive measures on $\mathbb{R}^d$ which do not charge $\{0\}$ and are finite on the complement of any neighborhood of 0; then $\eta_n(f) \to \eta(f)$ for all $f \in C_1(\mathbb{R}^d)$ implies $\eta_n(f) \to \eta(f)$ for all $f \in C_2(\mathbb{R}^d)$.
For a fixed $g \in C_1(\mathbb{R}^d)$, we define:

$$\begin{aligned} Z^{i,j} &= V^i V^j - \widetilde{C}^{i,j}, \\ N^g_t &= \sum_{s \leqslant t} g(\Delta X_s) - g * \nu_t, \\ Z^{n,i,j} &= V^{n,i} V^{n,j} - \widetilde{C}^{n,i,j}, \\ N^g_t &= \sum_{s \leqslant t} g(\Delta X^n_s) - g * \nu^n_t. \end{aligned}$$

We can claim that V^n is a local martingale. Indeed, since X^n has (B^n, C^n, ν^n) for triplet of predictable characteristics, we use Theorem 2.21 page 80 in [16] where, in virtue of 2.4 page 76, we have

$$X^n(H) = X^n - \check{X}^n(H) = X'^n.$$

We deduce that

$$M^n(H) = X^n(H) - B^n - X_0^n = X'^n - B^n - X^n = V^n.$$

Note that the jumps of $X^n(H)$ are bounded. Hence, X'^n is a special semi-martingale. In a similar way, because of Theorem 2.21, $M^n(H)^i M^n(H)^j - \widetilde{C}^{i,j}$ is a local martingale. Then, Z^n is a local martingale.
Taking $C^+(\mathbb{R}^d) = C_1(\mathbb{R}^d)$ where $C^+(\mathbb{R}^d)$ is defined by 2.20 page 80, we deduce that for any $g \in C_1(\mathbb{R}^d)$, $g * \mu^{X^n} - g * \nu^n$ is a local martingale. But, recall that

$$\mu^{X^n}(\omega, dt, dx) = \sum_s I_{\Delta X_s^n \neq 0} \delta_{s, \Delta X_s^n}(dt, dx).$$

Hence,

$$g * \mu_t^{X^n}(w) = \int_{[0,t]\times\mathbb{R}^d} g(x) \mu^{X^n}(\omega, dt, dx) = \sum_{s \leqslant t} g(\Delta X_s^n)$$

and $N^{n,g}$ is a local martingale.

In order to prove that X is a P-semimartingale with characteristics (B, C, ν), it suffices, in virtue of Theorem 2.21 page 80, to verify the following conditions:

(a) V is a local martingale,

(b) Z is a local martingale,

(c) N^g is a local martingale.

Condition(a). Since we can choose a sequence $T^n \in D$ converging to ∞, it suffices to prove that for any $T \in D$, V^T is a local martingale. But, since we also have $S_a(\alpha) \to \infty$ as $a \to \infty$, we shall prove that $M_t = V^i_{t\wedge T \wedge S_a}$, $i = 1, \cdots, d$ are local martingales.

From (ii), we deduce that there exists $K > 0$ such that P-a.s., $\widetilde{C}^{ii}_{T\wedge S_a(\alpha)}(\alpha) \leqslant K$. Let define the stopping time

$$T^n = inf\{t : \widetilde{C}^{ii}_{T\wedge S_a(X^n)} \geqslant K + 1\}.$$

We shall apply Proposition 1.12 p 484, which is a corollary of Theorem 1.4 p 482, with $Y^n = X^n$, $M^n_t = V^{n,i}_{t\wedge T\wedge S^n_a \wedge T^n}$, $Y = X$ and M. The needed conditions are fulfilled. Indeed, Y^n is càdlàg, and 1.4(ii) is verified: $\mathcal{L}(Y^n) = \mathcal{L}(X^n) \to P = \mathcal{L}(X) = \mathcal{L}(Y)$. We shall prove that M^n is a uniformly integrable martingale, i.e. 1.12(i') holds. First, we recall that M^n is a local martingale. Moreover, from 2.4 page 76, $V^n = M^n(H)$ comes from the canonical decomposition of the semi-martingale $X^n(H)$ whose the jumps are bounded. It follows from Theorem 4.24 page 44 that V^n has bounded jumps (as $B^n(H)$) and is a locally square integrable martingale: there exists a sequence of stopping times R^p increasing to ∞ such that $V^{nR^p} \in \mathcal{H}_2$. We can choose R^p such that $Z^{n\,R^p}$ is a uniformly integrable martingale. Then, from

$$Z^{n,ii\,R^p}_{T\wedge S^n_a \wedge T^n} = \left(V^{n,i\,R^p}_{T\wedge S^n_a\wedge T^n}\right)^2 - \widetilde{C}^{n,ii\,R^p}_{T\wedge S^n_a\wedge T^n}$$

we deduce that

$$E\left(V^{n,i\,R^p}_{T\wedge S^n_a\wedge T^n}\right)^2 = E\widetilde{C}^{n,ii\,R^p}_{T\wedge S^n_a\wedge T^n} \leqslant K + 1 + const.$$

The latter inequality comes from the definition of the stopping time T^n and the fact that the jumps of $\widetilde{C}^{i,j}$ are bounded by a constant which only depends on H. Indeed, recall that:

$$\tilde{C}^{n,i,j} = C^{n,i,j} + (H^i H^j) * \nu^n - \Sigma_{s\leqslant .}\Delta B^{n,i}_s \Delta B^{n,j}_s$$

where $C^{n,i,j}$ is continuous, $B^n = B^n(H)$ has bounded jumps, as already shown, whereas:

$$\begin{array}{rcl} |\Delta(H^i H^j) * \nu^n_t| & = & \lim\limits_{t^p \nearrow t} \left| \int_{]t^p,t]\times\mathbb{R}^d} H^i(x) H^j(x) \nu^n(w, ds, dx) \right| \\ & \leqslant & C\nu^n(w, \{t\} \times \mathbb{R}^d) \leqslant C \end{array}$$

where C is a constant (see 1.17 page 76). From now on, using the Doob inequality, we deduce that

$$E\left(\sup_{t\leqslant R^p} M^n_t\right)^2 \leqslant 4E\left(V^{n,i\,R^p}_{T\wedge S^n_a\wedge T^n}\right)^2 \leqslant const$$

and finally, as $p \to \infty$,

$$E\left(\sup_t M^n_t\right)^2 \leqslant const.$$

Then, M is a uniformly integrable martingale: $(1.4(i))$ replaced by $(1.12(i'))$ holds.

We shall prove that $(1.4(iii))$ holds. First, we prove that the mapping $\alpha \to X'_{t\wedge T\wedge S_a}(\alpha)$ is continuous where

$$X'_t = X_t - \sum_{s\leqslant t} \left[\Delta X_s - H(\Delta X_s)\right].$$

Recall that $x - h(x) = 0$ on a neighborhood of 0. Using Theorem 2.8 p 305 we deduce that the mapping

$$\alpha \to \sum_{s\leqslant t} \Delta X_s(\alpha) - H(\Delta X_s(\alpha))$$

is continuous. Moreover,

$$X_{t\wedge S_a(\alpha)}(\alpha) = \alpha(t \wedge S_a(\alpha)) = \alpha^{S_a}(t) = X_t(\alpha^{S_a}).$$

It follows that $X'_{t\wedge S_a(\alpha)}(\alpha) = X'_t(\alpha^{S_a})$. According to the proof of Proposition 1.17 page 485, the following sets

$$\begin{array}{rcl} \widetilde{V} & = & \{a > 0 : P\left(\alpha : a \in V(\alpha)\right) > 0\}, \\ \widetilde{V}' & = & \{a > 0 : P\left(\alpha : a \in V'(\alpha)\right) > 0\} \end{array}$$

are at most countable, and we recall that:

$$\begin{array}{rcl} V(\alpha) & = & \{a > 0 : S_a(\alpha) < S_{a+}(\alpha)\}, \\ V'(\alpha) & = & \{a > 0 : \Delta\alpha(S_a(\alpha)) \neq 0 \text{ and } |\Delta\alpha(S_a(\alpha)-)| = a\}. \end{array}$$

Then, we choose a out of $\widetilde{V} \cup \widetilde{V}'$ in order to have for each fixed a:

$$P\left(\alpha : a \in V(\alpha) \cup V'(\alpha)\right) = 0$$

and we apply Proposition 2.12 page 305 which claims that the mapping $\alpha \to \alpha^{S_a}$ is continuous at each point α such that $a \notin V(\alpha) \cup V'(\alpha)$. We deduce that, P-a.s., the mapping

$$X'_{t\wedge T\wedge S_a} : \alpha \to \alpha^{S_a} \to X'_{t\wedge T}(\alpha^{S_a})$$

is continuous. Indeed, from what precedes, it suffices to note that $t \wedge T \in D$ where $D \subset \mathbb{R}^+ \backslash J(X)$. It follows that, P-a.s.(α), $\Delta\alpha(t\wedge T) = 0$ and using Theorem 2.3 page 303, we deduce that the mapping $\alpha \to \alpha(t) = X_t(\alpha)$ is P-a.s. continuous.

In a similar way, we can claim that the mapping $\alpha \to B_{t\wedge T\wedge S_a}(\alpha)$ is P-a.s. continuous. For this, it suffices to apply Theorem 3.42 page 511 and Proposition 2.11 page 305. From all what precedes, we can conclude that for any $t \in D$, $\alpha \to M_t(\alpha)$ is P-a.s. continuous and $(1.4(iii))$ holds.

We shall prove that $(1.4(iv))$ holds. In virtue of the hypothesis $(i)b$,

$$\widetilde{C}^{n,ii}_{T\wedge S_a \circ X^n} - \widetilde{C}^{ii}_{T\wedge S_a} \circ X^n \to_{P^n} 0.$$

Moreover, recall that P-a.s., $\widetilde{C}^{ii}_{T\wedge S_a(\alpha)}(\alpha) \leqslant K$. Then,

$$P^n\left(\widetilde{C}^{n,ii}_{T\wedge S_a\circ X^n} > K+1\right) \to 0$$

as $n \to \infty$. In virtue of Proposition 2.17 page 79, $\widetilde{C}^{n,ii}$ is an increasing process. It follows that $\lim_n P^n(T^n < T) = 0$.

Moreover, on the set $\{T^n \geqslant T\}$, we have:

$$\begin{aligned} M^n_t &= V_{t\wedge T\wedge S_a\circ X^n} = X'^{i}_{t\wedge T\wedge S_a}(X^n) - B^{n,i}_{t\wedge T\wedge S^n_a} - X^{n,i}_0, \\ M_t \circ X^n &= V_{t\wedge T\wedge S_a} \circ X^n = X'^{i}_{t\wedge T\wedge S_a}(X^n) - B^{i}_{t\wedge T\wedge S_a}(X^n) - X^i_0. \end{aligned}$$

It follows that,

$$M^n_t - M_t \circ X^n = B^i_{t\wedge T\wedge S_a} \circ X^n - B^{n,i}_{t\wedge T\wedge S^n_a} + X^i_0 - X^{n,i}_0.$$

Recall that, by convention (see page 3), $\Delta X_0 = X_0 - X_{0-} = 0$, $\Delta X^{n,i}_0 = 0$. Then $0 \in \mathbb{R}^+\backslash J(X)$ and, according to Proposition 3.14 page 313, we can assume that $X^n_0 \to X_0$ where X is the canonical process (by hypothesis, we also have noted $X^n_0 = X_0$). Finally, using hypothesis $(i)a$, we can claim that $(1.4(iv))$ holds. Then, applying Theorem 1.4 page 482, we conclude that $M \circ Y$ is a martingale, i.e. $V^i_{t\wedge T\wedge S_a} \circ X$ is a martingale. Since X is the identity process, it follows that $V^i_{t\wedge T\wedge S_a}$ is also a martingale and finally, we deduce that V is a local martingale.

Condition(b). In a similar way, we shall prove that $Z^{i,j,T\wedge S_a}$ is a local martingale. We consider a constant K such that P-a.s. (α),

$$\widetilde{C}^{ii}_{t\wedge T\wedge S_a}(\alpha) + \widetilde{C}^{ij}_{t\wedge T\wedge S_a}(\alpha) \leqslant K$$

and the stopping time

$$T^n = \inf\{t : \widetilde{C}^{n,ii}_{t\wedge S^n_a} + \widetilde{C}^{n,ij}_{t\wedge S^n_a} \geqslant K+1\}.$$

We shall apply Proposition 1.12 p 484 with $Y^n = X^n$, $M_t^n = Z^{n,i,j}_{t\wedge T^n\wedge T\wedge S_a^n}, Y = X$ and $M_t = Z^{i,j}_{t\wedge T\wedge S_a}$. First, we prove that (1.12(i')) holds. We note that, as for (a), we have

$$\widetilde{C}^{n,ii}_{t\wedge T\wedge T^n\wedge S_a^n} + \widetilde{C}^{n,ij}_{t\wedge T\wedge T^n\wedge S_a^n} \leqslant const$$

and we recall that

$$M_t^n = V^{n,i}_{t\wedge T^n\wedge T\wedge S_a^n} V^{n,j}_{t\wedge T^n\wedge T\wedge S_a^n} - \widetilde{C}^{n,ij}_{t\wedge T\wedge T^n\wedge S_a^n}.$$

Moreover, in virtue of Lemma 3.34 page 382, there exists two constants K_1 and K_2 such that

$$E\sup_t \left(V^{n,i}_{t\wedge T^n\wedge T\wedge S_a^n}\right)^4 \leqslant (K_1\gamma + K_2)\sqrt{E\left(\widetilde{C}^{n,ii}_{t\wedge T\wedge T^n\wedge S_a^n}\right)^2}$$

where

$$\gamma = \sup_{t,\omega}\left|\Delta V^{n,i}_{t\wedge T^n\wedge T\wedge S_a^n}(\omega)\right|$$

is bounded (see previous remark for (a)) and

$$\widetilde{C}^{n,ii}_{t\wedge T\wedge T^n\wedge S_a^n} = \langle V^{n,i}, V^{n,i}\rangle_{t\wedge T^n\wedge T\wedge S_a^n}.$$

We deduce that

$$E\sup_t \left(V^{n,i}_{t\wedge T^n\wedge T\wedge S_a^n}\right)^4 \leqslant const$$

and using the Cauchy–Schwarz inequality:

$$E\sup_t \left(V^{n,i}_{t\wedge T^n\wedge T\wedge S_a^n}\right)^2 \left(V^{n,j}_{t\wedge T^n\wedge T\wedge S_a^n}\right)^2 \leqslant const.$$

It follows that

$$E\sup_t (M_t^n)^2 \leqslant const$$

and M is a uniform integrable martingale, i.e. $(1.4(i))$ replaced by $(1.12(i'))$ holds.

We shall prove that $(1.4(iii))$ holds. Recall that

$$M_t = V^i_{t\wedge T\wedge S_a} V^j_{t\wedge T\wedge S_a} - \widetilde{C}^{i,j}_{t\wedge T\wedge S_a}$$

where we have already shown that the mapping $\alpha \to V^i_{t\wedge T\wedge S_a} V^j_{t\wedge T\wedge S_a}(\alpha)$ is P-a.s. continuous (see (a)). But, in virtue of Theorem 3.42 page 511, we can claim that the mapping $\alpha \to \widetilde{C}^{i,j}_{t\wedge T\wedge S_a}(\alpha)$ is also continuous. Then, $(1.4(iii))$ holds.

We shall prove that $(1.4(iv))$ holds. We also have $\lim_n P^n(T^n < T) = 0$. Moreover, we have

$$\begin{aligned} M_t^n - M_t\circ X^n &= V^{n,i}_{t\wedge T\wedge T^n\wedge S_a^n}\left(V^{n,j}_{t\wedge T\wedge T^n\wedge S_a^n} - V^j_{t\wedge T\wedge S_a}\circ X^n\right) \\ &+ V^j_{t\wedge T\wedge S_a}\circ X^n\left(V^{n,i}_{t\wedge T\wedge T^n\wedge S_a^n} - V^i_{t\wedge T\wedge S_a}\circ X^n\right) \\ &+ \widetilde{C}^{i,j}_{t\wedge T\wedge S_a}\circ X^n - \widetilde{C}^{n,i,j}_{t\wedge T\wedge T^n\wedge S_a^n}. \end{aligned}$$

We have already shown that $\left(V^{n,i}_{t\wedge T\wedge T^n\wedge S_a^n}\right)_t$ is uniformly integrable and

$$\left(V^{n,j}_{t\wedge T\wedge T^n\wedge S_a^n} - V^j_{t\wedge T\wedge S_a}\circ X^n\right) \to_{P^n} 0.$$

It follows that

$$V^{n,i}_{t\wedge T\wedge T^n\wedge S^n_a}\left(V^{n,j}_{t\wedge T\wedge T^n\wedge S^n_a} - V^j_{t\wedge T\wedge S_a}\circ X^n\right) \to_{P^n} 0.$$

Moreover, on the set $\{T^n \geqslant T\}$, we have P-a.s.

$$V^{n,i}_{t\wedge T\wedge T^n\wedge S^n_a} - V^i_{t\wedge T\wedge S_a}\circ X^n = B^i_{t\wedge T\wedge S_a}\circ X^n - B^{n,i}_{t\wedge T\wedge S^n_a}$$

which converges to 0 in probability according to the hypothesis $(i)a$). Always on $\{T^n \geqslant T\}$, we have:

$$V^j_{t\wedge T\wedge S_a}\circ X^n = \left(V^j_{t\wedge T\wedge S_a}\circ X^n - V^{n,j}_{t\wedge T\wedge S^n_a}\right) + V^{n,j}_{t\wedge T\wedge T^n\wedge S^n_a}$$

where the first term of the right hand side converges to 0 in probability, whereas the second term is uniformly integrable. Then, we deduce that

$$V^j_{t\wedge T\wedge S_a}\circ X^n\left(V^{n,i}_{t\wedge T\wedge T^n\wedge S^n_a} - V^i_{t\wedge T\wedge S_a}\circ X^n\right) \to_{P^n} 0$$

Finally, using the hypothesis $(i)b$), we deduce that $M^n_t - M_t\circ X^n \to_{P^n} 0$ and the condition (1.4(iv)) holds. We can conclude that $Z^{i,j}$ is a local martingale.

Condition(c). We shall prove that $M_t = N^g_{t\wedge T\wedge S_a}$ is a local martingale. Recall that there exists a constant K such that $g * \nu_{T\wedge S_a} \leqslant K$. We consider

$$\begin{aligned} T^n &= \inf\{t : g * \nu^n_{t\wedge S_a} \geqslant K+1\}, \\ Y^n &= X^n, \quad Y = X, \quad M^n_t = N^{n,g}_{t\wedge T\wedge S^n_a\wedge T^n}. \end{aligned}$$

We can write

$$\begin{aligned} M^n_t &= \sum_{s\leqslant t\wedge T\wedge S^n_a\wedge T^n} g(\Delta X^n_s) - g*\nu^n_{t\wedge T\wedge S^n_a\wedge T^n}, \\ &= g*(\mu^{X^n} - \nu^n)_{t\wedge T\wedge S^n_a\wedge T^n}, \\ &= gI_{[0,T^n\wedge T\wedge S^n_a]} * (\mu^{X^n} - \nu^n)_t \end{aligned}$$

where $gI_{[0,T^n\wedge T\wedge S^n_a]}$ is $\widetilde{\mathcal{P}}$-measurable. Then, according to Lemma 4.3.6, we have

$$C(gI_{[0,T^n\wedge T\wedge S^n_a]})_t \leqslant g^2 I_{[0,T^n\wedge T\wedge S^n_a]} * \nu^n_t \leqslant g^2 * \nu^n_{t\wedge T^n\wedge T\wedge S^n_a}$$

where C is defined in 4.3.6. Moreover, $g^2*\nu^n_{t\wedge T^n\wedge T\wedge S^n_a}$ is bounded on $[0, T^n[$ whereas, because of the bound of g, we have:

$$g^2 * \nu^n_{T^n} - g^2 * \nu^n_{T^n-} = \int_{\mathbb{R}^d} g^2(x)\nu^n(\omega, \{T^n(\omega)\}\times dx) \leqslant const.$$

It follows that $C(gI_{[0,T^n\wedge T\wedge S^n_a]})_t \leqslant const$ and using 1.33 p 73, we have

$$\langle M^n, M^n\rangle_t = C(gI_{[0,T^n\wedge T\wedge S^n_a]})_t \leqslant const.$$

We can conclude that $E(M^n_\infty)^2 = E\langle M^n, M^n\rangle_\infty \leqslant const$ and (1.4(i)) replaced by (1.12(i')) p 484 in Proposition 1.12 of [16] holds.

Condition (1.4(ii)) is naturally verified. In a similar way as for (b), (1.4(iii)) holds. Moreover, on the set $\{T^n \geqslant T\}$, we have

$$\begin{aligned} M_t^n - M_t \circ X^n &= N^{n,g}_{t\wedge T\wedge S_a^n} - N^g_{t\wedge T\wedge S_a} \circ X^n, \\ &= g * \nu_{t\wedge T\wedge S_a} \circ X^n - g * \nu^n_{t\wedge T\wedge S_a} \end{aligned}$$

which converges, by hypothesis, to 0 in probability. Then, (1.4(iv)) holds and we can conclude, using Proposition 1.4 p 482 in [16] that N^g is a local-martingale. Finally, we can conclude about Theorem 4.3.5.

Recall that, if W is $\widetilde{\mathcal{P}}$-measurable, we define $C(W)$ by the formula 1.31 p 73 in [16]:

$$C(W)_t = (W - \widehat{W})^2 * \nu_t + \sum_{s\leqslant t}(1-a_s)(\widehat{W}_s)^2$$

where $a_s = \nu\left(\omega, \{s\} \times \mathbb{R}^d\right) \in \{0,1\}$.

Lemma 4.3.6. *If W is $\widetilde{\mathcal{P}}$-measurable, then $C(W)$ is also given by the formula*

$$C(W)_t = W^2 * \nu_t - \sum_{s\leqslant t} \widehat{W}_s^2.$$

Proof. According to Proposition 1.14 p 68,

$$D = \{(\omega,t) : \nu\left(\omega, \{t\} \times \mathbb{R}^d\right) = 1\} = \{(\omega,t) : a_t(\omega) > 0\}$$

is a random set. Then, for each fixed ω, the set $\{t : (w,t) \in D\}$ is countable and according to 1.14,

$$W * \nu_t = \sum_{s\leqslant t} W(s,\beta_s) I_D(s) = \sum_{s\leqslant t} W(s,\beta_s) a_s$$

where β is an $\mathbb{R}^d$-valued optional process. Moreover, we can write

$$C(W)_t = W^2 * \nu_t - 2W\widehat{W} * \nu_t + \widehat{W}^2 * \nu_t + \sum_{s\leqslant t} \widehat{W}_s^2 - \sum_{s\leqslant t} a_s \widehat{W}_s^2$$

where

$$\begin{aligned} \widehat{W}^2 * \nu_t(\omega) &= \int_{[0,t]\times\mathbb{R}^d} \widehat{W}^2(\omega,s)\nu(\omega,ds,dx), \\ \widehat{W}(\omega,s) &= \int_{\mathbb{R}^d} W(\omega,s,x)\nu(w,\{s\}\times dx). \end{aligned}$$

Since $\widehat{W}(\omega,s) = 0$ if $a_s = 0$, we deduce that

$$\widehat{W}^2 * \nu_t(\omega) = \sum_{s\leqslant t} \widehat{W}_s^2 a_s$$

and in a similar way, we have

$$W\widehat{W} * \nu_t = \sum_{s\leqslant t} \widehat{W}_s^2.$$

So, we can conclude about the lemma.

Chapter 5

Leland's Approximations when the Volatility is not Constant

In the previous chapters, we have applied the Leland method to pricing contingent claims under proportional transaction costs in the case where the volatility parameter, for the better, depends on t, the current time. From now on, we take interest in the model where the volatility varies also according to the price of the risky asset and we prove that the convergence in probability always holds if $\alpha \in]1/4, 1/2]$ under reasonable assumptions.

5.1 Theorems

We consider the standard two-asset model with the time horizon $T = 1$ assuming that it is specified under the martingale measure. The non-risky asset is the *numéraire*, and the price of the risky asset is given by the stochastic equation

$$dS_t = S_t \sigma(t, S_t) dW_t$$

where W is a Wiener process. Note that S is a strictly positive and continuous martingale verifying, in virtue of Theorem 2.3 p 107 in [10],

$$E \sup_{t \in [0,1]} S_t^{2m} < \infty, \quad \forall m \in \mathbb{R}.$$

We assume that $\sigma(t, x)$ is a strictly positive and continuous function on $[0, 1] \times \mathbb{R}^+$ verifying

$$0 < \underline{\sigma} \leqslant \sigma(t, x) \leqslant \overline{\sigma}$$

where $\underline{\sigma}$, $\overline{\sigma}$ are two constants.

In the model with proportional transaction costs and a finite number of revisions, the current value of the portfolio process at time t is described as

$$V_t^n = V_0^n + \int_0^t H_u^n dS_u - \sum_{t_i \leqslant t} k_n S_{t_i} |H_{i+1}^n - H_i^n| \tag{5.1.1}$$

where H^n is a piecewise-constant process with $H^n = H_i^n$ on the interval $]t_{i-1}, t_i]$, $t_i = t_i^n$, $i \leqslant n$, are the revision dates, and H_i^n are $\mathcal{F}_{t_{i-1}}$-measurable random variables. We assume that the transaction costs coefficient is

$$k = k_n = k_0 n^{-\alpha}, \qquad \alpha \in]1/4, 1/2], \tag{5.1.2}$$

and the dates t_i are defined by a strictly increasing function $g \in C^2[0,1]$ with $g(0) = 0$, $g(1) = 1$ so that $t_i = g(i/n)$. Let denote by f the inverse of g. The "enlarged volatility", in general depending on n, is given by the formula

$$\widehat{\sigma}^2(t,x) = \sigma^2(t,x) + \sigma(t,x)\gamma_n(t) \tag{5.1.3}$$

where

$$\gamma_n(t) = k_n n^{\frac{1}{2}} \sqrt{8/\pi}\sqrt{f'(t)}.$$

We shall use the following hypothesis on the "cadence" of revisions:

Assumption (G): $g' > 0$ and f'' is bounded.

We use the abbreviations $\widehat{H}_t = \widehat{C}_x(t, S_t)$ and $\widehat{h}_t = \widehat{C}_{xx}(t, S_t)$ where $\widehat{C}$ is the "Leland Strategy" defined later by the PDE

$$(\mathbf{e}) = \begin{cases} \widehat{C}_t(t,x) & + & \frac{1}{2}\widehat{\sigma}^2(t,x)x^2\widehat{C}_{xx}(t,x) = 0, \quad (x,t) \in]0,\infty[\otimes[0,1[\\ \widehat{C}(1,x) & = & h(x), \quad x \in]0,\infty[. \end{cases}$$

Of course, we define $H_i^n := \widehat{H}_{t_{i-1}}$.

Our hypothesis on the pay-off function is as follows:

Assumption ($\widetilde{H}$): h is a continuous function on $[0,\infty[$ which is once differentiable except the points $K_1 < \cdots < K_p < \cdots$ where h' admits right and left limits. Moreover, h verifies the Lipschitz condition $|h(x) - h(y)| \leqslant L|x - y|$ (h' is bounded).

Now, we give some hypotheses on σ in order to ensure the existence of a solution for the following PDE (e) ($k_0 = 4$ is sufficient for our needs).

Assumption (E): There exists some positive constant K such that for $1 \leqslant k \leqslant k_0$

$$\begin{array}{ll} a) & |\sigma(t,x) - \sigma(t',x')| \leqslant K\left(|x - x'| + |t - t'|\right), \\ b) & \dfrac{\partial^k}{\partial x^k}\sigma(t,x), \dfrac{\partial}{\partial t}\sigma(t,x) \text{ are continuous }, \\ c) & \left|x^k \dfrac{\partial^k}{\partial x^k}\sigma(t,x)\right| + \left|\dfrac{\partial^2}{\partial x^2}\sigma(t,x)\right| + \left|\dfrac{\partial^2}{\partial x \partial t}\sigma(t,x)\right| \leqslant K, \\ d) & \sigma(t,x) + x\sigma_x(t,x) \geqslant \mathit{const} > 0 \end{array}$$

We shall prove later the following results:

Theorem 5.1.1. *Assume that $\alpha \in]1/4, 1/2]$ and the conditions (E), (G), $(\widetilde{H})$ hold. Moreover, suppose that $\widehat{C}_{xx} \geqslant 0$. Then, V_1^n converges in probability to $h(S_1)$.*

Theorem 5.1.2. *Assume that $\alpha = 1/2$ and the conditions (E), (G), $(\widetilde{H})$ hold. Then, V_1^n converges in probability to*

$$h(S_1) + \frac{1}{2}\int_0^1 \sigma(t,S_t)\gamma(t)S_t^2\left(\widehat{C}_{xx}(t,S_t) - |\widehat{C}_{xx}(t,S_t)|\right)dt$$

where $\gamma(t) = k_0\sqrt{8/\pi}\sqrt{f'(t)}$.

Remark 5.1.3. *In the case where $\alpha = 1/2$, $\widehat{C}$ does not depend on n.*

Remark 5.1.4. *In the case where h is convex and $\widehat{\sigma}$ does not depend on t, we can prove that $\widehat{C}_{xx} \geqslant 0$ (Lemma 5.6.7).*

5.2 The Leland Strategy

In the Black–Scholes model, the hedging portfolio is $C(t, S_t)$ where C is the solution of the PDE

$$(\mathbf{e_0}) = \begin{cases} C_t(t,x) & + \quad \frac{1}{2}\sigma^2(t,x)x^2 C_{xx}(t,x) = 0, \quad (x,t) \in]0,\infty[\times[0,1[\\ C(1,x) & = \quad h(x), \quad x \in]0,\infty[\end{cases}$$

It exactly replicates the contingent claim $h(S_1)$ and verifies:

$$C(t, S_t) = Eh(S_1) + \int_0^t C_x(u, S_u) dS_u.$$

Under transaction costs, Leland suggested in his famous paper [21] to substitute the volatility σ by an artificially enlarged one, $\widehat{\sigma}$. The idea is to consider the following PDE

$$(\mathbf{e}) = \begin{cases} u_t(t,x) & + \quad \frac{1}{2}\widehat{\sigma}^2(t,x)x^2 u_{xx}(t,x) = 0, \quad (x,t) \in]0,\infty[\otimes[0,1[\\ u(1,x) & = \quad h(x), \quad x \in]0,\infty[\end{cases}$$

and to define $\widehat{\sigma}$ in order to take transaction costs into account. Precisely, the Ito Formula implies that the possible smooth solution $\widehat{C}$ of (e) verifies

$$\widehat{C}(t, S_t) \quad = \quad \widehat{C}(0, S_0) + \int_0^t \widehat{C}_x(u, S_u) dS_u + \frac{1}{2}\int_0^t \left[\sigma^2(u, S_u) - \widehat{\sigma}^2(u, S_u)\right] S_u^2 \widehat{C}_{xx}(u, S_u) du.$$

Then, $\widehat{C}$ may be a portfolio process as defined above provided that the last term of the right hand side of the previous formula corresponds to the transaction costs, i.e. we want to make equal the two following increments :

$$\frac{1}{2}\left[\sigma^2(u, S_u) - \widehat{\sigma}^2(u, S_u)\right] S_u^2 \widehat{C}_{xx}(u, S_u)\Delta u = -k_0 n^{-\alpha} \left|\widehat{C}_x(u + \Delta u, S_{u+\Delta u}) - \widehat{C}_x(u, S_u)\right| S_{u+\Delta u}.$$

For this, we write

$$\begin{aligned} \widehat{C}_x(u + \Delta u, S_{u+\Delta u}) - \widehat{C}_x(u, S_u) \quad &= \quad \widehat{C}_{xt}(u, S_u)\Delta_u + \widehat{C}_{xx}(u, S_u)\left(S_{u+\Delta u} - S_u\right), \\ &\simeq \quad \widehat{C}_{xx}(u, S_u)\left(S_{u+\Delta u} - S_u\right) \end{aligned}$$

where

$$S_{u+\Delta u} - S_u = \sigma(u, S_u) S_u \left(W_{u+\Delta u} - W_u\right).$$

Assuming that $\widehat{C}_{xx} \geqslant 0$, we deduce the equality

$$\frac{1}{2}\left[\sigma^2(u, S_u) - \widehat{\sigma}^2(u, S_u)\right]\Delta u = -k_0 n^{-\alpha}\sigma(u, S_u)\left|W_{u+\Delta u} - W_u\right| \frac{S_{u+\Delta u}}{S_u}.$$

Then, considering the conditional expectation knowing $\mathcal{F}_u$, the fact that

$$E|W_{\Delta u}| = \sqrt{\Delta u}\sqrt{\frac{2}{\pi}},$$

and
$$\frac{S_{u+\Delta u}}{S_u} = 1 + \sigma(u, S_u)\,(W_{u+\Delta u} - W_u)$$
we obtain, considering only the main terms, that
$$\frac{1}{2}\left[\sigma^2(u, S_u) - \widehat{\sigma}^2(u, S_u)\right]\Delta u = -k_0 n^{-\alpha}\sigma(u, S_u)\sqrt{\Delta_u}\sqrt{\frac{2}{\pi}}.$$
But, from $u = g(t)$ we deduce that $\Delta u = g'(t)\Delta t = g'(f(u))\Delta t$ where $\Delta t = 1/n$. So, we can conclude that
$$\widehat{\sigma}^2(u, S_u) = \sigma^2(u, S_u) + k_0 n^{1/2-\alpha}\sqrt{\frac{8}{\pi}}\sigma(u, S_u)\sqrt{f'(u)}.$$

Proposition 5.2.1. *Under the assumptions* **(E)** *and* $(\widetilde{H})$, *the PDE* (**e**) *has a unique solution.*

Proof. Note that we can't immediately conclude about the existence of a solution for (**e**) because our operator is not uniformly parabolic on $]0,\infty[\otimes[0,1[$. That's why, we shall bring the problem back to one for which the domaine verifies the needed uniform parabolicity .

In virtue of Lemma 5.6.1, we consider the unique solution $\widehat{S}_{x,s}(t)$ of the stochastic equation defined on $[s,1]$ for all $s \in [0,1]$ by :
$$\begin{cases} d\widehat{S}_{x,s}(t) &=& \widehat{\sigma}(t, \widehat{S}_{x,s}(t))\widehat{S}_{x,s}(t)dW_t \\ \widehat{S}_{x,s}(s) &=& x \end{cases}$$
verifying
$$E\sup_{s\leqslant t\leqslant 1}\widehat{S}^2_{x,s}(t) \leqslant C^*(1+x^2)$$
where C^* is a constant. We define $g(x,t) = Eh(\widehat{S}_{x,t}(1))$ which verifies
$$\begin{aligned} |g(x,t)| &\leqslant const\left(1 + E|\widehat{S}_{x,t}(1)|\right) \leqslant const\left(1 + (E\widehat{S}^2_{x,t}(1))^{1/2}\right) \\ &\leqslant const\,(1+|x|). \end{aligned}$$
Since h' is bounded, we obtain, using the Cauchy-Schwarz inequality and the Lipschitz condition verified by h, that
$$|g(x,t) - g(y,u)| \leqslant L\sqrt{E\left(\widehat{S}_{x,t}(1) - \widehat{S}_{y,u}(1)\right)^2}.$$

From Lemma 3.3 p 112 with Condition (A') p 113 [10], we deduce the existence of a constant C_R such that
$$|g(x,t) - g(y,u)| \leqslant C_R\sqrt{(x-y)^2 + |t-u|}$$
if $|x|, |y| \leqslant R$. It follows that g is continuous.
Using the notations of page 138 [10], written for t replaced by $1-t$, we consider the

following sets for $m \in \mathbb{N}^*$:

$$\begin{aligned}
Q_m &=]\frac{1}{m}, m[\times]0,1[,\\
B_m &=]\frac{1}{m}, m[\times\{1\},\\
T_m &=]\frac{1}{m}, m[\times\{0\},\\
S_m &= \{\frac{1}{m}, m\} \times [0,1[.
\end{aligned}$$

For each $y \in \partial Q_m$, it is easy to observe that there exists a closed ball K_y^m such that $K_y^m \cap Q_m = \emptyset$ and $K_y^m \cap \overline{Q_m} = \{y\}$. Then, the function W_y proposed p 134 [10] defines a *barrier* for each $y \in S_m \subseteq \partial Q_m$. At last, we have $g(x,t) = Eh(\widehat{S}_{x,1}(1)) = h(x)$ if $(x,t) \in \overline{B_m} \cap \overline{S_m}$. We can deduce that, under the assumptions **(E)** and **($\widetilde{H}$)**, the Dirichlet problem

$$(\mathbf{D_m}) = \left\{ \begin{array}{lcll} u_t(t,x) & + & \frac{1}{2}\widehat{\sigma}^2(t,x)x^2u_{xx}(t,x) = 0 & (x,t) \in Q_m \cup T_m \\ u(T,x) & = & h(x) & x \in B_m \\ u(t,x) & = & g(x,t) & (x,t) \in S_m \end{array} \right.$$

has a unique solution u_m according to Theorem 3.6 p 138 [10]. Indeed, g and h are continuous whereas, $\overline{Q_m}$ being bounded, the following condition holds:

$$\left|\widehat{\sigma}^2(t,x)x^2 - \widehat{\sigma}^2(t,\overline{x})\overline{x}^2\right| \leqslant const(m)|x-\overline{x}|.$$

We note that u_m is assumed continuous on $\overline{Q_m}$, whereas the derivatives are continuous on $Q_m \cup T_m$. Moreover, we also have

$$\left|\widehat{\sigma}^2(t,x)x^2 - \widehat{\sigma}^2(\overline{t},\overline{x})\overline{x}^2\right| \leqslant c(m)\left(|x-\overline{x}| + |t-\overline{t}|\right).$$

Then, Theorem 5.2 p 147 [10] asserts that u_m has the representation

$$u_m(x,t) = Eg(\widehat{S}_{x,t}(\tau),\tau)I_{\tau<1} + Eh(\widehat{S}_{x,t}(1))I_{\tau=1},$$

where $h(x) = g(x,1)$ and τ is a stopping time. It follows that $u_m(x,t) = Eg(\widehat{S}_{x,t}(\tau),\tau)$. But we have

$$g(\widehat{S}_{x,t}(\tau),\tau) = Eh\left(\widehat{S}_{\widehat{S}_{x,t}(\tau),\tau}(1)\right)$$

where we have clearly $\widehat{S}_{\widehat{S}_{x,t}(\tau),\tau}(1) = \widehat{S}_{x,t}(1)$. It follows that $u_m(x,t) = g(x,t)$. Finally, we can deduce that we have a unique solution $u(x,t) = g(x,t)$ to the PDE

$$(\mathbf{e}) = \left\{ \begin{array}{lcll} u_t(t,x) & + & \frac{1}{2}\widehat{\sigma}^2(t,x)x^2u_{xx}(t,x) = 0, & (x,t) \in]0,\infty[\times[0,1[\\ u(1,x) & = & h(x), \quad x \in]0,\infty[& \end{array} \right.$$

Indeed, from what precedes, it is easy to show that g verifies (e). Moreover, if we consider $v(t,y) = u(t,e^y)$, then we deduce easily that v verifies the following uniformly parabolic PDE

$$\left\{ \begin{array}{lcll} v_t(t,y) & + & \frac{1}{2}\widehat{\sigma}^2(t,e^y)v_{yy}(t,y) & -\frac{1}{2}\widehat{\sigma}^2(t,e^y)v_y(t,y) = 0, \quad (x,t) \in \mathbb{R}\times[0,1[\\ v(1,y) & = & h(e^y), \quad x \in \mathbb{R}. & \end{array} \right.$$

It suffices to apply Corollary 4.2 page 140 [10] to conclude to the uniqueness of the solution v. Hence, u is also unique.

From now on, we define the Leland Strategy as the unique solution $\widehat{C}(t,x)$ of (**e**) given by:

$$\widehat{C}(t,x) = Eh(\widehat{S}_{x,t}(1)). \tag{5.2.4}$$

Let define

$$\widehat{\Lambda}(t,x) := (\widehat{\sigma}(t,x) + x\widehat{\sigma}_x(t,x))\,\widehat{\sigma}(t,x)$$

and, in virtue of Lemma 5.6.2, we consider the solution $\widetilde{S}_{x,t}$ of the sde:

$$\left\{\begin{array}{rcl} d\widetilde{S}_{x,t}(u) & = & \widehat{\sigma}(u,\widetilde{S}_{x,t}(u))\widetilde{S}_{x,t}(u)dW_u + \widehat{\Lambda}(u,\widetilde{S}_{x,t}(u))\widetilde{S}_{x,t}(u)du \\ \widetilde{S}_{x,t}(t) & = & x. \end{array}\right.$$

Then, we have:

Lemma 5.2.2. $\widehat{C}_x(t,x) = Eh'\left(\widetilde{S}_{x,t}(1)\right)$.

Proof. We write:

$$\begin{array}{rcl} \widehat{C}(t,x) - \widehat{C}(t,x_0) & = & Eh(\widehat{S}_{x,t}(1)) - Eh(\widehat{S}_{x_0,t}(1)), \\ \widehat{C}(t,x) - \widehat{C}(t,x_0) & = & E\displaystyle\int_0^1 \frac{d}{d\mu}h\left(\widehat{S}_{x_0,t}(1) + \mu(\widehat{S}_{x,t}(1) - \widehat{S}_{x_0,t}(1))\right)d\mu. \end{array}$$

Since h' exists out of a countable set, we can claim that

$$\frac{\widehat{C}(t,x) - \widehat{C}(t,x_0)}{x - x_0} = E\int_0^1 h'\left(\widehat{S}_{x_0,t}(1) + \mu(\widehat{S}_{x,t}(1) - \widehat{S}_{x_0,t}(1)\right)\frac{\widehat{S}_{x,t}(1) - \widehat{S}_{x_0,t}(1)}{x - x_0}d\mu.$$

Under the assumption (**E**), we apply Theorem 5.12 p120 [10] and we deduce that $\partial\widehat{S}_{x,t}(1)/\partial x$ exists in the L^2 sense, i.e.:

$$\frac{\widehat{S}_{x,t}(1) - \widehat{S}_{x_0,t}(1)}{x - x_0} \to \frac{\partial\widehat{S}_{x_0,t}(1)}{\partial x} \text{ in } L^2.$$

Indeed, it suffices to verify that Condition (A) page 108 [10] holds for the sde of Lemma 5.6.1. First, we have $|\widehat{\sigma}(t,x)x| \leqslant \mathit{const}\,|x|$ and secondly:

$$|\widehat{\sigma}(t,x)x - \widehat{\sigma}(t,\overline{x})\overline{x}| \leqslant |\widehat{\sigma}(t,x)|\,|x - \overline{x}| + |\overline{x}\,(\widehat{\sigma}(t,x) - \widehat{\sigma}(t,\overline{x}))|$$

where

$$\widehat{\sigma}(t,x) - \widehat{\sigma}(t,\overline{x}) = \widehat{\sigma}_x(t,x_0)(x - \overline{x}), \quad x_0 \in [x,\overline{x}].$$

Then, we write:

$$|\overline{x}\,(\widehat{\sigma}(t,x) - \widehat{\sigma}(t,\overline{x}))| \leqslant |\widehat{\sigma}(t,x) - \widehat{\sigma}(t,\overline{x})|\,|\overline{x} - x_0| + |x_0\widehat{\sigma}_x(t,x_0)|\,|x - \overline{x}|$$

where, from

$$2\widehat{\sigma}_x(t,x)\widehat{\sigma}(t,x) = 2\sigma_x(t,x)\sigma(t,x) + \gamma_n(t)\sigma_x(t,x)$$

we deduce that $|x_0\widehat{\sigma}_x(t,x_0)|$ is bounded. It follows that there exists a constant c such that for all x, $\overline{x}$

$$|\widehat{\sigma}(t,x)x - \widehat{\sigma}(t,\overline{x})\overline{x}| \leqslant c\,|x-\overline{x}|.$$

Since σ_x is continuous, Condition (A) is well verified.

Furthermore, we have:

$$\frac{\partial \widehat{S}_{x,t}(u)}{\partial x} = 1 + \int_t^u \frac{\widehat{\Lambda}\left(s,\widehat{S}_{x,t}(s)\right)}{\widehat{\sigma}\left(s,\widehat{S}_{x,t}(s)\right)} \frac{\partial \widehat{S}_{x,t}(s)}{\partial x} dW_s$$

which is a strictly positive martingale (see Lemma 5.6.3). Note that, as in the proof of the next lemma, we claim that the distribution of $\widehat{S}_{x_0,t}(1)$ is of density relatively to the Lebesgue measure. It follows that, out of the null-set $\widehat{S}_{x_0,t}(1) \in \{K_p : p \in \mathbb{N}^*\}$, we have almost surely:

$$\int_0^1 h'\left(\widehat{S}_{x_0,t}(1) + \mu(\widehat{S}_{x_n,t}(1) - \widehat{S}_{x_0,t}(1)\right) d\mu \to h'(\widehat{S}_{x_0,t}(1))$$

provided that x_n is sufficiently near to x_0 and x_n is a subsequence such that

$$\frac{\widehat{S}_{x_n,t}(1) - \widehat{S}_{x_0,t}(1)}{x - x_0} \to \frac{\partial \widehat{S}_{x_0,t}(1)}{\partial x} \text{ a.s.}$$

Since h' is bounded, it follows that

$$\widehat{C}_x(t,x) = Eh'(\widehat{S}_{x,t}(1))\frac{\partial \widehat{S}_{x,t}(1)}{\partial x}.$$

Finally, we note $d\overline{P} = \frac{\partial \widehat{S}_{x,t}(1)}{\partial x} dP$ in order to have

$$\widehat{C}_x(t,x) = \overline{E}h'(\widehat{S}_{x,t}(1)).$$

The Girsanov theorem ((5.1) p 190 [20]) asserts that the process

$$B_u = W_u - W_t - \int_t^u \frac{\widehat{\Lambda}\left(s,\widehat{S}_{x,t}(s)\right)}{\widehat{\sigma}\left(s,\widehat{S}_{x,t}(s)\right)} du$$

is a standard Brownian motion under $\overline{P}$. Moreover, $\widehat{S}_{x,t}$ verifies the sde

$$d\widehat{S}_{x,t}(u) = \widehat{\sigma}(u,\widehat{S}_{x,t}(u))\widehat{S}_{x,t}(u)dB_u + \widehat{\Lambda}(t,\widehat{S}_{x,t}(u))\widehat{S}_{x,t}(u)du.$$

Since $\widehat{\sigma}$ and $\widehat{\Lambda}$ are bounded, the sde admits a unique strong solution, hence a unique weak solution. We can conclude that $\widehat{C}_x(t,x) = Eh'(\widetilde{S}_{x,t}(1))$.

Lemma 5.2.3. *We have:*

$$\widehat{C}_x(t,x) = \int_{-\infty}^{\infty} h'(e^z)\Gamma^*(\ln x, t, z, 1)dz$$

where $\Gamma^*(x,t,z,\tau)$ is the fundamental solution of the operator:

$$\frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2}+\widehat{\sigma}_b(t,x)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}$$

with

$$\begin{aligned}\widehat{\sigma}_a(t,x) &= \widehat{\sigma}(t,e^x),\\ \widehat{\sigma}_b(t,x) &= \widehat{\Lambda}(t,e^x)-\frac{1}{2}\widehat{\sigma}^2(t,e^x).\end{aligned}$$

Proof. We define $\widehat{\eta}_{x,t}(u)=\ln\widetilde{S}_{e^x,t}(u)$ which verifies the following sde:

$$\left\{\begin{array}{lcl} d\widehat{\eta}_{x,t}(u) &=& \widehat{\sigma}_a(u,\widehat{\eta}_{x,t}(u))dW_u+\widehat{\sigma}_b(u,\widehat{\eta}_{x,t}(u))du\\ \widehat{\eta}_{x,t}(t) &=& x\end{array}\right.$$

Indeed, it suffices to apply the Ito formula to $\exp(\widehat{\eta}_{x,t})$ where $\widehat{\eta}_{x,t}$ is the solution of the previous sde. According to Lemma 5.6.4, $\widehat{\eta}_{x,t}$ is a Markov process of transition density function $\Gamma^*(x,t,z,1)$, the fundamental solution of the operator:

$$\frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2}+\widehat{\sigma}_b(t,x)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}.$$

This means that:

$$P(\widehat{\eta}_{x,t}(u)\in dz)=\Gamma^*(x,t,z,u)dz$$

and it follows that

$$\widehat{C}_x(t,x)=\int_{-\infty}^{\infty}h'(e^z)\Gamma^*(\ln x,t,z,1)dz.$$

5.3 Estimation of the Derivatives of Γ^*.

In all this section, we suppose that the assumptions (**E**), (**G**) and (**H**) hold. Let define for $0\leqslant\tau\leqslant t\leqslant 1$, $\Gamma(x,t,z,\tau)=\Gamma^*(x,1-t,z,1-\tau)$ which is the fundamental solution of the operator:

$$\frac{1}{2}\widehat{\sigma}_a^2(1-t,x)\frac{\partial^2}{\partial x^2}+\widehat{\sigma}_b(1-t,x)\frac{\partial}{\partial x}-\frac{\partial}{\partial t}.$$

By definition, Γ is the function such that, for every continuous function f, we have:

$$\tfrac{1}{2}\widehat{\sigma}_a^2(1-t,x)\tfrac{\partial^2}{\partial x^2}\Gamma(x,t,z,\tau)+\widehat{\sigma}_b(1-t,x)\tfrac{\partial}{\partial x}\Gamma(x,t,z,\tau)-\tfrac{\partial}{\partial t}\Gamma(x,t,z,\tau)=0,$$

$$\int_{-\infty}^{\infty}\Gamma(x,t,z,\tau)f(z)dz\longrightarrow f(x)\text{ as }t\downarrow\tau.$$

There exists some estimations of the derivatives of Γ [11] but, unfortunately, they are too imprecise for our needs because of $\widehat{\sigma}$ depends on n. That's why, we propose to repeat the calculus of Chapter 9 [11] in the case where $\alpha<1/2$. In order to be clear, we shall specify with brackets the indexation of [11] if necessary.

5.3.1 The Parametrix

For more convenience, we note $\widehat{\sigma}_a^2(1-t,x) = \widehat{\sigma}_a^2(t,x)$ and $\widehat{\sigma}_b(1-t,x) = \widehat{\sigma}_b(t,x)$ which won't have impact on the result. We first construct a fundamental solution $Z(x,t,\xi,\tau)$ for the parabolic system:

$$[2.1]: \quad \frac{\partial u}{\partial t} = \frac{1}{2}\widehat{\sigma}_a^2(t)\frac{\partial^2}{\partial x^2} + \widehat{\sigma}_b(t)\frac{\partial}{\partial x}.$$

We associate the following linear ordinary differential equation:

$$[2.2]: \qquad \begin{aligned} \frac{\partial v}{\partial t} &= \left(-\frac{1}{2}\widehat{\sigma}_a^2(t)\zeta^2 + \widehat{\sigma}_b(t)\zeta i\right) v \\ v(\tau) &= 1. \end{aligned}$$

Obviously, we have

$$v(t,\zeta,\tau) = \exp\left\{\int_\tau^t -\frac{1}{2}\widehat{\sigma}_a^2(s)\zeta^2 + \widehat{\sigma}_b(s)\zeta i ds\right\}.$$

From the hypotheses, we deduce some strictly positive constants m and M such that for n sufficiently large:

$$\begin{aligned} mn^{1/2-\alpha} &\leqslant \widehat{\sigma}_a^2 \leqslant Mn^{1/2-\alpha}, \\ mn^{1/2-\alpha} &\leqslant \widehat{\sigma}_b \leqslant Mn^{1/2-\alpha}. \end{aligned}$$

We deduce that for $\beta \geqslant 0$,

$$|v(t,\alpha+i\beta,\tau)| \leqslant \exp\left\{-\frac{m\rho_t^\tau}{2}\alpha^2 + \frac{M\rho_t^\tau}{2}\beta^2 - m\rho_t^\tau\beta\right\}$$

and for $\beta \leqslant 0$,

$$|v(t,\alpha+i\beta,\tau)| \leqslant \exp\left\{-\frac{m\rho_t^\tau}{2}\alpha^2 + \frac{M\rho_t^\tau}{2}\beta^2 - M\rho_t^\tau\beta\right\}.$$

where $\rho_t^\tau = n^{1/2-\alpha}(t-\tau)$. We define the fundamental solution as follows:

$$[2.4]: \quad Z(x,t,\xi,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha(x-\xi)}v(t,\alpha,\tau)d\alpha.$$

From [11], we have for all β

$$Z(x,t,\xi,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\alpha+i\beta)(x-\xi)}v\left(t,\alpha+i\beta,\tau\right)d\alpha.$$

It follows that there exists a constant c such that for $\beta \leqslant 0$,

$$|Z(x,t,\xi,\tau)| \leqslant \frac{c}{\sqrt{\rho_t^\tau}}e^{-\beta(x-\xi)+\frac{1}{2}M\rho_t^\tau\beta^2 - M\rho_t^\tau\beta}.$$

So, choosing

$$\beta = 1 + \frac{x-\xi}{M\rho_t^\tau} \leqslant 0$$

we deduce that

$$|Z(x,t,\xi,\tau)| \leqslant \frac{c}{\sqrt{\rho_t^\tau}} \exp\left\{-\frac{1}{2}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

In the case where $\beta = 1+\frac{x-\xi}{M\rho_t^\tau} \geqslant 0$, we write $x-\xi = M\rho_t^\tau\beta - M\rho_t^\tau$ and we deduce that:

$$|Z(x,t,\xi,\tau)| \leqslant \frac{c}{\sqrt{\rho_t^\tau}} \exp\left\{\frac{1}{2}M\rho_t^\tau\left(-\beta^2+2(1-m/M)\beta\right)\right\}.$$

Moreover, for $\beta \geqslant 4(1-m/M)$, we have

$$-\beta^2+2(1-m/M)\beta \leqslant -\beta^2/2.$$

So, in all cases, we can deduce that

$$(5.3.5) \qquad |Z(x,t,\xi,\tau)| \leqslant \frac{ce^{4M\rho_t^\tau}}{\sqrt{\rho_t^\tau}} \exp\left\{-\frac{1}{4}M\rho_t^\tau\beta^2\right\},$$
$$|Z(x,t,\xi,\tau)| \leqslant \frac{ce^{4M\rho_t^\tau}}{\sqrt{\rho_t^\tau}} \exp\left\{-\frac{1}{4}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

In a similar way, if $k \geqslant 0$, there exists a constant c_k such that

$$(5.3.6) \qquad [2.5]: |D_x^k Z(x,t,\xi,\tau)| \leqslant \frac{c_k e^{4M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+1}{2}}} \exp\left\{-\frac{1}{5}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Indeed,

$$D_x^k Z(x,t,\xi,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[i(\alpha+i\beta)]^k e^{i(\alpha+i\beta)(x-\xi)} v\left(t,(\alpha+i\beta),\tau\right) d\alpha$$

and

$$|D_x^k Z(x,t,\xi,\tau)| \leqslant \frac{c_k^1 e^{4M\rho_t^\tau}}{\sqrt{\rho_t^\tau}}|\beta|^k \exp\left\{-\frac{1}{4}M\rho_t^\tau\beta^2\right\} + \frac{c_k^2 e^{4M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+1}{2}}} \exp\left\{-\frac{1}{4}M\rho_t^\tau\beta^2\right\}.$$

5.3.2 The Parametrix for Equations with Parameters

We consider the fundamental solution $Z(x-\xi,t,y,\tau)$ for the parabolic system (with y fixed):

$$[2.1]: \quad \frac{\partial u}{\partial t} = \frac{1}{2}\widehat{\sigma}_a^2(t,y)\frac{\partial^2}{\partial x^2}+\widehat{\sigma}_b(t,y)\frac{\partial}{\partial x}.$$

From the previous section, we deduce the following inequalities:

$$(5.3.7) \quad [3.3]: |D_x^k Z(x-\xi,t,y,\tau)| \leqslant \frac{c_k e^{4M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+1}{2}}} \exp\left\{-\frac{1}{5}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}$$

where c_k is independent of y. In a same way, we have:

Lemma 5.3.1.

$$|D_y^k Z(x-\xi,t,y,\tau)| \leqslant \frac{c_k e^{c_k M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{1}{2}}} \exp\left\{-\frac{1}{5}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}$$

Proof. First, we can easily show that for $p \geqslant 1$, there exists a constant $c(p)$ such that

$$(5.3.8)\quad \left|\frac{\partial^p}{\partial y^p} v(y,t,\alpha+i\beta,\tau)\right| \leqslant c(p) \sum_{l=1}^{p} \left(\rho_t^\tau\left(\alpha^2+\beta^2+|\alpha|+|\beta|\right)\right)^l v(y,t,\alpha+i\beta,\tau).$$

Moreover,

$$D_y^k Z(x-\xi,t,y,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\alpha+i\beta)(x-\xi)} \frac{\partial^k}{\partial y^k} v(y,t,\alpha+i\beta,\tau)d\alpha.$$

Then, taking $\beta = 1+\frac{(x-\xi)}{M\rho_t^\tau}$, we deduce that

$$|D_y^k Z(x-\xi,t,y,\tau)| \leqslant const\, e^{4M\rho_t^\tau} \exp\left\{-\frac{1}{4}M\rho_t^\tau\beta^2\right\}\Sigma(k)$$

where

$$\Sigma(k) = \sum_{p=1}^{k}\left[\frac{(\rho_t^\tau)^p}{(\rho_t^\tau)^{p+1/2}} + \frac{(\rho_t^\tau)^p}{(\rho_t^\tau)^{p/2+1/2}} + \frac{(\rho_t^\tau)^p\,\beta^{2p}}{(\rho_t^\tau)^{1/2}} + \frac{(\rho_t^\tau)^p\,|\beta|^p}{(\rho_t^\tau)^{1/2}}\right].$$

Using the fact that $|X|e^{-|X|}$ is bounded, we can conclude about the lemma.

5.3.3 Construction of the Fundamental Solution; the Cauchy Problem

We note $\Gamma(x,t,\xi,\tau)$ the fundamental solution of

$$[1.6]:\quad \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2} + \widehat{\sigma}_b(t,x)\frac{\partial}{\partial x}.$$

From [11], we have

$$[4.4]:\Gamma(x,t,\xi,\tau) = Z(x-\xi,t,\xi,\tau) + \int_\tau^t\int_{-\infty}^{\infty} Z(x-y,t,y,\sigma)\Phi(y,\sigma,\xi,\tau)dyd\sigma$$

where, for $\varsigma = \alpha+i\beta$, we define

$$\begin{aligned}
\Phi(x,t,\xi,\tau) &:= \Sigma_{k=1}^{\infty} K_k(x,t,\xi,\tau),\\
K_1(x,t,\xi,\tau) &:= \left[\frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2} + \widehat{\sigma}_b(t,x)\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right] Z(x-\xi,t,\xi,\tau),\\
K_k(x,t,\xi,\tau) &:= \int_\tau^t\int_{-\infty}^{\infty} K_1(x,t,y,\sigma)K_{k-1}(y,\sigma,\xi,\tau)dyd\sigma.
\end{aligned}$$

Note that

$$\begin{aligned}
K_1(x,t,\xi,\tau) &= -\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{i\varsigma(x-\xi)}\left[\widehat{\sigma}_a^2(t,x)-\widehat{\sigma}_a^2(t,\xi)\right]\varsigma^2 v\left(\xi,t,\varsigma,\tau\right)d\alpha\\
&\quad + \frac{i}{2\pi}\int_{-\infty}^{\infty} e^{i\varsigma(x-\xi)}\left[\widehat{\sigma}_b(t,x)-\widehat{\sigma}_b(t,\xi)\right]\varsigma v\left(\xi,t,\varsigma,\tau\right)d\alpha.
\end{aligned}$$

We have the following inequalities:

Lemma 5.3.2. *There exists some constants* c_k *for* $0 \leqslant k \leqslant k_0$ *such that*

$$\left|\frac{\partial^k}{\partial x^k} K_1(x,t,\xi,\tau)\right| \leqslant \frac{c_k n^{1/2-\alpha} e^{5M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+2}{2}}} \exp\left\{-\frac{1}{6}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Proof. We have, with $\varsigma = \alpha + i\beta$,

$$\begin{aligned}\frac{\partial^k}{\partial x^k} K_1(x,t,\xi,\tau) &= \sum_{p=0}^{k-1} C_k^p \int_{-\infty}^{\infty} \varsigma^{p+2} e^{i\varsigma(x-\xi)} \frac{\partial^{k-p}}{\partial x^{k-p}} \widehat{\sigma}_a^2(t,x) v(\xi,t,\varsigma,\tau) d\alpha \\ &\quad + \sum_{p=0}^{k-1} \widetilde{C}_k^p \int_{-\infty}^{\infty} \varsigma^{p+1} e^{i\varsigma(x-\xi)} \frac{\partial^{k-p}}{\partial x^{k-p}} \widehat{\sigma}_b(t,x) v(\xi,t,\varsigma,\tau) d\alpha \\ &\quad + c\int_{-\infty}^{\infty} \varsigma^{k+2} e^{i\varsigma(x-\xi)} \left(\widehat{\sigma}_a^2(t,x) - \widehat{\sigma}_a^2(t,\xi)\right) v(\xi,t,\varsigma,\tau) d\alpha \\ &\quad + d\int_{-\infty}^{\infty} \varsigma^{k+1} e^{i\varsigma(x-\xi)} \left(\widehat{\sigma}_b(t,x) - \widehat{\sigma}_b(t,\xi)\right) v(\xi,t,\varsigma,\tau) d\alpha\end{aligned}$$

where c, d are constants whereas $C_k^p, \widetilde{C}_k^p$ are some constants depending on p and k. From the hypotheses, we deduce that there exists a constant c such that

$$\begin{aligned}|\widehat{\sigma}_a^2(t,x) - \widehat{\sigma}_a^2(t,\xi)| &\leqslant cn^{1/2-\alpha}|x-\xi|, \\ |\widehat{\sigma}_b(t,x) - \widehat{\sigma}_b(t,\xi)| &\leqslant cn^{1/2-\alpha}|x-\xi|.\end{aligned}$$

Furthermore, we have

$$\left|\frac{\partial^k}{\partial x^k}\widehat{\sigma}_a^2(t,x)\right| + \left|\frac{\partial^k}{\partial x^k}\widehat{\sigma}_b(t,x)\right| \leqslant c(k) n^{1/2-\alpha} \tag{5.3.9}$$

where $c(k)$ is a constant. Indeed, recall that $x^k \frac{\partial^k}{\partial x^k}\sigma(t,x)$ is assumed bounded. Always using

$$\beta = 1 + \frac{x-\xi}{M\rho_t^\tau},$$

it follows that:

$$\left|\frac{\partial^k}{\partial x^k} K_1(x,t,\xi,\tau)\right| \leqslant c(k)\, n^{1/2-\alpha} e^{-\frac{1}{5}M\rho_t^\tau\beta^2 + 4M\rho_t^\tau} \left(\Sigma(p) + \Theta_k\right)$$

where

$$\begin{aligned}\Sigma(p) &= \sum_{p=0}^{k-1}\left[\frac{1}{(\rho_t^\tau)^{(p+2)/2}} + \frac{1}{(\rho_t^\tau)^{(p+3)/2}} + \frac{|\beta|^{p+1}}{(\rho_t^\tau)^{1/2}} + \frac{|\beta|^{p+2}}{(\rho_t^\tau)^{1/2}}\right], \\ \Theta_k &= \frac{1}{(\rho_t^\tau)^{(k+1)/2}} + \frac{1}{(\rho_t^\tau)^{(k+2)/2}} + |\beta|^{k+1} + |\beta|^{k+2}.\end{aligned}$$

Indeed, in order to dominate the last two terms of the previous sums (for $p = k$), we use the inequality

$$|X| e^{-b(\frac{X}{a}+a)^2} \leqslant const\,(b)\,(a + a^2)$$

with $X = x - \xi$ and $a = \sqrt{M\rho_t^\tau}$. From now on, it is easy to conclude using the boundedness of $|X|e^{-|X|}$.

Moreover, we have:

Lemma 5.3.3. *There exists a constant c_k such that:*

$$\left|\frac{\partial^k}{\partial x^k}K_1(x,t,y+x,\tau)\right| \leqslant \frac{c_k e^{5M\rho_t^\tau}}{t-\tau}\exp\left\{-\frac{1}{6}\left(\frac{y}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Proof. First, we deduce easily from the hypothesis, that

$$\left|\frac{\partial}{\partial y}v(y,t,\alpha+i\beta,\tau)\right| \leqslant c\rho_t^\tau\left(\alpha^2+\beta^2+|\alpha|+|\beta|\right)|v(y,t,\alpha+i\beta,\tau)|$$

and we can estimate the successive derivatives in a similar way. Indeed, it suffices to use 5.3.9 in order to obtain 5.3.8. Moreover, we can find a constant $c(k)$ such that

$$\Theta_a(k,x,y) = \frac{\partial^k}{\partial x^k}\left[\widehat{\sigma}_a^2(t,x)-\widehat{\sigma}_a^2(t,x+y)\right]$$

verifies $|\Theta_a(k,x,y)| \leqslant cn^{1/2-\alpha}|y|$, and analogously,

$$\Theta_b(k,x,y) = \frac{\partial^k}{\partial x^k}\left[\widehat{\sigma}_b(t,x)-\widehat{\sigma}_b(t,x+y)\right]$$

is such that $|\Theta_b(k,x,y)| \leqslant cn^{1/2-\alpha}|y|$. Secondly,

$$\begin{aligned}\frac{\partial^k}{\partial x^k}K_1(x,t,y+x,\tau) = \quad & \frac{-1}{4\pi}\sum_{p=0}^{k-1}C_k^p\int_{-\infty}^{\infty}e^{-i\varsigma y}\Theta_a(k-p,x,y)\varsigma^2\frac{\partial^p}{\partial y^p}v(y+x,t,\varsigma,\tau)d\alpha \\ & +\frac{i}{2\pi}\sum_{p=0}^{k-1}C_k^p\int_{-\infty}^{\infty}e^{-i\varsigma y}\Theta_b(k-p,x,y)\varsigma^2\frac{\partial^p}{\partial y^p}v(y+x,t,\varsigma,\tau)d\alpha\end{aligned}$$

where $\varsigma = \alpha+i\beta$. We deduce that

$$\left|\frac{\partial^k}{\partial x^k}K_1(x,t,y+x,\tau)\right| \leqslant c(k)e^{\beta y}n^{1/2-\alpha}|y|\sum_{p=0}^{k}\int_{-\infty}^{\infty}\left(\alpha^2+\beta^2+|\alpha|+|\beta|\right)\left|\frac{\partial^p}{\partial y^p}v(y+x,t,\alpha+i\beta,\tau)\right|d\alpha.$$

Then, if $\beta \leqslant 0$ (the case $\beta \geqslant 0$ is similar), we deduce that

$$\left|\frac{\partial^k}{\partial x^k}K_1(x,t,y+x,\tau)\right| \leqslant c(k)\,e^{\beta y+\frac{1}{2}M\rho_t^\tau\beta^2-M\rho_t^\tau\beta}n^{1/2-\alpha}|y|\sum_{p=0}^{k}\sum_{l=1}^{p}\rho(n,\beta,l,t,\tau)$$

where

$$\rho(n,\beta,l,t,\tau) = (\rho_t^\tau)^l\int_{-\infty}^{\infty}\left(\alpha^2+\beta^2+|\alpha|+|\beta|\right)^{l+1}|v(y+x,t,\varsigma,\tau)|\,d\alpha.$$

We choose $\beta = 1-\frac{y}{M\rho_t^\tau}$ and we can deduce that:

$$\left|\frac{\partial^k}{\partial x^k}K_1(x,t,y+x,\tau)\right| \leqslant c_k n^{1/2-\alpha}|y|e^{-\frac{1}{4}M\rho_t^\tau\beta^2+4M\rho_t^\tau}\Sigma(k)$$

where

$$\Sigma(k)=\sum_{p=0}^{k}\sum_{l=1}^{p}\left[\frac{(\rho_t^\tau)^l}{(\rho_t^\tau)^{(2l+3)/2}}+\frac{(\rho_t^\tau)^l}{(\rho_t^\tau)^{(l+2)/2}}+\frac{(\rho_t^\tau)^l\,\beta^{2l+2}}{(\rho_t^\tau)^{1/2}}+\frac{(\rho_t^\tau)^l\,|\beta|^{l+1}}{(\rho_t^\tau)^{1/2}}\right].$$

Since $|X|e^{-|X|}$ is bounded, we can replace β by $1/\sqrt{\rho_t^\tau}$. Finally, we use the inequality

$$|y|e^{-b(\frac{-y}{a}+a)^2}\leqslant const\,(b)\,(a+a^2)$$

with $a=\sqrt{M\rho_t^\tau}$ in order to conclude about the lemma.

We write now $K_2(x,t,\xi,\tau)=K_{21}(x,t,\xi,\tau)+K_{22}(x,t,\xi,\tau)$ with:

$$\begin{aligned}
K_{21}(x,t,\xi,\tau) &= \int_\tau^{\tau+\frac{t-\tau}{2}}\int_{-\infty}^{\infty}K_1(x,t,y,\sigma)K_1(y,\sigma,\xi,\tau)dyd\sigma,\\
K_{22}(x,t,\xi,\tau) &= \int_{\tau+\frac{t-\tau}{2}}^{t}\int_{-\infty}^{\infty}K_1(x,t,y+x,\sigma)K_1(y+x,\sigma,\xi,\tau)dyd\sigma,
\end{aligned}$$

after a change of variable in the second integral. Using the two previous lemmas, we obtain the following inequality.

Lemma 5.3.4.

$$\left|\frac{\partial^k}{\partial x^k}K_2(x,t,\xi,\tau)\right|\leqslant\frac{c_k n^{1/2-\alpha}e^{7M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+1}{2}}}\exp\left\{-\frac{1}{7}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Proof. First, $\frac{\partial^k}{\partial x^k}K_{22}(x,t,\xi,\tau)$ is equal to

$$\sum_{p=0}^{k}C_k^p\int_{\tau+\frac{t-\tau}{2}}^{t}\int_{-\infty}^{\infty}\frac{\partial^{k-p}}{\partial x^{k-p}}K_1(x,t,y+x,\sigma)\frac{\partial^p}{\partial x^p}K_1(y+x,\sigma,\xi,\tau)dyd\sigma.$$

Using the function f of Lemma 5.6.6 with $a=\sqrt{Mn^{1/2-\alpha}}$, we obtain that $\left|\frac{\partial^k}{\partial x^k}K_{22}(x,t,\xi,\tau)\right|$ is bounded by

$$c(k)\sum_{p=0}^{k}\frac{n^{1/2-\alpha}e^{5M\rho_t^\tau}}{(n^{1/2-\alpha})^{(p+2)/2}}\int_{\tau+\frac{t-\tau}{2}}^{t}\int_{-\infty}^{\infty}\frac{e^{-f(t,\sigma,\tau,x,x+y,\xi)/6}}{(t-\sigma)\,(\sigma-\tau)^{(p+2)/2}}dyd\sigma.$$

Note that $\sigma\geqslant\tau+(t-\tau)/2$ implies that $\sigma-\tau\geqslant(t-\tau)/2$. Then, we use the change of variable

$$z=\frac{-y}{\sqrt{M\rho_t^\sigma}}+\sqrt{M\rho_t^\sigma}$$

and we deduce, using the first assertion of Lemma 5.6.6 , that

$$\left|\frac{\partial^k}{\partial x^k}K_{22}(x,t,\xi,\tau)\right|\leqslant c(k)\sum_{p=0}^{k}\frac{n^{1/2-\alpha}e^{5M\rho_t^\tau}}{(\rho_t^\tau)^{(p+1)/2}}\exp\left\{-\frac{1}{7}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}$$

since

$$\int_{\tau+\frac{t-\tau}{2}}^{t}\frac{d\sigma}{\sqrt{t-\sigma}\sqrt{\sigma-\tau}}dy$$

is bounded.

We obtain a similar inequality for $\frac{\partial^k}{\partial x^k}K_{21}(x,t,\xi,\tau)$ but in this case, it's not necessary to use a change of variable. Then, we can conclude about the lemma.

Following the same scheme, we shall obtain inductively some constants $u_{p,k} \geqslant 6$ such that:

Lemma 5.3.5.

$$\left|\frac{\partial^k}{\partial x^k}K_p(x,t,\xi,\tau)\right| \leqslant \frac{c_{k,p}n^{1/2-\alpha}e^{u_{p,k}M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+3-p}{2}}}\exp\left\{-\frac{1}{u_{p,k}}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Proof. We know that the lemma holds for $p=1,2$. Assume that it holds for any k and $1,\cdots,p-1$ where $p-1\geqslant 2$. We note

$$K_p(x,t,\xi,\tau) = K_p^a(x,t,\xi,\tau)+K_p^b(x,t,\xi,\tau)$$

where

$$\frac{\partial^k}{\partial x^k}K_p^a(x,t,\xi,\tau) = \int_\tau^{\tau+\frac{t-\tau}{2}}\int_{-\infty}^{\infty}\frac{\partial^k}{\partial x^k}K_1(x,t,y,\sigma)K_{p-1}(y,\sigma,\xi,\tau)dyd\sigma.$$

In virtue of the previous lemmas and the first assertion of Lemma 5.6.6, we obtain some constants $c(k,p)$ and $u_{p,k}$ such that $\left|\frac{\partial^k}{\partial x^k}K_p^a(x,t,\xi,\tau)\right|$ is bounded by the product of the two following terms:

$$c(k,p)e^{u_{p,k}M\rho_t^\tau}\left(n^{1/2-\alpha}\right)^2\exp\left\{-\frac{1}{u_{p,k}}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}$$

and

$$\int_\tau^{\tau+\frac{t-\tau}{2}}\int_{-\infty}^{\infty}\frac{\exp\left\{-\frac{1}{u_{p,k}}\left((y-\xi)/\sqrt{M\rho_\sigma^\tau}+\sqrt{M\rho_\sigma^\tau}\right)^2\right\}}{\left(n^{1/2-\alpha}(t-\sigma)\right)^{(k+2)/2}\left(n^{1/2-\alpha}(\sigma-\tau)\right)^{(4-p)/2}}dyd\sigma.$$

In the present case, we use the property

$$\frac{1}{\left(n^{1/2-\alpha}(t-\sigma)\right)^{(k+2)/2}} \leqslant \frac{const}{\left(n^{1/2-\alpha}(t-\tau)\right)^{(k+2)/2}}.$$

After the change of variable

$$z=\frac{y-\xi}{\sqrt{M\rho_\sigma^\tau}}+\sqrt{M\rho_\sigma^\tau},$$

it suffices to estimate

$$\int_\tau^{\tau+\frac{t-\tau}{2}}\frac{d\sigma}{\left(n^{1/2-\alpha}(\sigma-\tau)\right)^{(3-p)/2}} \leqslant const\left(n^{1/2-\alpha}\right)^{(p-3)/2}(t-\tau)^{(p-1)/2}.$$

It follows that $\left|\frac{\partial^k}{\partial x^k}K_p^a(x,t,\xi,\tau)\right|$ verifies the inequality of the lemma. In a similar way, $\frac{\partial^k}{\partial x^k}K_p^b(x,t,\xi,\tau)$ is equal to

$$\sum_{j=0}^{k}C_k^j\int_{\tau+\frac{t-\tau}{2}}^{t}\int_{-\infty}^{\infty}\frac{\partial^{k-j}}{\partial x^{k-j}}K_1(x,t,y+x,\sigma)\frac{\partial^j}{\partial x^j}K_{p-1}(y+x,\sigma,\xi,\tau)dyd\sigma.$$

Choosing a constant

$$\widetilde{u}_{p,k} \geqslant 2 \max_{j=0,\cdots,k} u_{p-1,j},$$

using the induction hypothesis and the first assertion of Lemma 5.6.6, we deduce that there is a constant $c(k,p)$ such that $\left|\frac{\partial^k}{\partial x^k} K_p^b(x,t,\xi,\tau)\right|$ is bounded by the product of the two following terms:

$$c(k,p) e^{\widetilde{u}_{p,k} M \rho_t^\tau} n^{1/2-\alpha} \exp\left\{-\frac{1}{\widetilde{u}_{p,k}} \left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}$$

and

$$\sum_{j=0}^{k} \int_{\tau+\frac{t-\tau}{2}}^{t} \int_{-\infty}^{\infty} \frac{\exp\left\{-\frac{1}{\widetilde{u}_{p,k}} \left((x-y)/\sqrt{M\rho_t^\sigma} + \sqrt{M\rho_t^\sigma}\right)^2\right\}}{(t-\sigma)\left(n^{1/2-\alpha}(\sigma-\tau)\right)^{(j+4-p)/2}} dy d\sigma. \tag{5.3.10}$$

From now on, we take in consideration the fact that

$$\frac{1}{\left(n^{1/2-\alpha}(\sigma-\tau)\right)^{(j+3-p)/2}} \leqslant \frac{const}{\left(n^{1/2-\alpha}(t-\tau)\right)^{(j+3-p)/2}}$$

and we use the change of variable

$$z = \frac{x-y}{\sqrt{M\rho_t^\sigma}} + \sqrt{M\rho_t^\sigma}$$

in order to dominate 5.3.10 by

$$const(k) \sum_{j=0}^{k} \frac{1}{\left(n^{1/2-\alpha}(t-\tau)\right)^{(j+3-p)/2}}.$$

Then, from the boundedness of $|X|e^{-|X|}$, we deduce that $\frac{\partial^k}{\partial x^k} K_p^b(x,t,\xi,\tau)$ also verifies the needed inequality and we can conclude about the lemma.

In particular, we can easily find an increasing sequence $a_k > 6$ such that:

$$\left|\frac{\partial^k}{\partial x^k} K_{k+3}(x,t,\xi,\tau)\right| \leqslant c_k n^{1/2-\alpha} e^{(a_k - \frac{1}{2^k}) M \rho_t^\tau} \exp\left\{-\frac{1}{a_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}.$$

We deduce inductively that there exists a constant B_k such that:

Lemma 5.3.6. $\left|\frac{\partial^k}{\partial x^k} K_{k+3+p}(x,t,\xi,\tau)\right|$ *is bounded by*

$$\frac{(B_k)^{p+1} n^{1/2-\alpha} \left(\rho_t^\tau\right)^{p/2} e^{(a_k - \frac{1}{2^k}) M \rho_t^\tau}}{\Gamma(1+p/2)} \exp\left\{-\frac{1}{a_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Proof. We shall argue by applying a double induction, i.e., we assume that the result is true for $1, \cdots, k-1$ and for any p, and we show it for k. For this, we know from the last remark that the lemma holds for $p = 0$. Then, by induction, we assume that it's true for $1, \cdots, p$ and we shall prove it for $p+1$.

We split $K_{k+3+p+1}$ as $K^1_{k+3+p+1} + K^2_{k+3+p+1}$ with:

$$\frac{\partial^k}{\partial x^k} K^1_{k+3+p+1}(x,t,\xi,\tau) = \int_\tau^{\tau+\frac{t-\tau}{2}} \int_{-\infty}^{\infty} \frac{\partial^k}{\partial x^k} K_1(x,t,y,\sigma) K_{k+3+p}(y,\sigma,\xi,\tau) dy d\sigma.$$

From the previous inequalities and the induction hypothesis, it follows that there exists a common constant c_k independent of p such that $\left|\frac{\partial^k}{\partial x^k} K^1_{k+3+p+1}(x,t,\xi,\tau)\right|$ is bounded by the product of the two following terms:

$$\frac{c_k (B_0)^{k+p+1} e^{(a_k - \frac{1}{2^k}) M\rho_t^\tau} \left(n^{1/2-\alpha}\right)^2}{\Gamma(1+\frac{k+p}{2})} \exp\left\{ -\frac{1}{a_k} \left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau} \right)^2 \right\}$$

and

$$\int_\tau^{\tau+\frac{t-\tau}{2}} \int_{-\infty}^{\infty} \frac{e^{-(\frac{1}{6}-\frac{1}{a_k})\left((x-y)/\sqrt{M\rho_t^\sigma} + \sqrt{M\rho_t^\sigma}\right)^2}}{(\rho_t^\sigma)^{\frac{k+2}{2}}} (\rho_\sigma^\tau)^{\frac{k+p}{2}} dy d\sigma.$$

Moreover, we have $t-\sigma \geqslant (t-\tau)/2$ and $\sigma - \tau \leqslant (t-\tau)/2$. Then, we obtain from the last integral, after a change of variable, a constant d_k independent of p such that $\left|\frac{\partial^k}{\partial x^k} K^1_{k+3+p+1}(x,t,\xi,\tau)\right|$ is bounded by the product of the two following terms:

$$\frac{d_k c_k (B_0)^{k+p+1} e^{(a_k - \frac{1}{2^k}) M\rho_t^\tau} \left(n^{1/2-\alpha}\right)^{1+\frac{p+1}{2}}}{\Gamma(1+\frac{k+p}{2})} \exp\left\{ -\frac{1}{a_k} \left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau} \right)^2 \right\}$$

and

$$\int_\tau^t (\sigma-\tau)^{\frac{p}{2}} (t-\sigma)^{-1/2} d\sigma = (t-\tau)^{\frac{p+1}{2}} \Gamma(1+\frac{p}{2}) \Gamma(\frac{1}{2}) / \Gamma(1+\frac{p+1}{2}).$$

We use Lemma 5.6.5 and we choose B_k such that

$$B_k \geqslant \max\{2 c_k d_k B_0^k, B_0\}.$$

It follows that $\left|\frac{\partial^k}{\partial x^k} K^1_{k+3+p+1}(x,t,\xi,\tau)\right|$ is dominated by

$$\frac{B_k^{p+1} n^{1/2-\alpha} (\rho_t^\tau)^{\frac{p+1}{2}} e^{(a_k - \frac{1}{2^k}) M\rho_t^\tau}}{\Gamma(1+\frac{p+1}{2})} \exp\left\{ -\frac{1}{a_k} \left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau} \right)^2 \right\}.$$

Furthermore, $\frac{\partial^k}{\partial x^k} K^2_{k+3+p+1}(x,t,\xi,\tau)$ is equal to

$$\sum_{m=0}^{k} C_k^m \int_{\tau+\frac{t-\tau}{2}}^t \int_{-\infty}^{\infty} \frac{\partial^m}{\partial x^m} K_1(x,t,y+x,\sigma) \frac{\partial^{k-m}}{\partial x^{k-m}} K_{k+3+p}(y+x,\sigma,\xi,\tau) dy d\sigma.$$

In order to use the induction hypothesis, we write

$$k+3+p = (k-m)+3+p+m$$

if $m \neq 0$. So, we deduce a constant e_k independent of p such that

$$\left|\frac{\partial^k}{\partial x^k} K^2_{k+3+p+1}(x,t,\xi,\tau)\right|$$

is bounded by

$$\sum_{m=0}^{k} e_k c_k n^{1/2-\alpha} (B_{k-m})^{p+m+1} \left(\rho_t^\tau\right)^{m/2} e^{-\frac{2^m}{2^k} M\rho_t^\tau} \left(n^{1/2-\alpha}\right)^{p/2} \Theta^a(k,p)\Theta^b(m)$$

where

$$\begin{aligned}
\Theta^a(k,p) &= \int_{\tau+\frac{t-\tau}{2}}^{t} \int_{-\infty}^{\infty} \frac{e^{-\widetilde{a}_k\left((x-y)/\sqrt{M\rho_t^\sigma}+\sqrt{M\rho_t^\sigma}\right)^2}}{t-\sigma} (\sigma-\tau)^{\frac{p}{2}} dy d\sigma \\
\Theta^b(m) &= \frac{e^{a_k M\rho_t^\tau}}{\Gamma(1+\frac{p+m}{2})} \exp\left\{-\frac{1}{a_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}
\end{aligned}$$

and $\widetilde{a}_k = 1/6 - 1/a_k$. Using the boundedness of $|x|^m e^{-x^2}$, we deduce a constant f_k independent of p such that $\left(\rho_t^\tau\right)^{m/2} \exp\left\{-M\rho_t^\tau/2^k\right\} \leqslant f_k$. So,

$$\left|\frac{\partial^k}{\partial x^k} K^2_{k+3+p+1}(x,t,\xi,\tau)\right|$$

is dominated by

$$\sum_{m=0}^{k} \quad c_k d_k e_k f_k n^{1/2-\alpha} (B_{k-m})^{p+m+1} \left(n^{1/2-\alpha}\right)^{\frac{p+1}{2}} e^{-\frac{1}{2^k} M\rho_t^\tau} \widetilde{\Theta}^a(k,p)\widetilde{\Theta}^b(k)$$

where

$$\begin{aligned}
\widetilde{\Theta}^a(k,p) &= \int_{\tau+\frac{t-\tau}{2}}^{t} \frac{(\sigma-\tau)^{\frac{p}{2}}}{\sqrt{t-\sigma}} d\sigma \frac{e^{a_k M\rho_t^\tau}}{\Gamma(1+\frac{p+m}{2})} \\
\widetilde{\Theta}^b(k) &= \exp\left\{-\frac{1}{a_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.
\end{aligned}$$

Thus, it is enough to choose B_k verifying

$$B_k \geqslant \max\left\{4\sum_{m=1}^{k} c_k d_k e_k f_k (B_{k-m})^m ; 4c_k d_k e_k f_k ; B_0, \cdots, B_{k-1}\right\}$$

in order to obtain that $\left|\frac{\partial^k}{\partial x^k} K^2_{k+3+p+1}(x,t,\xi,\tau)\right|$ is bounded by

$$\frac{(B_k)^{p+2} n^{1/2-\alpha} \left(\rho_t^\tau\right)^{\frac{p+1}{2}} e^{(a_k-\frac{1}{2^k})M\rho_t^\tau}}{\Gamma(1+\frac{p+1}{2})} \exp\left\{-\frac{1}{a_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

We conclude that the recurrence is well verified for p and finally for k. Indeed, we can initiate it by reproducing the last reasoning when $k=0$.

Henceforth, we can deduce the following result:

Lemma 5.3.7. *There exists some constants $A_k > a_k$ and $C_k > 0$ such that:*

$$\left|\frac{\partial^k}{\partial x^k}\Phi(x,t,\xi,\tau)\right| \leqslant \frac{C_k n^{1/2-\alpha} e^{A_k M \rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+2}{2}}} \exp\left\{-\frac{1}{A_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Proof. Recall that

$$\Phi(x,t,\xi,\tau) = \sum_{i=1}^{\infty} K_i(x,t,\xi,\tau).$$

So, we write:

$$\left|\frac{\partial^k}{\partial x^k}\Phi(x,t,\xi,\tau)\right| \leqslant \Sigma_1 + \Sigma_2$$

where

$$\begin{aligned}
\Sigma_1 &= \sum_{i=1}^{k+2}\left|\frac{\partial^k}{\partial x^k}K_i(x,t,\xi,\tau)\right|, \\
\Sigma_2 &= \sum_{p=0}^{\infty}\left|\frac{\partial^k}{\partial x^k}K_{k+3+p}(x,t,\xi,\tau)\right|.
\end{aligned}$$

Using Lemma 5.3.5 we deduce that

$$\Sigma_1 \leqslant \sum_{i=1}^{k+2}\frac{c_{k,i} n^{1/2-\alpha} e^{u_{i,k} M \rho_t^\tau}}{(\rho_t^\tau)^{(k+3-i)/2}} \exp\left\{-\frac{1}{u_{i,k}}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Then, it is easy to find some constants A_k, C_k such that

$$\Sigma_1 \leqslant \frac{1}{2}\frac{C_k n^{1/2-\alpha} e^{A_k M \rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+2}{2}}} \exp\left\{-\frac{1}{A_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Finally, in virtue of Lemma 5.3.6, we have, up to a multiplier constant $c(k)$,

$$\Sigma_2 \leqslant \frac{n^{1/2-\alpha} e^{a_k M \rho_t^\tau}}{(\rho_t^\tau)^{(k+2)/2}} \exp\left\{-\frac{1}{a_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\} \sum_{p=0}^{\infty}\frac{(B_k)^p\,(\rho_t^\tau)^{p/2}}{\Gamma(1+p/2)}$$

because

$$(\rho_t^\tau)^{(k+2)/2}\exp\left\{-\frac{1}{2^k}M\rho_t^\tau\right\} \leqslant const\,(k).$$

Moreover, using the Stirling formula

$$\Gamma(1+x) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x}, \quad x \to \infty$$

and splitting the sum in the right hand side of the last inequality in two parts ($p \in 2\mathbb{N}$ or not), we deduce a constant c such that

$$\sum_{p=0}^{\infty}\frac{(B_k)^p\,(\rho_t^\tau)^{p/2}}{\Gamma(1+p/2)} \leqslant c\exp\{\widetilde{B}_k\rho_t^\tau\}$$

and we can conclude about the lemma.

Lemma 5.3.8. *We have some constant A_k such that*

$$\left|\frac{\partial^k}{\partial x^k}\Gamma(x,t,\xi,\tau)\right| \leqslant \frac{C_k e^{A_k M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+1}{2}}} \exp\left\{-\frac{1}{A_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Proof. Recall that:

$$[4.4] : \Gamma(x,t,\xi,\tau) = Z(x-\xi,t,\xi,\tau) + \Upsilon(x,t,\xi,\tau)$$

where

$$\begin{aligned}\Upsilon(x,t,\xi,\tau) &= \int_\tau^t \int_{-\infty}^{\infty} Z(x-y,t,y,\sigma)\Phi(y,\sigma,\xi,\tau)dyd\sigma,\\ &= \Upsilon^1(x,t,\xi,\tau)+\Upsilon^2(x,t,\xi,\tau).\end{aligned}$$

We have

$$\frac{\partial^k}{\partial x^k}\Upsilon^1(x,t,\xi,\tau) = \int_\tau^{\tau+\frac{t-\tau}{2}} \int_{-\infty}^{\infty} \frac{\partial^k}{\partial x^k} Z(x-y,t,y,\sigma)\Phi(y,\sigma,\xi,\tau)dyd\sigma.$$

Using 5.3.7 and the function of Lemma 6.2.11 with $a=\sqrt{Mn^{1/2-\alpha}}$, we deduce the existence of a constant C_k such that $\left|\frac{\partial^k}{\partial x^k}\Upsilon^1(x,t,\xi,\tau)\right|$ is bounded by:

$$C_k e^{A_0 M\rho_t^\tau} n^{1/2-\alpha} \int_\tau^{\tau+\frac{t-\tau}{2}} \int_{-\infty}^{\infty} \frac{\exp\{-f(t,\sigma,\tau,x,y,\xi)/A_0\}}{(\rho_t^\sigma)^{\frac{k+1}{2}}\rho_\sigma^\tau}dyd\sigma$$

and finally

$$\left|\frac{\partial^k}{\partial x^k}\Upsilon^1(x,t,\xi,\tau)\right| \leqslant \frac{C_k e^{A_0 M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k}{2}}}\frac{1}{\sqrt{n^{1/2-\alpha}}}\int_\tau^{\tau+\frac{t-\tau}{2}} \frac{1}{\sqrt{\sigma-\tau}} I_a(1/A_0)d\sigma.$$

From now on, it suffices to use Lemma 5.6.6 and the boundedness of $|X|e^{-|X|}$ to conclude that there is a constant $\widetilde{A}_k$ such that:

$$\left|\frac{\partial^k}{\partial x^k}\Upsilon^1(x,t,\xi,\tau)\right| \leqslant \frac{C_k e^{\widetilde{A}_k M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k}{2}}} \exp\left\{-\frac{1}{\widetilde{A}_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}}+\sqrt{M\rho_t^\tau}\right)^2\right\}.$$

Using the change of variable $x-y=z$, we deduce that $\frac{\partial^k}{\partial x^k}\Upsilon^2(x,t,\xi,\tau)$ is equal to:

$$\sum_{p=0}^k C_k^p \int_{\tau+\frac{t-\tau}{2}}^t \int_{-\infty}^{\infty} \frac{\partial^p}{\partial y^p} Z(z,t,x-z,\sigma)\frac{\partial^{k-p}}{\partial x^{k-p}}\Phi(x-z,\sigma,\xi,\tau)dyd\sigma.$$

Applying again the last change of variable and Lemma 5.3.1, we obtain that $\left|\frac{\partial^k}{\partial x^k}\Upsilon^2(x,t,\xi,\tau)\right|$ is bounded by

$$c(k)e^{A_k M\rho_t^\tau} n^{1/2-\alpha} \sum_{p=0}^k \int_{\tau+\frac{t-\tau}{2}}^t \int_{-\infty}^{\infty} \frac{\exp\{-f(t,\sigma,\tau,x,y,\xi)/A_k\}}{\sqrt{\rho_t^\sigma}\,(\rho_\sigma^\tau)^{(k-p+2)/2}}dyd\sigma.$$

Then, we use the change of variable

$$z = \frac{y-\xi}{\sqrt{M\rho_\sigma^\tau}} + \sqrt{M\rho_\sigma^\tau}$$

and the first assertion of Lemma 5.6.6 in order to have

$$\left|\frac{\partial^k}{\partial x^k}\Upsilon^2(x,t,\xi,\tau)\right| \leqslant c(k)\frac{e^{\widetilde{A}_k M\rho_t^\tau}}{(\rho_t^\tau)^{(k+1)/2}} \exp\left\{-\frac{1}{\widetilde{A}_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}.$$

From the two previous inequalities and 5.3.6, we can conclude about the lemma.

Since we have:

$$\frac{\partial}{\partial t}\Gamma(x,t,\xi,\tau) = \frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2}\Gamma(x,t,\xi,\tau) + \widehat{\sigma}_b(t,x)\frac{\partial}{\partial x}\Gamma(x,t,\xi,\tau),$$

we easily deduce from the previous inequalities the following lemma.

Corollary 5.3.9. *There exists some constants C_k and D_k such that:*

$$\left|\frac{\partial^{k+1}}{\partial x^k \partial t}\Gamma(x,t,\xi,\tau)\right| \leqslant \frac{C_k n^{1/2-\alpha} e^{D_k M\rho_t^\tau}}{(\rho_t^\tau)^{\frac{k+3}{2}}} \exp\left\{-\frac{1}{D_k}\left(\frac{x-\xi}{\sqrt{M\rho_t^\tau}} + \sqrt{M\rho_t^\tau}\right)^2\right\}.$$

5.3.4 Conclusion

We note:

$$\rho_\tau(t) = n^{1/2-\alpha}(\tau - t)$$

where $t \in [0,\tau[$. From the previous inequalities, we get the following

Lemma 5.3.10. *There exists some constants C_k and A_k such that*

$$\begin{aligned}
\left|\frac{\partial^k}{\partial x^k}\Gamma^*(x,t,\xi,\tau)\right| &\leqslant \frac{C_k e^{A_k M\rho_\tau(t)}}{\rho_\tau(t)^{\frac{k+1}{2}}} \exp\left\{-\frac{1}{A_k}\left(\frac{x-\xi}{\sqrt{M\rho_\tau(t)}} + \sqrt{M\rho_\tau(t)}\right)^2\right\}, \\
\left|\frac{\partial^{k+1}}{\partial x^k \partial t}\Gamma^*(x,t,\xi,\tau)\right| &\leqslant \frac{C_k n^{1/2-\alpha} e^{A_k M\rho_\tau(t)}}{\rho_\tau(t)^{\frac{k+3}{2}}} \exp\left\{-\frac{1}{A_k}\left(\frac{x-\xi}{\sqrt{M\rho_\tau(t)}} + \sqrt{M\rho_\tau(t)}\right)^2\right\}.
\end{aligned}$$

5.4 Estimates

5.4.1 Explicit Formulae

Recall that, in virtue of Lemma 5.2.2, we have :

$$\widehat{C}_x(t,x) = \int_{-\infty}^{\infty} h'(e^y)\Gamma^*(\ln x, t, y, 1)dy \tag{5.4.11}$$

and by the change of variable $z = e^y$, we obtain:

$$\widehat{C}_x(t,x) = \int_0^\infty \frac{h'(z)}{z}\Gamma^*(\ln x, t, \ln z, 1)dz, \tag{5.4.12}$$

$$\widehat{C}_{xx}(t,x) = \frac{1}{x}\int_0^\infty \frac{h'(z)}{z}\frac{\partial}{\partial x}\Gamma^*(\ln x, t, \ln z, 1)dz, \tag{5.4.13}$$

$$\widehat{C}_{xxx}(t,x) = -\frac{1}{x}\widehat{C}_{xx}(t,x) + \frac{1}{x^2}\int_0^\infty \frac{h'(z)}{z}\frac{\partial^2}{\partial^2 x}\Gamma^*(\ln x, t, \ln z, 1)dz, \tag{5.4.14}$$

$$\widehat{C}_{xt}(t,x) = \int_0^\infty \frac{h'(z)}{z}\frac{\partial}{\partial t}\Gamma^*(\ln x, t, \ln z, 1)dz, \tag{5.4.15}$$

$$\widehat{C}_{xxt}(t,x) = \int_0^\infty \frac{h'(z)}{z}\frac{\partial^2}{\partial t\partial x}\Gamma^*(\ln x, t, \ln z, 1)dz. \tag{5.4.16}$$

5.4.2 Inequalities

For all the sequence, we note

$$\rho_t^2 = n^{1/2-\alpha}(1-t).$$

Lemma 5.4.1. *There exists some constants A, $C > 0$ such that:*

$$\left|\widehat{C}_{xx}(t,x)\right| \leqslant \frac{Ce^{A\rho_t^2}}{x\rho_t}, \tag{5.4.17}$$

$$\left|\widehat{C}_{xxx}(t,x)\right| \leqslant \frac{Ce^{A\rho_t^2}}{x^2\rho_t^2}, \tag{5.4.18}$$

$$\left|\widehat{C}_{xt}(t,x)\right| \leqslant \frac{Ce^{A\rho_t^2}}{1-t}, \tag{5.4.19}$$

$$\left|\widehat{C}_{xxt}(t,x)\right| \leqslant \frac{Cn^{1/2-\alpha}e^{A\rho_t^2}}{x\rho_t^3}, \tag{5.4.20}$$

$$\left|\widehat{C}_{xxxt}(t,x)\right| \leqslant \frac{Cn^{1/2-\alpha}e^{A\rho_t^2}}{x^2\rho_t^4}. \tag{5.4.21}$$

Proof. From Lemma 5.3.10, it follows that there exists some constants C, A such that

$$\left|\widehat{C}_{xx}(t,x)\right| \leqslant C\frac{e^{A\rho_t^2}}{x\rho_t^2}\int_0^\infty \frac{1}{z}\exp\left\{-\frac{1}{A}\left(\frac{\ln(x/z)}{\rho_t}+\rho_t\right)^2\right\}dz.$$

Considering the change of variable

$$y = \frac{\ln(x/z)}{\rho_t} + \rho_t$$

verifying $dz = -\rho_t z dy$, we obtain that

$$\left|\widehat{C}_{xx}(t,x)\right| \leqslant const\,\frac{e^{A\rho_t^2}}{x\rho_t}\int_{-\infty}^\infty e^{-\frac{y^2}{A}}dy$$

and the result follows. The same reasoning gives us similar inequalities for the other derivatives.

Note that the exponential term in the previous formulae is embarrassing if $\alpha < 1/2$. That's why, for $\alpha \neq 1/2$, we put together revision dates in intervals whose the breadth is comparable to $1/n^{1/2-\alpha}$. We fix: $j_{-1} := j_0 := n$, and for $1 \leqslant p \leqslant p_n$, we define inductively

$$j_p := j_p^n = \min\left\{i : t_i^n \geqslant t_{j_{p-1}} - \frac{1}{n^{1/2-\alpha}}\right\}$$

until the last term $t_{j_{p_n}}$ verifies:

$$\frac{1}{n^{1/2-\alpha}} \leqslant t_{j_{p_n}} < \frac{2}{n^{1/2-\alpha}}.$$

Moreover, we define $t_{j_{p_n+1}} := t_{j_{p_n+2}} := 0$. We consider the number $N(n)$ of sub-intervals of $[0,1]$ generated by the sequence $(t_{j_p})_{p=0,\cdots,p_n}$. Since there exists a constant c such that $\Delta t_i \leqslant c\, n^{-1}$, we deduce that for $n \geqslant n_0$ large enough, we have

$$t_{j_p} - t_{j_{p+1}} \geqslant \frac{1}{2n^{1/2-\alpha}}.$$

It follows that $N(n) \leqslant 2n^{1/2-\alpha}$.

Lemma 5.4.2. *Assume that $\alpha \neq 1/2$. There exists a constant $\widetilde{C}$ independent of n and p such that for all $p = 0, \cdots, p_n$, we have:*

$$\left|\widehat{C}(t_{j_p}, x)\right| \leqslant \widetilde{C}\,(p+1)(1+|x|).$$

Proof. We shall argue inductively. The result is true by hypothesis with $p = 0$. So, we suppose that $\left|\widehat{C}(t_{j_p}, x)\right| \leqslant \widetilde{C}\,(p+1)(1+|x|)$ and we prove it for $p+1$. It is obvious that $\widehat{C}$ is the solution of the following PDE:

$$(\mathbf{e^P}) = \begin{cases} \widehat{C}_t(t,x) & + \ \frac{1}{2}\widehat{\sigma}^2(t,x)x^2\widehat{C}_{xx}(t,x) = 0, \quad (x,t) \in]0,\infty[\times[0,t_{j_p}[\\ \widehat{C}(t_{j_p},x) & = \ h_p(x), \quad x \in]0,\infty[\end{cases}$$

where $h_p(x) = \widehat{C}(t_{j_p}, x)$ verifies $|h_p'(x)| \leqslant \|h'\|_\infty$ because of Lemma 5.2.2.

From 5.2.4, we have

$$\widehat{C}(t_{j_{p+1}}, x) = E h_p(\widehat{S}_{x,t_{j_{p+1}}}(t_{j_p}))$$

and

$$\left|\widehat{C}(t_{j_{p+1}}, x) - \widehat{C}(t_{j_p}, x)\right| \leqslant \|h'\|_\infty E\left|\widehat{S}_{x,t_{j_{p+1}}}(t_{j_p}) - x\right|.$$

Moreover,

$$\widehat{S}_{x,t_{j_{p+1}}}(t) = x + \int_{t_{j_{p+1}}}^t \widehat{\sigma}(u, \widehat{S}_{x,t_{j_{p+1}}}(u))\widehat{S}_{x,t_{j_{p+1}}}(u)dW_u$$

is such that:

$$\widehat{S}^2_{x,t_{j_{p+1}}}(t) \leqslant 2x^2 + 2\left(\int_{t_{j_{p+1}}}^t \widehat{\sigma}(u, \widehat{S}_{x,t_{j_{p+1}}}(u))\widehat{S}_{x,t_{j_{p+1}}}(u)dW_u\right)^2.$$

It follows that there is a constant c such that:

$$E\widehat{S}^2_{x,t_{j_{p+1}}}(t) \leqslant 2x^2 + 2cn^{1/2-\alpha}\int_{t_{j_{p+1}}}^t E\widehat{S}^2_{x,t_{j_{p+1}}}(u)du.$$

The Gronwall lemma implies that:

$$E\widehat{S}^2_{x,t_{j_{p+1}}}(t_{j_p}) \leqslant 2x^2\left(1+e^{2cn^{1/2-\alpha}(t_{j_p}-t_{j_{p+1}})}\right).$$

We deduce the inequality

$$\left|\widehat{C}(t_{j_{p+1}},x)-\widehat{C}(t_{j_p},x)\right| \leqslant \widetilde{C}|x|$$

where $\widetilde{C}$ is a judicious constant. It follows that

$$\left|\widehat{C}(t_{j_{p+1}},x)\right| \leqslant \widetilde{C}(p+2)(1+|x|).$$

Since $\widehat{C}$ is the solution of the PDE ($\mathbf{e^p}$), with the essential inequality

$$\|h_p'(x)\| \leqslant \|h'\|_\infty,$$

we deduce from Lemmas 5.3.10 and 5.4.1 the following result.

Corollary 5.4.3. *Assume that $\alpha \neq 1/2$. There exists a constant C such that for any $p=0,\cdots,p_n$ and $t\in[t_{j_{p+2}},t_{j_p}[$, we have:*

$$\left|\widehat{C}_{xx}(t,x)\right| \leqslant \frac{C}{x\sqrt{n^{1/2-\alpha}(t_{j_p}-t)}}, \tag{5.4.22}$$

$$\left|\widehat{C}_{xxx}(t,x)\right| \leqslant \frac{C}{x^2n^{1/2-\alpha}(t_{j_p}-t)}, \tag{5.4.23}$$

$$\left|\widehat{C}_{xt}(t,x)\right| \leqslant \frac{C}{t_{j_p}-t}, \tag{5.4.24}$$

$$\left|\widehat{C}_{xxt}(t,x)\right| \leqslant \frac{C}{x\sqrt{n^{1/2-\alpha}}(t_{j_p}-t)^{3/2}}, \tag{5.4.25}$$

$$\left|\widehat{C}_{xxxt}(t,x)\right| \leqslant \frac{C}{x^2n^{1/2-\alpha}\left(t_{j_p}-t\right)^2}. \tag{5.4.26}$$

5.5 Proofs of Theorems 5.1.1 and 5.1.2

We essentially present the proof for $\alpha<1/2$ because for $\alpha=1/2$, the latter is similar but more easy since we don't need to introduce the sequence $(t_{j_p})_{p=0,\cdots,p_n}$ and use Corollary 5.4.3.

We recall a classical result, representing the difference $V_1^n-h(S_1)$ in a convenient form.

Lemma 5.5.1. *We have $V_1^n-h(S_1)=F_1^n+F_2^n+F_3^n$ where*

$$F_1^n := \int_0^1 (H_t^n-\widehat{H}_t)dS_t - k_n|H^n_{t_n}-H^n_{t_{n-1}}|S_{t_n}, \tag{5.5.27}$$

$$F_2^n := \frac{1}{2}\int_0^1 \sigma(t,S_t)\gamma_n(t)S_t^2|\widehat{C}_{xx}(t,S_t)|dt - k_n\sum_{i=1}^{n-1}|H^n_{t_i}-H^n_{t_{i-1}}|S_{t_i}, \tag{5.5.28}$$

$$F_3^n := \frac{1}{2}\int_0^1 \sigma(t,S_t)\gamma_n(t)S_t^2\left(\widehat{C}_{xx}(t,S_t)-|\widehat{C}_{xx}(t,S_t)|\right)dt. \tag{5.5.29}$$

Our objective is to prove that F_1, F_2 converge to zero in probability.

Lemma 5.5.2. *We have* $P - \lim F_1^n = 0$.

Proof. It is obvious that $k_n|H^n_{t_n} - H^n_{t_{n-1}}|S_{t_n} \to 0$. So, we note

$$\begin{aligned} \widetilde{F}_1^n &= \int_0^{t_{n-1}} (H_t^n - \widehat{H}_t)dS_t, \\ \dot{F}_1^n &= \int_{t_{n-1}}^1 (H_t^n - \widehat{H}_t)dS_t \end{aligned}$$

with:

$$E\left(\widetilde{F}_1^n\right)^2 = \int_0^{t_{n-1}} \sum_{i=1}^n E\sigma^2(t, S_t)S_t^2 \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t)\right)^2 1_{]t_{i-1},t_i]}(t)dt$$

and

$$E\left(\widetilde{F}_1^n\right)^2 \leqslant const \int_0^{t_{n-1}} \sum_{i=1}^n ES_t^2 \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t)\right)^2 1_{]t_{i-1},t_i]}(t)dt.$$

Note that for $\alpha < 1/2$

$$\int_{t_{j_1}}^{t_{n-1}} \sum_{i=1}^n ES_t^2 \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t)\right)^2 1_{]t_{i-1},t_i]}(t)dt \leqslant const\, \frac{1}{n^{1/2-\alpha}} \to 0.$$

Otherwise, since $|\widehat{C}_x(t, S_t)| \leqslant \|h'\|$, it suffices to prove that for each fixed $t \leqslant t_{j_1}$, respectively $t \leqslant t_{n-1}$ if $\alpha = 1/2$,

$$ES_t^2 \left(\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t)\right)^2 \to 0$$

and apply the Lebesgue theorem. Using again this latter, it suffices to prove that a.s.(ω),

$$\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t) \to 0$$

since $\sup_t S_t^2$ is integrable. The case $\alpha = 1/2$ is obvious because $\widehat{C}$ does not depend on n. Otherwise, we have:

$$\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t, S_t) = \widehat{C}_{xt}(\theta_i, S_{t_{i-1}})(t_{i-1} - t) + \widehat{C}_{xx}(t, \widetilde{S}_t)(S_{t_{i-1}} - S_t)$$

where $\theta_i \in [t_{i-1}, t]$ and $\widetilde{S}_t \in [S_{t_{i-1}}, S_t]$.

From the condition $t_{i-1} < t \leqslant t_i \leqslant t_{j_1}$, we deduce that there exists m_n verifying the inequality $t_{j_{m_n+1}} \leqslant t_{i-1} < t \leqslant t_i \leqslant t_{j_{m_n}} < t_{j_{m_{n-1}}}$. Indeed, it suffices to choose $m_n = \max\{k \geqslant 0 : t_{j_k} \geqslant t_i\}$. Then, from Corollary 5.4.3, we deduce some constants c_1, c_2 such that:

$$|\widehat{C}_{xt}(\theta_i, S_{t_{i-1}})(t_{i-1} - t)| \leqslant c_1 \frac{\Delta t_i}{t_{j_{m_{n-1}}} - \theta_i}$$

with

$$t_{j_{m_{n-1}}} - \theta_i \geqslant \frac{c_2}{n^{1/2-\alpha}}.$$

It follows that:

$$|\widehat{C}_{xt}(\theta_i, S_{t_{i-1}})(t_{i-1}-t)| \leqslant c\frac{n^{1/2-\alpha}}{n} \to 0.$$

In a same way, there exists a constant $c(\omega)$ depending on $\omega \in \Omega$ such that:

$$|\widehat{C}_{xx}(t, \widetilde{S}_t)(S_{t_{i-1}} - S_t)| \leqslant c(\omega)\frac{|S_{t_{i-1}} - S_t|}{\sqrt{n^{1/2-\alpha}(t_{j_{m_{n-1}}} - t)}} \leqslant c(\omega)\sqrt{n^{1/2+\alpha}}|S_{t_{i-1}} - S_t|.$$

But, from [10] page 112, there exists a constant c such that

$$n^{1/2+\alpha}E(S_{t_{i-1}} - S_t)^2 \leqslant c\, n^{1/2+\alpha}\Delta t_i \leqslant cn^{\alpha-1/2} \to 0.$$

Then, we can conclude that $E\left(\widetilde{F}_1^n\right)^2 \to 0$ whereas it is simpler to prove that $E\left(\dot{F}_1^n\right)^2 \to 0$. By the Ito Formula, we get that

$$\widehat{C}_x(t, S_t) = \widehat{C}_x(0, S_0) + M_t^n + A_t^n$$

where

$$\begin{aligned} M_t^n &:= \int_0^t \sigma(u, S_u)S_u\widehat{C}_{xx}(u, S_u)dW_u, \\ A_t^n &:= \int_0^t \left[\widehat{C}_{xt}(u, S_u) + \frac{1}{2}\sigma^2(u, S_u)S_u^2\widehat{C}_{xxx}(u, S_u)\right]du. \end{aligned}$$

We write $F_2^n = \sum_{i=1}^5 L_i$ where

$$\begin{aligned} L_1^n &:= \frac{1}{2}\int_0^1 \sigma(t, S_t)\gamma_n(t)S_t^2|\widehat{h}_t|dt - \frac{1}{2}\int_0^1\sum_{i=1}^{n-1}\sigma(t_{i-1}, S_{t_{i-1}})\gamma_n(t_{i-1})S_{t_{i-1}}^2|\widehat{h}_{t_{i-1}}|I_{]t_{i-1},t_i]}(t)dt \\ L_2^n &:= \sum_{i=1}^{n-1}\sigma(t_{i-1}, S_{t_{i-1}})|\widehat{h}_{t_{i-1}}|S_{t_{i-1}}^2\left(\frac{1}{2}\gamma_n(t_{i-1})\Delta t_i - k_n\, n^{1/2}\sqrt{\Delta t_i f'(t_{i-1})}|\Delta W_{t_i}|\right), \\ L_3^n &:= k_n\sum_{i=1}^{n-1}\sigma(t_{i-1}, S_{t_{i-1}})S_{t_{i-1}}^2|\widehat{h}_{t_{i-1}}|n^{1/2}\sqrt{\Delta t_i f'(t_{i-1}}|\Delta W_{t_i}| - k_n\sum_{i=1}^{n-1}S_{t_{i-1}}|\Delta M_{t_i}|, \\ L_4^n &:= k_n\sum_{i=1}^{n-1}S_{t_{i-1}}|\Delta M_{t_i}| - k_n\sum_{i=1}^{n-1}S_{t_{i-1}}|\Delta\widehat{H}_{t_i}|, \\ L_5^n &:= -k_n\sum_{i=1}^{n-1}\Delta S_{t_i}|\Delta\widehat{H}_{t_i}| \end{aligned}$$

Lemma 5.5.3. *We have $P - \lim\, L_1^n = 0$.*

Proof. We note $L_1^n = \sum_{k=1}^6 L_{1k}^n$ and we prove that $P - \lim\, L_{1k}^n = 0$.

$$L_{11}^n = \frac{1}{2}\int_0^1\sum_{i=1}^{n-1}\left[\sigma(t, S_t) - \sigma(t_{i-1}, S_t)\right]\gamma_n(t)S_t^2|\widehat{h}_t|I_{]t_{i-1},t_i]}(t)dt$$

Using the hypothesis on σ, and Lemma 5.4.3, we deduce a constant c_ω depending on $\omega \in \Omega$ such that:

$$|L_{11}^n| \leqslant c_\omega \frac{\sqrt{n^{1/2-\alpha}}}{n} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}} \frac{dt}{\sqrt{t_{j_p} - t}} \leqslant c_\omega \frac{\sqrt{n^{1/2-\alpha}}}{n} N(n) \frac{1}{\sqrt{n^{1/2-\alpha}}}.$$

It follows that $P - |L_{11}^n| \to 0$.

$$L_{12}^n = \frac{1}{2} \int_0^1 \sum_{i=1}^{n-1} \left[\sigma(t_{i-1}, S_t) - \sigma(t_{i-1}, S_{t_{i-1}})\right] \gamma_n(t) S_t^2 |\widehat{h}_t| I_{]t_{i-1},t_i]}(t) dt$$

In a same way, we deduce a constant c_ω depending on $\omega \in \Omega$ such that $|L_{12}^n| \leqslant c_\omega \widetilde{L}_{12}^n$ where

$$\widetilde{L}_{12}^n = \sqrt{n^{1/2-\alpha}} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}} \sum_{i=1}^{n-1} \frac{|S_t - S_{t_{i-1}}|}{\sqrt{t_{j_p} - t}} I_{]t_{i-1},t_i]}(t) dt.$$

But, since $E|S_t - S_{t_{i-1}}| \leqslant \sqrt{t - t_{i-1}}$, we obtain that

$$E\widetilde{L}_{12}^n \leqslant const \frac{\sqrt{n^{1/2-\alpha}}}{\sqrt{n}} \sum_{p=0}^{p_n} \int_0^{t_{j_p} - t_{j_{p+1}}} \frac{du}{\sqrt{u}} \leqslant const \frac{n^{1/2-\alpha}}{n^{1/2}} \to 0.$$

It follows that $P - |L_{12}^n| \to 0$.

$$L_{13}^n = \frac{1}{2} \int_0^1 \sum_{i=1}^{n-1} \left[\gamma_n(t) - \gamma_n(t_{i-1})\right] \sigma(t_{i-1}, S_{t_{i-1}}) S_t^2 |\widehat{h}_t| I_{]t_{i-1},t_i]}(t) dt$$

Using the hypothesis on f we deduce that

$$|\gamma_n(t) - \gamma_n(t_{i-1})| \leqslant const\, n^{1/2-\alpha} \Delta t_i.$$

So, there exists a constant c_ω such that:

$$|L_{13}^n| \leqslant c_\omega \frac{n^{-1/2-\alpha}}{\sqrt{n^{1/2-\alpha}}} \sum_{p=0}^{p_n} \int_0^{t_{j_p} - t_{j_{p+1}}} \frac{du}{\sqrt{u}} \leqslant c_\omega\, n^{-1/2-\alpha} \to 0.$$

$$L_{14}^n = \frac{1}{2} \int_0^1 \sum_{i=1}^{n-1} \left[S_t^2 - S_{t_{i-1}}^2\right] \gamma_n(t_{i-1}) \sigma(t_{i-1}, S_{t_{i-1}}) |\widehat{h}_t| I_{]t_{i-1},t_i]}(t) dt$$

A reasoning similar to the one used for L_{12}^n leads to $P - |L_{14}^n| \to 0$.

$$L_{15}^n = \frac{1}{2} \int_0^1 \sum_{i=1}^{n-1} \left[|\widehat{C}_{xx}(t, S_t)| - |\widehat{C}_{xx}(t, S_{t_{i-1}})|\right] \gamma_n(t_{i-1}) \sigma(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 I_{]t_{i-1},t_i]}(t) dt$$

Using $||a| - |b|| \leqslant |a - b|$ and

$$\widehat{C}_{xx}(t, S_t) - \widehat{C}_{xx}(t, S_{t_{i-1}}) = \widehat{C}_{xxx}(t, \widetilde{S}_i) \left(S_t - S_{t_{i-1}}\right)$$

where $\widetilde{S}_i \in [S_t, S_{t_{i-1}}]$, we deduce that there exists a constant c_ω such that: $|L_{15}^n| \leqslant c_\omega \widetilde{L}_{15}^n$ where

$$\widetilde{L}_{15}^n = \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \sum_{i=1}^{n-1} \frac{|S_t - S_{t_{i-1}}|}{t_{j_{p-1}} - t} I_{]t_{i-1},t_i]}(t)dt.$$

But we have:

$$E\widetilde{L}_{15}^n \leqslant \frac{const}{\sqrt{n}} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \frac{1}{t_{j_{p-1}} - t} dt$$

where

$$\begin{aligned} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \frac{1}{t_{j_{p-1}} - t} dt &= \int_{\Delta_p^-}^{\Delta_p} \frac{1}{u} du = \ln\left(\frac{\Delta_p}{\Delta_p^-}\right) \leqslant const \ln n, \\ \Delta_p &= t_{j_{p-1}} - t_{j_{p+1}}, \quad \Delta_p^- = t_{j_{p-1}} - t_{j_p} \wedge t_{n-1} \end{aligned}$$

because $\Delta_p \leqslant 3/n^{1/2-\alpha}$ and $t_{j_{p-1}} - t_{j_p} \wedge t_{n-1} \geqslant c/n$. It follows that

$$E\widetilde{L}_{15}^n \leqslant \frac{const\, n^{1/2-\alpha} \ln n}{\sqrt{n}} \to 0$$

and $P - |L_{15}^n| \to 0$.

$$L_{16}^n = \frac{1}{2} \int_0^1 \sum_{i=1}^{n-1} \left[|\widehat{C}_{xx}(t, S_{t_{i-1}})| - |\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})| \right] \gamma_n(t_{i-1}) \sigma(t_{i-1}, S_{t_{i-1}}) S_{t_{i-1}}^2 I_{]t_{i-1},t_i]}(t)dt$$

In a same way, we write:

$$\widehat{C}_{xx}(t, S_{t_{i-1}}) - \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) = \widehat{C}_{xxt}(\widetilde{t}_i, S_{t_{i-1}})(t - t_{i-1})$$

where $\widetilde{t}_i \in [t_{i-1}, t_i]$. Note that, if $t \in [t_{i-1}, t_i] \cap [t_{j_{p+1}}, t_{j_p}]$, we have $t_{j_p} \geqslant t_i$ and $t_i \leqslant t_{n-1}$ implies that

$$\frac{t_{j_{p-1}} - t_{i-1}}{t_{j_{p-1}} - t_i} = 1 + \frac{t_i - t_{i-1}}{t_{j_{p-1}} - t_i} \leqslant 1 + \frac{c\,n}{n} \leqslant const.$$

So, there exists a constant c_ω such that:

$$|L_{16}^n| \leqslant c_\omega n^{1/2-\alpha} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \frac{\Delta t_i}{\sqrt{n^{1/2-\alpha}} (t_{j_{p-1}} - t)^{3/2}} dt.$$

For $p = 0$, the term of the previous sum verifies

$$n^{1/2-\alpha} \int_{t_{j_1}}^{t_{n-1}} \frac{\Delta t_i}{\sqrt{n^{1/2-\alpha}} (1-t)^{3/2}} dt \leqslant const \sqrt{\frac{n^{1/2-\alpha}}{n}} \ln n \to 0$$

whereas, for $p \geqslant 1$,

$$t_{j_{p-1}} - t \geqslant \frac{const}{n^{1/2-\alpha}}.$$

It follows that it is sufficient to estimate

$$\frac{n^{1/2-\alpha}}{n} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}} \frac{dt}{t_{j_{p-1}} - t} \leqslant const \frac{n^{2(1/2-\alpha)}}{n} \to 0.$$

Then $P - |L_{16}^n| \to 0$.

Lemma 5.5.4. *We have* $P - \lim L_2^n = 0$.

Proof. Recall that we have a constant c such that $\Delta t_i \geqslant c n^{-1}$. We have:

$$L_2^n = \sum_{i=1}^{n-1} \left| \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \right| S_{t_{i-1}}^2 \xi_i$$

where

$$\xi_i = \frac{1}{2}\gamma_n(t_{i-1})\Delta t_i - k_n\, n^{1/2}\sqrt{\Delta t_i f'(t_{i-1})}|\Delta W_{t_i}|$$

is independent of $\mathcal{F}_{t_{i-1}}$ and verifies $E\xi_i = 0$, and

$$E\xi_i^2 = (1 - 2/\pi)k_n^2 n f'(t_{i-1})(\Delta t_i)^2.$$

It follows that

$$E\left(L_2^n\right)^2 = \sum_{i=1}^{n-1} E\widehat{C}_{xx}^2(t_{i-1}, S_{t_{i-1}})S_{t_{i-1}}^4 E\xi_i^2$$

and

$$E\left(L_2^n\right)^2 \leqslant const\, \frac{n^{-2\alpha}}{n^{1/2-\alpha}} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \frac{dt}{t_{j_{p-1}} - t} \leqslant const\, n^{-2\alpha} \ln n.$$

Then, we deduce that $P - |L_2^n| \to 0$.

Lemma 5.5.5. *We have* $P - \lim L_3^n = 0$.

Proof. Since g'' is supposed bounded, we deduce that

$$n^{1/2}\sqrt{\Delta t_i f'(t_{i-1})} = 1 + \varepsilon_n$$

where $\varepsilon_n = O(n^{-1})$. Then, we write $L_3^n = A^n + B^n$ with

$$\begin{aligned} A^n &= k_n\varepsilon_n \sum_{i=1}^{n-1} \sigma(t_{i-1}, S_{t_{i-1}})S_{t_{i-1}}^2 |\widehat{h}_{t_{i-1}}||\Delta W_{t_i}|, \\ B^n &= k_n \sum_{i=1}^{n-1} \sigma(t_{i-1}, S_{t_{i-1}})S_{t_{i-1}}^2 |\widehat{h}_{t_{i-1}}||\Delta W_{t_i}| - S_{t_{i-1}}|\Delta M_{t_i}|. \end{aligned}$$

By independence, we deduce a constant c such that

$$E\,|A_n| \leqslant c\, n^{1/2-\alpha}\varepsilon_n \sum_{i=1}^{n-1} E\left|\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})\right| S_{t_{i-1}}^2 \Delta t_i,$$

$$E\,|A_n| \leqslant c\, \sqrt{n^{1/2-\alpha}}\varepsilon_n \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \frac{dt}{\sqrt{t_{j_p} - t}} \leqslant c\, n^{1/2-\alpha}\varepsilon_n \to 0.$$

Moreover, $|B_n| \leqslant D_1^n + D_2^n + D_3^n$ where D_i^n, $i = 1, 2, 3$ are defined as follows.

$$D_1^n = k_n \sum_{i=1}^{n-1} \left| \int_{t_{i-1}}^{t_i} \xi_i(t) dW_t \right|$$

with

$$\xi_i(t) = S^2_{t_{i-1}}\left[\sigma(t_{i-1}, S_{t_{i-1}}) - \sigma(t_{i-1}, S_t)\right]\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}).$$

Then, there exists a constant c such that

$$\|D_1^n\|_2 \leqslant c\,n^{-\alpha}\sum_{i=1}^{n-1}\left(\int_{t_{i-1}}^{t_i} E\xi_i^2(t)dt\right)^{1/2}.$$

We note $I_p^n = I_p := [t_{j_{p+1}}, t_{j_p}\wedge t_{n-1}]$. Using the hypothesis on σ and the Cauchy-Scwharz inequality, we obtain that

$$\begin{aligned}
\|D_1^n\|_2 &\leqslant c\,n^{-\alpha}\sum_{p=0}^{p_n}\sum_{t_{i-1},t_i\in I_p^n}\left(\int_{t_{i-1}}^{t_i}\frac{\Delta t_i dt}{n^{1/2-\alpha}(t_{j_p}-t_{i-1})}\right)^{1/2},\\
\|D_1^n\|_2 &\leqslant c\,\frac{n^{-\alpha}}{\sqrt{n^{1/2-\alpha}}}\sum_{p=0}^{p_n}\sum_{t_{i-1},t_i\in I_p^n}\frac{\Delta t_i}{\sqrt{t_{j_p}-t_{i-1}}},\\
\|D_1^n\|_2 &\leqslant c\,\frac{n^{-\alpha}}{\sqrt{n^{1/2-\alpha}}}\sum_{p=0}^{p_n}\int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}}\frac{dt}{\sqrt{t_{j_p}-t}} \leqslant c\,n^{-\alpha}\to 0.
\end{aligned}$$

We have

$$D_2^n = k_n\sum_{i=1}^{n-1}\left|\int_{t_{i-1}}^{t_i}\widetilde{\xi}_i(t)dW_t\right|$$

where

$$\widetilde{\xi}_i(t) = S^2_{t_{i-1}}\left[\sigma(t_{i-1}, S_t) - \sigma(t, S_t)\right]\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}).$$

Following the previous reasoning, we also obtain that $\|D_2^n\|_2 \to 0$. From now on, we deal with

$$D_3^n = k_n\sum_{i=1}^{n-1}\left|\int_{t_{i-1}}^{t_i}\mathcal{X}_i(t)dW_t\right|$$

where

$$\mathcal{X}_i(t) = \left[S_{t_{i-1}}\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) - S_t\widehat{C}_{xx}(t, S_t)\right]\sigma(t, S_t).$$

In a similar way, we have

$$\|D_3^n\|_2 \leqslant c\,n^{-\alpha}\sum_{i=1}^{n-1}\left(\int_{t_{i-1}}^{t_i} E\mathcal{X}_i^2(t)dt\right)^{1/2}$$

where, using the Ito formula in the similar way as in the proof of Lemma 2.4.6, Chapter 2, we obtain a constant c such that

$$E\mathcal{X}_i^2(t) \leqslant \frac{c\,\Delta t_i}{n^{2(1/2-\alpha)}(t_{j_{p-1}}-t_i)^2} + \frac{c\,(\Delta t_i)^2}{n^{1/2-\alpha}(t_{j_{p-1}}-t_i)^3}$$

under the condition $t_{j_{p+1}} \leqslant t_{i-1} \leqslant t \leqslant t_i \leqslant t_{j_p}\wedge t_{n-1}$.

Recall that if $t\in[t_{i-1}, t_i]\cap[t_{j_{p+1}}, t_{j_p}]$, there is a constant c such that

$$\frac{t_{j_{p-1}}-t_{i-1}}{t_{j_{p-1}}-t_i} \leqslant c.$$

Then, it suffices to analyse the two following sums:

$$\begin{aligned} S_n^1 &= n^{-\alpha}\sum_{p=0}^{p_n}\sum_{t_{i-1},\,t_i\in I_p^n}\frac{\Delta t_i}{n^{1/2-\alpha}(t_{j_{p-1}}-t_{i-1})},\\ S_n^2 &= \frac{n^{-\alpha}}{n^{1/2}n^{1/2(1/2-\alpha)}}\sum_{p=0}^{p_n}\sum_{t_{i-1},\,t_i\in I_p^n}\frac{\Delta t_i}{(t_{j_{p-1}}-t_{i-1})^{3/2}}. \end{aligned}$$

First, we have

$$S_n^1\leqslant \frac{n^{-\alpha}}{n^{1/2-\alpha}}\sum_{p=0}^{p_n}\int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}}\frac{dt}{t_{j_{p-1}}-t}\leqslant const\,n^{-\alpha}\ln n\to 0.$$

Secondly, we have for $p=0$,

$$t_{j_{p-1}}-t_{i-1}\geqslant\frac{const}{n}.$$

It follows that the first term of the sum S_n^2 is less than

$$const\,n^{-\alpha}\int_{t_{j_1}}^{t_{n-1}}\frac{dt}{1-t}\leqslant const\,n^{-\alpha}\ln n\to 0.$$

Otherwise, for $p\geqslant 1$, we use the inequality

$$t_{j_{p-1}}-t\geqslant\frac{const}{n^{1/2-\alpha}}$$

and it suffices to estimate

$$\frac{n^{-\alpha}}{n^{1/2}}\sum_{p=1}^{p_n}\int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}}\frac{dt}{t_{j_{p-1}}-t}\leqslant\frac{const\,n^{-\alpha}n^{1/2-\alpha}}{n^{1/2}}\ln n\to 0.$$

Finally, we can claim that $P-\lim L_3^n=0$.

Lemma 5.5.6. *We have* $P-\lim L_4^n=0$.

Proof. Using again the inequality $||a|-|b||\leqslant|a-b|$ we get that

$$|L_n^4|\ \leqslant\ k_n\sum_{i=1}^{n-1}S_{t_{i-1}}|\Delta A_{t_i}|\leqslant c(\omega)\,(I_n+J_n)$$

where

$$\begin{aligned} I_n &= k_n\int_0^1|\widehat{C}_{xt}(u,S_u)|du,\\ J_n &= k_n\int_0^1\sigma^2S_u^2|\widehat{C}_{xxx}(u,S_u)|du. \end{aligned}$$

We have

$$I_n\leqslant c(\omega)\,n^{-\alpha}\sum_{p=0}^{p_n}\int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}}\frac{dt}{t_{j_{p-1}}-t}\leqslant c_\omega n^{1/2-2\alpha}\ln n\to 0$$

and

$$J_n \leqslant c(\omega)\, \frac{n^{-\alpha}}{n^{1/2-\alpha}} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \frac{dt}{t_{j_{p-1}}-t} \leqslant c\, n^{-\alpha} \ln n \to 0.$$

Then, we can conclude that $P-\lim\, L_4^n = 0$.

Lemma 5.5.7. *We have* $P-\lim\, L_5^n = 0$.

Proof. We use the Taylor expansion

$$\widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \widehat{C}_x(t_i, S_{t_i}) = \widehat{C}_{xt}(\theta_i, S_{t_{i-1}})(t_{i-1}-t_i) + \widehat{C}_{xx}(t_i, \widetilde{S}_{t_i})(S_{t_{i-1}} - S_{t_i})$$

where $\theta_i \in [t_{i-1}, t_i]$ and $\widetilde{S}_{t_i} \in [S_{t_{i-1}}, S_t]$. Then, $|L_5^n| \leqslant c(w)(A_n + B_n)$ with

$$\begin{aligned} A_n &= \frac{n^{-\alpha}}{\sqrt{n^{1/2-\alpha}}} \sum_{p=0}^{p_n} \sum_{t_{i-1},\, t_i \in I_p^n} \frac{(\Delta S_{t_i})^2}{\sqrt{t_{j_{p-1}} - t_{i-1}}}, \\ B_n &= n^{-\alpha} \sum_{p=0}^{p_n} \sum_{t_{i-1},\, t_i \in I_p^n} \frac{|\Delta S_{t_i}|\, \Delta t_i}{t_{j_{p-1}} - t_{i-1}}. \end{aligned}$$

We deduce a constant c such that

$$EA_n \leqslant c \frac{n^{-\alpha}}{\sqrt{n^{1/2-\alpha}}} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \frac{dt}{\sqrt{t_{j_p} - t}} \leqslant c\, n^{-\alpha} \to 0$$

and

$$B_n \leqslant \frac{n^{-\alpha}}{n^{1/2}} \sum_{p=0}^{p_n} \int_{t_{j_{p+1}}}^{t_{j_p}\wedge t_{n-1}} \frac{dt}{t_{j_{p-1}} - t} \leqslant c\, n^{-2\alpha} \ln n \to 0.$$

Then, we can conclude that $P-\lim\, L_5^n = 0$.

5.6 Appendix

Lemma 5.6.1. *The stochastic equation defined on* $[s,1]$ *for all* $s \in]0,1[$ *by :*

$$\begin{cases} d\widehat{S}_{x,s}(t) &= \widehat{\sigma}(t, \widehat{S}_{x,s}(t))\widehat{S}_{x,s}(t) dW_t \\ \widehat{S}_{x,s}(s) &= x \end{cases}$$

has a unique solution verifying, for a constant C^*,

$$E \sup_{s\leqslant t\leqslant 1} \widehat{S}^2_{x,s}(t) \leqslant C^*(1+x^2).$$

Proof. It suffices to apply Theorem 2.2 p104 [10]. For this, we verify the following conditions with $\overline{\sigma}(t,x) = \widehat{\sigma}(t,x)x$ (depending on n).

Since f' is bounded, there exists a constant c_n such that

$$|\overline{\sigma}(t,x)| \leqslant c_n |x|.$$

Moreover, if $|x|$, $|\overline{x}| \leqslant N$,

$$|\overline{\sigma}(t,x) - \overline{\sigma}(t,\overline{x})| \leqslant |\widehat{\sigma}(t,x)|\, |x - \overline{x}| + N\, |\widehat{\sigma}(t,x) - \widehat{\sigma}(t,\overline{x})|\, .$$

Since $\widehat{\sigma}(t,x)$ is bounded from below by a strictly positive constant, we have

$$|\widehat{\sigma}(t,x)-\widehat{\sigma}(t,\overline{x})| \leqslant const \left|\widehat{\sigma}^2(t,x)-\widehat{\sigma}^2(t,\overline{x})\right|$$

where

$$\widehat{\sigma}^2(t,x)-\widehat{\sigma}^2(t,\overline{x}) = \sigma^2(t,x)-\sigma^2(t,\overline{x})+\gamma_n(t)\left(\sigma(t,x)-\sigma(t,\overline{x})\right).$$

From the hypothesis on σ we deduce a constant $K_n(N)$ such that for $|x|$, $|\overline{x}| \leqslant N$,

$$|\overline{\sigma}(t,x)-\overline{\sigma}(t,\overline{x})| \leqslant K_n(N)|x-\overline{x}|.$$

Then, we can conclude about the lemma.

Lemma 5.6.2. *Assume that $t \in [0,1]$. Then, the stochastic equation:*

$$\begin{cases} d\widetilde{S}_{x,t}(u) &= \widehat{\sigma}(u,\widetilde{S}_{x,t}(u))\widetilde{S}_{x,t}(u)dW_u + \widehat{\Lambda}(u,\widetilde{S}_{x,t}(u))\widetilde{S}_{x,t}(u)du \\ \widetilde{S}_{x,t}(t) &= x \end{cases}$$

has a unique solution on $[t,1]$.

Proof. It suffices to apply Theorem 2.2 p104 [10]. For this, we verify the needed conditions with:

$$\begin{aligned} \widetilde{\sigma}(t,x) &= \widehat{\sigma}(t,x)x, \\ b(t,x) &= \widehat{\Lambda}(t,x)x. \end{aligned}$$

From

$$\widehat{\sigma}^2(t,x) = \sigma^2(t,x)+\gamma_n(t)\sigma(t,x)$$

we deduce that

$$2\widehat{\sigma}_x(t,x)\widehat{\sigma}(t,x) = 2\sigma_x(t,x)\sigma(t,x)+\gamma_n(t)\sigma_x(t,x).$$

Recall that

$$\widehat{\Lambda}(t,x) = \widehat{\sigma}^2(t,x)+\widehat{\sigma}_x(t,x)\widehat{\sigma}(t,x)x.$$

Then, from the boundedness of $|x\sigma_x(t,x)|$, it is easy to deduce that there exists a constant c such that $|b(t,x)| \leqslant c\,|x|$. Furthermore, it is clear that there exists a constant $\widetilde{c}$ such that $|\widetilde{\sigma}(t,x)| \leqslant \widetilde{c}\,|x|$.

Finally, we suppose that $|x|, |\overline{x}| \leqslant N$. We have

$$|b(t,x)-b(t,\overline{x})| \leqslant \left|\widehat{\Lambda}(t,x)\right||x-\overline{x}|+|\overline{x}|\left|\widehat{\Lambda}(t,x)-\widehat{\Lambda}(t,\overline{x})\right|$$

where $\left|\widehat{\Lambda}(t,x)\right|$ is bounded and $|\overline{x}| \leqslant N$, whereas

$$\widehat{\Lambda}(t,x)-\widehat{\Lambda}(t,\overline{x}) = \widehat{\sigma}^2(t,x)-\widehat{\sigma}^2(t,\overline{x})+x\widehat{\sigma}_x(t,x)\widehat{\sigma}(t,x)-\overline{x}\widehat{\sigma}_x(t,\overline{x})\widehat{\sigma}(t,\overline{x}).$$

But, from the hypotheses on σ and f, we deduce a constant c such that the following inequality $|\widehat{\sigma}^2(t,x)-\widehat{\sigma}^2(t,\overline{x})| \leqslant c\,|x-\overline{x}|$ holds. Moreover

$$x\widehat{\sigma}_x(t,x)\widehat{\sigma}(t,x) = x\sigma_x(t,x)\sigma(t,x)+\frac{1}{2}\gamma_n(t)x\sigma_x(t,x).$$

Since the next expression is bounded, we first write $x=(x-\overline{x})+\overline{x}$ and finally, we estimate

$$\sigma_x(t,x)\sigma(t,x)-\sigma_x(t,\overline{x})\sigma(t,\overline{x}) \;=\; \sigma(t,x)\left(\sigma_x(t,x)-\sigma_x(t,\overline{x})\right)+\sigma_x(t,\overline{x})\left(\sigma(t,x)-\sigma(t,\overline{x})\right)$$

where

$$\begin{aligned}|\sigma_x(t,x)-\sigma_x(t,\overline{x})| &\leqslant const\,|x-\overline{x}|,\\ |\sigma(t,x)-\sigma(t,\overline{x})| &\leqslant const\,|x-\overline{x}|.\end{aligned}$$

because $\sigma_{xx}(t,x)$ is bounded. Then, we can conclude that for $|x|,|\overline{x}|\leqslant N$,

$$|b(t,x)-b(t,\overline{x})|\leqslant const(N)\,|x-\overline{x}|.$$

In a similar way, it is easy to prove that

$$|\widetilde{\sigma}(t,x)-\widetilde{\sigma}(t,\overline{x})|\leqslant const(N)\,|x-\overline{x}|.$$

Lemma 5.6.3. *The local martingale*

$$\frac{\partial\widehat{S}_{x,t}(u)}{\partial x}=1+\int_t^u\frac{\widehat{\Lambda}\left(s,\widehat{S}_{x,t}(s)\right)}{\widehat{\sigma}\left(s,\widehat{S}_{x,t}(s)\right)}\frac{\partial\widehat{S}_{x,t}(s)}{\partial x}dW_s$$

is a strictly positive martingale.

Proof. The Doleans–Dade formula give us

$$\frac{\partial\widehat{S}_{x,t}(u)}{\partial x}=\exp\left\{\int_t^1\Lambda^*\left(v,\widehat{S}_{x,t}(v)\right)dW_v-\frac{1}{2}\int_t^1\Lambda^{*2}\left(v,\widehat{S}_{x,t}(v)\right)dv\right\}$$

where $\Lambda^*=\widehat{\Lambda}/\widehat{\sigma}$. Since $\widehat{\Lambda}$ is bounded, we deduce that there exists a constant c such that

$$\left(\frac{\partial\widehat{S}_{x,t}(u)}{\partial x}\right)^2\leqslant c\,N_u$$

where

$$N_u=\exp\left\{\int_t^u 2\Lambda^*\left(v,\widehat{S}_{x,t}(v)\right)dW_v-\frac{1}{2}\int_t^1 4\Lambda^{*2}\left(v,\widehat{S}_{x,t}(v)\right)dv\right\}$$

is a strictly positive locale martingale verifying

$$dN_u=2N_u\Lambda^*\left(u,\widehat{S}_{x,t}(u)\right)dW_u.$$

Using the Fatou lemma, we deduce that the latter is integrable and finally

$$\sup_u E\left(\frac{\partial\widehat{S}_{x,t}(u)}{\partial x}\right)^2<\infty.$$

So, we can conclude about the lemma.

Lemma 5.6.4. *The process* $\widehat{\eta}_{x,t}$ *is a Markov process of transition density function* $\Gamma^*(x,t,z,\tau)$*, the fundamental solution of the operator:*

$$\frac{1}{2}\widehat{\sigma}_a^2(t,x)\frac{\partial^2}{\partial x^2}+\widehat{\sigma}_b(t,x)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}.$$

Proof. In virtue of Theorem 5.4 p 149 [10], it suffices to verify the needed conditions.

Condition (A_1) is well verified since $\widehat{\sigma}_a^2(t,x)\geqslant const>0$.

Let verify Condition $(B_1)(i)$. First, $\widehat{\sigma}_a^2(t,x) = \widehat{\sigma}^2(t,x)$ and $\widehat{\sigma}_b(t,x)$ are bounded. Secondly, suppose $|x|,|\overline{x}|\leqslant N$. Then

$$\left|\widehat{\sigma}^2(t,e^x)-\widehat{\sigma}^2(t',e^{\overline{x}})\right|\leqslant\left|\widehat{\sigma}^2(t,e^x)-\widehat{\sigma}^2(t',e^{x})\right|+\left|\widehat{\sigma}^2(t',e^x)-\widehat{\sigma}^2(t',e^{\overline{x}})\right|$$

where, as already shown, $|\widehat{\sigma}^2(t',e^x)-\widehat{\sigma}^2(t',e^{\overline{x}})|\leqslant c|x-\overline{x}|$ whereas

$$\left|\widehat{\sigma}^2(t,e^x)-\widehat{\sigma}^2(t',e^{x})\right|\leqslant\left|\sigma^2(t,e^x)-\sigma^2(t',e^{x})\right|+\left|\sqrt{f'(t)}\sigma(t,e^x)-\sqrt{f'(t')}\sigma(t',e^x)\right|.$$

Since $f'>0$ and f'' are bounded, there exists a constant c such that

$$\left|\sqrt{f'(t)}-\sqrt{f'(t')}\right|\leqslant c|t-t'|.$$

It follows that if $|x|,|\overline{x}|\leqslant N$,

$$\left|\widehat{\sigma}_a^2(t,e^x)-\widehat{\sigma}_a^2(t',e^{\overline{x}})\right|\leqslant C(N)\ \left(|t-t'|+|x-\overline{x}|\right).$$

In a similar way, since we suppose that $\widehat{\sigma}_{x,t}$ is bounded, we have

$$\left|\widehat{\sigma}_b(t,e^x)-\widehat{\sigma}_b(t',e^{\overline{x}})\right|\leqslant const(N)\ \left(|t-t'|+|x-\overline{x}|\right).$$

Finally, since $x\widehat{\sigma}_x(t,x)$ is bounded, we deduce that Condition $(B_1)(ii)$ holds, i.e. for any $x,\overline{x}$,

$$\left|\widehat{\sigma}_a^2(t,e^x)-\widehat{\sigma}_a^2(t,e^{\overline{x}})\right|\leqslant C\ \left(|x-\overline{x}|\right).$$

Lemma 5.6.5. *For any* $n,p\in\mathbb{N}$*, we have*

$$\frac{\Gamma(1+\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(1+\frac{n+p}{2})}\leqslant 2.$$

Proof. From the formula $\Gamma(1+z)=z\Gamma(z)$, it is easy to prove that

$$\Gamma(1+\frac{n+p}{2})\geqslant\Gamma(1+\frac{n}{2})\text{ if }p\text{ is even.}$$

Otherwise, $\Gamma(1+\frac{n+p}{2})\geqslant\Gamma(1+\frac{n+1}{2})$. It suffices to use the Bessel function

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt$$

verifying

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

provided that $Re(p),Re(q)>0$ with $p=1+n/2$ and $q=1/2$ to conclude.

Now, we propose a lemma similar to Lemma 6 p 252 [11]:

Lemma 5.6.6. *Let consider*

$$f(t,\sigma,\tau,x,y,\xi)=\left(\frac{x-y}{a\sqrt{t-\sigma}}+a\sqrt{t-\sigma}\right)^2+\left(\frac{y-\xi}{a\sqrt{\sigma-\tau}}+a\sqrt{\sigma-\tau}\right)^2$$

where $a>0$ *and*

$$I_{a,k}=\int_{-\infty}^{\infty}\frac{1}{\sqrt{t-\sigma}\sqrt{\sigma-\tau}}\exp\{-kf(t,\sigma,\tau,x,y,\xi)\}dy.$$

Then

$$f(t,\sigma,\tau,x,y,\xi)\geqslant\left(\frac{x-\xi}{a\sqrt{t-\tau}}+a\sqrt{t-\tau}\right)^2$$

and for all $\varepsilon\in]0,1[$,

$$I_{a,k}\leqslant\exp\left\{-k(1-\varepsilon)\left(\frac{x-\xi}{a\sqrt{t-\tau}}+a\sqrt{t-\tau}\right)^2\right\}\frac{\sqrt{2}a}{\sqrt{t-\tau}\sqrt{k\varepsilon}}.$$

Proof. For the first assertion, it suffices to use Lemma 6 p 252 [11]. Indeed, we have:

$$\frac{(x-y)^2}{t-\sigma}+\frac{(y-\xi)^2}{\sigma-\tau}\geqslant\frac{(x-\xi)^2}{t-\tau}.$$

Secondly, we deduce that

$$I_{a,k}\leqslant\exp\left\{-k(1-\varepsilon)\left(\frac{x-\xi}{a\sqrt{t-\tau}}+a\sqrt{t-\tau}\right)^2\right\}\int_{-\infty}^{\infty}\frac{\exp\{-k\varepsilon f(t,\sigma,\tau,x,y,\xi)\}}{\sqrt{t-\sigma}\sqrt{\sigma-\tau}}dy.$$

If $\tau\leqslant\sigma\leqslant\tau+(t-\tau)/2$, we have $t-\sigma\geqslant(t-\tau)/2$. So, it suffices to use the inequality

$$f(t,\sigma,\tau,x,y,\xi)\geqslant\left(\frac{y-\xi}{a\sqrt{\sigma-\tau}}+a\sqrt{\sigma-\tau}\right)^2$$

to conclude. The case $\tau+(t-\tau)/2\leqslant\sigma\leqslant t$ is similar.

Theorem 5.1.1 holds provided that $\widehat{C}_{xx}\geqslant 0$. The convexity propagation is a subject of first importance ([25], [22]). We prove that this condition is guaranteed if h is a convex function and $\widehat{\sigma}(t,x)=\widehat{\sigma}(x)$.

Lemma 5.6.7. *Assume that* h *is a convex function verifying the condition* $(\widetilde{H})$*. If* $\widehat{\sigma}$ *does not depend on* t*, then* $\widehat{C}_{xx}\geqslant 0$.

In virtue of the Tanaka–Meyer formula, we have:

$$h\left(\widehat{S}_{x,t}(1)\right)=h(x)+\int_t^1 h'_-\left(\widehat{S}_{x,t}(u)\right)dW_u+\frac{1}{2}\int_{\mathbb{R}}L_1^u\mu(du)$$

where h'_- is the left derivative and

$$\mu=h''(u)du+\sum_i[h'_+(K_i)-h'_+(K_i)]\delta_{K_i},$$

δ_{K_i} is the Dirac measure. Moreover, $(L_s^u)_{s\in[t,1]}$ is a continuous and positive semi-martingale verifying

$$\int_{\mathbb{R}} g(u)L_s^u du = \int_t^s g\left(\widehat{S}_{x,t}(u)\right) d\langle \widehat{S}_{x,t}\rangle_u, \quad s \in [t,1]$$

for any positive and bounded measurable functions g. It follows that

$$\begin{aligned} h\left(\widehat{S}_{x,t}(1)\right) &= h(x) + \int_t^1 h'_-\left(\widehat{S}_{x,t}(u)\right) dW_u + \frac{1}{2}\sum_i [h'_+(K_i) - h'_-(K_i)] L_1^{K_i} \\ &\quad + \frac{1}{2}\int_t^1 h''\left(\widehat{S}_{x,t}(u)\right) \widehat{\sigma}^2\left(\widehat{S}_{x,t}(u)\right) \widehat{S}_{x,t}^2(u) du. \end{aligned} \tag{5.6.30}$$

Recall that

$$\left(\widehat{S}_{x,t}(u) - K\right)^+ = (x-K)^+ + \int_t^u I_{\widehat{S}_{x,t}(s)>K} d\widehat{S}_{x,t}(s) + \frac{1}{2}L_u^K.$$

Then,

$$\frac{1}{2}EL_1^{K_i} = \widehat{C}^i(t,x) - (x-K_i)^+$$

where $\widehat{C}^i(t,x)$ is the solution of **(e)** with $h(x) = (x-K_i)^+$. Having computed expectations, we deduce from 5.6.30 that

$$\widehat{C}_t(t,x) = \sum_i \alpha_i \widehat{C}_t^i(t,x) - \frac{1}{2}E\left(h''\left(\widehat{S}_{x,t}(1)\right)\widehat{\sigma}^2\left(\widehat{S}_{x,t}(1)\right)\widehat{S}_{x,t}^2(1)\right) \tag{5.6.31}$$

where $\alpha_i = h'_+(K_i) - h'_+(K_i) \geqslant 0$. Indeed, to obtain derivatives, we note that we have $\widehat{S}_{x,t}(u) = \overline{S}_{x,0}(u-t)$ where $\overline{S}_{x,0}$ verifies

$$d\overline{S}_{x,0}(v) = \widehat{\sigma}\left(\overline{S}_{x,0}(v)\right)\overline{S}_{x,0}(v)dW_v, \quad v \in [0, 1-t]$$

and we use the change of variable $v = u - t$.

We first prove the lemma for $h(x) = (x-K)^+$ where K is a constant. For this, we define:

$$\begin{aligned} h_n(x) &:= 0, \qquad x \in [0, K-1/n] \\ &:= n\left(x - K + 1/n\right)^2/4, \qquad x \in [K-1/n, K+1/n] \\ &:= x - K, \qquad x \in [K+1/n, \infty[. \end{aligned}$$

This latter is a continuous and convex function and verifies

$$\begin{aligned} 0 \leqslant h_n(x) - h(x) &\leqslant \frac{1}{4n}, \\ |h'_n(x) - h'(x)| &\leqslant I_{[K-1/n, K+1/n]}(x). \end{aligned}$$

It follows that $\widehat{C}_x^n(t,x) \to \widehat{C}_x(t,x)$ where $\widehat{C}^n, \widehat{C}$ are the solutions of **(e)** respectively with terminal conditions h_n and $h(x) = (x-K)^+$. Indeed, it suffices to recall Lemma 5.2.2. Since h_n is a C^1-function, we deduce from 5.6.31 (with $\alpha_i = 0$ and $h''_n \geqslant 0$) that $\widehat{C}_{xx}^n(t,x) \geqslant 0$ and $x \to \widehat{C}_x^n$ is increasing. Then, $x \to \widehat{C}_x$ is also increasing and finally $\widehat{C}_{xx} \geqslant 0$.

In the general case, since $h'' \geqslant 0$ and

$$\widehat{C}_t(t,x) = -\frac{1}{2}\widehat{\sigma}^2(t,x)x^2\widehat{C}_{xx}(t,x),$$

it suffices to apply 5.6.31 where, as already shown with $h(x) = (x-K_i)^+$, $\widehat{C}_t^i(t,x) \leqslant 0$.

Deuxième partie

Arbitrage Theory

Chapter 6

Arbitrage Theory for a Continuous Time Model

We consider the continuous-time model developed in [19] for markets with transaction costs. In the latter, the strategies generating the value processes are expressed in physical units of assets. The ones which are bounded from below in sense of partial ordering generated by the solvency cône are considered as admissible. Here, by a slightly different approach of the admissibility condition, we can suggest and characterize a " No Generalized Arbitrage " (NGA) criteria. Moreover, we give a version of hedging theorem for European options but also a dual description of the set of initial endowments from which we can start a portfolio process hedging a given American option. The latter is deduced from our joint work with Dimitri De Vallière and Yuri Kabanov [6] about the hedging of American options [1]. Finally, we propose to define hedging " minimal prices ".

6.1 Introduction

6.1.1 The Standard Discrete-Time Model

All processes are given on a fixed stochastic basis $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\leqslant T}, P)$ satisfying the usual conditions and $t = 0, 1, \cdots, T$. A finite time horizon T is fixed and the initial σ-algebra is trivial. We suppose that the agent portfolio contains d assets. Their quotes are given in units of a fixed *numéraire* which not be a traded security. At time t, they are expressed by the vector of prices $S_t = (S_t^1, \cdots, S_t^d)$; its components are strictly positive and adapted.
The agent's positions can be described either by the vector of "physical" quantities $\widehat{V}_t = (\widehat{V}_t^1, \cdots, \widehat{V}_t^d)$ or by the vector $V_t = (V_t^1, \cdots, V_t^d)$ of values invested in each asset; they are related as follows:

$$\widehat{V}_t^i = V_t^i / S_t^i, \quad i \leqslant d.$$

In the considered market, any asset can be exchanged to any other. At time t, the increase of the value of the ith position in one unit of the numéraire by changing the value of the jth position requires diminishing the value of the latter in $1 + \lambda_t^{j,i}$ units of the numéraire.

[1]Bruno Bouchard and Jean-François Chassagneux, by an other approach, produce a similar result [2].

We assume that the matrix of transaction cost coefficients is adapted and has positive components whereas the diagonal is zero.

The portfolio evolution can be described by the initial condition $V_{-0} = v$ and the increments at dates $t \geqslant 0$:

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i, \tag{6.1.1}$$

$$\Delta B_t^i = \sum_{i=1}^{d} \Delta L_t^{j,i} - \sum_{i=1}^{d} (1+\lambda_t^{i,j}) \Delta L_t^{i,j}, \tag{6.1.2}$$

where $L_t^{i,j} \in L^0(\mathbb{R}_+, \mathcal{F}_t)$ represents the accumulated net amount transferred from the position i to the position j at the date t. The first term in the right-hand side of 6.1.1 is due to the price increments while the second corresponds to the agent's own actions at the date t after knowledge of the new prices.

Note that any $\Delta L_t \in L^0(\mathbb{R}_+^d, \mathcal{F}_t)$ defines the $\mathcal{F}_t$-measurable random variable ΔB_t with values in the set $-M_t$ where

$$M_t := \left\{ x \in \mathbb{R}^d : \exists a \in \mathbb{R}_+^d \text{ such that } x = \sum_{i=1}^{d} [(1+\lambda_t^{i,j}) a^{i,j} - a^{j,i}], i \leqslant d \right\}.$$

Reciprocally, a measurable selection argument shows that any portfolio increment $\Delta B_t \in L^0(-M_t, \mathcal{F}_t)$ is generated by a certain $\Delta L_t \in L^0(\mathbb{R}_+^d, \mathcal{F}_t)$. So, we can decide to choose B as the control strategy.

It is convenient to consider the dynamics of the portfolio in "physical units". Indeed, it is given by the following formula:

$$\Delta \widehat{V}_t = \Delta \widehat{B}_t, \quad \Delta \widehat{B}_t \in -\widehat{M}_t$$

where, for a set A_t, we note $\widehat{A}_t = \{\widehat{x} : x \in A_t\}$ with $\widehat{x}^i = x^i / S_t^i$, $i = 1, \cdots, d$. An important concept in the above setting is the *solvency cone*

$$K_t = M_t + \mathbb{R}_+^d,$$

i.e. the set of portfolios which can be converted at time t, paying transaction costs, to portfolios without short position.

For this model, an arbitrage theory is already developed in [19] as well as hedging theorems for European and American options [1].

6.1.2 The Continuous Time Model

The continuous-time model with efficient market friction suggested in [19] is inspired by the previous one. It requires the continuity of the price processes and transaction cost coefficients. Of course, theses conditions are fulfilled in the traditional case of Brownian motion and constant transaction coefficients.

All processes are given on a fixed stochastic basis $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \leqslant T}, P)$ satisfying the usual conditions. A finite time horizon T is fixed and the initial σ-algebra is trivial. In a financial context, the continuous-time model is defined by a continuous semi-martingale $S = (S^1, \cdots, S^d) \in \operatorname{int} \mathbb{R}_+^d$ with $S_0 = \mathbf{1} = (1, \cdots, 1)$ considered as the price process. The transaction costs are represented by an adapted and continuous matrix-valued process $\Lambda = (\Lambda^{i,j})$ verifying $\Lambda^{i,j} \geqslant 0$ and $\Lambda^{i,i} = 0$. We shall assume that at each instant at least one $\Lambda^{i,j} \neq 0$ in order to have $K_t = M_t$ where the solvency cone K and the cone M are defined as in the discrete-time model (see [19]).

The portfolio processes are controlled by the class of strategies $\mathcal{B}$, the set of all right-continuous d-dimensional adapted processes B of bounded variations such that $dB_t = \dot{B}_t d\|B\|_t$ where $\dot{B}_t \in -L^0(K_t, \mathcal{F}_t)$ and $\|B\|$ is the total variation. The choice of the norm is of no importance since all norms are equivalent in a finite-dimensional space. Recall that $B \in \mathcal{B}$ is such that ΔB_t represents the variation of the portfolio expressed in numéraire at date t due to the trader.

So, it is easy to deduce that the dynamic of the process $\widehat{V} = \widehat{V}^{v,B}$, value of a self-financing portfolio defined by the strategy B and the initial endowment v, is given by:

$$\widehat{V}_t^i = v^i + \int_{]0,t]} \frac{dB_u^i}{S_u^i}.$$

It follows that the portfolio processes $\widehat{V}$ are of bounded variations, right-continuous and verifies $d\widehat{V}/d\|\widehat{V}\| \in -\widehat{K}$ a.s.

The last properties lead us to consider the arbitrage not only for the cone $G = \widehat{K}$ but for more general $\mathcal{C}$*-valued process* $G = (G_t)_{t\geqslant 0}$ as defined later.

We note $G_t^* = \{y \in \mathbb{R}_+^d : yx \geqslant 0, \forall x \in G_t\}$ and $\mathcal{M}_0^T(G^*)$ is the set of all martingales $(Z_t)_{t\in[0,T]}$ such that $Z_t \in G_t^*$ a.s.

6.2 Generalized Arbitrage in Abstract Setting

We consider a $\mathcal{C}$*-valued process* $G = (G_t)_{t\in[0,T]}$ defined by a countable sequence of adapted d-dimensional processes $\xi^k = (\xi_t^k)$ such that for every t and ω only a finite but non-zero number of $\xi_t^k(\omega)$ are different from zero and $G_t(\omega) = cone\{\xi_t^k(\omega),\, k \in \mathbb{N}\}$, i.e. $G_t(\omega)$ is a polyhedral cone generated by the finite set $\{\xi_t^k(\omega),\, k \in \mathbb{N}\}$.

We suppose that G dominates the constant process $\mathbb{R}_+^d$, all cones G_t are proper, i.e. $G_t \cap (-G_t) = \{0\}$ or, equivalently, $\operatorname{int} G_t^* \neq \emptyset$.

We assume that the generators of G are continuous processes and we add the following assumption about the generators of G_t^*:

Assumption (G):There is a countable family of continuous adapted processes (ζ^k) such that for each ω only a finite number of vectors ζ^k are different from zero and $G_t^* = cone\{\zeta_t^k : k \in \mathbb{N}\}$ for every t.

The next hypothesis, used for hedging theorems, is a requirement that the set $\mathcal{M}_T^0(G^*)$ is rich enough (see [19]) and is fulfilled for the model with constant transaction costs admitting an equivalent martingale measure:

Assumption (B): Let $\xi \in L^0(\mathbb{R}^d, \mathcal{F}_t)$. If the scalar product $Z_t\xi \geqslant 0$ for all $Z \in \mathcal{M}_T^0(G^*)$, then $\xi \in L^0(G_t, \mathcal{F}_t)$.

Let $\mathcal{X} = \mathcal{X}_T^0$ be the set of all càdlàg processes X of bounded variations with $X_0 = 0$ such that $dX = \dot{X} d\|X\|$ with $\dot{X}_t \in L^0(-G_t, \mathcal{F}_t)$ for all $t \in [0,T]$ and let $\mathcal{X}^x = x + \mathcal{X}$, $x \in \mathbb{R}^d$. We denote by $\mathcal{X}_b^x = \mathcal{X}_{b,T}^x$ where $\mathcal{X}_{b,t}^x$ is the subset of $\mathcal{X}_t^x$ formed by the processes X on $[0,t]$ such that $X_s + \kappa_X \mathbf{1} \in L^0(G_s, \mathcal{F}_s)$, with $\kappa_X \geqslant 0$, and $X_t + \kappa_X \mathbf{1} \in L^0(\mathbb{R}_+^d, \mathcal{F}_t)$. Such processes $X \in \mathcal{X}_b^x$ are called admissible. Our admissibility condition is more restrictive than the one proposed in [19] where it is only assumed that $X_s + \kappa_X \mathbf{1} \in L^0(G_s, \mathcal{F}_s)$ but legitimate. Indeed, we can cite [8] in which short sales are ruled out. Finally, we put $\mathcal{X}^x(t) = \{X_t : X \in \mathcal{X}^x_t\}$ and $\mathcal{X}^x_b(t) = \{X_t : X \in \mathcal{X}^x_{b,t}\}$. It is easy to show the following property:

Lemma 6.2.1. *We have: $\mathcal{X}(t) \subseteq \mathcal{X}(T)$ and $\mathcal{X}_b(t) \subseteq \mathcal{X}_b(T)$ for all $t \in [0, T]$.*

Proof. Let $\xi = X_t \in \mathcal{X}_b(t)$ where $X \in \mathcal{X}_{b,t}$.

We define the stopped process $Y = X^t$ such that $\|Y\| = \|X\|^t$. Then, $dY_s = \dot{Y}_s d\|Y\|_s$ where $\dot{Y}_s = \dot{X}_s I_{s \leqslant t}$ verifies $\dot{Y}_s \in L^0(-G_s, \mathcal{F}_s)$ for all s and $Y_s + \kappa_X \mathbf{1} \in L^0(\mathbb{R}^d_+, \mathcal{F}_s)$ for $s \geqslant t$. It follows that $\xi \in \mathcal{X}_b(T)$.

We shall propose an arbitrage theory inspired by [3]. For this, we introduce some notations.

If $x \in \mathbb{R}^d_+$, we note $x \geqslant 0$.

Let be $R_T \subseteq \mathcal{X}(T)$ verifying $L^\infty(-G_t, \mathcal{F}_t) \subseteq R_T$, $\forall t \in [0, T]$. We define the set $A := R_T - L^0(\mathbb{R}^d_+)$ and

$$\overline{R}^F_T := \left\{\xi = \lim \xi^n \, a.s. : \xi^n \in R_T \text{ and there exists } k \geqslant 0 \text{ such that } \xi^n + k\mathbf{1} \geqslant 0\right\}.$$

If $C \in \overline{R}^F_T$, we note $\Upsilon(C) = \mathbf{1} + C - \operatorname{essinf} C$ where $\operatorname{essinf} C$ is a constant defined by

$$(\operatorname{essinf} C)^i = \operatorname{essinf} C^i, \quad i = 1, \cdots, d.$$

Observe that for $C \in \overline{R}^F_T$, there exists $k \geqslant 0$ such that $C \geqslant -k\mathbf{1}$. So, $\operatorname{essinf} C$ is well defined and $\Upsilon(C) \geqslant \mathbf{1}$ a.s. We note, for $a, b \in \mathbb{R}^d$, a/b and $a \times b$ the vectors whose components are respectively a_i/b_i and $a_i b_i$. Finally, we define for $C \in \overline{R}^F_T$:

$$\begin{aligned}
A_N(C) &= \{X/\Upsilon(C) : X \in A\}, \\
A^\infty_N(C) &= A_N(C) \cap L^\infty, \\
\overline{A^\infty_N}^w(C) &= \overline{A^\infty_N(C)} \text{ closure in } \sigma(L^\infty, L^1), \\
\mathcal{R} &= \left\{Z \in \mathcal{M}^0_T(G^* \backslash \{0\}) : E(Z_T X)^- \geqslant E(Z_T X)^+, \forall X \in R_T\right\}, \\
\mathcal{R}(C) &= \big\{Z \in \mathcal{M}^0_T(G^* \backslash \{0\}) : E|Z_T X| < \infty \text{ and } E Z_T X \leqslant 0 \\
&\qquad \text{if } X \in R_T \text{ verifies } X \geqslant -\alpha C - \beta \mathbf{1} \text{ where } \alpha, \beta \geqslant 0\big\}.
\end{aligned}$$

Définition 6.1. *We say that G satisfies the No General Arbitrage property NGA if for all $C \in \overline{R}^F_T$,*

$$\overline{A^\infty_N}^w(C) \cap L^0(\mathbb{R}^d_+) = \{0\}.$$

We shall prove later the following results:

Theorem 6.2.2. *Suppose that $R_T = \mathcal{X}_b(T)$, then*

$$\overline{A^\infty_N}^w(0) \cap L^0(\mathbb{R}^d_+) = \{0\} \Leftrightarrow \mathcal{M}^0_T(G^* \backslash \{0\}) \neq \emptyset.$$

Theorem 6.2.3. *Assume that $\overline{A^\infty_N}^w(C) \cap L^0(\mathbb{R}^d_+) = \{0\}$. Then $\mathcal{R}(C) \neq \emptyset$.*

Theorem 6.2.4. *Assume that $\mathcal{R} \neq \emptyset$ then the NGA condition holds.*

Corollary 6.2.5. *Assume that there exists $C_0 \in \overline{R}_T^F$ such that $\mathcal{R}(C_0) = \mathcal{R}$. Then*

$$\text{(NGA)} \Leftrightarrow \mathcal{R} \neq \emptyset.$$

Corollary 6.2.6. *Suppose that $R_T = \mathcal{X}_b(T)$, then we have*

$$(NGA) \Leftrightarrow \mathcal{M}_T^0(G^*\backslash\{0\}) \neq \emptyset.$$

Remark 6.2.7. *We can partially follow the proof of Corollary 6.2.5 in order to have for $R_T = \mathcal{X}_b(T)$,*

$$\overline{A_N^\infty}^w(0) \cap L^0(\mathbb{R}_+^d) = \{0\} \Leftrightarrow \mathcal{M}_T^0(G^*\backslash\{0\}) \neq \emptyset.$$

In the case where $R_T = \mathcal{X}_b(T)$, we can extend to the continuous time the concept introduced in the discrete model. We say that G satisfies :

Weak No Arbitrage property NA^w if for all $t \in [0, T]$,

$$\mathcal{X}_b(t) \cap L^0(G_t, \mathcal{F}_t) \subseteq L^0(\partial G_t, \mathcal{F}_t).$$

Proposition 6.2.8. *Assume that $R_T = \mathcal{X}_b(T)$, then:*

$$NA^w \Leftrightarrow \mathcal{X}_b(T) \cap L^0(\mathbb{R}_+^d, \mathcal{F}_t) = \{0\}.$$

Remark 6.2.9. $(NGA) \Rightarrow (NA^w)$.

For the following definition inspired from [3], we consider a random variable F, considered as the contingent claim expressed in physical units, verifying $-k_F\mathbf{1} \leqslant F \leqslant \overline{k}_F\mathbf{1}$ for some constants $k_F, \overline{k}_F \geqslant 0$. In a financial context, such a pay-off exits: we can cite for example $F(S_t) = (S_t - K)^+/S_t$.

Définition 6.2. *A real number x is a fair price of F if the extended model*

$$\big(\Omega, \mathcal{F}, P, R_T + \{h(F - x) : h \in \mathbb{R}\}\big)$$

satisfies the NGA condition.

Note that there exists new cones

$$G'_t = G_t + \{h(F - x) : h \in \mathbb{R}\}I_{\{t=T\}}$$

corresponding to the extended model. Indeed, if we note

$$R'_T = R_T + \{h(F - x) : h \in \mathbb{R}\},$$

we have the following inclusion.

Lemma 6.2.10. *We have $R'_T \subseteq \mathcal{X}(G')(T)$.*

Proof. We consider $X'_T = X_T + h(F-x)$ where $X \in \mathcal{X}$. It suffices to define

$$Y_s = X_s + h(F-x)I_{\{s=T\}}$$

which verifies:

$$\begin{aligned} \|Y_s\| &= \|X\|_s + (\|\Delta X_T + h(F-x)\| - \|\Delta X_T\|)\, I_{\{s=T\}}, \\ \dot{Y}_s &= \dot{X}_s 1_{\{s<T\}} + \frac{\Delta X_T + h(F-x)}{\|\Delta X_T + h(F-x)\|} I_{\{s=T\}} I_{\Delta X_T + h(F-x) \neq 0}. \end{aligned}$$

It follows that $\dot{Y}_s \in -G'_s$ a.s. and $X'_T \in \mathcal{X}(G')(T)$. Moreover, if we suppose that $R_T = \mathcal{X}_b(T)$, then there exists $k_X \geqslant 0$ such that $X_T + k_X\mathbf{1} \geqslant 0$ and $X_t + k_X\mathbf{1} \in G_t$. But we also have $-k_F\mathbf{1} \leqslant F \leqslant \overline{k}_F\mathbf{1}$, so $Y \in \mathcal{X}_{b,T}(G')$ and $X'_T \in \mathcal{X}_b(G')(T)$.

We define I_F the set of all fair prices for F verifying $-k_f\mathbf{1} \leqslant F \leqslant \overline{k}_F\mathbf{1}$.

Theorem 6.2.11. *Assume that there exists $C_0 \in \overline{R}_T^F$ such that $\mathcal{R}(C_0) = \mathcal{R}$ and the NGA condition holds. Then, we have:*

$$I_F = \{x \in \mathbb{R}^d : \exists Z \in \mathcal{R} \text{ such that } Z_0 x = EZ_T F\}.$$

For $Z \in \mathcal{M}_0^T(G^*\backslash\{0\})$, we define

$$x^{(Z)} = \frac{Z_0}{\|Z_0\|^2} EZ_T F.$$

This latter is such that $EZ_T F = Z_0 x^{(Z)}$ and $x^{(Z)} \leqslant \sqrt{d}\,\overline{k}_F\mathbf{1}$. So, in virtue of Theorem 6.2.11, we deduce that $x^{(Z)} \in I_F$. Moreover, for $x \in I_F$, we note $Z^x \in \mathcal{M}_0^T(G^*)$ verifying $EZ_T^x F = Z_0^x x$ and $|Z_0^x| = 1$. Finally, we give the following definition, in order to propose minimal prices for European options as we shall see later.

Définition 6.3. *Assume that there exists $C_0 \in \overline{R}_T^F$ such that $\mathcal{R}(C_0) = \mathcal{R}$ and the NGA condition holds. If $-k_f\mathbf{1} \leqslant F \leqslant \overline{k}_F\mathbf{1}$, we define:*

$$\begin{aligned} \alpha_I &= \sup\left\{EZ_T F : Z \in \mathcal{M}_0^T(G^*) \text{ and } |Z_0| = 1\right\}, \\ M_I &= \{Z_0 : Z_0 = \lim Z_0^{x_n} \text{ with } Z_0^{x_n} x_n \to \alpha_I\}. \end{aligned}$$

Note that we have clearly

$$\alpha_I = \sup\{Z_0^x x : x \in I_F\} = \sup\left\{Z_0 x^{(Z)} : Z \in \mathcal{M}_0^T(G^*) \text{ and } |Z_0| = 1\right\}.$$

6.3 Hedging Theorem For European Options

In this section, we only consider the case $R_T = \mathcal{X}_b(T)$.
Let L_b^0 be the cone in $L^0(\mathbb{R}^d)$ formed by random variables ξ verifying $\xi + k\mathbf{1} \geqslant 0$ a.s. for some $k \geqslant 0$. We are given a non-null random variable $F \in L_b^0$ considered as a contingent claim. We define the convex set

$$\Gamma_F = \left\{x \in \mathbb{R}^d : F \in \mathcal{X}_b^x(T)\right\}$$

and the closed convex set

$$D_F = \left\{x \in \mathbb{R}^d : Z_0 x \geqslant E Z_T F \quad \forall Z \in \mathcal{M}_0^T(G^*)\right\}.$$

We denote by $\mathcal{D} = \mathcal{D}(G)$ the subset of $\mathcal{M}_0^T(G^*)$ formed by martingales Z such that not only $Z_\tau \in L^0(\operatorname{int} G_\tau^*)$ for any stopping times τ but also $Z_{\tau-} \in L^0(\operatorname{int} G_{\tau-}^*)$ for any predictable times $\tau \in [0, T]$.

Note that $\mathcal{D} \neq \emptyset$ implies the (NGA) condition. We shall recall the following version of hedging theorem (the only difference from [19] is the model):

Theorem 6.3.1. *Assume $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold, then $\Gamma_F = D_F$.*

Remark 6.3.2. *It is easy to show that if the NGA condition holds then*

$$D_F = \left\{x \in \mathbb{R}^d : Z_0 x \geqslant E Z_T F \quad \forall Z \in \mathcal{M}_0^T(G^* \backslash \{0\})\right\}.$$

Indeed, for $Z \in \mathcal{M}_0^T(G^*)$, it suffices to consider $Z^n = Z + \frac{1}{n}\widetilde{Z}$ where $\widetilde{Z} \in \mathcal{M}_0^T(G^* \backslash \{0\})$.

We define minimal prices for the European option defined by a contingent claim F:

Lemma 6.3.3. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold. If $x_1 \in \Gamma_F \neq \emptyset$, we can define minimal prices $p_F \in \{x \leqslant x_1 : x \in \Gamma_F\}$ according to the partial ordering generated by $\mathbb{R}_+^d$.*

Proof. Let consider $x_1 \in \Gamma_F$. Suppose that for all $p \in \mathbb{N}$, there exists $x_p \in \Gamma_F$ verifying $x_p \leqslant x_1$ and $x_p^i \leqslant -p$ where $i \in \{1, \cdots, d\}$. Then for $Z \in \mathcal{M}_0^T(G^* \backslash \{0\})$,

$$Z_0 x_p \leqslant -p \min_i Z_0^i + d \max_i Z_0^i \max_i |x_1^i|.$$

Moreover, we have $Z_0 x_p \geqslant E Z_T F \geqslant -k_F Z_0 \mathbf{1}$. This leads to a contradiction if we get p converged to ∞. So, there exists $p \in \mathbb{N}$ such that $x \leqslant x_1$ and $x \in \Gamma_F$ implies that $x \geqslant -p\mathbf{1}$. Using the Zorn lemma, it follows that the set $\{x \leqslant x_1 : x \in \Gamma_F\}$ has, at least, a minimal element p_F.

From the definition, it is clear that $p_F \in \Gamma_F$ and we have the following characterization:

Theorem 6.3.4. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold. If $x_1 \in \Gamma_F \neq \emptyset$, then the two following conditions are equivalent:*

a) $p_F \in \{x \leqslant x_1 : x \in \Gamma_F\}$ is a minimal price.

b) There exists a sequence $Z^n \in \mathcal{M}_0^T(G^* \backslash \{0\})$ verifying $Z_0^n \to Z_0$ where $|Z_0| = 1$ and $E Z_T^n F \to Z_0 p_F$.

For the sequence, we suppose that $-k_F \mathbf{1} \leqslant F \leqslant \overline{k}_F \mathbf{1}$. In this case, we have obviously $\overline{k}_F \mathbf{1} \in \Gamma_F$ and it is natural to define minimal prices as minimal elements

$$p_F \in \{x \leqslant \overline{k}_F \mathbf{1} : x \in \Gamma_F\}$$

according to the Zorn lemma. We note that for all $x \in \Gamma_F$ and $Z_0 \in M_I$, we have $Z_0 x \geqslant \alpha_I$. From 6.3.4, we deduce easily the following corollaries.

Corollary 6.3.5. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold. Suppose that $-k_F \mathbf{1} \leqslant F \leqslant \overline{k}_F \mathbf{1}$ and $p_F \in \{x \leqslant \overline{k}_F \mathbf{1} : x \in \Gamma_F\}$ verifies $Z_0 p_F = \alpha_I$ where $Z_0 \in M_I$. Then p_F is a minimal price of Γ_F.*

Corollary 6.3.6. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold. Suppose that $-k_F \mathbf{1} \leqslant F \leqslant \overline{k}_F \mathbf{1}$ and $e = \lim \nearrow x_n$ where $x_n \in I_F$. If $e \notin I_F$, then e is a minimal price of Γ_F.*

6.4 Hedging Theorem For American Options

In this section, we only consider the case $R_T = \mathcal{X}_b(T)$.
We are given a cadlag process F considered as a contingent claim, defined on $(\Omega_T, \mathcal{O})$ where $\Omega_T = \Omega \times [0,T]$, $\mathcal{O}$ is the optional σ-field . We assume that $F_- \geqslant F$ and there exists $k_F \geqslant 0$ such that $F_t + k_F\mathbf{1} \geqslant 0$ for all t. We define the convex set

$$\Gamma_F^a := \left\{x \in \mathbb{R}^d : \exists X \in \mathcal{X}_b \text{ such that } x + X_\tau \geqslant F_\tau \text{ for all stopping time } \tau\right\}$$

and the closed convex set

$$D_F^a := \left\{x \in \mathbb{R}^d : E_\mu \eta F \leqslant x E_\mu \eta,\ \forall \eta \in \mathcal{P}_0^T(G^*, \mu), \forall \mu \in \nu_T\right\}$$

where

$$\mathcal{P}_0^T(G^*, \mu) := \left\{\eta \in L^1(\Omega_T, P \otimes \mu, \mathbb{R}_+^d) : Z_u^{\eta,\mu} \in G_u^*,\ \forall u \geqslant 0\right\}$$

and ν_T is the set of all positive finite measures on $[0,T]$. Moreover, E_μ means the expectation on Ω_T under the measure $P \otimes \mu$ whereas

$$Z_u^{\eta,\mu} = E\left(\int_{u+}^T \eta_t d\mu(t)|\mathcal{F}_u\right)$$

is a cadlag version.

We shall prove the following version of hedging theorem:

Theorem 6.4.1. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold, then $\Gamma_F^a = D_F^a$.*

Note that we can also produce an analogous theorem for the initial X-model introduced in [19], following the same reasoning [6].

It is easy to show the following lemma.

Lemma 6.4.2. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold, then*

$$\Gamma_F^a = \left\{x \in \mathbb{R}^d : E_\mu \eta F \leqslant x E_\mu \eta,\ \forall \eta \in \mathcal{P}_0^{T+}(G^*, \mu),\ \forall \mu \in \nu_T\right\}$$

where

$$\mathcal{P}_0^{T+}(G^*, \mu) = \mathcal{P}_0^T(G^*, \mu) \cap L^1(\Omega_T, P \otimes \mu, \operatorname{int} \mathbb{R}_+^d).$$

Note that $\mathcal{M}_0^T(G^* \backslash \{0\}) \subseteq \mathcal{P}_0^{T+}(G^*, \mu)$.

Since $F \geqslant -k_F\mathbf{1}$, $x \in \Gamma_F^a = D_F^a$ implies that $x \geqslant -k_F\mathbf{1}$ and we can define, using the Zorn lemma, minimal prices $p_F^a \in \Gamma_F^a$.

Theorem 6.4.3. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold. Then, the two following conditions are equivalent:*

a) The price p_F^a is a minimal price of Γ_F^a.

b) There exists a sequence $\mu^n \in \nu_T$, $\eta^n \in \mathcal{P}_0^{T+}(G^*, \mu^n)$ verifying $E_{\mu^n}\eta^n \to Z_0$ where $|Z_0| = 1$ and $E_{\mu^n}\eta^n F \to Z_0 p_F^a$.

Proof. This is similar to Theorem 6.3.4. Indeed, $E_{\mu^n}\eta^n \in G_0^*$.

With the hypothesis $-k_F\mathbf{1} \leqslant F \leqslant \overline{k}_F\mathbf{1}$, we define:

$$\begin{aligned} \alpha_a &= \sup\left\{E_\mu \eta F : \mu \in \nu_T,\ \eta \in \mathcal{P}_1^{T+}(G^*, \mu)\right\}, \\ M_a &= \{Z_0 : Z_0 = \lim E_{\mu^n}\eta^n \text{ and } E_{\mu^n}\eta^n F \to \alpha_a\} \end{aligned}$$

where
$$\mathcal{P}_1^{T+}(G^*,\mu)=\left\{\eta\in\mathcal{P}_0^{T+}(G^*,\mu):|E_\mu\eta|=1\right\}.$$
Note that for all $x\in\Gamma_F^a$ and $Z_0\in M_a$, we have $Z_0x\geqslant\alpha_a$. From 6.4.3, we deduce easily the following corollary.

Corollary 6.4.4. *Assume that $\mathcal{D}\neq\emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold.*
Suppose that $-k_F\mathbf{1}\leqslant F\leqslant\overline{k}_F\mathbf{1}$. If $p_F^a\in\Gamma_F^a$ verifies $Z_0p_F^a=\alpha_a$ where $Z_0\in M_a$, then p_F^a is a minimal price of Γ_F^a.

6.5 Proofs

6.5.1 Proof of Proposition 6.2.8

First we assume that the condition NA^w holds and we consider $\xi\in\mathcal{X}_b(T)\cap L^0(\mathbb{R}_+^d)$.

Then $\xi\in\mathcal{X}_b(T)\cap L^0(G_T,\mathcal{F}_T)\subseteq L^0(\partial G_T,\mathcal{F}_T)$. So, if $\xi\in\mathbb{R}_+^d\setminus\{0\}$ on a non-null set , then $\xi\in\operatorname{int}G_T$ on the latter because the domination of $\mathbb{R}_+^d$ by G means that $\mathbb{R}_+^d\setminus\{0\}\subseteq\operatorname{int}G_T$. This contradicts the hypothesis $\xi\in\partial G_T$ a.s.

Suppose that $\mathcal{X}_b(T)\cap L^0(\mathbb{R}_+^d)=\{0\}$. Then, $\mathcal{X}_b(t)\cap L^0(\mathbb{R}_+^d,\mathcal{F}_t)=\{0\}$ for all t. Let $\xi\in\mathcal{X}_b(t)\cap L^0(G_t,\mathcal{F}_t)$ and suppose that $\xi\in\operatorname{int}G_t$ on a non-null set. By a measurable selection argument, we deduce the existence of $X^+\in L^\infty(\mathbb{R}_+^d,\mathcal{F}_t)\setminus\{0\}$ such that $Z_t=\xi-X^+\in L^0(G_t,\mathcal{F}_t)$.

We first suppose that $\xi\in L^\infty$. Then, $-Z_t\in L^\infty(-G_t,\mathcal{F}_t)$ and we deduce that $X^+\in\mathcal{X}_b(t)\cap L^0(\mathbb{R}_+^d,\mathcal{F}_t)=\{0\}$ which leads to a contradiction. So, we have
$$\mathcal{X}_b(t)\cap L^\infty(G_t,\mathcal{F}_t)\subseteq L^0(\partial G_t,\mathcal{F}_t).$$
Otherwise, we define $\xi^n=\xi 1_{\|\xi\|\leqslant n}$ and we show that $\xi^n\in\mathcal{X}_b(t)$. Indeed, let the process be $X_s^n=X_s-\xi 1_{\|\xi\|>n}I_{s=t}$ where $X\in\mathcal{X}_{b,t}$. It is such that $X_t=\xi$. We have
$$\|X^n\|_s=\|X\|_s+\left(\|\Delta X_t-\xi 1_{\|\xi\|>n}\|-\|\Delta X_t\|\right)I_{s=t}$$
and
$$\dot{X}_s^n=\dot{X}_sI_{s<t}+\frac{\Delta X_t-\xi 1_{\|\xi\|>n}}{\|\Delta X_t-\xi 1_{\|\xi\|>n}\|}I_{\{\Delta X_t-\xi 1_{\|\xi\|>n}\neq 0\}}1_{s=t}\in L^0(-G_s,\mathcal{F}_s).$$
Furthermore, we can easily verify that there exists $\kappa_n\geqslant 0$ such that $X_s^n+\kappa_n\mathbf{1}\in L^0(G_s,\mathcal{F}_s)$ and $X_t^n+\kappa_n\mathbf{1}\in L^0(\mathbb{R}_+^d,\mathcal{F}_t)$ since we have $X_t^n=\xi^n\in L^\infty$. It follows that $\xi^n\in\mathcal{X}_b(t)\cap L^\infty(G_t,\mathcal{F}_t)\subseteq L^0(\partial G_t,\mathcal{F}_t)$ and we deduce that $\xi\in L^0(\partial G_t,\mathcal{F}_t)$ as $n\to\infty$.

6.5.2 Proofs of Theorems 6.2.3 and 6.2.4

We need some auxiliary results.

Let recall the following lemma that we can find in [19]:

Lemma 6.5.1. *Let $\mathcal{G}$ be a family of measurable sets such that any non-null set Γ has the non-null intersection with an element of $\mathcal{G}$. Then, there is at most countable subfamily of sets $\{\Gamma_i\}$ of full measure.*

We can deduce the following theorem:

Theorem 6.5.2. *Let $\mathcal{C}$ be a convex cone in L^∞ closed in $\sigma(L^\infty, L^1)$ containing $L^\infty(\mathbb{R}^d_-)$ and such that $\mathcal{C} \cap L^\infty(\mathbb{R}^d_+) = \{0\}$. Then, there exists $\rho \in L^0(\operatorname{int}\mathbb{R}^d_+)$ verifying $E\rho X \leqslant 0$ for all $X \in \mathcal{C}$ and $E|\rho| < \infty$.*

Proof. From the Hahn-Banach theorem, we deduce that for any element $x \in L^\infty(\mathbb{R}^d_+) \setminus \{0\}$, there exists $Z_x \in L^1$ such that $EZ_x\xi < EZ_x x$, for all $\xi \in \mathcal{C}$. We note e^i the vector whose only the ith component is non-null and equal to unit. Taking $\xi = \alpha e^i Z_x^i 1_{-M \leqslant Z_x^i < 0} \in \mathcal{C}$, for all $\alpha, M \geqslant 0$, we deduce that $Z_x \geqslant 0$. Moreover, $Z_x \neq 0$. So, we can assume that $EZ_x \leqslant \mathbf{1}$ and $EZ_x x > 0$. Let define

$$\mathcal{G}^k = \left\{\{Z_x^k \neq 0\},\, x \in L^\infty(\mathbb{R}^d_+) \setminus \{0\}\right\}.$$

Then, for all Γ such that $P(\Gamma) \neq 0$, we have $P(\Gamma \cap \{Z_x^k \neq 0\}) \neq 0$ where $x = e^k 1_\Gamma$. Indeed, $EZ_x.x > 0$. We deduce from the previous lemma a countable family $Z_{x_{k,i}}$ such that

$$P\left(\cup_i \left\{Z^k_{x_{k,i}} \neq 0\right\}\right) = 1.$$

Defining $\rho = \sum_{k,i} 2^{-k-i} Z_{x_{k,i}}$, it is obvious that $E|\rho| < \infty$ and we can easily verify that for any k, we have $\rho^k > 0$ on the set

$$N^c = \bigcap_{k=1}^d \bigcup_i \left\{Z^k_{x_{k,i}} \neq 0\right\}$$

of full measure. So $\rho \in \operatorname{int}\mathbb{R}^d_+$ a.s. and, from what precedes , we have $E\rho\xi \leqslant 0$ for any $\xi \in \mathcal{C}$.

Lemma 6.5.3. *For all $C \in \overline{R}^F_T$, we have $\mathcal{R} \subseteq \mathcal{R}(C)$.*

Proof. Let consider $C \in \overline{R}^F_T$ and $Z \in \mathcal{R}$. We have $C = \lim X_n$ where $X_n \in R_T$ verifies $X_n \geqslant -k\mathbf{1}$ with $k \geqslant 0$. Since $Z \in \mathcal{R}$, we have $E(Z_T X_n)^- \geqslant E(Z_T X_n)^+$ whereas $Z_T X_n \geqslant -kZ_T\mathbf{1}$ implies that $E(Z_T X_n)^- \leqslant kE|Z_T\mathbf{1}| < \infty$. Thus $E|Z_T X_n| \leqslant 2kE|Z_T\mathbf{1}|$ and $E|Z_T C| < \infty$ by the Fatou Lemma.

From now on, we suppose that $X \in R_T$ verifies $X \geqslant -\alpha C - \beta\mathbf{1}$ where $\alpha, \beta \geqslant 0$. Then $Z_T X \geqslant -\alpha Z_T C - \beta Z_T \mathbf{1}$ and it follows that $E(Z_T X)^- < \infty$. So, $Z \in \mathcal{R}$ implies that $E(Z_T X)^+ \leqslant E(Z_T X)^- < \infty$ and $EZ_T X \leqslant 0$. We can conclude that $Z \in \mathcal{R}(C)$.

Lemma 6.5.4. *Let consider $C_0 \in \overline{R}^F_T$ such that there exists $\rho \in L^0(\operatorname{int}\mathbb{R}^d_+)$ verifying $E|\rho| < \infty$ and $E\rho X \leqslant 0$ for all $X \in \overline{A^\infty_N}^w(C_0)$. Assume that $X \in R_T$ verifies $X/\Upsilon(C_0) \geqslant -a\mathbf{1}$ where $a \geqslant 0$. Then we have:*

$$E\left|\rho\frac{X}{\Upsilon(C_0)}\right| < \infty, \quad E\rho\frac{X}{\Upsilon(C_0)} \leqslant 0.$$

Proof. For all $c \in \mathbb{R}_+$, we define $(X - c\mathbf{1})^+$ and $X \wedge c\mathbf{1}$ the random variables whose components are respectively $(X^i - c)^+$ and $X^i \wedge c$.

Then, $X \wedge c\mathbf{1} = X - (X - c\mathbf{1})^+ \in A$. It follows that $\zeta_c = X \wedge c\mathbf{1}/\Upsilon(C_0) \in A_N(C_0)$. Moreover, $-a\mathbf{1} \leqslant \zeta_c \leqslant c\mathbf{1}$. It follows that $\zeta_c \in \overline{A^\infty_N}^w(C_0)$ and $E\rho\zeta_c \leqslant 0$. But we have

$$\rho\zeta_c = \sum_i \rho^i \zeta_c^i \geqslant -a\sum_i \rho^i$$

where $E\rho^i < \infty$. Then, we can apply the Fatou lemma as $c \to \infty$ to conclude about the lemma.

Corollary 6.5.5. *Assume that* $\overline{A_N^\infty}(C_0) \cap L^0(\mathbb{R}_+^d) = \{0\}$. *Then* $\mathcal{R}(C_0) \neq \emptyset$.

Proof. First, we claim that $L^\infty(\mathbb{R}_-^d) \subseteq \overline{A_N^\infty}^w(C_0)$. Indeed, if $X \in L^\infty(\mathbb{R}_-^d)$, we have $\Upsilon(C_0) \times X \in L(\mathbb{R}_-^d) \subseteq A$. It follows that $X \in \overline{A_N^\infty}^w(C_0)$. In virtue of Theorem 6.5.2, we deduce the existence of $\rho \in L^0(\operatorname{int}\mathbb{R}_+^d)$ such that $E|\rho| < \infty$ and $E\rho X \leqslant 0$ for all $X \in \overline{A_N^\infty}^w(C_0)$.

Let define the cadlag version martingale Z by $Z_T = \rho/\Upsilon(C_0) \in \operatorname{int}\mathbb{R}_+^d$ and $Z_t = E(Z_T|\mathcal{F}_t)$ verifying $E|Z_T| < \infty$. We shall prove that $Z \in \mathcal{M}_T^0(G^*\backslash\{0\})$. Suppose that $Z_t \notin G_t^*$ on a non-null set. By a measurable selection argument, we can find $X_t \in G_t$ a.s. verifying $|X| \leqslant 1$, $Z_tX_t \leqslant 0$ and $Z_tX_t < 0$ on a non-null set. Thus $-X_t \in R_T$ and verifies $-X_t/\Upsilon(C_0) \geqslant -\mathbf{1}$. From Lemma 6.5.4, it follows that $EZ_tX_t \geqslant 0$ in contradiction with the inequality $EZ_tX_t < 0$. We can conclude that $Z \in \mathcal{M}_0^T(G^*\backslash\{0\})$.

Let show that $Z \in \mathcal{R}(C_0)$. If $X \in R_T$ verifies $X \geqslant -\alpha C_0 - \beta\mathbf{1}$ where $\alpha, \beta \geqslant 0$, we have:

$$\frac{X^i}{\Upsilon(C_0)^i} \geqslant -\alpha + \frac{\alpha(1 - essinf C_0^i) - \beta}{\Upsilon(C_0)^i} \geqslant -\alpha$$

provided that $\alpha(1 - essinf C_0^i) - \beta \geqslant 0$ and otherwise

$$\frac{\alpha(1 - essinf C_0^i) - \beta}{\Upsilon(C_0)^i} \geqslant \alpha(1 - essinf C_0^i) - \beta.$$

It follows that there exists $a \geqslant 0$ such that $X/\Upsilon(C_0) \geqslant -a\mathbf{1}$. In virtue of Lemma 6.5.4, we deduce that $E|Z_TX| < \infty$ and $EZ_TX \leqslant 0$. Thus, $Z \in \mathcal{R}(C_0)$.

Corollary 6.5.6. *Assume that* $\mathcal{R} \neq \emptyset$. *Then the NGA condition holds.*

Proof. Suppose that $\mathcal{R} \neq \emptyset$ and consider $Z \in \mathcal{R} \subseteq \mathcal{R}(C)$ for some $C \in \overline{R}_T^F$.
If $Y \in \overline{A_N^\infty}^w(C) \cap L^0(\mathbb{R}_+^d)$, we have $Y = \lim Y_n$ in $\sigma(L^\infty, L^1)$ where

$$Y_n = \frac{X_n - \varepsilon_n}{\Upsilon(C)}, \qquad X_n \in R_T,\ \varepsilon_n \geqslant 0.$$

We deduce that $X_n \geqslant -\|Y_n\|_\infty\Upsilon(C)$. So, $Z \in \mathcal{R}(C)$ implies that $E|Z_TX_n| < \infty$ and $EZ_TX_n \leqslant 0$. From the proof of Lemma 6.5.3, we know that $E|Z_TC| < \infty$. Moreover, the components of $Z_T \times \Upsilon(C)$ verify $0 \leqslant Z_T^i\Upsilon(C)^i \leqslant Z_T\Upsilon(C)$.

So, $Z_T \times \Upsilon(C) \in L^1$ and $EZ_T \times \Upsilon(C)Y_n \to EZ_T \times \Upsilon(C)Y \leqslant 0$. We deduce that $Y = 0$ and the (NGA) condition holds.

6.5.3 Proof of Corollary 6.2.6

From [19], we recall the following lemma.

Lemma 6.5.7. *If* $Z \in \mathcal{M}_0^T(G^*)$ *and* $X \in \mathcal{X}_b^x$, *then* ZX *is a supermartingale and*

$$E(-Z\dot{X}).\|X\|_T \leqslant Z_0x - EZ_TX_T.$$

So, we easily deduce that in the case where $R_T = \mathcal{X}_b(T)$, we have

$$\mathcal{M}_T^0(G^*\backslash\{0\}) = \mathcal{R} = \mathcal{R}(C)$$

for all $C \in \overline{R}_T^F$. Thus, we can conclude about Corollary 6.2.6.

6.5.4 Proof of Theorem 6.2.11

If we define, for the extended model, $C_0' = C_0 + F - x$, we have obviously $C_0' \in \overline{R}_T^F{}'$.

Since $x \in I_F$ implies that the NGA condition holds for the extended model, we deduce, from Theorem 6.5.2, the existence of a random variable $\rho \in L^0(\mathrm{int}\,\mathbb{R}^d_+)$ verifying $E|\rho| < \infty$ and $E\rho X \leqslant 0$ for all $X \in \overline{A_N^\infty}^w(C_0', G')$. From Lemma 6.5.4, we deduce that if $X \in A_T'$ verifies $X/\Gamma(C_0') \geqslant -a\mathbf{1}$ where $a \geqslant 0$, then we have $E\rho X/\Gamma(C_0') \leqslant 0$. We define the martingale

$$Z_t = E\left(\frac{\rho}{\Gamma(C_0')}|\mathcal{F}_t\right).$$

From the hypothesis

$$L^\infty(-G_t, \mathcal{F}_t) \subseteq R_T \subseteq R_T',$$

and it is easy to see that $Z \in \mathcal{R}(C_0)$ in virtue of Lemma 6.5.4. Moreover, since we assume that F is bounded, we finally have from what precedes $EZ_T(F - x) = 0$ because $F - x$ and $x - F$ belong to A_T'.

Reciprocally, suppose that $Z_0 x = EZ_T F$ for some $Z \in \mathcal{R}$. First, we consider an element $Y = X + h(F - x) \in R_T'$ verifying $Y \geqslant -\alpha\mathbf{1}$ for some constant $\alpha \geqslant 0$. We deduce easily that $E(Z_T X)^- < \infty$ and finally, $Z \in \mathcal{R}$ implies that $EZ_T X \leqslant 0$. So, $E|Z_T Y| < \infty$ and $EZ_T Y \leqslant 0$. It follows that for all $C' \in \overline{R}_T^F{}'$, we have $E|Z_T C'| < \infty$ and $EZ_T C' \leqslant 0$ because of the Fatou lemma.

From now on, we define $\overline{Z}_T = \Upsilon(C') \times Z_T$. From what precedes, we have $E|\overline{Z}_T| < \infty$. We deduce that if $Y = X + h(F - x) \in R_T'$ verifies the inequality $Y/\Upsilon(C') \geqslant -\alpha\mathbf{1}$, then using the previous reasoning we obtain that

$$E|\overline{Z}_T \frac{Y}{\Upsilon(C')}| < \infty, \quad E\overline{Z}_T \frac{Y}{\Upsilon(C')} \leqslant 0.$$

Finally,

$$E\overline{Z}_T \frac{Y - V}{\Upsilon(C')} \leqslant 0$$

provided that

$$\frac{Y - V}{\Upsilon(C')} \geqslant -\alpha\mathbf{1}$$

and $V \geqslant 0$. It follows that $E\overline{Z}_T X \leqslant 0$ for all $X \in A_N^\infty(C')$ and finally for all $X \in \overline{A_N^\infty}^w(C')$. But, if we consider $X \in \overline{A_N^\infty}^w(C') \cap L^0(\mathbb{R}^d_+)$, we have obviously $\overline{Z}_T X \geqslant 0$. So, $\overline{Z}_T X = 0$ and $X = 0$ since $\overline{Z}_T \in \mathrm{int}\,\mathbb{R}^d_+$. It follows that the NGA condition is verified for the extended model and $x \in I_F$.

6.5.5 Proof of Theorem 8.1.2

From [19], we recall the following result.

Lemma 6.5.8. *Assume that $\mathcal{D} \neq \emptyset$ and $(\mathbf{G})$ holds. Let $\widetilde{R}$ be a subset of $\mathcal{X}_b$. Suppose that there exists a constant κ such that $X_T + \kappa\mathbf{1} \geqslant 0$ for all $X \in \widetilde{R}$. Then there exists a probability measure $Q \sim P$ with bounded density such that*

$$\sup_{X \in \widetilde{R}} E_Q \|X\|_T < \infty.$$

We also recall the following Komlós theorem [19]:

Theorem 6.5.9. *Let (ξ_n) be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ bounded in L^1, i.e. with $\sup_n E|\xi_n| < \infty$. Then, there exists a random variable $\xi \in L^1$ and a subsequence (ξ_{n_k}) Césaro convergent to ξ a.s., that is $k^{-1}\sum_{i=1}^k \xi_{n_i} \to \xi$ a.s. Moreover, the subsequence (ξ_{n_k}) can be chosen in such way that any its further subsequence is also Césaro convergent to ξ a.s.*

We note ν_T the space of positive finite measures on $[0, T]$ with the topology of weak convergence in probabilistic sense. An optional measure is a ν_T-valued random variable such that the process $\mu_t(w) = \mu(w, [0, t])$ is adapted. Then, we get the following lemma [19].

Lemma 6.5.10. *Let μ^n be optional random measures with $\sup_n E\mu_T^n < \infty$. Then, there exists an optional random measure μ such that $\mu_T \in L^1$ and a subsequence $\mu^{n'}$ such that all its further subsequences are Césaro convergent in ν_T to μ a.s.*

We define

$$Q_T = \{q_k^n = \frac{kT}{2^n}, k \leqslant 2^n, k, n \in \mathbb{N}\}.$$

The following result can be found in [19]. We recall the proof. For more details, we refer the reader to Lemma 6.6.1 where similar arguments are used.

Lemma 6.5.11. *Assume $\mathcal{D} \neq \emptyset$, $(\mathbf{B})$ and $(\mathbf{G})$ hold. Let consider a sequence $X^n \in \mathcal{X}_b^x$ verifying $X^n(T) + k\mathbf{1} \geqslant 0$ a.s. for a constant $k \geqslant 0$ and $X^n(T)$ converges almost surely. Then, there exists $X \in \mathcal{X}_b^x$, a subsequence $X^{n'}$ such that $X^{n'}(T)$ are Césaro convergent to $X(T)$.*

Proof. Let $X^n \in \mathcal{X}_b^x$ be a sequence with $X_T^n + \kappa\mathbf{1} \geqslant 0$ converging to U a.s.

We note, for each component:

$$X_n^i = \overline{X}_n^i - \underline{X}_n^i$$

where $\overline{X}_n^i$ and $\underline{X}_n^i$ are two increasing processes such that

$$\|X_n\| = \sum_{i=1}^d \|X_n^i\|, \quad \|X_n^i\| = \overline{X}_n^i + \underline{X}_n^i.$$

Applying Lemmas 6.5.8 and 6.5.10, we may assume that there exists a subsequence n' such that, for all subsequences, each components of $\overline{X}_{n'}$ and $\underline{X}_{n'}$ are Césaro-convergent a.s. in ν_T to increasing processes $\overline{X}^i$ and $\underline{X}^i$. It follows that $X_{n'}(T)$ is Césaro-convergent to $X(T) = \overline{X}_T - \underline{X}_T$ a.s. Otherwise, we can assume that $X_{n'}(t)$ is Césaro-convergent to $\widetilde{X}(t)$, for all points of Q_T and we note $X_t = \lim_{s \searrow t; s \in Q_T} \widetilde{X}_s$. Since all the processes $\zeta^k X^{n'}$ are decreasing, we deduce from the continuity of ζ^k that the process $\zeta^k X$ is also decreasing. Thus, $X \in \mathcal{X}^x$.

It remains to check that, for all t, $X_t + k\mathbf{1} \in G_t$ for some $k \geqslant 0$. For this, we consider $Z \in \mathcal{M}_T^0(G^*)$. In virtue of Lemma 8.1.1, the prelimit processes $Z(X^n + \kappa\mathbf{1})$ are supermartingales, positive at the terminal date. It follows that

$$E1_\Gamma Z_t(X_t^n + \kappa\mathbf{1}) \geqslant 0 \quad \forall \Gamma \in \mathcal{F}_t.$$

Therefore, $Z_t(X_t^n + \kappa\mathbf{1}) \geqslant 0$. Condition $\mathbf{B}$ implies that $X_t^n + \kappa\mathbf{1} \in G_t$. It follows that $X_t + \kappa\mathbf{1} \in G_t$, at least for $t \in \mathbb{Q}_T$ but, by continuity, for all points of $[0, T]$. Moreover, we have obviously $X_T + \kappa\mathbf{1} \geqslant 0$. So, we can deduce that $X \in \mathcal{X}_b^x$.

We deduce easily from the previous lemma, the following corollary.

Corollary 6.5.12. *Assume $\mathcal{D} \neq \emptyset$, $(\mathbf{B})$ and $(\mathbf{G})$ hold. Then, $\mathcal{X}_b(T)$ is Fatou-closed.*

From [19], we recall the following result:

Lemma 6.5.13. *The set $\mathcal{X}_b^x(T) \cap L^\infty$ is Fatou-dense in $\mathcal{X}_b^x(T)$.*

In virtue of the Bipolar Theorem ([19]), the previous lemma and Corollary 6.5.12, we deduce a dual description of $\mathcal{X}_b^x(T)$.

Lemma 6.5.14. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{B})$ and $(\mathbf{G})$ hold, then*

$$\mathcal{X}_b^x(T) = \left\{ \xi \in L_b^0 : E\xi\eta \leqslant \sup_{X \in \mathcal{X}_b^x(T)} EX\eta \quad \forall \eta \in L^1(\mathbb{R}_+^d) \right\}.$$

Corollary 6.5.15. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{B})$ and $(\mathbf{G})$ hold, then $\Gamma_F = D_F$.*

Proof. In virtue of Lemma 8.1.1, it is easy to deduce that $\Gamma_F \subseteq D_F$.

Assume that $x \in D_F$ and suppose that $x \notin \Gamma_F$. This means that $F - x \notin \mathcal{X}_b(T)$. Then, from Lemma 6.5.14, we deduce the existence of $\eta \in L^1(\mathbb{R}_+^d)$ verifying: $E\eta X < E(F - x)\eta$, $\forall X \in \mathcal{X}_b(T)$. Since $L^\infty(-G_t, \mathcal{F}_t) \subseteq \mathcal{X}_b(T)$, we have $EZ_tX \leqslant 0$, $\forall X \in L^\infty(-G_t, \mathcal{F}_t)$ where $Z_t = E(\eta|\mathcal{F}_t)$. It follows that Z belongs to $\mathcal{M}_0^T(G^*)$ and verifies in particular $Z_0 x < EZ_TF$. So, $x \notin D_F$ which leads to a contradiction.

6.5.6 Proof of Theorem 6.3.4

Assume that $p_F \in \Gamma_F$ is a minimal price. Thus, $p_F - \frac{1}{n}\mathbf{1} \notin \Gamma_F$ which implies that there exists $Z^n \in \mathcal{M}_T^0(G^* \backslash \{0\})$ verifying $|Z_0^n| = 1$ and

$$Z_0^n(p_F - \frac{1}{n}\mathbf{1}) < EZ_T^nF \leqslant Z_0^n p_F.$$

Since there exists a subsequence such that $Z_0^n \to Z_0$, we deduce easily that EZ_T^nF converges to $Z_0 p_F$.

Reciprocally, if x verifies $x \leqslant p_F$ and $x \in \Gamma_F$, then $Z_0^n p_F \geqslant Z_0^n x \geqslant EZ_T^nF$ and as $n \to \infty$, we obtain that $Z_0(p_F - x) = 0$. Moreover, $p_F - x \geqslant 0$ and $Z_0 \in G_0^* \backslash \{0\} \subseteq \operatorname{int} \mathbb{R}_+^d$ since G dominates $\mathbb{R}_+^d$. It follows that $x = p_F$ and p_F is a minimal price.

6.5.7 Proof of Corollary 6.3.6

We consider he extended model defined in Definition 6.2 for e and we shall prove that $e \in I_F$ if $e \notin \Gamma_F$. Note that $\mathcal{R}' = \mathcal{R}'(0)$. So, it suffices to prove that

$$\overline{A_N'^\infty}^w(0) \cap L^0(\mathbb{R}_+^d) = \{0\}$$

in order to have $\mathcal{R}' \neq \emptyset$ and prove the NGA condition for the extended model.

Let be $\xi \in \overline{A_N'^\infty}^w(0) \cap L^0(\mathbb{R}_+^d)$. We have $\xi = \lim(r_T^m - \varepsilon^m)$ where $\varepsilon^m \geqslant 0$ and r_T^m has the representation $r_T^m = X_T^m + h_m(F - e) \in R_T'^e$ [2] with $X_T^m \in \mathcal{X}_b(T)$. Considering $r_T^{m,n} = X_T^m + h_m(F - x_n) \in R_T'^{x_n}$, we can assume that $\xi = \lim(r_T^m - \varepsilon^m)$ where $r_T^m = X_T^m + h_m(F - x_m) \in R_T'^{x_m}$ and $x_m \nearrow e$. In the case where there exists a subsequence

[2] We note by $R_T'^e$ the set of the terminal values of portfolios in the extended model for e

such that $h_m \geqslant 0$, we have the inequality $r_T^m \leqslant X_T^m + h_m(F - x_{m_0})$ for $m \geqslant m_0$. Since $\widetilde{Z}_0 x_{m_0} = E\widetilde{Z}_T F$ for some $\widetilde{Z} \in \mathcal{M}_T^0(G^*\backslash\{0\})$, we deduce from Lemma 8.1.1 that $Er_T^m \widetilde{Z}_T \leqslant 0$ and $E\widetilde{Z}_T \xi \leqslant 0$ as $n \to \infty$. Thus $\xi = 0$.

Otherwise, assume that $h_m \leqslant 0$.

If there exists $Z \in \mathcal{M}_T^0(G^*\backslash\{0\})$ such that $Z_0 e \leqslant EZ_T F$, we deduce that

$$h_m EZ_T(F - x_m) \leqslant h_m EZ_T(F - e) \leqslant 0.$$

So, we also have $\xi = 0$ and $e \in I_F$.
Finally, we can suppose that $e \in \Gamma_F$. Since there exists some $Z^n \in \mathcal{M}_T^0(G^*\backslash\{0\})$ such that $EZ_T^n F = Z_0^n x_n \to Z_0 e$ with $Z_0^n \to Z_0$ and $|Z_0^n| = 1$, it suffices to apply Theorem 6.3.4 to conclude that e is a minimal price.

6.6 Proof of Theorem 6.4.1

In this section, we only consider the case $R_T = \mathcal{X}_b(T)$. We recall the Campi–Schachermayer model Y defined as follows (see [19]).

6.6.1 $\mathcal{Y}$-Model

We are given on the interval $[0,T]$ two set-valued processes $G = (G_t)$ and $G^* = (G_t^*)$ where $G_t = \operatorname{cone}\{\xi_t^k : \ k \in \mathbb{N}\}$ and $G_t^* = \operatorname{cone}\{\zeta_t^k : \ k \in \mathbb{N}\}$. It is assumed that the generating processes are càdlàg, adapted, and for each ω only a finite number of $\xi_t(\omega)$, $\xi_{t-}^k(\omega)$, $\zeta_t^k(\omega)$ and $\zeta_{t-}^k(\omega)$ are different from zero, i.e. all cones are polyhedral. We put $G_{t-}^* = \operatorname{cone}\{\zeta_{t-}^k : \ k \in \mathbb{N}\}$.

We define the portfolio processes following the paper [6].

Let Y be a d-dimensional predictable process of bounded variation starting from zero and having trajectories with left and right limits (French abbreviation: *làdlàg*). Put $\Delta Y := Y - Y_-$, as usual, and $\Delta^+ Y := Y_+ - Y$ where $Y_+ = (Y_{t+})$. Define the right-continuous processes

$$Y_t^d = \sum_{s \leqslant t} \Delta Y_s, \qquad Y_t^{d,+} = \sum_{s \leqslant t} \Delta^+ Y_s$$

(the first is predictable while the second is only adapted) and, at last, the continuous one:

$$Y^c := Y - Y^d - Y_-^{d,+}.$$

Let $\mathcal{Y}$ be the set of such processes Y satisfying the following conditions:

1) $\dot{Y}^c \in -G \, dP \, d||Y^c||$-a.e.;

2) $\Delta^+ Y_\tau \in -G_\tau$ a.s. whatever is a stopping time $\tau \leqslant T$;

3) $\Delta Y_\sigma \in -G_{\sigma-}$ a.s. whatever is a predictable time $\sigma \leqslant T$.

Let $\mathcal{Y}^x := x + \mathcal{Y}$, $x \in \mathbb{R}^d$. We denote by $\mathcal{Y}_b^x$ the subset of $\mathcal{Y}^x$ formed by the processes Y such that $Y_t + \kappa_Y \mathbf{1} \in L^0(G_t, \mathcal{F}_t)$, $t \leqslant T$, for some $\kappa_Y \in \mathbb{R}$. In the financial context (where $G = \widehat{K}$) the elements of $\mathcal{Y}_b^x$ are the *admissible* portfolio processes.

We associate with Y the following right-continuous adapted process of bounded variation:

$$\bar{Y} := Y^c + Y^d + Y^{d,+},$$

i.e. $\bar{Y} = Y + \Delta^+ Y = Y_+$. Since the generators are right-continuous, the process $\bar{Y}$ inherits the boundedness from below of Y (by the same constant process $\kappa_Y \mathbf{1}$).

We formulate for our needs, the following lemma

Lemma 6.6.1. *Let A^n be a sequence of predictable and increasing làdlàg processes on $[0,T]$ such that $\sup_n EA^n_T < \infty$. Then, there exists a predictable and increasing làdlàg process A and a subsequence A^{n_k} such that all its further subsequences $A^{\widetilde{n}_k}_t$ are Césaro convergent to A_t for any $t \in [0,T]$.*

Proof. We note $Q_T = \{t_k,\, k \in \mathbb{N}\backslash\{0\}\}$. It is clear that $\sup_n EA^n_{t_1} < \infty$. From Theorem 6.5.9, we deduce a subsequence $n^{(1)}$ such that, for all its subsequences, we have the convergence $\overline{A}^{n^{(1)}_k}_{t_1} \to \widetilde{A}_{t_1}$ on a set $\bigwedge_1$ of full measure where

$$\overline{A}^{n^{(1)}_k}_{t_1} = \frac{1}{k}(A^{n^{(1)}_1}_{t_1} + \cdots + A^{n^{(1)}_k}_{t_1}).$$

In a similar way, since we also have $\sup_k EA^{n^{(1)}_k}_{t_2} < \infty$, we deduce from $n^{(1)}$ a subsequence $n^{(2)}$ such that the convergence $\overline{A}^{n^{(2)}_k}_{t_2} \to \widetilde{A}_{t_2}$ holds on a set $\bigwedge_2$ of full measure for all its subsequences. Following this scheme, we inductively construct a sequence $n^{(p)}$ extracted from $n^{(p-1)}$ such that the convergence $\overline{A}^{n^{(p)}_k}_{t_p} \to \widetilde{A}_{t_p}$ holds on a set $\bigwedge_p$ of full measure for all its subsequences. Then, we deduce that the subsequence $m_p = n^{(p)}_p$ verifies the convergence $\overline{A}^{m_p}_t \to \widetilde{A}_t$ for any $t \in Q_T$ on the set $\bigwedge = \bigcap_p \bigwedge_p$ of full measure for all its subsequences. Indeed, let $\widetilde{n}_p = n^{(k_p)}_{k_p}$ be a subsequence of m_p and $t = t_{k_0} \in Q_T$. By hypothesis, the convergence $\overline{A}^{n^{(k_0)}_k}_t \to \widetilde{A}_t$ holds not only for the sequence $n^{(k_0)}$ but also for its subsequences. Moreover, if $p \geqslant k_0$, then $k_p \geqslant p \geqslant k_0$. It follows that $n^{(k_p)}$ is a subsequence extracted from $n^{(k_0)}$ and finally $\widetilde{n}$ too. We can conclude that $\overline{A}^{\widetilde{n}_p}_t \to \widetilde{A}_t$.

We shall prove that the mapping $s \to \widetilde{A}_s$ is a.s. increasing on Q_T. For this, it suffices to argue on the set $\bigwedge$ of full measure. If $t_1 \leqslant t_2$, then $A^{m_i}_{t_1} \leqslant A^{m_i}_{t_2}$ for $i \leqslant p$ by hypothesis on A. It follows that $\overline{A}^{m_p}_{t_1} \leqslant \overline{A}^{m_p}_{t_2}$ and $\widetilde{A}_{t_1} \leqslant \widetilde{A}_{t_2}$ as $p \to \infty$.We define the process $A_t = \lim_{s \nearrow t;\, s \in Q_T} \widetilde{A}_s$ on $[0,T]$ and we prove that the mapping $s \to A_s$ is a.s. increasing. It suffices to argue on the set $\bigwedge$ of full measure and use the increase of $\widetilde{A}$. Indeed, if $s_1 < t_1 < s_2 < t_2$ where $s_1,\, s_2 \in Q_T$, then $\widetilde{A}_{s_1} \leqslant \widetilde{A}_{s_2}$ and we get that $A_{t_1} \leqslant A_{t_2}$ as $s_i \to t_i$, $i = 1, 2$.

Moreover, A is a.s. left-continuous. Indeed, for t_0 fixed and any arbitrary small $\varepsilon > 0$, we have $A_{t_0} - \varepsilon \leqslant \widetilde{A}_s \leqslant A_{t_0}$ provided that $s \in Q_T$ verifies $t_0 - r \leqslant s \leqslant t_0$ where $r > 0$ is near to 0. It follows that $t_0 - r/2 \leqslant t \leqslant t_0$ implies that $A_t = \lim_{s \to t; s \geqslant t_0 - r} \widetilde{A}_s$. Then, it is clear that $A_{t_0} - \varepsilon \leqslant A_t \leqslant A_{t_0}$ and finally $A_t \to A_{t_0}$ as $t \nearrow t_0$. Note that A is clearly predictable.

We shall prove that $\overline{A}^{m_p}_{t_0} \to A_{t_0}$ provided that A is continuous at t_0. Note that $A_{t_0} \leqslant \widetilde{A}_{t_0}$ and by continuity at t_0, for any $\varepsilon > 0$ arbitrary small, we have $A_{t_0} \leqslant \widetilde{A}_{t_0} \leqslant A_t \leqslant A_{t_0} + \varepsilon$ provided that $t \in]t_0, t_0 + r[$ where $r > 0$ is near to 0. Furthermore, $|\overline{A}^{m_p}_{t_0} - \widetilde{A}_{t_0}| \leqslant \varepsilon$ for p large enough. It follows that $|\overline{A}^{m_p}_{t_0} - A_{t_0}| \leqslant 2\varepsilon$ and we can conclude.

Note that $A^+_t = \lim_{s \searrow t} A_s$ is right-continuous and has the same jumps as A. Then, we can claim that there exists stopping times τ_k exhausting the jumps of A. By similar arguments, that is a diagonal procedure, we deduce from the sequence m_p a subsequence

n_p such that $\overline{A}_t^{n_p} \to \overline{A}_t$ at each point t of continuity for A and $\overline{A}_{\tau_k}^{n_p} \to \overline{A}_{\tau_k}$ on a set $\overline{\bigwedge}$ of full mesure where $\overline{A}_t = A_t$ at each point t of continuity for A. Clearly, $\overline{A}^{n_p} \to \overline{A}$ and finally we can claim that $\overline{A}$ is a predictable and increasing process. We can conclude about the lemma. This latter serves to get the following (see [6]).

Lemma 6.6.2. *Let Y^n be a sequence of $\mathcal{Y}_b$ such that there is a threshold k at time T, i.e. $Y_T^n + k\mathbf{1} \in L^0(G_T, \mathcal{F}_T)$. Suppose that $\mathcal{D}(G) \neq \emptyset$ and* **B** *holds. Then, there exists $Y \in \mathcal{Y}_b$ and a subsequence n_k such that a.s.(ω), $Y_t^{n_k}$ is Césaro convergent to Y_t for all $t \in [0,T]$.*

We observe that the conditions of this model are verified by the $\mathcal{C}$-valued process $G = (G_t)_{t\in[0,T]}$ of our X-model. We can deduce that $Y \in \mathcal{Y} \Rightarrow \bar{Y} \in \mathcal{X}$. Indeed, $X = \bar{Y}$ is such that a version of its density $\dot{X}$ is given by the following formula

$$\dot{X} = \limsup_n \sum_k \frac{Y_{t_{k+1}+} - Y_{t_k+}}{||Y_+||_{t_{k+1}} - ||Y_+||_{t_k}} I_{]t_k,t_{k+1}]}.$$

Then, we can conclude using the next lemma from [6].

Let $G_{s,t}(\omega)$ denotes the closure of cone $\{G_r(\omega) : \ s \leqslant r < t\}$ and let

$$G_{s,t+}(\omega) := \cap_{\varepsilon>0} G_{s,t+\varepsilon}(\omega), \qquad G_{s-,t}(\omega) := \cap_{\varepsilon>0} G_{s-\varepsilon,t}(\omega)$$

with an obvious change when $s = 0$.

We assume [3] in all this chapter that $G_{t,t+} = G_t$ and $G_{t-,t} = G_{t-}$.

Lemma 6.6.3. *Let Y be a predictable process of bounded variation. Then*

$$Y \in \mathcal{Y} \quad \Leftrightarrow \quad Y_\sigma - Y_\tau \in L^0(G_{\sigma,\tau}) \text{ for all stopping times } \sigma, \tau, \ \sigma \leqslant \tau \leqslant T.$$

6.6.2 Proof

First, we give an obvious result for more convenience.

Lemma 6.6.4. *We have*

$$\Gamma_F^a = \left\{x \in \mathbb{R}^d : \exists X \in \mathcal{X}_{b,T} \text{ such that } x + X_\tau \geqslant F_\tau \text{ for all stopping time } \tau \text{ and } x + X_T = F_T\right\}.$$

Proof. Let consider $X \in \mathcal{X}_{b,T}$ verifying $x + X_\tau \geqslant F_\tau$ for all stopping times τ. We have $x + X_T = F_T + Y_T$ with $Y_T \geqslant 0$. It suffices to consider

$$X'_t = X_t - Y_T I_T(t)$$

in order to have $X' \in \mathcal{X}_{b,T}$ with $x + X'_T = F_T$ and $x + X'_\tau \geqslant F_\tau$.

Lemma 6.6.5. *We have $\Gamma_F^a \subseteq D_F^a$.*

Proof. Suppose that $X \in \mathcal{X}_{b,T}$ verifies $x + X_t \geqslant F_t$ for all t and consider $\eta \in \mathcal{P}_0^T(G^*, \mu)$. We have chosen a càdlàg version of the martingale

$$M_t = E\left(\int_0^T \eta_s d\mu(s) | \mathcal{F}_t\right)$$

[3] The property $G_{t,t+} = G_t$ is considered as obvious in [19] and [9] but the proof is not produced and seems to be an open problem.

such that the process

$$Y_t = M_t - \int_0^t \eta_s d\mu(s)$$

is a semi-martingale verifying by hypothesis $Y_t \in L^0(G_t^*)$ and $Y_T = 0$. We recall that the formula of integration by parts is:

$$[X, Y] = XY - \int X_- dY - \int Y_- dX.$$

So, we obtain for all stopping times τ_n,

$$\int_0^{\tau_n} X_{t-}\eta_t d\mu(t) = \int_0^{\tau_n} X_{t-} dM_t + \int_0^{\tau_n} Y_t dX_t - X_{\tau_n} Y\tau_n$$

where τ_n is chosen such that the process $X_-.M^{\tau_n}$ is a martingale. Recall that Y verifies $Y_t \in G_t^*$. So, we have

$$E\int_0^{\tau_n} Y_t dX_t = E\int_0^{\tau_n} Y_t \dot{X}_t d\|X\|_t \leqslant 0$$

It follows that

$$E\int_0^{\tau_n} X_{t-}\eta_t d\mu(t) \leqslant -EX_{\tau_n} Y\tau_n.$$

Since F verifies $F_t \leqslant F_{t-}$ we have $F_t\eta_t \leqslant x\eta_t + X_{t-}\eta_t$ and

$$E\int_0^{\tau_n} F_t\eta_t d\mu(t) \leqslant xE\int_0^{\tau_n} \eta_t d\mu(t) - EX_{\tau_n} Y\tau_n.$$

Note that there is a constant c such that $-EX_{\tau_n} Y\tau_n \leqslant -cEY_{\tau_n}\mathbf{1}$ where

$$EY_{\tau_n}\mathbf{1} = E_\mu \eta I_{]\tau_n, T]} \to 0.$$

Moreover, F being bounded from below, it suffices to use the Fatou lemma to conclude. We note

$$L_b^0(\Omega) = \left\{\xi \in L^0(\Omega) : \exists k_\xi \geqslant 0 \text{ such that } \xi + k_\xi \mathbf{1} \geqslant 0\right\}.$$

With

$$\mu^n = \sum_{k=0}^{2^n} \delta_{q_k^n},$$

we define the following sets of $L_b^0(\Omega_T, P \otimes \mu^n, \mathbb{R}^d)$:

$$A_0^n = \left\{\xi : \xi_{q_k^n} \in \mathcal{F}_{q_k^n}, \exists Y \in \mathcal{Y}_b \text{ such that } Y_{q_k^n} \geqslant \xi_{q_k^n}, \forall k \leqslant 2^n\right\}.$$

Lemma 6.6.6. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{B})$ and $(\mathbf{G})$ hold, then A_0^n is Fatou-closed.*

Proof. We assume that $\xi^m \to \xi$ $P \otimes \mu^n$ a.s. (ω, t) with $\xi^m \in A_0^n$ verifying $\xi^m + p\mathbf{1} \geqslant 0$ $P \otimes \mu^n$ a.s. (ω, t) for some constant $p \geqslant 0$. Then, we deduce that $\xi_{q_k^n}^m \to \xi_{q_k^n}$ P a.s.(ω) for all $k \leqslant 2^n$ and $\xi_T^m + p\mathbf{1} \geqslant 0$ a.s.(ω). It follows that $\xi_{q_k^n} \in \mathcal{F}_{q_k^n}$.

By hypothesis, there exists $Y^m \in \mathcal{Y}_b$ such that $Y_{q_k^n}^m \geqslant \xi_{q_k^n}^m$ a.s. (ω) and $Y_T^m + p\mathbf{1} \geqslant 0$. From now on, it suffices to apply Lemma 6.6.2 to conclude that there exists $Y \in \mathcal{Y}_b$ and a subsequence m_p such that for all t, $Y_t^{m_p}$ is Césaro convergent to Y_t.

It follows that $Y_{q_k^n} \geqslant \xi_{q_k^n}$ a.s. (ω).

Lemma 6.6.7. *$A_0^n \cap L^\infty$ is Fatou-dense in A_0^n.*

Proof. Let consider $\xi \in A_0^n$ verifying $\xi + p\mathbf{1} \geqslant 0$ for some constant $p \geqslant 0$. There exists $Y \in \mathcal{Y}_b$ such that $Y_{q_k^n} \geqslant \xi_{q_k^n}$ a.s. (ω) for all k. It suffices to consider $Y^m = Y$ and $\xi^m = \xi \wedge m$ in order to have $\xi^m \in A_0^n$, with $\xi^m \to \xi$ $P \otimes \mu^n$ a.s. (ω, t).

In virtue of the Bipolar Theorem ([19]) and the previous lemma, we deduce a dual description of A_0^n.

Corollary 6.6.8. *If $\mathcal{D} \neq \emptyset$, $(\mathbf{B})$ and $(\mathbf{G})$ hold, then*

$$A_0^n = \left\{ \xi \in L_b^0 : E_{\mu^n} \xi \eta \leqslant \sup_{X \in A_0^n} E_{\mu^n} X \eta,\ \forall \eta \in L^1(\Omega_T, P \otimes \mu^n, \mathbb{R}_+^d) \right\}.$$

Finally, we can prove Theorem 6.4.1.

Corollary 6.6.9. *Assume that $\mathcal{D} \neq \emptyset$, $(\mathbf{B})$ and $(\mathbf{G})$ hold. Then,*

$$D_F^a \subseteq \Gamma_F^a.$$

Proof. We consider $x \in D_F^a$ and we first suppose that there exists $n \in \mathbb{N}$ such that $F - x \notin A_0^n$. Then, we deduce from the previous corollary the existence of $\eta^n \in L^1(\Omega_T, P \otimes \mu^n, \mathbb{R}_+^d)$ such that:

$$E \int_0^T (F_t - x)\eta_t^n d\mu^n(t) > E \int_0^T \xi_t \eta_t^n d\mu^n(t)$$

for all $\xi \in A_0^n$. We deduce that for all $\xi \in A_0^n$,

$$E \int_0^T \xi_t \eta_t^n \mu^n(t) \leqslant 0.$$

Considering, for any $u > 0$ and $N^u \in L^\infty(-G_u)$, the process $\xi(t) = N^u I_{]u,T]}(t) \in A_0^n \cap \mathcal{Y}_b$, we deduce that $\eta^n \in \mathcal{P}_0^T(G^*, \mu^n)$. This leads to a contradiction since $0 \in A_0^n$ implies that

$$E \int_0^T F_t \eta_t^n d\mu^n(t) > xE \int_0^T \eta_t^n d\mu^n(t)$$

whereas $x \in D_F^a$ and $\eta^n \in \mathcal{P}_0^T(G^*, \mu^n)$.

From now on, we can assume that $F - x \in A_0^n$ for all $n \in \mathbb{N}$. It follows that for all $n \in \mathbb{N}$, there exists $Y^n \in \mathcal{Y}_b$ such that $x + Y_{q_k^n}^n \geqslant F_{q_k^n}$ a.s.(ω) for all $k \leqslant 2^n$, and $Y_T^n + p\mathbf{1} \geqslant 0$ where $p \geqslant 0$. Then, it suffices to use again Lemma 6.6.2 to conclude that there exists $Y \in \mathcal{Y}_b$ such that a subsequence of Y_t^n is Césaro-convergent to Y_t for all $t \in Q_T$. Moreover, for each fixed $t \in Q_T$, there exists n_t, k_t such that

$$t = \frac{k_t T}{2^{n_t}},\ n \geqslant n_t \Rightarrow t = \frac{2^{n-n_t} k_t T}{2^n}.$$

Then, $n \geqslant n_t$ implies that $x + Y_t^n \geqslant F_t$ a.s. (ω) because $F - x \in A_0^n$. We deduce that $x + Y_t \geqslant F_t$ a.s. (ω), for each $t \in Q_T$.

Furthermore, if we consider any stopping time τ, we have $\tau = \lim \searrow \tau^n$ everywhere on Ω with:

$$\tau^n = \sum_{k=0}^{2^n - 1} \frac{(k+1)T}{2^n} 1_{\{\frac{kT}{2^n} \leqslant \tau < \frac{(k+1)T}{2^n}\}}.$$

Obviously, we have $x + Y_{\tau^n} \geqslant F_{\tau^n}$ a.s. (ω) and we deduce, from the fact that F is càdlàg, that $x + Y_{\tau+} \geqslant F_\tau$ a.s. (ω) for all stopping times τ where $Y_+ \in \mathcal{X}_b$. Indeed, we have already observe that $\bar{Y} = Y_+ \in \mathcal{X}$. Moreover, because of F, it is clear that Y_+ is bounded from below.

Chapter 7

No Free Lunch Arbitrage in the $\mathcal{Y}$-Model

We consider the continuous-time model $\mathcal{Y}$ of financial market with proportional transaction costs. In a recent paper [6], a dual description of the set of initial endowments of self-financing portfolios super-replicating American-type contingent claim is proved under some assumptions. The hypotheses used in order to show the hedging theorem of European options are the same. We suggest to link these conditions with the absence of arbitrage when the market verifies an hypothesis fulfilled if transaction costs and risky assets are jump processes.

7.1 Introduction and Formulation of the Main Results.

In the present paper, we investigate the problem of no-arbitrage using the approach of Campi and Schachermayer. The model (see [6]) is the following:

We shall work in a general "abstract" setting where we are given on the interval $[0, T]$ two set-valued processes $G = (G_t)$ and $G^* = (G^*_t)$ where $G_t = \text{cone}\,\{\xi^k_t : \ k \in \mathbb{N}\}$ and $G^*_t = \text{cone}\,\{\zeta^k_t : \ k \in \mathbb{N}\}$. It is assumed that the generating processes are càdlàg, adapted, and for each ω only a finite number of $\xi_t(\omega)$, $\xi^k_{t-}(\omega)$, $\zeta^k_t(\omega)$ and $\zeta^k_{t-}(\omega)$ are different from zero, i.e. all cones are polyhedral. We put $G^*_{t-} = \text{cone}\,\{\zeta^k_{t-} : \ k \in \mathbb{N}\}$.

Standing hypotheses. Throughout the paper we shall assume that all cones G_t contain $\mathbb{R}^d_+$ and are proper, i.e. $G_t \cap (-G_t) = \{0\}$ or, equivalently, $\text{int}\, G^*_t \neq \emptyset$. In the financial setting, the cones G_t are the solvency cones $\widehat{K}_t$ provided that the portfolio positions are expressed in physical units:

$$\widehat{K}_t = cone\{\pi^{ij}_t e_i - e_j,\, e_i,\, 1 \leqslant i, j \leqslant d\}$$

where

$$\pi^{ij}_t = (1 + \lambda^{i,j}_t) S^j_t / S^i_t$$

and S^i_t are the risky assets whereas $\lambda^{i,j}_t$ are the transaction costs.
This hypothesis means that we are working assuming **efficient friction**.

We denote $\mathcal{D}(G)$ the subset of $\mathcal{M}_0^T(G^*)$ formed by martingales Z such that not only $Z_\tau \in L^0(\operatorname{int} G^*_\tau)$ for any stopping time $\tau \in [0,T]$ but also $Z_{\tau-} \in L^0(\operatorname{int} G^*_{\tau-})$ for any predictable time $\tau \in [0,T]$.

In a similar way, if I is a countable set of stopping times, we note $\mathcal{D}^I(G)$ the subset of $\mathcal{M}_0^T(G^*)$ formed by martingales Z such that not only $Z_\tau \in L^0(\operatorname{int} G^*_\tau)$ for any stopping time $\tau \in I$ but also $Z_{\tau-} \in L^0(\operatorname{int} G^*_{\tau-})$ for any predictable time $\tau \in I$. For more convenience, we note I^p the subset of all predictable times of I.

Moreover, we assume the following condition:

D'. If $\mathcal{D}^I(G) \neq \emptyset$ for any countable set I of stopping times , then $\mathcal{D}(G) \neq \emptyset$.

In a financial context, where the cones are defined as above, we can observe that this assumption is verified in the following case:

Lemma 7.1.1. *Assume that the risky asset $S = (S^i)_{i=1,\cdots,d}$ and the transaction costs $\lambda = (\lambda^{i,j})_{i,j=1,\cdots,d}$ are jump processes having the form*

$$X_t = \sum_n X_n I_{[T^n, T^{n+1}[}$$

where T^n are stopping times totally inaccessible and X_n are random variables $\mathcal{F}_{T^n}$-measurable. Moreover, suppose that there exists $\varepsilon_0 > 0$ such that $\tau^{n+1} - \tau^n \geqslant \varepsilon_0$ where $(\tau^n)_n$ is the set of all stopping times exhausting the jumps of the processes S and λ. Then, the condition **D'** *holds.*

Proof. We consider the countable set I of all stopping times

$$\tau^{n,\epsilon} = \tau^n + \epsilon, \quad \epsilon \in \mathbb{Q}_+$$

where τ^n exhaust the jumps of the processes defining the risky asset and transaction costs. We can suppose that $0,\, T \in I$. Then, assuming the required hypothesis for **D'**, we can define $Z := Z^I$ and we prove that Z belongs to $\mathcal{D}(G)$. In the contrary case, suppose that there exists a stopping time $\tau \in [0,T]$ such that $Z_\tau N_\tau = 0$ where $N_\tau \in L^\infty(G_\tau, \mathcal{F}_\tau) \setminus \{0\}$. We can assume that $\tau \neq \tau^n$ because of Z. We deduce that $Z_\tau N_\tau I_{\tau^n < \tau < \tau^{n,\epsilon}} = 0$ for any n and $\epsilon \in]0, \epsilon_0[$. It follows that $E Z_{\tau^{n,\epsilon}} N_\tau I_{\tau^n < \tau < \tau^{n,\epsilon}} = 0$ where $Z_{\tau^{n,\epsilon}} \in \operatorname{int} G^*_{\tau^{n,\epsilon}}$ and $G^*_{\tau^{n,\epsilon}} = G^*_{\tau^n}$ whereas $N_\tau \in G_{\tau^n} \setminus \{0\}$ on the set $\tau^n \leqslant \tau < \tau^{n,\epsilon} < \tau^{n+1}$. We deduce that $I_{\tau^n \leqslant \tau < \tau^{n,\epsilon} < \tau^{n+1}} = 0$ for any n, ϵ which leads to a contradiction. The reasoning is similar if we assume that $Z_{\tau-} N_{\tau-} = 0$ since we suppose in this case that τ is predictable. Indeed, the stopping times τ^n are assumed totally inaccessible and we can assume that $\tau \neq \tau^n$. It follows that we can repeat the previous reasoning.

We define the portfolio processes following the paper [9].

Let Y be a d-dimensional predictable process of bounded variation starting from zero and having trajectories with left and right limits (French abbreviation: *làdlàg*). Put $\Delta Y := Y - Y_-$, as usual, and $\Delta^+ Y := Y_+ - Y$ where $Y_+ = (Y_{t+})$. Define the right-continuous processes

$$Y_t^d = \sum_{s \leqslant t} \Delta Y_s, \qquad Y_t^{d,+} = \sum_{s \leqslant t} \Delta^+ Y_s$$

(the first is predictable while the second is only adapted) and, at last, the continuous one:

$$Y^c := Y - Y^d - Y_-^{d,+}.$$

Let $\mathcal{Y}$ be the set of such process Y satisfying the following conditions:

1) $\dot{Y}^c \in -G\ dP\, d||Y^c||$-a.e.;

2) $\Delta^+ Y_\tau \in -G_\tau$ a.s. whatever is a stopping time $\tau \leqslant T$;

3) $\Delta Y_\sigma \in -G_{\sigma-}$ a.s. whatever is a predictable time $\sigma \leqslant T$.

Let $\mathcal{Y}^x := x + \mathcal{Y}$, $x \in \mathbb{R}^d$. We denote by $\mathcal{Y}_b^x$ the subset of $\mathcal{Y}^x$ formed by the processes Y such that $Y_t + \kappa_Y \mathbf{1} \in L^0(G_t, \mathcal{F}_t)$, $t \leqslant T$, for some $\kappa_Y \in \mathbb{R}$. In the financial context (where $G = \widehat{K}$) the elements of $\mathcal{Y}_b^x$ are the *admissible* portfolio processes. We note

$$\mathcal{Y}_b^x(T) := \{Y_T : Y \in \mathcal{Y}_b^x\}.$$

We associate with Y the following right-continuous adapted process of bounded variation:

$$\bar{Y} := Y^c + Y^d + Y^{d,+},$$

i.e. $\bar{Y} = Y + \Delta^+ Y = Y_+$. Since the generators are right-continuous, the process $\bar{Y}$ inherits the boundedness from below of Y (by the same constant process $\kappa_Y \mathbf{1}$).
Fix $\xi \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ with $\xi + \kappa\mathbf{1} \in L^0(G_T, \mathcal{F}_T)$ where κ is constant and define the convex set

$$\Gamma := \{x \in \mathbb{R}^d :\ \xi \in \mathcal{Y}_b^x(T)\}$$

and the closed convex set

$$D := \{x \in \mathbb{R}^d :\ Z_0 x \geqslant E Z_T \xi \ \ \forall Z \in \mathcal{M}_0^T(G^*)\}.$$

The next hypothesis is a requirement that the set $\mathcal{M}_0^T(G^*)$ is rich enough.

B. Let $\xi \in L^0(\mathbb{R}^d, \mathcal{F}_t)$. If the scalar product $Z_t \xi \geqslant 0$ for all $Z \in \mathcal{M}_0^T(G^*)$, then $\xi \in L^0(G_t, \mathcal{F}_t)$.

Then, recall the hedging theorem of European options for the considered model.

Theorem 7.1.2. *Assume that $\mathcal{D}(G) \neq \emptyset$ and* **B** *holds. Then $\Gamma = D$.*

Our objective is to prove that the hypotheses of this theorem are equivalent to the conditions of No Free Lunch Arbitrage (NFL) (in the literature, the notion of No Free Lunch Arbitrage is usually defined using a closure of the set of incomes). For this, we define the following sets

$$\gamma_t(B) := \left\{\xi \in L^0(\mathbb{R}^d, \mathcal{F}_t) : Z_t \xi \leqslant 0, \forall Z \in \mathcal{M}_0^T(G^*)\right\}, \quad t \in [0, T]$$

and we say that the model satisfies the (NFL) condition if the following statements hold:

$(i)\ \overline{\mathcal{Y}_b^0(T)^\infty}^w \cap L^0(G_\tau, \mathcal{F}_\tau) = \{0\}$,

$(ii)\ \overline{\mathcal{Y}_b^0(T)^\infty}^w \cap L^0(G_{\tau-}, \mathcal{F}_{\tau-}) = \{0\}$,

$(iii)\ \overline{\mathcal{Y}_b^0(T)^\infty}^w \cap \gamma_t(B) \subset L^0(-G_t, \mathcal{F}_t)$

where $\overline{\mathcal{Y}_b^0(T)^\infty}^w$ means the closure in $\sigma(L^\infty, L^1)$ of the set $\mathcal{Y}_b^0(T) \cap L^\infty$, τ is any stopping time $\tau \in [0, T]$, assumed predictable for (ii). Then, we can establish that the (NFL) condition is equivalent to the hypotheses used for hedging theorems.

Theorem 7.1.3. *Assume that the assumption* $\mathbf{D}'$ *holds. Then, the following conditions are equivalent:*

(a) *(NFL) condition holds.*

(b) *$\mathcal{D}(G) \neq \emptyset$ and the condition* **B** *holds.*

(c) *$\mathcal{Y}_b^0(T)$ is Fatou-closed and (NFL) condition holds.*

7.2 Proof of Theorem 7.1.3

First, the implication $(b) \Rightarrow (c)$ is obvious. Indeed, the Fatou-closure under (b) was shown in [6]. Moreover, if we consider $Y_T \in \overline{\mathcal{Y}_b^0(T)^\infty}^w \cap L^0(G_\tau, \mathcal{F}_\tau)$, choosing a fixed $Z \in \mathcal{D}(G)$, we have $Z_\tau Y_T \geqslant 0$ since $Z_\tau \in L^0(\operatorname{int} G^*_\tau, \mathcal{F}_\tau)$. But, recall from [6], the following lemma:

Lemma 7.2.1. *If $Z \in \mathcal{M}_0^T(G^*)$ and $Y \in \mathcal{Y}_b^x$, then both processes $Z\bar{Y}$ and ZY are supermartingale and*

$$E(-Z\dot{\bar{Y}}) \cdot ||\bar{Y}||_T \leqslant Z_0 x - EZ_T\bar{Y}_T. \tag{7.2.1}$$

Then, we write $Y_T = \lim_n Y_T^n$ and from $EZ_T Y_T^n \leqslant 0$, we deduce that $EZ_T Y_T \leqslant 0$ which implies that $Z_\tau Y_T = 0$ and $Y_T = 0$. From the condition **B** , we can easily deduce (iii). Obviously, (c) implies (a). So, it remains to prove the implication $(a) \Rightarrow (b)$. For this, we need some preliminary lemmas:

Lemma 7.2.2. *Assume that (i) holds. We consider a fixed constant $k > 0$ and a countable set I of stopping times. Then, for any càdlàg process ζ verifying $\zeta_\tau \in L^0(G_\tau, \mathcal{F}_\tau)$ for any stopping time τ, there exists a martingale Z^ζ such that*

(1) $Z_t^\zeta \xi \geqslant 0$ *for any* $t \leqslant T$, $\xi \in L^0(G_t, \mathcal{F}_t)$,

(2) $\zeta_\tau I_{\{Z_\tau^\zeta \zeta_\tau = 0\}} = 0$ *for any* $\tau \in I$,

(3) $\|Z_t^\zeta\|_1 \leqslant k$.

Lemma 7.2.3. *Assume that (ii) holds. We consider a fixed constant $k > 0$ and a countable set I of stopping times. Then, for any càdlàg process ζ verifying $\zeta_t \in L^0(G_t, \mathcal{F}_t)$, $\forall t$ and $\zeta_{\tau-} \in L^0(G_{\tau-}, \mathcal{F}_{\tau-})$ for any predictable time τ, there exists a martingale $Z^{\zeta-}$ such that*

(1) $Z_{t-}^{\zeta-} \xi \geqslant 0$ *for any* $t \leqslant T$, $\xi \in L^0(G_{t-}, \mathcal{F}_{t-})$,

(2) $Z_t^{\zeta-} \xi \geqslant 0$ *for any* $t \leqslant T$, $\xi \in L^0(G_t, \mathcal{F}_t)$,

(3) $\zeta_{\tau-} I_{\{Z_{\tau-}^{\zeta-} \zeta_{\tau-} = 0\}} = 0$ *for any* $\tau \in I^p$,

(4) $\|Z_t^{\zeta-}\|_1 \leqslant k$.

Proofs. Proofs of the previous lemmas are similar. That's why we only give the second. We define

$$\mathcal{Z}_T := \{\eta \in L^1(\mathbb{R}^d) : E\eta\xi \leqslant 0, \forall \xi \in \overline{\mathcal{Y}_b^0(T)^\infty}^w\}.$$

For each $\eta \in \mathcal{Z}_T$, we consider the martingale $Z_t^\eta = E(\eta|\mathcal{F}_t)$ and we first show that it verifies (1). In the contrary case, there exists $t \leqslant T$, $\xi \in L^\infty(G_{t-}, \mathcal{F}_{t-})$ such that $Z_{t-}^\eta \xi < 0$ on a non-null set. We define $\widetilde{\xi} = -\xi I_\Gamma$ where $\Gamma = \{Z_{t-}^\eta \xi < 0\}$. So, $\widetilde{\xi} \in \overline{\mathcal{Y}_b^0(T)^\infty}^w$. Moreover, $EZ_{t-}^\eta \widetilde{\xi} > 0$ which implies that $E\eta\widetilde{\xi} > 0$ in contradiction with the fact that $\eta \in \mathcal{Z}_T$. We can prove (2) in a similar way. In order to prove (3), we define, for each predictable time $\tau \in I^p$,

$$c_\tau = \sup_{\eta \in \mathcal{Z}_T} P\left(Z_{\tau-}^\eta \zeta_{\tau-} > 0\right).$$

There exists a sequence $\eta^n \in \mathcal{Z}_T$ such that $P(Z_{\tau-}^{\eta^n} \zeta_{\tau-} > 0)$ increases to c_τ. Obviously, we can choose $\|\eta^n\|_1 \leqslant k$. We note

$$\eta^*(\tau) = \sum_{n=1}^\infty 2^{-n} \eta^n \in L^1.$$

We prove easily that $\eta^*(\tau) \in \mathcal{Z}_T$ and, since τ is predictable,

$$Z_{\tau-}^{\eta^*(\tau)} = \sum_{n=1}^{\infty} 2^{-n} Z_{\tau-}^{\eta^n}.$$

It follows that

$$P(Z_{\tau-}^{\eta^n}\zeta_{\tau-} > 0) \leqslant P(Z_{\tau}^{\eta^*(\tau-)}\zeta_{\tau-} > 0) \leqslant c_\tau$$

and $c_\tau = P(Z_{\tau-}^{\eta^*(\tau)}\zeta_{\tau-} > 0)$. We note $I^p = \{\tau^m : m \in \mathbb{N}^*\}$ and we define

$$\eta^* = \sum_{m=1}^{\infty} 2^{-m}\eta^*(\tau^m)$$

verifying $\|\eta^*\|_1 \leqslant k$. We still have $\eta^* \in \mathcal{Z}_T$. Moreover, for any $\tau \in I^p$, $c_\tau = P(Z_{\tau-}^{\eta^*}\zeta_{\tau-} > 0)$.

We shall prove that Z^{η^*} verifies (3). In the contrary case, there exists $\tau \in I^p$ and $a > 0$ such that $\gamma_\tau^a = \zeta_{\tau-} I_{\{Z_{\tau-}^{\eta^*}\zeta_{\tau-}=0;|\zeta_{\tau-}|<a\}} \neq 0$. But, $\gamma_\tau^a \in L^\infty(G_{\tau-}, \mathcal{F}_{\tau-})$ and (ii) implies that $\gamma_\tau^a \notin \overline{\mathcal{Y}_b^0(T)^\infty}^w$. Thanks to the Hahn–Banach theorem, we deduce the existence of $\eta \in L^1(\mathcal{F}_T)$ such that

$$E\eta Y < E\eta\gamma_\tau^a, \forall Y \in \overline{\mathcal{Y}_b^0(T)^\infty}^w$$

which implies that $E\eta Y \leqslant 0$ for all Y and $\eta \in \mathcal{Z}_T$. Moreover, $E\gamma_\tau^a\eta > 0$ and finally $EZ_{\tau-}^\eta\gamma_\tau^a > 0$ since it is easy to show that $Z^\eta \in \mathcal{M}_T^0(G^*)$. It follows that $Z_{\tau-}^\eta\gamma_\tau^a > 0$ on a non-null set Γ. From now on, we define $\widetilde{Z} = Z^{\eta^*} + Z^\eta$. We observe that $F = \{Z_{\tau-}^{\eta^*}\zeta_{\tau-} > 0\}$ verifies $F \cup \Gamma \subset G$ where $G = \{\widetilde{Z}_{\tau-}\zeta_{\tau-} > 0\}$. Moreover, it is obvious that $F \cap \Gamma = \emptyset$. It follows that

$$c_\tau = P(F) < P(F \cup \Gamma) \leqslant P(G) \leqslant c_\tau$$

which leads to a contradiction. It suffices to consider $Z^\zeta = Z^{\eta^*}$ to conclude.

Corollary 7.2.4. *Assume that the conditions (i) and (ii) hold. We consider a fixed constant $k > 0$ and a countable set I of stopping times. Then, there exists a martingale Z^I such that:*

(1) $Z_{t-}^I\xi \geqslant 0$, $\forall t \leqslant T$, $\xi \in L^0(G_{t-}, \mathcal{F}_{t-})$,

(2) $Z_t^I\xi \geqslant 0$, $\forall t \leqslant T$, $\xi \in L^0(G_t, \mathcal{F}_t)$,

(3) $\zeta_\tau I_{\{Z_\tau^I\zeta_\tau=0\}} = 0$, $\forall \tau \in I$, $\zeta = \sum_n \alpha^n\xi^n$, $\alpha_\tau^n \in L_+^0(\mathcal{F}_\tau)$,

(4) $\zeta_{\tau-} I_{\{Z_{\tau-}^I\zeta_{\tau-}=0\}} = 0$, $\forall \tau \in I^p$, $\zeta_{\tau-} = \sum_n \alpha^n\xi_{\tau-}^n$, $\alpha_\tau^n \in L_+^0(\mathcal{F}_{\tau-})$,

(5) $\|Z^I\|_1 \leqslant k$.

Proof. It suffices to apply the two previous lemmas for each cone generator ξ^k in order to respectively obtain Z^{ξ^k} and Z^{ξ^k-}. Then, we consider:

$$Z^I := \sum_{k=1}^{\infty} 2^{-k-1} Z^{\xi^k} + \sum_{k=1}^{\infty} 2^{-k-1} Z^{\xi^k-}$$

and it is easy to prove that Z verifies conditions (1) and (2). Let prove (4). First, we choose a fixed generator $\zeta_{\tau-} = \xi_{\tau-}^n$. Then, the equality $Z_{\tau-}^I\zeta_{\tau-} = 0$ implies that $Z_{\tau-}^{\xi^n-}\xi_{\tau-}^n = 0$ and finally $\xi_{\tau-}^n = 0$ by hypothesis on Z^{ξ^n-}. Secondly, if $\zeta_{\tau-} = \alpha_{\tau-}^n\xi_{\tau-}^n$ we have, by the same reasoning, $Z_{\tau-}^{\xi^n-}\xi_{\tau-}^n = 0$ on $\alpha_{\tau-}^n \neq 0$ and we can conclude. Finally, if $\zeta_{\tau-} = \sum_n \alpha_{\tau-}^n\xi_{\tau-}^n$, we obtain that for any k, $Z^{\xi^k-}_{\tau-}\zeta_{\tau-} = 0$ which implies that $Z^{\xi^n-}_{\tau-}\alpha^n_{\tau-}\xi^n_{\tau-} = 0$ and we can conclude as previously. We can similarly prove (3).

It is easy to deduce the following lemma:

Corollary 7.2.5. *If the conditions (i), (ii) and* **D'** *hold, then $\mathcal{D}(G) \neq \emptyset$.*

Proof. Indeed, for any countable family I of stopping times, it suffices to consider the martingale Z^I produced by the previous lemma in order to ensure the required condition for $\mathbf{D}'$.

Thanks to the last corollary, it remains to prove that the condition **(B)** holds. For this, we consider for any $F \in L^\infty$,

$$\overline{\Gamma}_F := \{x \in \mathbf{R}^d : \ F - x \in \overline{\mathcal{Y}^0_b(T)^\infty}^w\}$$

and the closed convex set

$$D_F := \{x \in \mathbf{R}^d : \ Z_0 x \geqslant EZ_T F \ \ \forall Z \in \mathcal{M}^T_0(G^*)\}.$$

It is easy to prove that $\overline{\Gamma}_F \subset D_F$. Indeed, it suffices to use Lemma 7.2.1. For the converse, we consider $x \in D_F$ and we suppose that $F-x \notin \overline{\mathcal{Y}^0_b(T)^\infty}^w$. Using the Hahn–Banach theorem, we deduce (see proof of Lemma 7.2.3) the existence of a martingale $Z^\eta \in \mathcal{M}^T_0(G^*)$ such that $EZ^\eta(F-x) > 0$ and we can conclude that $x \notin \overline{\Gamma}_F$ implies that $x \notin D_F$. It follows that $\overline{\Gamma}_F = D_F$. From now on, suppose that $\xi \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ verifies $Z_t\xi \geqslant 0$ for any $Z \in \mathcal{M}^T_0(G^*)$. We can assume that $\xi \in L^\infty$. It follows that $0 \in D_{-\xi}$ and finally $-\xi \in \overline{\mathcal{Y}^x_b(T)^\infty}^w$. We deduce that

$$-\xi \in \overline{\mathcal{Y}^x_b(T)^\infty}^w \cap \gamma_t(B) \subset L^0(-G_t, \mathcal{F}_t)$$

and we can conclude.

Chapter 8

Asymptotic Arbitrage in Large Financial Markets

Yu. Kabanov and D.O. Kramkov have defined several notions of asymptotic arbitrage for a large financial market described by a sequence of standard general models without friction of continuous trading [17]. Moreover, they link the absence of asymptotic arbitrage to the notion of contiguity. Here, we propose analogous results when the market is subjected to transaction costs. We deal with the traditional model of discret trading [19] but also with the continuous model of Chapter 6.

8.1 Introduction

We fix a sequence T^n of positive numbers. We define a *large financial market* as a sequence of markets whose the time horizons are T^n, the dimensions are $d(n)$ and all described by the same model among the two following.

In the two cases, for each n, we assume that we are given a stochastic basis $B^n = (\Omega, \mathcal{F}^n, F^n = (\mathcal{F}^n_t), P^n)$. The latter satisfies the usual conditions and the initial σ-algebra is trivial (up to P^n-null sets).

8.1.1 Large Financial Market of Continuous Trading

We consider a *$\mathcal{C}$-valued process* $G = (G_t)_{0 \leqslant t \leqslant T}$ defined by a countable sequence of adapted d-dimensional processes $\xi^k = (\xi^k_t)$ such that for every t and ω only a finite but non-zero number of $\xi^k_t(\omega)$ are different from zero and $G_t(\omega) = cone\{\xi^k_t(\omega),\, k \in \mathbb{N}\}$, i.e. $G_t(\omega)$ is a polyhedral cone generated by the finite set $\{\xi^k_t(\omega),\, k \in \mathbb{N}\}$.
We suppose that G dominates the constant process $\mathbb{R}^d_+$, all cones G_t are proper, i.e. $G_t \cap (-G_t) = \{0\}$ or, equivalently, $\operatorname{int} G^*_t \neq \emptyset$.
We assume that the generators of G are continuous processes and we add the following assumption about the generators of G^*_t:
Assumption (G):There is a countable family of continuous adapted processes (ζ^k) such that for each ω only a finite number of vectors ζ^k are different from zero and $G^*_t = cone\{\zeta^k_t : k \in \mathbb{N}\}$ for every t.

Recall the following hypothesis:

Assumption (B): Let $\xi \in L^0(\mathbb{R}^d, \mathcal{F}_t)$. If the scalar product $Z_t\xi \geqslant 0$ for all $Z \in \mathcal{M}_T^0(G^*)$, then $\xi \in L^0(G_t, \mathcal{F}_t)$.

Let $\mathcal{X} = \mathcal{X}_T^0$ be the set of all càdlàg processes X of bounded variations with $X_0 = 0$ such that $dX = \dot{X}d\|X\|$ with $\dot{X}_t \in L^0(-G_t, \mathcal{F}_t)$ for all $t \in [0,T]$ and let $\mathcal{X}^x = x + \mathcal{X}$, $x \in \mathbb{R}^d$. We denote by $\mathcal{X}_b^x = \mathcal{X}_{b,T}^x$ the subset of $\mathcal{X}^x$ formed by the processes X such that $X_t + \kappa_X \mathbf{1} \in L^0(G_t, \mathcal{F}_t)$ where $\kappa_X \geqslant 0$. Finally, we put $\mathcal{X}^x(t) = \{X_t : X \in \mathcal{X}_t^x\}$ and $\mathcal{X}_b^x(t) = \{X_t : X \in \mathcal{X}_{b,t}^x\}$.

For the following, $\mathcal{X}_b^x$ is interpreted as the set of incomes, i.e. terminal values at date T^n of portfolios starting from x. In order to make definitions uniform between the two models of this chapter, we note $\widehat{R}_T = \mathcal{X}_b^0$.

We recall the main results for our needs:

Lemma 8.1.1. *If $Z \in \mathcal{M}_0^T(G^*)$ and $X \in \mathcal{X}_b^x$, then ZX is a supermartingale and*

$$E(-Z\dot{X}).\|X\|_T \leqslant Z_0x - EZ_TX_T.$$

Let L_b^0 be the cone in $L^0(\mathbb{R}^d)$ formed by random variables ξ verifying $\xi + k\mathbf{1} \geqslant 0$ a.s. for some $k \geqslant 0$. We are given a non-null random variable $F \in L_b^0$ considered as a contingent claim. We define the convex set

$$\Gamma_F = \left\{x \in \mathbb{R}^d : F \in \mathcal{X}_b^x(T)\right\}$$

and the closed convex set

$$D_F = \left\{x \in \mathbb{R}^d : Z_0x \geqslant EZ_TF \quad \forall Z \in \mathcal{M}_0^T(G^*)\right\}.$$

We denote by $\mathcal{D} = \mathcal{D}(G)$ the subset of $\mathcal{M}_0^T(\operatorname{int} G^*)$ formed by martingales Z such that not only $Z_\tau \in L^0(\operatorname{int} G_\tau^*)$ for any stopping time τ but also $Z_{\tau-} \in L^0(\operatorname{int} G_{\tau-}^*)$ for all predictable times $\tau \in [0,T]$.

We recall the following version of hedging theorem:

Theorem 8.1.2. *Assume $\mathcal{D} \neq \emptyset$, $(\mathbf{G})$ and $(\mathbf{B})$ hold, then $\Gamma_F = D_F$.*

8.1.2 Large Financial Market of Discrete Trading

For each $T > 0$, we consider a sequence of $\mathcal{C}$-*valued process* $G = (G_t)_{t=0,\cdots,T}$ defined by a countable sequence of adapted d-dimensional processes $\xi^k = (\xi_t^k)$ such that for every t and ω only a finite but non-zero number of $\xi_t^k(\omega)$ are different from zero and $G_t(\omega) = cone\{\xi_t^k(\omega),\, k \in \mathbb{N}\}$, i.e. $G_t(\omega)$ is a polyhedral cone generated by the finite set $\{\xi_t^k(\omega),\, k \in \mathbb{N}\}$.

The set of incomes expressed in physical units and starting from zero is defined by:

$$\widehat{R}_T = \sum_{t=0}^{T} L^0(-G_t, \mathcal{F}_t).$$

We assume that G dominates $\mathbb{R}_+^d$, i.e., $\mathbb{R}_+^d \backslash \{0\} \subseteq \operatorname{int} G$. For the sequence, we give the main results, that we can find in [19], which require the notion of *No Robust Arbitrage* related to the existence of a martingale evolving in the interior of G^* :

Theorem 8.1.3. *Assume that G dominates $\mathbb{R}_+^d$. Then,*

$$NA^r \Leftrightarrow \mathcal{M}_0^T(\operatorname{ri} G^*) \neq \emptyset.$$

For a fix d-dimensional random variable F considered as a contingent claim expressed in physical units, we define the set

$$\Gamma_F = \left\{ x \in \mathbb{R}^d_+ : F \in x + \widehat{R}_T \right\}.$$

Let $\mathcal{Z}$ be the set of martingales from $\mathcal{M}_0^T(\operatorname{ri} G^*)$ such that $E(Z_T F)^- < \infty$. We recall the following lemma:

Lemma 8.1.4. *Let Z an $\mathbb{R}^d$-valued martingale and let $\Sigma_T := Z_T \sum_{s=0}^T \xi_s$ where $\xi_s \in L^0(\mathbb{R}^d, \mathcal{F}_s)$ are such that $Z_s \xi_s \leqslant 0$. If $E\Sigma_T^- < \infty$, then all products $Z_s\xi_s$ are integrable, Σ_T is integrable and $E\Sigma_T \leqslant 0$.*

We put

$$D_F = \left\{ x \in \mathbb{R}^d_+ : \sup_{Z \in \mathcal{Z}} E(Z_T F - Z_0 x) \leqslant 0 \right\}$$

and finally we give the following version of hedging theorem [1]:

Theorem 8.1.5. *Suppose that $\mathcal{M}_0^T(\operatorname{ri} G^*) \neq \emptyset$. Then $\Gamma_F = D_F$.*

8.2 Asymptotic Arbitrage

We fix a sequence T^n of positive numbers which are interpreted as time horizons.

Définition 8.1. *A sequence of incomes $\widehat{V}_{T^n}$ realizes an asymptotic arbitrage of first kind if there exists a subsequence n' and positive numbers $(x^{n'})_{n' \in \mathbb{N}}$ such that:*

8.1.*a*) $\widehat{V}_{T^{n'}} \in x^{n'} + \widehat{R}_{T^{n'}}$,

8.1.*b*) $\widehat{V}_{T^{n'}} \geqslant 0$,

8.1.*c*) $\max_{i \leqslant d(n')} x_i^{n'} \to 0$,

8.1.*d*) $\lim_{n \to \infty} P^{n'} \left(\widehat{V}_{T^{n'}} \geqslant \mathbf{1} \right) > 0$.

Note that x^n is an initial endowment for $\widehat{V}_{T^n}$ and this latter is not necessary unique. We add an assumption verified for each market belonging to either of large financial market defined above:

Assumption (H): For the large financial market of continuous trading, we suppose that for all n, the cones $(G_t^n)_{t \leqslant T^n}$ verify the hypotheses (B) and $\mathcal{D}(G^n) \neq \emptyset$.

For the large financial market of discrete trading, we suppose that for all n, the cones $(G_t^n)_{t \leqslant T^n}$ verify the (NA^r)-property, i.e. $\mathcal{M}_0^{T^n}(\operatorname{ri} G^{n*}) \neq \emptyset$.

In the case of continuous trading , we define the convex set:

$$\mathcal{Q}_c^n = \left\{ Q \sim P^n : \frac{dQ}{dP^n} = Z_{T^n}\mathbf{1},\ Z \in \mathcal{M}_0^{T^n}(G^{n*} \backslash \{0\}) \text{ with } Z_0 \mathbf{1} = 1 \right\},$$

whereas for the discrete model we consider:

$$\mathcal{Q}_d^n = \left\{ Q \sim P^n : \frac{dQ}{dP^n} = Z_{T^n}\mathbf{1},\ Z \in \mathcal{M}_0^{T^n}(\operatorname{ri} G^{n*}) \text{ with } Z_0 \mathbf{1} = 1 \right\}.$$

For the sequence, $\mathcal{Q}^n$ designates $\mathcal{Q}^n_c$ or $\mathcal{Q}^n_d$ according to the foreseen model. We define the upper and lower envelopes of the measures of $\mathcal{Q}^n$ as follows:

$$\overline{\mathcal{Q}}^n(A) = \sup_{Q \in \mathcal{Q}^n} Q(A), \qquad \underline{\mathcal{Q}}^n(A) = \inf_{Q \in \mathcal{Q}^n} Q(A)$$

and we recall the definition of contiguity:

Définition 8.2. *The sequence* (P^n) *is contiguous with respect to* $(\overline{\mathcal{Q}}^n)$ *and we note* $(P^n) \lhd (\overline{\mathcal{Q}}^n)$ *when the implication*

$$\lim_{n\to\infty} \overline{\mathcal{Q}}^n(A^n) = 0 \Rightarrow \lim_{n\to\infty} P^n(A^n) = 0$$

holds for any sequence $A^n \in \mathcal{F}^n = \mathcal{F}^n_{T^n}$, $n \geqslant 1$.

Now, we give the first result of this section:

Proposition 8.2.1. *Assume that* (H) *holds. Then, for the two models described above, the following conditions are equivalent:*

(a) *There is no asymptotic arbitrage of first kind (NAA1).*

(b) $P^n \lhd (\overline{\mathcal{Q}}^n)$.

(c) *There exists a sequence* $R^n \in \mathcal{Q}^n$ *such that* $(P^n) \lhd (R^n)$.

Définition 8.3. *A sequence of incomes* $\widehat{V}_{T_n}$ *realizes an asymptotic arbitrage of second kind if there exists a subsequence such that:*

8.3.$a)$ $\widehat{V}_{T_n} \leqslant \mathbf{1}$,

8.3.$b)$ $\lim_{n\to\infty} P^n\left(\widehat{V}_{T_n} \not\leq \varepsilon\mathbf{1}\right) = 0$, $\forall \varepsilon \in]0,1[$,

8.3.$c)$ *for all sequence of prices* $(x^n)_{n\in\mathbb{N}} \leqslant \mathbf{1}$ *such that* $\widehat{V}_{T_n} \in x^n + \widehat{R}_{T^n}$, *we have* $\lim_{n\to\infty} \max_{i\leqslant d(n)} x^n_i > 0$.

To formulate the next result, we give the following definition:

Définition 8.4. *The sequence* $(\overline{\mathcal{Q}}^n)$ *is said to be weakly contiguous with respect to* (P^n) *and we note* $(\overline{\mathcal{Q}}^n) \lhd_w (P^n)$ *if for any subsequence* n' *and any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $n'_0 \in \mathbb{N}$, *there is* $n' \geqslant n'_0$ *verifying* $\overline{\mathcal{Q}}^{n'}(A^{n'}) \leqslant \varepsilon$ *for any sequence* $A^{n'} \in \mathcal{F}^n$ *with the property* $\limsup_{n'} P^{n'}(A^{n'}) < \delta$.

Proposition 8.2.2. *Assume that* (H) *holds. Then, for the two models described above, the following conditions are equivalent:*

(a) *There is no asymptotic arbitrage of second kind (NAA2).*

(b) $(\overline{\mathcal{Q}}^n) \lhd (P^n)$.

(c) $(\overline{\mathcal{Q}}^n) \lhd_w (P^n)$.

(d) $\lim_{K\to\infty} \lim_n \inf \sup_{Q\in\mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right) = 0$.

Définition 8.5. *A sequence of incomes* $\widehat{V}_{T_n}$ *realizes a strong asymptotic arbitrage of first kind if there exists a subsequence of positive numbers* $(x^n)_{n\in\mathbb{N}}$ *such that:*

8.5.$a)$ $\widehat{V}_{T_n} \in x^n + \widehat{R}_{T^n}$,

8.5.$b)$ $\widehat{V}_{T_n} \geqslant 0$,

8.5.$c)$ $\max_{i\leqslant d(n)} x^n_i \to 0$,

8.5.$d)$ $\lim_n P^n\left(\widehat{V}_{T_n} \geqslant \mathbf{1}\right) = 1$.

Définition 8.6. *A sequence of incomes $\widehat{V}_{T_n}$ realizes a strong asymptotic arbitrage of second kind (SAA2) if there exists a subsequence such that:*

8.6.*a*) $\widehat{V}_{T_n} \leqslant \mathbf{1}$,

8.6.*b*) $\lim P^n\left(\widehat{V}_{T_n} \nleq \varepsilon \mathbf{1}\right) = 0$, $\forall \varepsilon \in]0,1[$,

8.6.*c*) *for all sequence of prices* $(x^n)_{n\in\mathbb{N}} \leqslant \mathbf{1}$ *verifying* $\widehat{V}_{T_n} \in x^n + \widehat{R}_{T^n}$, *we have* $\lim_{n\to\infty} \max_{i\leqslant d(n)} x_i^n = 1$.

Lemma 8.2.3. *Assume that (H) holds. If there exists a strong asymptotic arbitrage of first kind, then there is a strong asymptotic arbitrage of second kind.*

Proposition 8.2.4. *Assume that (H) holds. Then, for the two models described above, the following conditions are equivalent:*

(a) *There is a strong asymptotic arbitrage of first kind (SAA1).*

(b) $(P^n)\triangle(\overline{\mathcal{Q}}^n)$.

(c) $(\underline{\mathcal{Q}}^n)\triangle(P^n)$.

Proposition 8.2.5. *Assume that (H) holds. Then, for the two models described above, the following conditions are equivalent:*

(a) *There is a strong asymptotic arbitrage of second kind.*

(b) $(\overline{\mathcal{Q}}^n)\triangle(P^n)$.

8.3 Proofs

8.3.1 Proof of Proposition 8.2.1

We first assume that the model is of continuous trading.

For the equivalence $(b) \Leftrightarrow (c)$, it suffices to consult [17].

Assume (a) and let prove (b). Suppose that there exists a sequence $(A_n) \in \mathcal{F}^n$ such that $\overline{\mathcal{Q}}^n(A_n) \to 0$ and $P^n(A_n) \to \alpha > 0$. We consider $F^n = \mathbf{1}I_{A_n}$ as a contingent claim and $x_n = \overline{\mathcal{Q}}^n(A_n)\mathbf{1}$. For any $Z \in \mathcal{M}_0^{T^n}(G^{n*}\backslash\{0\})$ with $Z_0\mathbf{1} = 1$, we have obviously

$$Z_0 x_n \geqslant E Z_{T^n} F^n$$

which also holds for any $Z \in \mathcal{M}_0^{T^n}(G^{n*}\backslash\{0\})$. In virtue of Theorem 8.1.2, it follows that $F^n \in x^n + \widehat{R}_{T^n}$ and realizes an asymptotic arbitrage of first kind.

Assume (b) and let prove (a). We suppose that there exists an asymptotic arbitrage $\widehat{V}^n$ of first kind and we consider a sequence $Q^n \in \mathcal{Q}^n$ such that $dQ^n = Z_{T^n}\mathbf{1}dP^n$. Then, applying Lemma 8.1.1, we deduce that $0 \leqslant E Z_{T^n}\widehat{V}_{T^n}^n \leqslant Z_0^n x^n \leqslant \max_i x_i^n$. Moreover,

$$E Z_{T^n}\widehat{V}_{T^n}^n \geqslant E Z_{T^n}\widehat{V}_{T^n}^n I_{\widehat{V}_{T^n}^n \geqslant \mathbf{1}} \geqslant E Z_{T^n}\mathbf{1} I_{\widehat{V}_{T^n}^n \geqslant \mathbf{1}} \geqslant Q^n(\widehat{V}_{T^n}^n \geqslant \mathbf{1}).$$

It follows that $\overline{\mathcal{Q}}^n(\widehat{V}_{T^n}^n \geqslant \mathbf{1}) \leqslant \max_i x_i^n$ and $\overline{\mathcal{Q}}^n(\widehat{V}_{T^n}^n \geqslant \mathbf{1}) \to 0$ which implies that $P^n(\widehat{V}_{T^n}^n \geqslant \mathbf{1}) \to 0$ in contradiction with 8.1.d).

In the case of discrete trading, the proof is the same. Indeed, if $F \geqslant 0$, it is obvious that $\mathcal{Z}(F) = \mathcal{M}_0^T(\mathrm{ri}\, G^*)$. Moreover, if $\widehat{V}^n = x^n + \widehat{V}^0_{T^n} \geqslant 0$ with $\widehat{V}^0_{T^n} \in \widehat{R}_{T^n}$, we have $EZ_{T^n}\widehat{V}^0_{T^n} \leqslant 0$ for any $Z \in \mathcal{M}_0^T(\mathrm{ri}\, G^*)$. Indeed, we can write $\widehat{V}^0_{T^n} = \sum_{t=0}^{T^n} \gamma_t$ where a.s. $\gamma_t \in -G_t$. It follows that $Z_s\gamma_s \leqslant 0$ and from $Z_{T^n}\widehat{V}^0_{T^n} \geqslant -Z_{T^n}x^n$, we deduce that $E(Z_{T^n}\widehat{V}^0_{T^n})^- < \infty$. From now on, it suffices to apply Lemma 8.1.4 to conclude.

8.3.2 Proof of Proposition 8.2.2

We first assume that the model is of continuous trading.

Prove that $(a) \Rightarrow (b)$. Suppose that there exists a sequence $A_n \in \mathcal{F}^n$ such that $P^n(A_n) \to 0$ and $\overline{\mathcal{Q}}^n(A_n) \to \alpha > 0$. We define the contingent claim $F^n = \mathbf{1}I_{A_n}$ and we consider a price $x^n \leqslant \mathbf{1}$ for F^n. For any $Z \in \mathcal{M}_0^{T^n}(G^{n*}\backslash\{0\})$ with $Z_0\mathbf{1} = 1$, we deduce from Theorem 8.1.2 the inequality $Z_0x^n \geqslant EZ_{T^n}\mathbf{1}I_{A_n}$ which implies that $\max_i x_i^n \geqslant \overline{\mathcal{Q}}^n(A_n)$. It follows that there exists a subsequence such that $\max_i x_i^n \to a \in]0,1]$ and F^n is an asymptotic arbitrage of second kind.

Prove that $(b) \Rightarrow (a)$. Suppose that there exists an asymptotic arbitrage $\widehat{V}_{T^n}$ of second kind. Then, for any $Z \in \mathcal{M}_0^{T^n}(G^{n*}\backslash\{0\})$ with $Z_0\mathbf{1} = 1$, and $\varepsilon > 0$, we have obviously for $dQ^n = Z_{T^n}dP^n$,

$$Z_0\left(\overline{\mathcal{Q}}^n(\widehat{V}_{T^n} \leqslant \varepsilon\mathbf{1})\mathbf{1}\right) \geqslant Q^n(\widehat{V}_{T^n} \leqslant \varepsilon\mathbf{1}).$$

It follows that for any $Z \in \mathcal{M}_0^{T^n}(G^{n*}\backslash\{0\})$,

$$Z_0\left(\overline{\mathcal{Q}}^n(\widehat{V}_{T^n} \leqslant \varepsilon\mathbf{1})\mathbf{1}\right) \geqslant E_{P^n}Z_{T^n}\mathbf{1}I_{\widehat{V}_{T^n} \leqslant \varepsilon\mathbf{1}}.$$

In a similar way, we have

$$Z_0\left(\overline{\mathcal{Q}}^n(\widehat{V}_{T^n} \nleqslant \varepsilon\mathbf{1})\mathbf{1}\right) \geqslant E_{P^n}Z_{T^n}\mathbf{1}I_{\widehat{V}_{T^n} \nleqslant \varepsilon\mathbf{1}}.$$

So, we deduce that

$$y^n = \varepsilon\overline{\mathcal{Q}}^n(\widehat{V}_{T^n} \leqslant \varepsilon\mathbf{1})\mathbf{1} + \overline{\mathcal{Q}}^n(\widehat{V}_{T^n} \nleqslant \varepsilon\mathbf{1})\mathbf{1}$$

verifies

$$\begin{aligned}
Z_0y^n &\geqslant E_{P^n}Z_{T^n}\varepsilon\mathbf{1}I_{\widehat{V}_{T^n} \leqslant \varepsilon\mathbf{1}} + E_{P^n}Z_{T^n}\mathbf{1}I_{\widehat{V}_{T^n} \nleqslant \varepsilon\mathbf{1}},\\
Z_0y^n &\geqslant EZ_{T^n}\widehat{V}_{T^n}I_{\widehat{V}_{T^n} \leqslant \varepsilon\mathbf{1}} + EZ_{T^n}\widehat{V}_{T^n}I_{\widehat{V}_{T^n} \nleqslant \varepsilon\mathbf{1}} \geqslant EZ_{T^n}\widehat{V}_{T^n}.
\end{aligned}$$

We deduce that there exists a price $y^n(\varepsilon) \in \Gamma_{\widehat{V}_{T^n}}$ for any arbitrary ε. Let $\varepsilon^n \searrow 0$ be a sequence of $]0,1[$. By hypothesis,

$$P^m(\widehat{V}_{T^m} \nleqslant \varepsilon^n\mathbf{1}) \to 0,$$

as $m \to \infty$. So, we deduce k_n such that

$$P^{k_n}(\widehat{V}_{T^{k_n}} \nleqslant \varepsilon^n\mathbf{1}) \leqslant \frac{1}{n}.$$

But, from what precedes, there exists a price y^{k_n} verifying

$$\overline{\mathcal{Q}}^{k_n}(\widehat{V}_{T^{k_n}} \not\leqslant \varepsilon^n \mathbf{1}) + \varepsilon^n \geqslant \max_i y_i^{k_n}.$$

It follows that

$$\overline{\mathcal{Q}}^{k_n}(\widehat{V}_{T^{k_n}} \not\leqslant \varepsilon^n \mathbf{1}) \to \alpha > 0$$

whereas

$$P^{k_n}(\widehat{V}_{T^{k_n}} \not\leqslant \varepsilon^n \mathbf{1}) \to 0.$$

We deduce that $(\overline{\mathcal{Q}}^n) \lhd (P^n)$ fails.

In the case of discrete trading, the proof is the same because of 8.1.5. The end of the proof is deduced from Theorem 8.4.1.

8.3.3 Proof of Lemma 8.2.3

From a strong asymptotic arbitrage $\widehat{V}_{T^n}$ of first kind, we deduce a strong asymptotic arbitrage of second kind. Indeed, it suffices to consider $\widehat{V}^{(2)}_{T^n} = \mathbf{1} - \widehat{V}_{T^n}$. It is clear that $\widehat{V}^{(2)}_{T^n} \leqslant \mathbf{1}$ and

$$P^n\left(\widehat{V}^{(2)}_{T^n} \not\leqslant \varepsilon \mathbf{1}\right) = 1 - P^n\left(\widehat{V}^{(2)}_{T^n} \leqslant \varepsilon \mathbf{1}\right) \leqslant 1 - P\left(\widehat{V}_{T^n} \geqslant \mathbf{1}\right) \to 0.$$

Finally, if $y^n \leqslant \mathbf{1}$ is a price for $\widehat{V}^{(2)}_{T^n}$, then there exists $\mathcal{X}^{(2)}_{T^n} \in \widehat{R}_{T^n}$ such that $y^n + \mathcal{X}^{(2)}_{T^n} = \mathbf{1} - \widehat{V}_{T^n}$ where $\widehat{V}_{T^n} = x^n + \mathcal{X}_{T^n}$, $\mathcal{X}_{T^n} \in \widehat{R}_{T^n}$. We deduce that $y^n + x^n + \mathcal{X}^{(2)}_{T^n} + \mathcal{X}_{T^n} = \mathbf{1}$ and applying Lemmas 8.1.1 or 8.1.4, we obtain that $\max_i y_i^n \leqslant 1 \leqslant \max_i y_i^n + \max_i x_i^n$. It follows that $\max_i y_i^n \to 1$.

8.3.4 Proof of Proposition 8.2.4

First, we assume that there exists a strong asymptotic arbitrage $\widehat{V}_{T^n}$ of first kind. Using the implication $(b) \Rightarrow (a)$ in the proof 8.3.1, we deduce that $P^n(A_n) \to 1$ and $\overline{\mathcal{Q}}^n(A_n) \to 0$ where $A_n = \{\widehat{V}_{T^n} \geqslant \mathbf{1}\}$. So, $(P^n) \triangle (\overline{\mathcal{Q}}^n)$.

Reciprocally, if we suppose that $(P^n) \triangle (\overline{\mathcal{Q}}^n)$, then it suffices to use the implication $(a) \Rightarrow (b)$ with $\alpha = 1$ in the proof 8.3.1 in order to obtain a strong asymptotic arbitrage of first kind.

Note that we have obviously $(b) \Leftrightarrow (c)$.

8.3.5 Proof of Proposition 8.2.5

We first assume that there exists a strong asymptotic arbitrage $\widehat{V}_{T^n}$ of second kind. From the implication $(b) \Rightarrow (a)$ in the proof 8.3.2, with $\alpha = 1$, we deduce that $(\overline{\mathcal{Q}}^n) \triangle (P^n)$.

Reciprocally, if $(\overline{\mathcal{Q}}^n) \triangle (P^n)$, it suffices to use the implication $(a) \Rightarrow (b)$ in the proof 8.3.2 with $\alpha = 1$ in order to obtain a strong asymptotic arbitrage of second kind.

8.4 Appendix

We shall prove the following result:

Theorem 8.4.1. *The following conditions are equivalent:*

(*a*) $(\overline{\mathcal{Q}}^n) \lhd (P^n)$.

(*b*) $(\overline{\mathcal{Q}}^n) \lhd_w (P^n)$.

(*c*) $\lim_{\alpha \searrow 0} \lim_n \sup \inf_{Q\in\mathcal{Q}^n} H(\alpha, P^n, Q) = 1$.

(*d*) $\lim_{K\to\infty} \lim_n \inf \sup_{Q\in\mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right) = 0$.

Proof. We first prove that $(\overline{\mathcal{Q}}^n) \lhd (P^n)$ holds if and only if for any subsequence n',

$$\lim_{\delta\searrow 0} \lim_{n'} \sup \sup_{g\in B^{n',\delta}} \sup_{Q\in\mathcal{Q}^{n'}} E_Q g = 0$$

where

$$B^{n,\delta} = \{g : E_{P^n} g \leqslant \delta,\ 0 \leqslant g \leqslant 1\}$$

is a closed convex in $\sigma\left(L^\infty(P^n), L^1(P^n)\right)$.

For this, we assume that $(\overline{\mathcal{Q}}^n) \lhd (P^n)$ holds and we suppose that there exists a strictly positive constant c such that

$$\lim_{\delta\searrow 0} \lim_{n'} \sup \sup_{g\in B^{n',\delta}} \sup_{Q\in\mathcal{Q}^{n'}} E_Q g > c.$$

We deduce a subsequence $(n_k)_k$, $g_k \in B^{n_k,\delta_k}$ and $Q^k \in \mathcal{Q}^{n_k}$ such that $E_{Q^k} g_k \geqslant c$ and $\delta_k \leqslant 1/k$. But, on an other hand, $g_k \in B^{n_k,\delta_k}$ implies that

$$P^{n_k}\left(g_k \geqslant \frac{1}{\sqrt{k}}\right) \leqslant \frac{1}{\sqrt{k}}.$$

Then, we deduce that $\lim_k P^{n_k}(A^{n_k}) = 0$ where $A^{n_k} = \left\{g_k \geqslant 1/\sqrt{k}\right\}$. Otherwise, $\lim_k Q^k(A^{n_k}) > 0$. Indeed, from $E_{Q^k} g_k \geqslant c$, we deduce that

$$\frac{1}{\sqrt{k}} + Q^k\left(g_k \geqslant \frac{1}{\sqrt{k}}\right) \geqslant c.$$

Then, from what precedes, we have a contradiction.

From now on, we assume that the second assertion of our claim holds and we suppose that $(\overline{\mathcal{Q}}^n) \lhd (P^n)$ fails. Then, we deduce a subsequence $(n_k)_k$, a sequence $(A^{n_k})_k$ and $\delta, \delta_0 > 0$ such that

$$\overline{\mathcal{Q}}^{n_k}(A^{n_k}) \geqslant const > 0, \quad P^{n_k}(A^{n_k}) < \delta < \delta_0$$

where δ_0 is chosen such that

$$\lim_{n_k} \sup \sup_{g\in B^{n_k,\delta}} \sup_{Q\in\mathcal{Q}^{n_k}} E_Q g \leqslant \frac{const}{2}$$

provided that $\delta < \delta_0$. Since $g^k = I_{A^{n_k}} \in B^{n_k,\delta}$, we deduce that

$$\sup_{g\in B^{n_k,\delta}} \sup_{Q\in\mathcal{Q}^{n_k}} E_Q g \geqslant \sup_{Q\in\mathcal{Q}^{n_k}} Q(A^{n_k}) = \overline{\mathcal{Q}}^{n_k}(A^{n_k}) \geqslant const$$

and

$$\lim_{n_k} \sup \sup_{g \in B^{n_k,\delta}} \sup_{Q \in \mathcal{Q}^{n_k}} E_Q g \geqslant const$$

which leads to a contradiction.

From what precedes, it suffices to prove that $(\overline{\mathcal{Q}}^n) \lhd_w (P^n)$ holds if and only if for any subsequence n',

$$\lim_{\delta \searrow 0} \lim_{n'} \sup \sup_{g \in B^{n',\delta}} \sup_{Q \in \mathcal{Q}^{n'}} E_Q g = 0.$$

Assume that the second assertion holds and the property (b) fails. Then, there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}^*$, there is a sequence $(A_k^n)_n$ verifying $\limsup_n P^n(A_k^n) < 1/2k$ and $\overline{\mathcal{Q}}^n(A_k^n) \geqslant \varepsilon$ provided that $n \geqslant n_0(k)$. We can choose $n_0(k)$ such that $P^n(A_k^n) < 1/k$ for $n \geqslant n_0(k)$. We deduce that $I_{A_k^n} \in B^{n,1/k}$ and for $n \geqslant n_0(k)$,

$$\sup_{Q \in \mathcal{Q}^n} \sup_{g \in B^{n,1/k}} E_Q\, g \geqslant \sup_{Q \in \mathcal{Q}^n} E_Q I_{A_k^n} \geqslant \varepsilon.$$

Then,

$$\lim_n \sup \sup_{Q \in \mathcal{Q}^n} \sup_{g \in B^{n,1/k}} E_Q\, g \geqslant \varepsilon$$

and

$$\lim_{\delta \searrow 0} \lim_n \sup \sup_{Q \in \mathcal{Q}^n} \sup_{g \in B^{n,\delta}} E_Q\, g \geqslant \varepsilon$$

which leads to a contradiction.

From now on, we suppose that the condition $(\overline{\mathcal{Q}}^n) \lhd_w (P^n)$ holds and there exists a constant $c > 0$ such that

$$\lim_{\delta \searrow 0} \lim_{n'} \sup \sup_{g \in B^{n',\delta}} \sup_{Q \in \mathcal{Q}^{n'}} E_Q g > c.$$

It follows that there exists a subsequence $(n_k)_k$, $\delta_k \leqslant 1/k$, $Q^k \in \mathcal{Q}^{n_k}$ and $g^k \in B^{n_k,\delta_k}$ verifying $E_{Q^k} g^k \geqslant c$ where $c > 0$. Moreover, from hypothesis, there exists $\delta > 0$ such that for any sequence $(A^{n_k})_k$ which verifies $\lim_k \sup P(A^{n_k}) < \delta$, we have for any $k_0 \in \mathbb{N}$ the existence of $k \geqslant k_0$ verifying $\overline{\mathcal{Q}}^{n_k}(A^{n_k}) < c/2$. But, $g^k \in B^{n_k,\delta_k}$ implies that $P^{n_k}(A^{n_k}) \leqslant 1/\sqrt{k}$ where $A^{n_k} = \{g^k \geqslant 1/\sqrt{k}\}$ verifies $\lim_k \sup P^{n_k}(A^{n_k}) = 0$. From hypothesis, we can deduce a subsequence $(n'_k)_k$ such that $\overline{\mathcal{Q}}^{n'_k}(A^{n'_k}) < c/2$ whereas $E_{Q^k} g^k \geqslant c$ implies that

$$\frac{1}{\sqrt{k}} + Q^{n_k}(A^{n_k}) \geqslant c.$$

Then, as $k \to \infty$, we obtain a contradiction and we can conclude that $(a) \Leftrightarrow (b)$.

We shall prove that $(b) \Rightarrow (d)$. For this, we consider $\varepsilon > 0$. We have

$$P^n\left(\frac{dQ}{dP^n} \geqslant K\right) \leqslant \frac{1}{K}$$

because of the Bienaymé–Tchebychev inequality. Then,

$$\sup_{Q \in \mathcal{Q}^n} P^n\left(\frac{dQ}{dP^n} \geqslant K\right) \to 0,$$

as $K \to \infty$, which implies, in virtue of (b), that for any $n_0 \in \mathbb{N}$, there exists $n \geqslant n_0$ such that
$$\sup_{Q \in \mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right) \leqslant \varepsilon$$
provided that K is large enough. Then, we obtain (d).

We shall prove that $(d) \Rightarrow (c)$. For this, we recall from [17] the following inequality:
$$d_H^2(\alpha, P^n, Q) \leqslant 8\alpha K + 4 \sup_{Q \in \mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right).$$
We deduce that
$$\lim_{\alpha \searrow 0} \liminf_n \sup_{Q \in \mathcal{Q}^n} d_H^2(\alpha, P^n, Q) = 0.$$
Since $d_H^2(\alpha, P^n, Q) = 1 - H(\alpha, P^n, Q)$, we obtain (c).

We shall prove that $(c) \Rightarrow (d)$. For this, we recall from [17], that for any $\alpha \in]0, 1/2]$, there exists $K \geqslant 4$ such that
$$\sup_{Q \in \mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right) \leqslant 8 \sup_{Q \in \mathcal{Q}^n} d_H^2(\alpha, P^n, Q).$$
Hence,
$$\lim_{K \to \infty} \liminf_n \sup_{Q \in \mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right) \leqslant 8 \liminf_n \sup_{Q \in \mathcal{Q}^n} d_H^2(\alpha, P^n, Q)$$
and, as $\alpha \to 0$, we obtain (d).

Finally, we prove that $(d) \Rightarrow (b)$. We write for $Q \in \mathcal{Q}^n$,
$$Q(A^n) = E_{P^n}\left(\frac{dQ}{dP^n} I_{\{\frac{dQ}{dP^n} \leqslant K\}} I_{A^n}\right) + E_{P^n}\left(\frac{dQ}{dP^n} I_{\{\frac{dQ}{dP^n} > K\}} I_{A^n}\right).$$
Then,
$$Q(A^n) \leqslant K P^n(A^n) + \sup_{Q \in \mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right)$$
and
$$\overline{\mathcal{Q}}^n(A^n) \leqslant K P^n(A^n) + \sup_{Q \in \mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right).$$
From now on, for any $\varepsilon > 0$, it suffice to fix K large enough in order to have
$$\liminf_n \sup_{Q \in \mathcal{Q}^n} Q\left(\frac{dQ}{dP^n} \geqslant K\right) \leqslant \varepsilon/4$$
and the result follows from Definition 8.4.

Chapter 9

Bibliography

[1] Bruno Bouchard, Emmanuel Temam. On the Hedging of American Options in Discrete Time Markets with Proportional Transaction Costs. Electronic Journal of Probability , (2005), 10, 746-760. 3, 120, 147

[2] Bruno Bouchard, Jean-François Chassagneux. Representation of Continuous Linear Forms on the set of Ladlag Processes and the Pricing of American Claims under Proportional Costs. Preprint. 119

[3] A.S Cherny. General Arbitrage Pricing Model: Probability Approach. Lecture Notes in Mathematics, 1899 (2007), p. 463-481. 122, 123

[4] Denis E. Approximate Hedging of Contingent Claims under Transaction Costs for General Pay-offs. Submitted to *Applied Mathematical Finance*.

[5] Denis E. Approximate Hedging of Contingent Claims under Transaction Costs for $\alpha \in]1/4, 1/2]$ when the Volatility is not Constant. Working Paper.

[6] Denis Emmanuel, De Vallière Dimitri , Kabanov Yuri. Hedging of American Options under Transaction Costs. Submitted to *Finance and Stochastics*. 3, 119, 126, 133, 135, 139, 142

[7] Denis Emmanuel, Yuri Kabanov. Functional Limit Theorem for Leland–Lott Hedging Strategy. Submitted to *Finance and Stochastics*. 69

[8] Michael A.H Dempster, Igor V.Evstigneev, Klaus R.Schenk-Hoppé. Exponential Growth of fixed-mix Strategies in Stationary Asset Markets. Finance and Stochastics. 7,263-276 (2003). 122

[9] Campi L., Schachermayer W. A super-replication Theorem in Kabanov's model of Transaction Costs. *Finance and Stochastics*, **10** (2006), 4, 579–596. 135, 140

[10] Friedman Avner. Stochastic Differential Equations and Applications. Volume 1. *Academic Press* , (1975). 79, 82, 83, 84, 104, 110, 111, 113

[11] Friedman Avner. Partial Differential Equations of Parabolic Type. *Library of Congress Catalog Card Number 64-7571* , (1964). 2, 86, 87, 89, 113, 114

[12] Gamys M., Yuri Kabanov. Mean Square Error for the Leland–Lott Hedging Strategy. Submitted to *Finance and Stochastics*.

[13] Granditz P., Schachinger W. Leland's Approach to Option Pricing: The Evolution of Discontinuity. *Mathematical Finance*, **11** (2001), 3, 347–355. 2, 8, 14, 27, 58

[14] Grannan E.R., Swindle G.H. Minimizing Transaction Costs of Hedging Strategies. *Mathematical Finance*, **6** (1996), 4, 341–364. 8, 63

[15] Hall.P, Heyde.C.C. Martingale Limit Theory and its Application. *Academic Press*. (1980). 2

[16] Jacod J., Shiryaev A.N. Limit Theorems for Stochastic Processes. Springer, Berlin–Heidelberg–New York, (1987). 2, 64, 67, 68, 69, 71, 72, 77, 78

[17] Kabanov Yuri, Kramkov.D.O. Asymptotic Arbitrage in Large Financial Markets. *Finance and Stochastics*. Springer-Verlag (1998). 3, 145, 149, 154

[18] Kabanov Yu.M., Safarian M. On Leland's strategy of Option Pricing with Transaction Costs. *Finance and Stochastics*, **1**, 3, (1997), 239-250. 1, 2, 8

[19] Kabanov Yuri, Safarian M. Markets with Transaction Costs. Mathematical Theory. Monograph. To appear. 3, 16, 19, 28, 41, 119, 120, 121, 122, 125, 126, 127, 129, 130, 131, 132, 133, 135, 137, 145, 146

[20] Karatzas.I, Shreve.S.E. Brownian Motion and Stochastic Calculus. *Springer Verlag*. 51, 70, 85

[21] Leland H. Option pricing and Replication with Transactions Costs, *Journal of Finance*, **XL** (1985), 5, 1283–1301. 1, 7, 81

[22] Lions P.L, Musiela M. Convexity of Solutions of Parabolic Equations. *ELSVIER*. C.R.Acad.Sci.Paris, Ser.I 342(2006),915-921. 114

[23] Liptser R. Sh., Shiryaev A. N. Theory of Martingales. Kluwer, Dordrecht, (1989).

[24] Lott K. Ein Verfahren zur Replikation von Optionen unter Transaktionkosten in stetiger Zeit, Dissertation. Universität der Bundeswehr München. Institut für Mathematik und Datenverarbeitung, (1993). 1, 7

[25] Claude Martini. Propagation of Convexity by Markovian and Martingalian Semi-groups. *Potential Analysis*10: 133-175, (1999). *Kluwer Academic Publishers.* 114

[26] Pergamenchtikov S. Limit Theorem for Leland's Strategy. *Annals of Applied Probability*, **13** (2003), 1099–1118. 2, 8, 63

[27] Sekine J., Yano J. Hedging Errors of Leland's Strategies with time-inhomogeneous Rebalancing. Preprint.

Résumé

Cette thèse aborde plusieurs problèmes qui se posent pour les marchés financiers soumis à des coûts de transaction.
Nous revisitons d'abord la méthode d'approximation des portefeuilles de couverture des options Européennes suggérée par Leland pour le call Européen. On met en évidence la convergence en probabilité des portefeuilles discrétisés vers le pay-off lorsque ce dernier est bien plus général. Dans le même esprit, on mesure la vitesse de convergence en estimant la moyenne de l'erreur quadratique. Cela nous conduit à formuler un théorème de convergence en loi de l'erreur d'approximation du type "central-limite". Toutefois, le modèle de Black et Scholes utilisé est critiquable dans la pratique puisque la volatilité est supposée constante. C'est pourquoi, nous proposons d'établir un théorème de convergence en probabilité analogue au précédent lorsque la volatilité ne dépend pas seulement du temps mais aussi de l'actif risqué sous-jacent.
Enfin, on s'intéresse à des marchés continus plus abstraits décrits par des cônes générés par les coûts de transactions. Nous formulons quelques notions d'arbitrage mais surtout on propose une description duale des prix de couverture des options américaines comme cela a déjà été fait pour les marchés discrétisés.

Mots-clés:
Coûts de transaction, approximations de Leland, couverture d'un portefeuille, théorème limite fonctionnel, options européennes et américaines, arbitrage.