La théorie des inégalités de martingales et applications à l'analyse harmonique

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Résumé

La théorie des martingales est une direction de recherche étroitement liée à beaucoup d'autres domaines tels que l'analyse harmonique, l'analyse stochastique et la théorie des espaces de Banach. Elle a aussi de nombreuses applications aux mathématiques finanères, à l'analyse de risques, à la théorie de l'information. Plus récemment, des chercheurs commencent à s'intéresser à d'autres espaces que les espaces L_p usuels avec $p \geq 1$. Par exemple, les L_p avec $0 < p < 1$, les espace de Hardy H_p $(0 < p < 1)$ et les espace de Lorentz $L_{p,q}$. Ces espaces ne font pas partie en général de l'étude classique mais ont des applications varées. D'autre part, par rapport aux fonctions, les martingales peuvent mieux reféter les processus, l'information et le rapprochement. C'est la raison principale pourquoi beaucoup de chercheurs portent leur attention à la théorie des martingales. A titre de comparaison, on a relativement plus de résultats sur les espaces de Lorentz $L_{p,q}$ dans l'analyse harmonique; mais ce n'est pas le cas pour les espaces $H^*_{p,q}$, $H^S_{p,q}$, et $H^s_{p,q}$ de martingales. C'est pourquoi nous recherchons à accomplir la recherche sur les espaces de Lorentz de martingales.

L'objectif de la présente thèse est d'étudier les espaces de Lorentz de martingales formés par les fonctions maximales, les fonctions carrées, les fonction carrées conditionnelles. On s'intéresse en particulier aux relations entre ces espaces, aux inégalités de martingales vérifiées par eux et à leur interpolation. Nos outils principales sont diverses décompositions atomiques, la transformation de martingales. A part des martingales commutatives, nous nous intéresser aussi aux celles non commutatives. La théorie des martingales non commutatives a connu un développement remarquable ces dernières années. Nous obtenons ici les inégalités de Burkholder-Gundy et les inégalités de Burkholder pour les martingales non commutatives dans des espaces de Lorentz.

Nous décrivons maintenant le contenu de la thèse. Dans le chapitre 1, nous donnons quelques résultats bien connus et élémentaires sur la théorie des martingales, introduisons plusieurs espaces de Lorentz de martingales, rappelons les decompositions atomiques des espaces de Hardy de martingales pondérés et les transformées de martingales. Dans le chapitre 2, nous étudions l'interpolation des espaces de Lorentz de martingales pondérés et identifions les espaces d'interpolation réels entre H_p et $H_{p,\infty}$. Dans le chapitre 3, nous étudions la bornitude des applications sous-linéaires en utilisant les decompositions atomiques des espaces de Lorentz de martingales. Nous y obtenons certaines inégalités de martingales et prouvons aussi le théorème d'interpolation de type restreint faible. Dans le chapitre 4, nous discutons de transformées de martingales à valeurs vectorielles sur les espaces de Lorentz. Avec ces transformées nous construisons des plongements des espaces de Lorentz de martingales à valeurs dans un espace de Banach. Ces plongements dependent des propriétés géométriques de l'espace de Banach en question telles que la convexité (ou lissité) uniforme, la propriété de Radon-Nikodym, etc. Dans le chapitre 5 nous obtenons des relation entre les mesures de Carleson et la norme BMO de martingales vectorielles, qui sont étroitement liées aux propriétés géométriques de l'espace de Banach sous-jacent. Dans le chapitre 6, nous démontrons les inégalités de Burkholder-Gundy et les inégalités de Burkholder, qui étendent certaines résultats de Junge et Xu.

Mots clés: Espaces de Lorentz de martingales; décomposition atomique; interpolation; inégalités pondérées; mesures de Carleson; BMO; martingales noncommutatives; inégalité de Burkholder; propriétés géométriques des espaces de Banach.

Contents

Chapter 1 Preliminaries

§ 1.1 Notations and classical results

Let $\{\Sigma_n\}_{n\geq 0}$ be a nondecreasing sequence of sub-σ-fields of Σ such that $\Sigma = \bigvee \Sigma_n$. We denote the expectation operator and the conditional expectation operator relative to Σ_n by E and E_n, respectively. For a martingale $f = (f_n)_{n\geq 0}$, we define $\Delta_n f = f_n - f_{n-1}, n \geq 0$ (with the convention that $f_{-1} = 0, \Sigma_{-1} = \{\Omega, \Phi\}$) and adopt the notions of its maximal function, quadratic function and conditional quadratic function as follows, respectively:

$$M_n(f) = \sup_{0\leq i\leq n} |f_i|, \quad M(f) = \sup_{n\geq 0} |f_n|,$$

$$S_n(f) = (\sum_{i=0}^{n} |\Delta_i f|^2)^{1/2}, \quad S(f) = (\sum_{n=0}^{\infty} |\Delta_n f|^2)^{1/2},$$

$$s_n(f) = (\sum_{i=0}^{n} E_{i-1}|\Delta_i f|^2)^{1/2}, \quad s(f) = (\sum_{n=0}^{\infty} E_{n-1}|\Delta_n f|^2)^{1/2}.$$

Denote by Λ the set of all non-decreasing, non-negative and adapted r.v. sequences $\rho = (\rho_n)_{n\geq 0}$ with $\rho_\infty = \lim_{n\to\infty} \rho_n$. We shall say a martingale $f = (f_n)_{n\geq 0}$ has predictable control in L_p if there is a sequence $\rho = (\rho_n)_{n\geq 0} \in \Lambda$ such that

$$|f_n| \leq \rho_{n-1}, \quad \rho_\infty \in L_p.$$

As usual, we define the following martingale spaces (see [35] and [59])

$$L_p = \{f = (f_n)_{n\geq 0} : \|f\|_p = \sup_n \|f_n\|_p < \infty\}$$

$$H_p^* = \{f = (f_n)_{n\geq 0} : \|f\|_{H_p} = \|M(f)\|_p < \infty\},$$

$$H_p^s = \{f = (f_n)_{n\geq 0} : \|f\|_{H_p^s} = \|s(f)\|_p < \infty\},$$

$$H_p^S = \{f = (f_n)_{n\geq 0} : \|f\|_{H_p^S} = \|S(f)\|_p < \infty\},$$

$$Q_p = \{f = (f_n)_{n\geq 0} : \exists (\rho_n)_{n\geq 0} \in \Lambda, s.t. S_n(f) \leq \rho_{n-1}, \rho_\infty \in L_p\},$$

$$\|f\|_{Q_p} = \inf_\rho \|\rho_\infty\|_p$$

$$D_p = \{f = (f_n)_{n\geq 0} : \exists (\rho_n)_{n\geq 0} \in \Lambda, s.t. |f_n| \leq \rho_{n-1}, \rho_\infty \in L_p\},$$

$$\|f\|_{D_p} = \inf_{\rho} \|\rho_\infty\|_p.$$

Remark The norms of Q_p and D_p are attainable respectively. For example, there exists $(\rho_n)_{n\geq 0} \in \Lambda, S_n(f) \leq \rho_{n-1}, \rho_\infty \in L_p$ such that $\|f\|_{Q_p} = \|\rho_\infty\|_p$, which is also called the optimal control.

Theorem 1.1.1(Burkholder-Gundy-Davis) For $1 \leq p < \infty$, we have

$$\|Mf\|_p \approx \|S(f)\|_p.$$

The Burkholder-Gundy-Davis inequality shows if $1 \leq p < \infty$ then $H_p^* = H_p^S$ with equivalent norm. Moreover, if $1 < p < \infty$, it is well known that $H_p^S = H_p^* = L_p$ with equivalent norms. $\{\Sigma_n\}_{n\geq 0}$ is called regular if there exists $R > 0$ such that

$$|f_n| \leq R|f_{n-1}| \quad (\forall n \in \mathbb{N}).$$

This condition is denoted by the R condition, and we refer to [59] for more details.

Theorem 1.1.2 If R condition holds, then for all $0 < p < \infty$, we have

$$\|Mf\|_p \approx \|S(f)\|_p \approx \|s(f)\|_p \approx \|f\|_{Q_p} \approx \|f\|_{P_p}.$$

§ 1.2 Atomic decompositions of Hardy martingale spaces

Let (Ω, Σ, P) be complete probability space and f a measure function defined on Ω. Its distribution function is

$$\lambda_f(t) = P(x : |f(x)| > t), \quad t \geq 0,$$

and its decreasing rearrangement function f^* is defined as

$$f^*(t) = \inf\{s > 0 : \lambda_f(t) \leq t\} \quad t \geq 0.$$

The Lorentz space $L_{p,q}(\Omega) = L_{p,q}, 0 < p < \infty, 0 < q \leq \infty$, consists of those measurable functions f with finite quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = (\frac{q}{p}\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t})^{1/q}, 0 < q < \infty,$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t), q = \infty.$$

It will be convenient for us to use an equivalent definition of $\|f\|_{p,q}$, namely

$$\|f\|_{p,q} = (q\int_0^\infty [tP(|f(x)| > t)^{1/p}]^q \frac{dt}{t})^{1/q}, 0 < q < \infty,$$

$$\|f\|_{p,\infty} = \sup_{t>0} tP(|f(x)| > t)^{1/p}, \qquad q = \infty.$$

To check that these two expressions are the same, simply make the substitution $y = P(|f(x)| > t)$ and then integrate by parts.

It is well known that if $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, then $L_{p,q}$ is a Banach space, and $\|f\|_{p,q}$ is equivalent to a norm. However, for other values of p and q, $L_{p,q}$ is only a quasi-Banach spaces. In particular, if $0 < q \le 1 \le p$ or $0 < q \le p < 1$ then $\|f\|_{p,q}$ is equivalent to a q-norm. Recall also that a quasi-norm $\|\cdot\|$ in X is equivalent to a p-norm, $0 < p < 1$, if there exists $c > 0$ such that for any $x_i \in X, i = 1, ..., n$

$$\|x_1 + \cdots + x_n\|^p \le c(\|x_1\|^p + \cdots + \|x_n\|^p).$$

For all these properties, and more on Lorentz spaces, see for example [5], [19] and [2]. The Holder inequality for Lorentz spaces is the following,

$$\|fg\|_{p,q} \le c\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}$$

for all $0 < p, q, p_1, q_1, p_2, q_2 \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Let ω be a strict positive r.v. on (Ω, Σ, P) and $\omega(A) = \int_A \omega dP$ for every $A \in \Sigma$. The distribution function of f with respect to ω is defined as

$$\lambda_{f,\omega}(y) = \omega(x \in \Omega : |f(x)| > y), \quad y > 0$$

the non-increasing rearrangement function of f with respect to ω is defined as

$$f^*_\omega(t) = \inf\{y : \lambda_{f,\omega}(y) \le t\}, \quad t > 0$$

and the average function of f with respect to ω is defined as

$$f^{**}_\omega(t) = \frac{1}{t}\int_0^t f^*_\omega(y)dy, \quad t > 0.$$

The weighted Lorentz spaces $L_{p,q;\omega}$ is defined as all of the r.v. f on (Ω, Σ, P) such that $\|f\|_{p,q;\omega} < \infty$, where

$$\|f\|_{p,q;\omega} = \begin{cases} (\frac{q}{p}\int_0^\infty [t^{1/p} f^*_\omega(t)]^q \frac{dt}{t})^{1/q}, & 0 < p < \infty, 0 < q < \infty \\ \\ \sup\limits_{t>0} t^{1/p} f^*_\omega(t), & 0 < p \le \infty, q = \infty. \end{cases}$$

Remark The Lorentz space $L_{p,q}$ increases as the second exponent q increases, namely, for $0 < p < \infty$ and $0 < q_1 < q_2 \leq \infty$ one has $L_{p,q_1} \subset L_{p,q_2}$. Moreover, one has $L_{r,s} \subset L_{p,q}$ for $0 < p < r \leq \infty$ and $0 < q, s \leq \infty$.

Then for $0 < p, q \leq \infty$, we define the weighted Lorentz martingale spaces as follows:

$$H^s_{p,q;\omega} = \{f = (f_n)_{n\geq 0} : \|f\|_{H^s_{p,q;\omega}} = \|s(f)\|_{p,q;\omega} < \infty\},$$

$$Q_{p,q;\omega} = \{f = (f_n)_{n\geq 0} : \exists(\rho_n)_{n\geq 0} \in \Lambda, s.t. S_n(f) \leq \rho_{n-1}, \rho_\infty \in L_{p,q;\omega}\},$$

$$\|f\|_{Q_{p,q;\omega}} = \inf_\rho \|\rho_\infty\|_{p,q;\omega}$$

$$P_{p,q;\omega} = \{f = (f_n)_{n\geq 0} : \exists(\rho_n)_{n\geq 0} \in \Lambda, s.t. |f_n| \leq \rho_{n-1}, \rho_\infty \in L_{p,q;\omega}\},$$

$$\|f\|_{P_{p,q;\omega}} = \inf_\rho \|\rho_\infty\|_{p,q;\omega}.$$

Remark If $p = q$, the weighted Lorentz martingale spaces are respectively reduced to the weighted Hardy martingale spaces.

In general, however, $\|.\|_{p,q;\omega}$ is not a norm since the Minkowski inequality may be fail. If replacing $f^*_\omega(t)$ by $f^{**}_\omega(t)$ in the above definition of $\|f\|_{p,q;\omega}$, we obtain a new norm $\|.\|_{(p,q);\omega}$ for every $q \geq 1$:

$$\|f\|_{(p,q);\omega} = \begin{cases} (\frac{q}{p}\int_0^\infty [t^{1/p} f^{**}_\omega(t)]^q \frac{dt}{t})^{1/q}, & 0 < p < \infty, 0 < q < \infty \\ \\ \sup\limits_{t>0} t^{1/p} f^{**}_\omega(t), & 0 < p \leq \infty, q = \infty. \end{cases}$$

We state a result in [5] as a lemma.

Lemma 1.2.1 If $1 < p \leq \infty, 0 < q \leq \infty$, then for a measurable function f,

$$\|f\|_{p,q;\omega} \leq \|f\|_{(p,q);\omega} \leq \frac{p}{p-1}\|f\|_{p,q;\omega}.$$

Now we consider atomic decompositions of Hardy martingale spaces. We refer to [97], [101] and [39] for some definitions of atoms.

Definition 1.2.2 A measurable function α is called $(1, p, \infty)$ atom, if there exists a stopping time τ such that:

(i) $a_n = E_n a = 0$, $n \leq \tau$,

(ii) $\|s(a)\|_\infty \leq P\{\tau < \infty\}^{-\frac{1}{p}}$.

Replacing (ii) by $(ii)'$ $\|S(a)\|_\infty \leq P\{\tau < \infty\}^{-\frac{1}{p}}$ or $(ii)''$ $\|M(a)\|_\infty \leq P\{\tau < \infty\}^{-\frac{1}{p}}$, we get the concept of $(2, p, \infty)$ atom or $(3, p, \infty)$ atom.

Definition 1.2.3 A measurable function a is called w-1-atom (or, w-2-atomic, w-3-atomic), if there exists a stopping time τ such that

(i) $a_n = E_n a = 0, \quad \forall n \leq \tau,$

(ii) $\|s(a)\|_\infty < \infty$ (or $(ii)\|S(a)\|_\infty < \infty, \quad (ii)\|M(a)\|_\infty < \infty$).

Lemma 1.2.4 Let $0 < p < \infty$. Then $f = (f_n)_{n\geq 0} \in wH_p^s$(or wQ_p, wD_p) if and only if there exists a sequence of w-1-atom (or, w-2-atomic, w-3-atomic) $(a^k)_{k\in Z}$ and the corresponding stopping time sequence $(\tau_k)_{k\in Z}$ such that

(i) $f_n = \sum_{k\in Z} E_n a^k, \quad \forall n \in N$

(ii) For some constant $A \geq 0$, $s(a^k) \leq A2^k$(or $S(a^k) \leq A2^k, M(a^k) \leq A2^k$),

and

$$\sup_{k\in Z} 2^{kp} P(\tau_k < \infty) < \infty.$$

Definition 1.2.5 A measurable function α is called a $(1, p, \infty)$ atom with respect to w, if there exists a stopping time τ such that

(i) $a_n = E_n a = 0$, $n \leq \tau$,

(ii) $\|s(a)\|_\infty \leq w\{\tau < \infty\}^{-\frac{1}{p}}$.

Replacing (ii) by $(ii)'$ $\|S(a)\|_\infty \leq w\{\tau < \infty\}^{-\frac{1}{p}}$ or $(ii)''$ $\|M(a)\|_\infty \leq w\{\tau < \infty\}^{-\frac{1}{p}}$, we get the concept of $(2, p, \infty)$ atom with respect to w or $(3, p, \infty)$ atom with respect to w.

Many atomic decomposition theorems of Hardy martingale spaces can be transformed to be of weighted Hardy martingale spaces. In the following we state some of them as Lemmas, their proofs are similar to those in [97] and [54], so here we only prove one of them and omit others.

Lemma 1.2.6 If $f = (f_n)_{n\geq 0}$ is in $H^s_{p;\omega}$, $0 < p < \infty$, then there exist a sequence $(a^k)_{k\in Z}$ of $(1, p, \infty)$-atoms with respect to ω and a sequence $\mu = (\mu_k)_{k\in Z} \in l_p$ of real numbers such that for every $n \in N$

$$\sum_{k\in Z} \mu_k E_n a^k = f_n, \qquad (1)$$

and

$$(\sum_{k\in Z} |\mu_k|^p)^{1/p} \leq C_p \|f\|_{H^s_{p;\omega}}, \quad (2)$$

where (1) is convergent in $H^s_{p;\omega}$.

Proof Assume $f \in H^s_{p;\omega}$. Considering the following stopping times

$$\tau_k = \inf\{n \in N : s_{n+1}(f) > 2^k\}, \qquad k \in Z.$$

It is obvious that the sequence of these stopping times is non-decreasing and easy to see

$$f_n = \sum_{k\in Z}(f_n^{\tau_{k+1}} - f_n^{\tau_k}).$$

Let $\mu_k = 2^k 3\omega(\tau_k \neq \infty)^{1/p}$ and

$$a_n^k = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

It is clear that, for a fixed k, (a_n^k) is a martingale. Since $s(f_n^{\tau_k}) \leq 2^k$ and $s(f_n^{\tau_{k+1}}) \leq 2^{k+1}$,

$$s(a_n^k) \leq \frac{1}{\mu_k}(s(f_n^{\tau_k}) + s(f_n^{\tau_{k+1}})) \leq \omega(\tau_k \neq \infty)^{-1/p}.$$

Consequently,(a_n^k) is L_2-bounded, so there exists L_2-bounded measurable function, also denoting by a^k, such that

$$E_n a^k = a_n^k, \qquad \forall n \geq 0.$$

$a_n^k = 0$ when $n \leq \tau_k$, thus a^k is really a $(1, p, \infty)$ atom. By Abel rearrangement we get

$$\begin{aligned}
\sum_{k\in Z}|\mu_k|^p &= 3^p\sum_{k\in Z}2^{kp}\omega(\tau_k \neq \infty) = 3^p\sum_{k\in Z}2^{kp}\omega(s(f) > 2^k) \\
&= \frac{3^p}{2^p-1}\sum_{k\in Z}[(2^p)^{k+1} - (2^p)^k]\omega(s^p(f) > (2^k)^p) \\
&= \frac{3^p}{2^p-1}\sum_{k\in Z}(2^p)^k\omega((2^p)^{k-1} < s^p(f) \leq (2^p)^k) \\
&\leq \frac{3^p}{2^p-1}\|f\|^p_{H^s_{p;\omega}}
\end{aligned}$$

which proves (2). Obviously,

$$f - \sum_{k=l}^{m}\mu_k a^k = (f - f^{\tau_{m+1}}) + f^{\tau_l}.$$

Because $s(f^{\tau_l}) \leq 2^l \longrightarrow 0$, as $l \longrightarrow -\infty$ and as $m \longrightarrow +\infty$

$$s^p(f - f^{\tau_{m+1}}) = [s^2(f) - s^2(f^{\tau_{m+1}})]^{p/2} \longrightarrow 0, \quad a.e.,$$

by the majorized convergence theorem, (1) holds in $H^s_{p;\omega}$ norm.

This finishes the proof.

Lemma 1.2.7 If $f = (f_n)_{n\geq 0}$ is in $Q_{p;\omega}, 0 < p < \infty$, then there exist a sequence $(a^k)_{k\in Z}$ of $(2, p, \infty)$-atoms with respect to ω and a sequence $\mu = (\mu_k)_{k\in Z} \in l_p$ such that for every $n \in N$

$$\sum_{k\in Z} \mu_k E_n a^k = f_n$$

and

$$(\sum_{k\in Z} |\mu_k|^p)^{1/p} \leq C_p \|f\|_{Q_{p;\omega}},$$

where the series is convergent in $Q_{p;\omega}$.

Lemma 1.2.8 If $f = (f_n)_{n\geq 0}$ is in $P_{p;\omega}, 0 < p < \infty$, then there exist a sequence $(a^k)_{k\in Z}$ of $(3, p, \infty)$-atoms with respect to ω and a sequence $\mu = (\mu_k)_{k\in Z} \in l_p$ such that for every $n \in N$

$$\sum_{k\in Z} \mu_k E_n a^k = f_n$$

and

$$(\sum_{k\in Z} |\mu_k|^p)^{1/p} \leq C_p \|f\|_{P_{p;\omega}},$$

where the series is convergent in $P_{p;\omega}$.

Now we introduce operator-valued martingale transform.

Burkholder's martingale transforms (see[10] and [11]) are defined by using scalar-valued multiplying sequences. One main tool in our proofs will be martingale transforms with operator-valued multiplying sequences, defined and studied in [69] and [68].

Definition 1.2.9 Let X_1 and X_2 be two Banach spaces. Let $L(X_1, X_2)$ denote the space of all bounded linear operators from X_1 to X_2. Let $\upsilon = \{\upsilon_n\}_{n\geq 1}$ be a predictable sequence such that $\upsilon_n \in L_\infty(L(X_1, X_2))$ and $\sup_{n\geq 1} \|\upsilon_n\|_{L_\infty(L(X_1,X_2))} \leq 1$. Then the martingale transform T associated to υ is defined as follows. For any X_1-valued martingale $f = \{f_n\}_{n\geq 1}$

$$(Tf)_n = \sum_{k=1}^{n} \upsilon_k df_k \,.$$

We refer to [69] and [68] for some basic results.

§ 1.3 Real interpolation spaces

In this section we introduce some properties and results of real interpolation. We refer to [5] and [57] for more details .

Suppose that A_0 and A_1 are two quasi-normed spaces embedded continuously in a topological space A. The interpolation spaces between A_0 and A_1 are defined by means of an interpolating function $K(t, f, A_0, A_1)$,

$$K(t, f, A_0, A_1) = \inf_{f=f_0+f_1} \{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\}.$$

For $0 < \theta < 1$, $0 < q \le \infty$，the interpolation spaces $(A_0, A_1)_{\theta,q}$ is defined as the space of all functions $f \in A_0 + A_1$ such that

$$\begin{aligned} \|f\|_{(A_0,A_1)_{\theta,q}} &= \left(\int_0^\infty (t^{-\theta} K(t, f, A_0, A_1))^q \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} < \infty, && q < \infty \\ \|f\|_{(A_0,A_1)_{\theta,\infty}} &= \sup_{t>0} t^{-\theta} K(t, f, A_0, A_1), && q = \infty \end{aligned}$$

Lemma 1.3.1 Let a quasilinear operator T defined on $A_0 + A_1$，if

$$T:\ A_0 \to B_0, \qquad T : A_1 \to B_1,$$

is bounded，then for $0 < \theta < 1$, $0 < q \le \infty$，

$$T:\ (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$$

is also bounded.

Lemma 1.3.2 (Reiteration theorem) Let $0 \le \theta_0 < \theta_1 \le 1$, $0 < q_0, q_1 \le \infty$，(A_0, A_1) be an interpolation couple. If $X_i = (A_0, A_1)_{\theta_i,q_i}, i = 0, 1$. then for $0 < \eta < 1$, $0 < q \le \infty$，

$$(X_0, X_1)_{\eta,q} = (A_0, A_1)_{\theta,q},$$

where $\theta = (1 - \eta)\theta_0 + \eta\theta_1$.

The following is the Hardy inequality.

Lemma 1.3.3 Let $0 < q \le \infty$，$0 < r < q$，$q_0 = \min(1, q)$，then for any nonnegative function f on $[0, \infty)$,

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f(x)\,\mathrm{d}x\right)^q t^r \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \le \left(\frac{q}{q-r}\right)^{\frac{1}{q_0}} \left(\int_0^\infty f(t)^q t^r \,\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}.$$

Chapter 2 Interpolation on Lorentz martingale spaces

§ 2.1 Introduction

Since its invention in the late 1950's, interpolation theory has been a tremendous development and applied to different fields of mathematics, for example partial differential equations, numerical analysis, approximation theory and so on. Moreover, it has also attracted considerable interest in itself. In particular, as well known, in 1958 Stein and Weiss [95] proved an interpolation theorem on L_p spaces which allows one to change measures simultaneously with changing exponents; in 1966 Calderón [16] and Hunt [41] proved an interpolation theorem on Lorentz spaces $L_{p,q}$. But in 1997 Ferreyra [34] gave an example to show that Weiss' result is not true in $L_{p,q}$ spaces. Thus it is worth to seek a such interpolation theorem for $L_{p,q}$ spaces. In recent years, real interpolation and weighted inequality theorem have been developed by [29], [21] and [98]. At the same time, weighted Lorentz spaces have been studied in [17], [18], [19] and other papers. We also mention the following interpolation theorem on weighted Lorentz spaces, which is proved by Moritoh, Niwa and Sobukawa [74] in 2006 (for the notations see section 2):

Theorem 2.1.1 Let $i = 0, 1$ and $1 \leq p_0 < p_1 \leq \infty$, $1 \leq r_i \leq \infty$ with $r_0 \neq r_1$, $0 \leq q_i, s_i \leq \infty$. Put $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$ for $0 < \theta < 1$. If υ, ω_i are two nonnegative measurable functions and T is a nonnegative sublinear operator from $L_{p_i,q_i;\upsilon}$ to $L_{r_i,s_i;\omega_i}$, then there exists a constant C such that

$$\|Tf\|_{r,\infty;\omega} \leq C\|f\|_{p,1;\upsilon},$$

where $\omega^{1/r} = \omega_0^{(1-\theta)/r_0}\omega_1^{\theta/r_1}$.

In this section, we prove several similar versions of this theorem on martingale Lorentz spaces over weighted measure spaces. Our proofs are different from those in [74], and the atomic decomposition method plays an important role.

§ 2.2 Interpolation on weighted Lorentz martingale spaces

Theorem 2.2.1 Let $i = 0, 1, 0 < p_i \leq q_i \leq \infty, 1 < r_i \leq \infty$ and υ, ω_i be nonnegative r.v. Put

$$1/r = (1-\theta)/r_0 + \theta/r_1, 1/p = (1-\theta)/p_0 + \theta/p_1, 0 < \theta < 1.$$

If T is a bounded linear operator from $H^s_{p_i,q_i;\upsilon}$ to $H^s_{r_i,\infty;\omega_i}$, then T is bounded from $H^s_{p,q;\upsilon}$ to $H^s_{r,\infty;\omega}$, i.e., there exists a constant C such that

$$\|Tf\|_{H^s_{r,\infty;\omega}} \leq C\|f\|_{H^s_{p,q;\upsilon}},$$

where $q \leq p \leq 1$ and $\omega^{1/r} = \omega_0^{(1-\theta)/r_0}\omega_1^{\theta/r_1}$.

Proof Assume that $f \in H^s_{p,q;\upsilon} \subset H^s_{p,\upsilon}$, from Lemma 1.2.6, there is a sequence $(a^k)_{k\in Z}$ of $(1, p, \infty)$-atoms with respect to υ and a sequence $\mu = (\mu_k)_{k\in Z} \in l_p$ such that $(1), (2)$ are true for $H^s_{p,\upsilon}$, and the series $\sum_{k\in Z} \mu_k a^k$ converges to f in $H^s_{p,\upsilon}$. Without loss of generality, we suppose $p_0 < p < p_1$, then $H^s_{p_1,q_1;\upsilon} \subset H^s_{p;\upsilon} \subset H^s_{p_0,q_0;\upsilon}$.Thus as $K \to \infty$

$$\| \sum_{|k|\leq K} \mu_k a^k - f \|_{H^s_{p_0,q_0;\upsilon}} \leq \| \sum_{|k|\leq K} \mu_k a^k - f \|_{H^s_{p;\upsilon}} \longrightarrow 0.$$

T is bounded from $H^s_{p_0,q_0;\upsilon}$ to $H^s_{r_0,\infty;\omega_0}$, so $\sum_{|k|\leq K} \mu_k T a^k$ converges to Tf in $H^s_{r_0,\infty;\omega_0}$ norm. Consequently

$$Tf = \sum_{k\in Z} \mu_k T a^k \qquad a.e.$$

Of course for every $n \in N$,

$$(Tf)_n = \sum_{k\in Z} \mu_k (T a^k)_n \qquad a.e.$$

Thus

$$\begin{aligned}
\|Tf\|_{H^s_{r,\infty;\omega}} &= \|s(Tf)\|_{r,\infty;\omega} \leq \|s(Tf)\|_{(r,\infty);\omega} \\
&\leq \sum_{k\in Z} |\mu_k| \|s(Ta^k)\|_{(r,\infty);\omega} \\
&\leq \frac{r}{r-1} \sum_{k\in Z} |\mu_k| \|s(Ta^k)\|_{r,\infty;\omega}. \qquad (2.1)
\end{aligned}$$

Now it is only to estimate $\|s(Ta^k)\|_{r,\infty;\omega}$. Since T is a bounded operator from $H^s_{p_i,q_i;\upsilon}$ to $H^s_{r_i,\infty;\omega_i}$, then

$$t^{1/r_i}(s(Ta^k))^*_{\omega_i}(t) \leq C\|s(a^k)\|_{p_i,q_i;\upsilon} \ , i = 0, 1.$$

By taking $t = \lambda_{s(Ta^k),\omega_i}(y)$, we have

$$y(\lambda_{s(Ta^k),\omega_i}(y))^{1/r_i} \leq C\|s(a^k)\|_{p_i,q_i;\upsilon}, i = 0, 1.(2.2)$$

Using Hölder inequality and $\omega^{1/r} = \omega_0^{(1-\theta)/r_0}\omega_1^{\theta/r_1}$, we obtain

$$\begin{aligned} y(\lambda_{s(Ta^k),\omega}(y))^{1/r} &= y(\int_{\{x:s(Ta^k)>y\}} \omega(x)dP)^{1/r} \\ &\leq y(\int_{\{x:s(Ta^k)>y\}} \omega_0(x)dP)^{(1-\theta)/r_0}(\int_{\{x:s(Ta^k)>y\}} \omega_1(x)dP)^{\theta/r_1} \\ &= [y(\lambda_{s(Ta^k),\omega_0}(y))^{1/r_0}]^{1-\theta}[y(\lambda_{s(Ta^k),\omega_1}(y))^{1/r_1}]^{\theta}. \qquad (2.3) \end{aligned}$$

From (2.2) and (2.3) ,

$$y(\lambda_{s(Ta^k),\omega}(y))^{1/r} \leq C\|s(a^k)\|^{1-\theta}_{p_0,q_0;\upsilon}\|s(a^k)\|^{\theta}_{p_1,q_1;\upsilon}. \qquad (2.4)$$

Thus we get

$$\|s(Ta^k)\|_{r,\infty;\omega} \leq C\|s(a^k)\|^{1-\theta}_{p_0,q_0;\upsilon}\|s(a^k)\|^{\theta}_{p_1,q_1;\upsilon}. \qquad (2.5)$$

Notice that $p_0 \leq q_0$, we have

$$\|a^k\|_{H^s_{p_0,q_0;\upsilon}} \leq \|a^k\|_{H^s_{p_0,p_0;\upsilon}} = \|a^k\|_{H^s_{p_0;\upsilon}} = (\int_{\Omega} |s(a^k)|^{p_0}\upsilon(x)dp)^{1/p_0}.$$

From the definition of $(1, p, \infty)$-atom,

$$\chi_{\{\tau_k \geq n\}}E_{n-1}|\Delta_n a^k|^2 = E_{n-1}[\chi_{\{\tau_k \geq n\}}|\Delta_n a^k|^2] = 0,$$

thus $s(a^k) = 0$ on set $\{\tau_k = \infty\}$, where τ_k is a stopping time with respect to the atom a^k. Consequently,

$$\begin{aligned} \int_{\Omega} |s(a^k)|^{p_0}\upsilon(x)dp &= \int_{\{\tau_k<\infty\}} |s(a^k)|^{p_0}\upsilon(x)dP \\ &\leq \|s(a^k)\|^{p_0}_{\infty}\upsilon(\{\tau_k < \infty\}). \end{aligned}$$

In other words ,

$$\|s(a^k)\|^{1-\theta}_{p_0,q_0;\upsilon} \leq \|s(a^k)\|^{1-\theta}_{\infty}\upsilon(\{\tau_k < \infty\})^{(1-\theta)/p_0}. \qquad (2.6)$$

It is the same for $\|s(a^k)\|_{p_1,q_1;\upsilon}$, i.e.

$$\|s(a^k)\|^{\theta}_{p_1,q_1;\upsilon} \leq \|s(a^k)\|^{\theta}_{\infty}\upsilon(\{\tau_k < \infty\})^{\theta/p_1}. \qquad (2.7)$$

Since a^k is $(1, 1, \infty)$-atom with respect to υ,

$$\|s(a^k)\|_{\infty} \leq \upsilon(\{\tau_k < \infty\})^{-1/p}.$$

Then from (2.5), (2.6), (2.7), we get

$$\|s(Ta^k)\|_{r,\infty;\omega} \leq C\|s(a^k)\|_\infty \upsilon(\{\tau_k < \infty\})^{(1-\theta)/p_0+\theta/p_1} \leq C.$$

Thus from (2.1), we get

$$\begin{aligned}\|Tf\|_{H^s_{r,\infty;\omega}} &\leq \frac{Cr}{r-1}\sum_{k\in Z}|\mu_k| \leq C(\sum_{k\in Z}|\mu_k|^p)^{1/p} \\ &\leq C\|f\|_{H^s_{p;\upsilon}} \leq C\|f\|_{H^s_{p,q;\upsilon}}.\end{aligned}$$

This finishes the proof of the theorem.

Theorem 2.2.2 Let $i = 0, 1, 0 < p_i \leq q_i \leq \infty, 1 < r_i \leq \infty$ and υ, ω_i are nonnegative r.v. Put

$$1/r = (1-\theta)/r_0 + \theta/r_1, 1/p = (1-\theta)/p_0 + \theta/p_1, 0 < \theta < 1.$$

If T is a bounded operator from $Q_{p_i,q_i;\upsilon}$ to $Q_{r_i,\infty;\omega_i}$, then T is bounded from $Q_{p,q;\upsilon}$ to $Q^s_{r,\infty;\omega}$, i.e., there exists a constant C such that

$$\|Tf\|_{Q_{r,\infty;\omega}} \leq C\|f\|_{Q_{p,q;\upsilon}},$$

where $q \leq p \leq 1$ and $\omega^{1/r} = \omega_0^{(1-\theta)/r_0}\omega_1^{\theta/r_1}$.

Proof We also assume $f \in Q_{p,q;\upsilon} \subset Q_{p;\upsilon}$, from Lemma 1.2.7, there exist a sequence $(a^k)_{k\in Z}$ of $(2, p, \infty)$-atoms with respect to υ and a sequence $\mu = (\mu_k)_{k\in Z} \in l_p$ such that (1) and (2) hold for $Q_{p;\upsilon}$, and the series $\sum_{k\in Z}\mu_k a^k$ converges to f in $Q^s_{p,\upsilon}$. Similarly to Theorem 3.1, we have

$$(Tf)_n = \sum_{k\in Z}\mu_k(Ta^k)_n \qquad a.e.$$

Thus

$$\begin{aligned}S_n(Tf) &= (\sum_{m=0}^{n}|\Delta_m(Tf)|^2)^{1/2} = (\sum_{m=0}^{n}|\sum_{k\in Z}\mu_k\Delta_m(Ta^k)|^2)^{1/2} \\ &\leq \sum_{k\in Z}|\mu_k|S_n(Ta^k). \quad (2.8)\end{aligned}$$

We set $\rho^k = (\rho^k_n)_{n\geq 0}$ is the optimal control of Ta^k, i.e., $\rho^k = (\rho^k_n)_{n\geq 0}$ is an r.v. sequence of non-decreasing, non-negative and adapted such that

$$S_n(Ta^k) \leq \rho^k_{n-1}, \quad \|Ta^k\|_{Q_{r_i,\infty;\omega_i}} = \|\rho^k_\infty\|_{r_i,\infty;\omega_i}. \quad (2.9)$$

From (2.8),

$$S_n(Tf) \leq \sum_{k\in Z}|\mu_k|\rho^k_{n-1}.$$

Thus by the definition of $Q_{r,\infty;\omega}$,

$$\begin{aligned}\|Tf\|_{Q_{r,\infty;\omega}} &\leq \|\sum_{k\in Z}|\mu_k|\rho_\infty^k\|_{r,\infty;\omega} \leq \|\sum_{k\in Z}|\mu_k|\rho_\infty^k\|_{(r,\infty);\omega} \\ &\leq \frac{r}{r-1}\sum_{k\in Z}|\mu_k|\|\rho_\infty^k\|_{r,\infty;\omega}. \qquad (2.10)\end{aligned}$$

To estimate $\|\rho_\infty^k\|_{r,\infty;\omega}$, in this time since T is bounded from $Q_{p_i,q_i;\upsilon}$ to $Q_{r_i,\infty;\omega_i}$, and by (2.9)

$$\|\rho_\infty^k\|_{r_i,\infty;w_i} = \|Ta^k\|_{Q_{r_i,\infty;\omega_i}} \leq C\|a^k\|_{Q_{p_i,q_i;\upsilon}}.$$

In other words,

$$y(\lambda_{\rho_\infty^k,\omega_i}(y))^{1/r_i} \leq C\|a^k\|_{Q_{p_i,q_i;\upsilon}}, \qquad (2.11)$$

By the Hölder inequality, we get

$$\begin{aligned}y(\lambda_{\rho_\infty^k;\omega}(y))^{1/r} &= y(\int_{\{x:\rho_\infty^k>y\}}\omega(x)dP)^{1/r} \\ &\leq y(\int_{\{x:\rho_\infty^k>y\}}\omega_0(x)dP)^{1-\theta/r_0}(\int_{\{x:\rho_\infty^k>y\}}\omega_1(x)dP)^{\theta/r_1} \\ &= [y(\lambda_{\rho_\infty^k,\omega_0}(y))^{1/r_0}]^{1-\theta}[y(\lambda_{\rho_\infty^k,\omega_1}(y))^{1/r_1}]^{\theta} \\ &\leq C\|a^k\|_{Q_{p_0,q_0;\upsilon}}^{1-\theta}\|a^k\|_{Q_{p_1,q_1;\upsilon}}^{\theta}. \qquad (2.12)\end{aligned}$$

And then

$$\|\rho_\infty^k\|_{r,\infty;\omega} \leq C\|a^k\|_{Q_{p_0,q_0;\upsilon}}^{1-\theta}\|a^k\|_{Q_{p_1,q_1;\upsilon}}^{\theta}. \quad (2.13)$$

We set $\xi_n^k = \chi_{\{\tau_k\leq n\}}\|S(a^k)\|_\infty$, and it is easy to see that ξ_n^k is non-decreasing, non-negative, and adapted. By the definition of atom, we have

$$\sum_{n=0}^{\infty}\chi_{\{\tau_k\geq n\}}|\Delta_n a^k|^2 = 0.$$

Then

$$\begin{aligned}S_n(a^k) = (\sum_{n=0}^{n}|\Delta_n a^k|^2)^{1/2} &\leq (\sum_{n=0}^{\infty}\chi_{\{\tau_k<n\}}|\Delta_n a^k|^2)^{1/2} \\ &\leq \chi_{\{\tau_k\leq n-1\}}\|S(a^k)\|_\infty = \xi_{n-1}^k.\end{aligned}$$

Consequently,

$$\|a^k\|_{Q_{p_0,q_0;\upsilon}}^{1-\theta} \leq \|\xi_\infty^k\|_{p_0,q_0;\upsilon}^{1-\theta} \leq \|\xi_\infty^k\|_{p_0;\upsilon}^{1-\theta} \leq (\int_\Omega \|S(a^k)\|_\infty^{p_0}\upsilon(x)dp)^{(1-\theta)/p_0}. \quad (2.14)$$

Now $S(a^k) = 0$ on the set $\{\tau_k = \infty\}$, where τ_k is a stopping time with respect to the atom a^k. Consequently,

$$\begin{aligned}\int_\Omega \|S(a^k)\|_\infty^{p_0}\upsilon(x)dp &= \int_{\{\tau_k<\infty\}}\|S(a^k)\|_\infty^{p_0}\upsilon(x)dp \\ &= \|S(a^k)\|_\infty^{p_0}\upsilon(\{\tau_k<\infty\}). \quad (2.15)\end{aligned}$$

(2.14) and (2.15) show,

$$\|a^k\|^{1-\theta}_{Q_{p_0,q_0;\upsilon}} \leq \|S(a^k)\|^{1-\theta}_\infty \upsilon(\{\tau_k < \infty\})^{(1-\theta)/p_0}.$$

Similarly

$$\|a^k\|^{\theta}_{Q_{p_1,q_1;\upsilon}} \leq \|S(a^k)\|^{\theta}_\infty \upsilon(\{\tau_k < \infty\})^{\theta/p_1}.$$

From the definition of $(2,p,\infty)-$atom we have

$$\|\rho^k_\infty\|_{r,\infty;\omega} \leq C\|S(a^k)\|_\infty \upsilon(\tau_k < \infty)^{1/p} \leq C.$$

From (2.10)

$$\begin{aligned}\|Tf\|_{Q_{r,\infty;\omega}} &\leq \frac{Cr}{r-1}\sum_{k\in Z}|\mu_k| \leq C(\sum_{k\in Z}|\mu_k|^p)^{1/p} \\ &\leq C\|f\|_{Q_{p;\upsilon}} \leq C\|f\|_{Q_{p,q;\upsilon}}.\end{aligned}$$

This finishes the proof of the theorem.

Theorem 2.2.3 Let p_i, q_i, r_i and p, r, υ, ω_i as in theorem 2.2.2, if T is a bounded operator from $P_{p_i,q_i;\upsilon}$ to $P_{r_i,\infty;\omega_i}$, then T is bounded from $P_{p,q;\upsilon}$ to $P^s_{r,\infty;\omega}$, i.e., there exists a constant C such that

$$\|Tf\|_{P_{r,\infty;\omega}} \leq C\|f\|_{P_{p,q;\upsilon}},$$

where $q \leq p \leq 1$ and $\omega^{1/r} = \omega_0^{(1-\theta)/r_0}\omega_1^{\theta/r_1}$.

Proof We suppose $f \in P_{p,q;\upsilon} \subset P_{p,\upsilon}$. From the Lemma 1.2.8, there exist a sequence $(a^k)_{k\in Z}$ of $(3,p,\infty)$ atoms with respect to $\upsilon(x)$ and a sequence $\mu = (\mu_k)_{k\in Z} \in l_p$ such that (1) and (2) hold for $P_{p;\upsilon}$, and the series $\sum_{k\in Z}\mu_k a^k$ converges to f in $P^s_{p,\upsilon}$. Similarly to Theorem 2.2.1, we have

$$(Tf)_n = \sum_{k\in Z}\mu_k(Ta^k)_n, \qquad a.e. \qquad (2.16)$$

Let $\rho^k = (\rho^k_n)_{n\geq 0}$ is the optimal control of Ta^k, i.e., $\rho^k = (\rho^k_n)_{n\geq 0}$ is a sequence of non-decreasing, non-negative and adapted functions such that

$$|(Ta^k)_n| \leq \rho^k_{n-1} \qquad , \qquad \|Ta^k\|_{P_{r_i,\infty;\omega_i}} = \|\rho^k_\infty\|_{r_i,\infty;\omega_i}. \qquad (2.17)$$

From (2.16),

$$|(Tf)_n| \leq \sum_{k\in Z}|\mu_k|\rho^k_{n-1}.$$

Thus by the definition of $P_{r,\infty;\omega}$ and Lemma 1.2.1

$$\begin{aligned}
\|Tf\|_{P_{r,\infty;\omega}} &\le \|\sum_{k\in Z}|\mu_k|\rho_\infty^k\|_{r,\infty;\omega} \\
&\le \|\sum_{k\in Z}|\mu_k|\rho_n^k\|_{(r,\infty);\omega} \\
&\le \frac{r}{r-1}\sum_{k\in Z}|\mu_k|\|\rho_\infty^k\|_{r,\infty;\omega}.
\end{aligned}$$

Similarly to Theorem 2.2.2, we get

$$\|\rho_\infty^k\|_{r,\infty;\omega} \le C\|a^k\|^{1-\theta}_{P_{p_0,q_0;\upsilon}}\|a^k\|^{\theta}_{P_{p_1,q_1;\upsilon}}.$$

We set $\xi_n^k = \chi_{\{\tau_k\ge n\}}\|a^{k^*}\|_\infty$. It is easy to see that ξ_n^k is non-decreasing, non-negative, adapted and from the atomic definition

$$a_n^k \le \chi_{\{\tau_k\le n-1\}}\|a^{k^*}\|_\infty + \chi_{\{\tau_k\ge n\}}\|a^{k^*}\|_\infty \le \xi_{n-1}^k.$$

Similarly to Theorem 2.2.2 we have

$$\|a^k\|^{1-\theta}_{P_{p_0,q_0;\upsilon}} \le \|a^{k^*}\|_\infty^{1-\theta}\upsilon(\{\tau_k<\infty\})^{(1-\theta)/p_0}.$$

and

$$\|a^k\|^{\theta}_{P_{p_1,q_1;\upsilon}} \le \|a^{k^*}\|_\infty^{\theta}\upsilon(\{\tau_k<\infty\})^{\theta/p_1}.$$

The rest of the proof is similar to the one in Theorem 2.2.2.

First we introduce a new space. Let $0<p\le\infty$, denote all of the scalar adapted process

$$V_p = \{\nu = (\nu_n)_{n\ge1} : \|\nu\|_{V_p} = \|\sup_n|\nu_n|\|_p < \infty\}.$$

With $\nu\in V_p$ as a multiplier, we define $T_\nu : f = (f_n)_{n\ge1} \to T_\nu f$, where

$$(T_\nu f)_n = \sum_1^n \nu_{k-1}\Delta_k f \quad , \quad n\ge1.$$

According to the theorems above, here we give some inequalities of martingale transform operator T_ν .

Theorem 2.2.4 Suppose that $0<p_i\le q_i\le\infty$, $1<r_i\le\infty, i=0,1$. Put

$$1/p=(1-\theta)/p_0+\theta/p_1, 1/r=(1-\theta)/r_0+\theta/r_1, 0<\theta<1.$$

If υ is strict positive weight function and $\nu\in V_{\frac{r_ip_i}{p_i-r_i};\upsilon}$, then T_ν is bounded from $H^s_{p,q;\upsilon}$ to $H^s_{r,\infty;\upsilon}$, i.e., $\exists C>0$ s.t.

$$\|T_\nu f\|_{H^s_{r,\infty;\upsilon}} \le C\|\nu\|_{V_{\frac{r_ip_i}{p_i-r_i};\upsilon}}\|f\|_{H^s_{p,q;\upsilon}}, q\le p\le 1.$$

Proof From $\nu \in V_{\frac{p_i-r_i}{r_ip_i};\upsilon}$, we know that T_ν is bounded from $H^s_{p_i;\upsilon}$ to $H^s_{r_i;\upsilon}$. In fact,

$$s(T_\nu f) = (\sum_1^\infty \nu_{n-1}^2 |\Delta_n f|^2)^{1/2} \le \sup_n |\nu_n(x)| s(f).$$

By the relation $1/r_i = 1/p_i + \frac{p_i-r_i}{r_ip_i}$ and Hölder inequality , we get

$$\|T_\nu f\|_{H^s_{r_i;\upsilon}} \le C\|\nu\|_{V_{\frac{r_ip_i}{p_i-r_i};\upsilon}} \|f\|_{H^s_{p_i;\upsilon}}.$$

Now we assume that $\omega_i = \upsilon$, $p_i = q_i$, then T_ν satisfies the conditions of Theorem 2.2.1. Consequently,

$$\|T_\nu f\|_{H^s_{r,\infty;\upsilon}} \le C\|\nu\|_{V_{\frac{r_ip_i}{p_i-r_i};\upsilon}} \|f\|_{H^s_{p,q;\upsilon}}.$$

The theorem is proved.

Under the conditions of Theorem 2.2.4 with $p = q$, we get

Corollary 2.2.5 T_ν is of $(H^s_{p;\upsilon}, wH^s_{r;\upsilon})$-type , i.e. the inequality

$$\lambda_{s(T_\nu f);\upsilon}(y) \le C(y^{-1}\|f\|_{H^s_{p;\upsilon}})^r, \quad y > 0$$

holds.

The following lemma can be found in [22].

Lemma 2.2.6 Let $0 < p \le \infty, 0 < q < \infty, \nu \in V_p$ and $1/r = 1/p + 1/q$. Then T_ν is of (P_q, P_r) and (Q_q, Q_r)-types with $\|T_\nu\| \le C\|\nu\|_{V_p}$.

Theorem 2.2.7 Suppose that $0 < p_i \le q_i \le \infty$, $1 < r_i \le \infty$, $i = 0, 1$. Put

$$1/p = (1-\theta)/p_0 + \theta/p_1, 1/r = (1-\theta)/r_0 + \theta/r_1, 0 < \theta < 1.$$

If υ is strict positive weight function and $\nu \in V_{\frac{r_ip_i}{p_i-r_i};\upsilon}$, then T_ν is bounded from $Q_{p,q;\upsilon}$ to $Q_{r,\infty;\upsilon}$, i.e., $\exists C > 0$ s.t.

$$\|T_\nu f\|_{Q_{r,\infty;\upsilon}} \le C\|\nu\|_{V_{\frac{r_ip_i}{p_i-r_i};\upsilon}} \|f\|_{Q_{p,q;\upsilon}}, q \le p \le 1.$$

Proof By $1/r_i = 1/p_i + \frac{p_i-r_i}{r_ip_i}$ and Lemma 2.2.6, we have

$$\|T_\nu f\|_{Q_{r_i;\upsilon}} \le C\|\nu\|_{V_{\frac{r_ip_i}{p_i-r_i};\upsilon}} \|f\|_{Q_{p_i;\upsilon}}.$$

Thus T_ν satisfies the conditions of Theorem 2.2.2.

Theorem 2.2.8 Suppose that $< p_i \le q_i \le \infty, 1 < r_i \le \infty, i = 0, 1$. Put

$$1/p = (1-\theta)/p_0 + \theta/p_1, 1/r = (1-\theta)/r_0 + \theta/r_1, 0 < \theta < 1.$$

If υ is strict positive weight function and $\nu \in V_{\frac{r_i p_i}{p_i - r_i};\upsilon}$, then T_ν is bounded from $P_{p,q;\upsilon}$ to $P_{r,\infty;\upsilon}$, i.e., $\exists C > 0$ s.t.

$$\|T_\nu f\|_{P_{r,\infty;\upsilon}} \leq C\|\nu\|_{V_{\frac{r_i p_i}{p_i - r_i};\upsilon}} \|f\|_{P_{p,q;\upsilon}}, q \leq p \leq 1.$$

Proof By the relation $1/r_i = 1/p_i + \frac{p_i - r_i}{r_i p_i}$ and Lemma 4.3, we get

$$\|T_\nu f\|_{P_{r_i;\upsilon}} \leq C\|\nu\|_{V_{\frac{r_i p_i}{p_i - r_i};\upsilon}} \|f\|_{P_{p_i;\upsilon}}, i = 0, 1.$$

Thus T_ν satisfies the conditions of Theorem 2.2.3.

§ 2.3 Real interpolation spaces between H_p and $H_{p,\infty}$

In harmonic analysis it is well known if $1/p = (1-\eta)/p_0 + \eta/p_1, 0 < \eta < 1$, then

$$(wL_{p_0}, L_{p_1})_{\theta,p} = L_p.$$

In this section we give similar version in the martingale setting; and the weak atomic decomposition is the main tool (see [39] and [101]).

Theorem 2.3.1 $(wH^s_{p_0}, H^s_\infty)_{\theta,p} = H^s_p$, $1/p = (1-\theta)/p_0, 0 < \theta < 1$.

Proof Suppose that $f \in wH^s_{p_0}$. For any fixed $y = (sf)^*(t^{p_0})$, choose $j \in Z$ such that $2^j \leq y < 2^{j+1}$. From Lemma 1.2.4 there exist a sequence $(a^k)_{k\in Z}$ of w-1-atoms and the corresponding stopping times $(\tau_k)_{k\in Z}$ such that

$$f_n = \sum_{k\in Z} a^k_n = \sum_{k=-\infty}^{j-1} a^k_n + \sum_{k=j}^{\infty} a^k_n =: g_n + h_n$$

and $s(a^k) \leq A2^k$ for some constant $A > 0$. Then

$$s(g) \leq \sum_{k=-\infty}^{j-1} s(a)^k \leq \sum_{k=-\infty}^{j-1} A2^k = A2^j \leq A(sf)^*(t^{p_0})$$

From the definition of w-1-atom,

$$\chi_{\{\tau_k \geq n\}} E_{n-1}|\Delta_n a^k|^2 = E_{n-1}[\chi_{\{\tau_k \geq n\}}|\Delta_n a^k|^2] = 0,$$

thus $s(a^k) = 0$ on set $\{\tau_k = \infty\}$. Recall that the stopping times

$$\tau_k = \inf\{n \in N : s_{n+1}(f) > 2^k\}(\inf \phi = \infty),$$

and $\tau_k \uparrow \infty (k \to \infty)$. Then $\{\tau_k < \infty\} = \{s(f) > 2^k\}$.

So we have

$$
\begin{aligned}
P(s(h) > y) &\leq P(s(h) > 0) \leq \sum_{k=j}^{\infty} P(s(a^k) > 0) \\
&= \sum_{k=j}^{\infty} P(\tau_k < \infty) \leq 2^{-jp_0} \sum_{k=j}^{\infty} 2^{kp_0} P(\tau_k < \infty) \\
&\leq cy^{-p_0} \sum_{k=j}^{\infty} 2^{kp_0} P(s(f) > 2^k) \\
&\leq cy^{-p_0} \int_{\{s(f) \geq 2^j\}} s(f)^{p_0} dP \\
&\leq cy^{-p_0} \int_{\{s(f) \geq y\}} s(f)^{p_0} dP
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|h\|_{wH^s_{p_0}}^{p_0} &\leq c \int_{\{s(f) \geq y\}} s(f)^{p_0} dP \leq c \int_{\{s(f) \geq (sf)^*(t^{p_0})\}} s(f)^{p_0} dP \\
&\leq c \int_0^{t^{p_0}} (sf)^*(x)^{p_0} dx
\end{aligned}
$$

By Hardy inequality and $\frac{1}{p} = \frac{1-\theta}{p_0}$, we obtain

$$
\begin{aligned}
\int_0^1 (t^{-\theta} \|h\|_{wH^s_{p_0}}^{p_0})^p \frac{dt}{t} &\leq c \int_0^1 t^{-\theta p} (\int_0^{t^{p_0}} (sf)^*(x)^{p_0} dx)^{p/p_0} \frac{dt}{t} \\
&= c \int_0^1 t^{(1-\theta)p/p_0} (\frac{1}{t} \int_0^t (sf)^*(x)^{p_0} dx)^{p/p_0} \frac{dt}{t} \\
&= c \int_0^1 t^{(1-\theta)p/p_0} (sf)^*(x)^p \frac{dt}{t} \\
&\leq c \int_0^1 (sf)^*(x)^p dt \\
&= c\|f\|_{H^s_p}^p
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_0^1 (t^{-\theta} \|g\|_{H^s_\infty}^{p_0})^p \frac{dt}{t} &\leq c \int_0^1 t^{(1-\theta)p} (sf)^*(t^{p_0})^p \frac{dt}{t} \\
&\leq c \int_0^1 (sf)^*(t)^p dt \leq c\|f\|_{H^s_p}^p
\end{aligned}
$$

By the definition of the functional K,

$$
K(t, f; wH^s_{p_0}, H^s_\infty) \leq \|h\|_{wH^s_{p_0}} + t\|g\|_{H^s_\infty}.
$$

Henceforth,

$$
\|f\|_{(wH^s_{p_0}, H^s_\infty)_{\theta,p}}^p = \int_0^1 (t^{-\theta} K(t, f; wH^s_{p_0}, H^s_\infty))^p \frac{dt}{t} \leq c\|f\|_{H^s_p}^p.
$$

To prove the converse consider the sublinear operator $T : f \mapsto s(f)$. By the definition $T : H^s_\infty \to L_\infty$ and $T : wH^s_{p_0} \to wL_{p_0}$ are bounded. Therefore, by the Lemma 2.4 and Lemma 2.5

$$T : (wH^s_{p_0}, H^s_\infty)_{\theta,p} \to (wL_{p_0}, L_\infty)_{\theta,p} = L_p$$

is bounded, too, that is to say $f \in (wH^s_{p_0}, H^s_\infty)_{\theta,p}$ implies

$$\|f\|_{H^s_p} = \|Tf\|_{L_p} \le c\|f\|^p_{(wH^s_{p_0}, H^s_\infty)_{\theta,p}}.$$

Theorem 2.3.2 $(wH^s_{p_0}, H^s_{p_1})_{\eta,p} = H^s_p$, $1/p = (1-\eta)/p_0 + \eta/p_1, 0 < \eta < 1$.
Proof Choose θ and θ_1 satisfying

$$1/p = (1-\theta)/p_0, \theta = \eta\theta_1, 1/p_1 = (1-\theta_1)/p_0.$$

Then by Theorem 2.3.1 and the Reiteration Theorem, we obtain

$$(wH^s_{p_0}, H^s_{p_1})_{\eta,p} = (wH^s_{p_0}, (wH^s_{p_0}, H^s_\infty)_{\theta_1,p})_{\eta,p} = (wH^s_{p_0}, H^s_\infty)_{\theta,p} = H^s_p$$

Theorem 2.3.3 $(wQ_{p_0}, Q_\infty)_{\theta,p} = Q_p$, $1/p = (1-\theta)/p_0, 0 < \theta < 1$.
Proof Suppose that $f \in Q_p \subset wQ_{p_0}$. Let $\beta = (\beta_n)_{n\ge 0}$ is the optimal control of $S_n(f)$, i.e., $\beta \in \Lambda, S_n(f) \le \beta_{n-1}, \|f\|_{Q_p} = \|\beta_\infty\|_p$. From Lemma 1.2.4 there exist a sequence $(a^k)_{k\in Z}$ of w-2-atoms and the corresponding stopping times $(\tau_k)_{k\in Z}$ such that

$$f_n = \sum_{k\in Z} a^k_n = \sum_{k=-\infty}^{j-1} a^k_n + \sum_{k=j}^{\infty} a^k_n =: g_n + h_n$$

and $S(a^k) \le A2^k$ for some constant $A > 0$. Remember that the stopping times

$$\tau_k = \inf\{n \in N : \beta^n > 2^k\}(\inf \phi = \infty),$$

and $\tau_k \uparrow \infty (k \to \infty)$.

Define

$$\lambda_n = \sum_{k\in Z} \chi_{\{\tau_k \le n\}} \|S(a^k)\|_\infty \quad (n \in N).$$

For any fixed $t \in [0,1]$, $y = \beta^*(t^{p_0})$, choose $j \in Z$ such that $2^j \le y < 2^{j+1}$.Then

$$\lambda_n = \sum_{k=-\infty}^{j-1} \chi_{\{\tau_k \le n\}} \|S(a^k)\|_\infty + \sum_{k=j}^{\infty} \chi_{\{\tau_k \le n\}} \|S(a^k)\|_\infty =: \lambda^{(1)}_n + \lambda^{(2)}_n$$

It is obvious that $(\lambda_n^{(1)})_{n\geq 0}$ and $(\lambda_n^{(2)})_{n\geq 0}$ are non-negative, nondecreasing and adapted sequences. From the definition of w-2-atoms, $S_{n+1}(a^k) = 0$ on the set $\{\tau_k > n\}$. Hence

$$\begin{aligned} S_{n+1}(g) &\leq \sum_{k=-\infty}^{j-1} S_{n+1}(a^k) = \sum_{k=-\infty}^{j-1} \chi_{\{\tau_k\leq n\}} S_{n+1}(a^k) \\ &\leq \sum_{k=-\infty}^{j-1} \chi_{\{\tau_k\leq n\}} \|S_{n+1}(a^k)\|_\infty = \lambda_n^{(1)}, \end{aligned}$$

and

$$\begin{aligned} S_{n+1}(h) &\leq \sum_{k=j}^{\infty} S_{n+1}(a^k) = \sum_{k=j}^{\infty} \chi_{\{\tau_k\leq n\}} S_{n+1}(a^k) \\ &\leq \sum_{k=j}^{\infty} \chi_{\{\tau_k\leq n\}} \|S_{n+1}(a^k)\|_\infty = \lambda_n^{(2)}. \end{aligned}$$

Thus

$$\|g\|_{Q_\infty} \leq \|\lambda_\infty^{(1)}\|_\infty \leq \sum_{k=-\infty}^{j-1} \|S(a^k)\|_\infty \leq \sum_{k=-\infty}^{j-1} A2^j \leq Ay = A\beta^*(t^{p_0}).$$

Now we shall estimate $\|h\|_{wQ_{p_0}}$.From the definition of w-2-atom,

$$\chi_{\{\tau_k\geq n\}} E|\Delta_n a^k|^2 = E[\chi_{\{\tau_k\geq n\}} |\Delta_n a^k|^2] = 0,$$

thus $S(a^k) = 0$ on set $\{\tau_k = \infty\}$, and noting that $\{\tau_k < \infty\} = \{\beta_\infty > 2^k\}$, we have

$$\begin{aligned} P(\lambda_\infty^{(2)} > y) &\leq P(\lambda_\infty^{(2)} > 0) \leq \sum_{k=j}^{\infty} P(\tau_k < \infty) \\ &\leq 2^{-jp_0} \sum_{k=j}^{\infty} 2^{kp_0} P(\tau_k < \infty) \\ &\leq cy^{-p_0} \sum_{k=j}^{\infty} 2^{kp_0} P(\beta_\infty > 2^k) \\ &\leq cy^{-p_0} \int_{\{\beta_\infty \geq 2^j\}} \beta_\infty^{p_0} dP \\ &\leq cy^{-p_0} \int_{\{\beta_\infty \geq y\}} \beta_\infty^{p_0} dP \end{aligned}$$

Hence

$$\begin{aligned} \|h\|_{wQ_{p_0}}^{p_0} &\leq c \int_{\{\beta_\infty \geq y\}} \beta_\infty^{p_0} dP \leq c \int_{\{\beta_\infty \geq \beta_\infty^*(t^{p_0})\}} \beta_\infty^{p_0} dP \\ &\leq c \int_0^{t^{p_0}} \beta_\infty^*(x)^{p_0} dx \end{aligned}$$

By Hardy inequality and $\frac{1}{p} = \frac{1-\theta}{p_0}$, we obtain

$$\begin{aligned}
\int_0^1 (t^{-\theta}\|h\|_{wQ_{p_0}}^{p_0})^p \frac{dt}{t} &\leq c\int_0^1 t^{-\theta p}(\int_0^{t^{p_0}} \beta_\infty^*(x)^{p_0}dx)^{p/p_0}\frac{dt}{t} \\
&= c\int_0^1 t^{(1-\theta)p/p_0}(\frac{1}{t}\int_0^t \beta_\infty^*(x)^{p_0}dx)^{p/p_0}\frac{dt}{t} \\
&= c\int_0^1 t^{(1-\theta)p/p_0}\beta_\infty^*(x)^p\frac{dt}{t} \\
&\leq c\int_0^1 \beta_\infty^*(x)^p dt \\
&= c\|f\|_{Q_p^s}^p
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^1 (t^{-\theta}\|g\|_{Q_\infty}^{p_0})^p \frac{dt}{t} &\leq c\int_0^1 t^{(1-\theta)p}\beta_\infty^*(t^{p_0})^p\frac{dt}{t} \\
&\leq c\int_0^1 \beta_\infty^*(t)^p dt \leq c\|f\|_{Q_p^s}^p
\end{aligned}$$

By the definition of the functional K,

$$K(t,f;wQ_{p_0},Q_\infty) \leq \|h\|_{wQ_{p_0}} + t\|g\|_{Q_\infty}.$$

Henceforth,

$$\|f\|_{(wQ_{p_0},Q_\infty)_{\theta,p}}^p = \int_0^1 (t^{-\theta}K(t,f;wQ_{p_0},Q_\infty))^p\frac{dt}{t} \leq c\|f\|_{Q_p}^p.$$

To prove the converse consider the sublinear operator $T : f \mapsto S(f)$. By the definition $T : Q_\infty \to L_\infty$ and $T : wQ_{p_0} \to wL_{p_0}$ are bounded. Therefore, by the interpolation property and Lemma 1.3.1

$$T : (wQ_{p_0},Q_\infty)_{\theta,p} \to (wL_{p_0},L_\infty)_{\theta,p} = L_p$$

is bounded, too, that is to say $f \in (wQ_{p_0},Q_\infty)_{\theta,p}$ implies

$$\|f\|_{Q_p} = \|Tf\|_{L_p} \leq c\|f\|_{(wQ_{p_0},Q_\infty)_{\theta,p}}^p.$$

Theorem 2.3.4 $(wQ_{p_0},wQ_{p_1})_{\eta,p} = Q_p$, $1/p = (1-\eta)/p_0 + \eta/p_1, 0 < \eta < 1$.

Proof Choose θ and θ_1 satisfying

$$1/p = (1-\theta)/p_0, \theta = \eta\theta_1, 1/p_1 = (1-\theta_1)/p_0.$$

Then by Theorem 2.3.3 and the Reiteration Theorem, we obtain

$$(wQ_{p_0}, Q_{p_1})_{\eta,p} = (wQ_{p_0}, (wQ_{p_0}, Q_\infty)_{\theta_1,p})_{\eta,p} = (wQ_{p_0}, Q_\infty)_{\theta,p} = Q_p$$

Theorem 2.3.5 $(wD_{p_0}, D_\infty)_{\theta,p} = D_p$, $1/p = (1-\theta)/p_0, 0 < \theta < 1$.

Proof Suppose that $f \in D_p \subset wD_{p_0}$. Let $\beta = (\beta_n)_{n\geq 0}$ is the optimal control of f_n, i.e., $\beta \in \Lambda, f_n \leq \beta_{n-1}, \|f\|_{D_p} = \|\beta_\infty\|_p$. From Lemma 1.2.4 there exist a sequence $(a^k)_{k\in Z}$ of w-3-atoms and the corresponding stopping times $(\tau_k)_{k\in Z}$ such that

$$f_n = \sum_{k\in Z} a_n^k = \sum_{k=-\infty}^{j-1} a_n^k + \sum_{k=j}^{\infty} a_n^k =: g_n + h_n$$

and $M(a^k) \leq A2^k$ for some constant $A > 0$. Remember that the stopping times

$$\tau_k = \inf\{n \in N : \beta^n > 2^k\}(\inf \phi = \infty),$$

and $\tau_k \uparrow \infty (k \to \infty)$.

Define

$$\lambda_n = \sum_{k\in Z} \chi_{\{\tau_k \leq n\}} \|M(a^k)\|_\infty \quad (n \in N).$$

For any fixed $t \in [0, 1]$, $y = \beta^*(t^{p_0})$, similarly to the proof of Theorem 2.3.3, we obtain

$$\int_0^1 (t^{-\theta}\|h\|_{wD_{p_0}}^{p_0})^p \frac{dt}{t} \leq c\|f\|_{D_p^s}^p, \qquad \int_0^1 (t^{-\theta}\|g\|_{D_\infty}^{p_0})^p \frac{dt}{t} \leq c\|f\|_{D_p^s}^p$$

By the definition of the functional K,

$$K(t, f; wD_{p_0}, D_\infty) \leq \|h\|_{wD_{p_0}} + t\|g\|_{D_\infty}.$$

Henceforth,

$$\|f\|_{(wD_{p_0}, D_\infty)_{\theta,p}}^p = \int_0^1 (t^{-\theta} K(t, f; wD_{p_0}, D_\infty))^p \frac{dt}{t} \leq c\|f\|_{D_p}^p.$$

The rest proof is similar to one of Theorem 2.3.3.

Theorem 2.3.6 $(wD_{p_0}, wD_{p_1})_{\eta,p} = D_p$, $1/p = (1-\eta)/p_0 + \eta/p_1, 0 < \eta < 1$.

Proof Choose θ and θ_1 satisfying

$$1/p = (1-\theta)/p_0, \theta = \eta\theta_1, 1/p_1 = (1-\theta_1)/p_0.$$

Then by Theorem 2.3.4 and the Reiteration Theorem, we obtain

$$(wD_{p_0}, D_{p_1})_{\eta,p} = (wD_{p_0}, (wD_{p_0}, D_\infty)_{\theta_1,p})_{\eta,p} = (wD_{p_0}, D_\infty)_{\theta,p} = D_p.$$

□

Chapter 3 Bounded operators on Lorentz martingale spaces

§ 3.1 Atomic decompositions of Lorentz martingale spaces

Now we can present the atomic decompositions for Lorenz martingale spaces.

Theorem 3.1.1 If the martingale $f \in H^s_{p,q}, 0 < p < \infty, 0 < q \leq \infty$ then there exist a sequence a^k of $(1, p, \infty)$ atoms and a positive real number sequence $(\mu_k) \in l_q$ such that

$$f_n = \sum_{k \in Z} \mu_k a^k_n, \forall n \in N$$

and

$$\|(\mu_k)_{k \in Z}\|_{l_q} \preceq \|f\|_{H^s_{p,q}}.$$

Conversely, if $0 < q \leq 1, q \leq p < \infty$, and the martingale f has the above decomposition, then $f \in H^s_{p,q}$ and

$$\|f\|_{H^s_{p,q}} \leq c \inf \|(\mu_k)_{k \in Z}\|_{l_q},$$

where the inf is taken over all the preceding decompositions of f.

Proof Assume that $f \in H^s_{p,q}, q \neq \infty$. Now considering the following stopping time for all $k \in Z$:

$$\tau_k = \inf\{n \in N : s_{n+1}(f) > 2^k\}(\inf \phi = \infty).$$

The sequence of these stopping times is obviously non-decreasing. It easy to see that

$$\begin{aligned}
\sum_{k \in Z}(f_n^{\tau_{k+1}} - f_n^{\tau_k}) &= \sum_{k \in Z}(\sum_{m=0}^{n} \chi_{\{m \leq \tau_{k+1}\}} \Delta_m f - \sum_{m=0}^{n} \chi_{\{m \leq \tau_k\}} \Delta_m f) \\
&= \sum_{k \in Z}(\sum_{m=0}^{n} \chi_{\{\tau_k < m \leq \tau_{k+1}\}} \Delta_m f) = f_n.
\end{aligned}$$

Let

$$\mu_k = 2^k 3 P(\tau_k < \infty)^{\frac{1}{p}},$$

and

$$a_n^k = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

If $\mu_k = 0$ then let $a_n^k = 0$. Then for a fixed k, (a_n^k) is a martingale. Since $s(f_n^{\tau_k}) \le 2^k, s(f_n^{\tau_{k+1}}) \le 2^{k+1}$,

$$s(a_n^k) \le \frac{s(f_n^{\tau_{k+1}}) + s(f_n^{\tau_k})}{\mu_k} \le P(\tau_k < \infty)^{-\frac{1}{p}}, \forall n \in N,$$

which implies (a_n^k) is a L_2-bounded martingale, so there exists $a^k \in L_2$ such that $E_n a^k = a_n^k$. If $n \le \tau_k$ then $a_n^k = 0$, so we get a^k is really a $(1, p, \infty)$ atom. And

$$\begin{aligned}
(\sum_{k\in Z} |\mu_k|^q)^{\frac{1}{q}} &= 3(\sum_{k\in Z}(2^k P(\tau_k < \infty)^{\frac{1}{p}})^q)^{\frac{1}{q}} = 3(\sum_{k\in Z}(2^k P(s(f) > 2^k)^{\frac{1}{p}})^q)^{\frac{1}{q}} \\
&\le c(\sum_{k\in Z}\int_{2^{k-1}}^{2^k} y^{q-1} dy P(s(f) > 2^k)^{\frac{q}{p}})^{\frac{1}{q}} \\
&\le c(\sum_{k\in Z}\int_{2^{k-1}}^{2^k} y^{q-1} P(s(f) > y)^{\frac{q}{p}} dy)^{\frac{1}{q}} \\
&\le c(\int_0^{\infty} y^{q-1} P(s(f) > y)^{\frac{q}{p}} dy)^{\frac{1}{q}} \\
&\le c\|f\|_{H^s_{p,q}}
\end{aligned}$$

If $q = \infty$, it only needs to make a standard rectification.

Conversely, if f has the above decomposition, then from $\|s(a^k)\|_\infty \le P(\tau_k < \infty)^{-\frac{1}{p}}$ and

$$P(s(a^k) > y) \le P(s(a^k) \ne 0) \le P(\tau_k < \infty),$$

we get

$$\begin{aligned}
\|a^k\|^q_{H^s_{p,q}} &= q\int_0^{\infty} y^{q-1} P(s(a^k) > y)^{\frac{q}{p}} dy = q\int_0^{P(\tau_k<\infty)^{-\frac{1}{p}}} y^{q-1} P(s(a^k) > y)^{\frac{q}{p}} dy \\
&\le P(\tau_k < \infty)^{\frac{q}{p}} \int_0^{P(\tau_k<\infty)^{-\frac{1}{p}}} y^{q-1} dy \le \frac{1}{q}.
\end{aligned}$$

For $0 < q \le 1, q \le p < \infty$, $\|\cdot\|_{p,q}$ is equivalent to a q-norm,

$$\|f\|^q_{H^s_{p,q}} \le \|\sum_{k\in Z} \mu_k s(a^k)\|^q_{p,q} \le \sum_{k\in Z} \mu_k^q \|s(a^k)\|^q_{p,q} \le c\sum_{k\in Z} \mu_k^q,$$

which gives the desired result.

Theorem 3.1.2 If the martingale $f \in Q_{p,q}, 0 < p < \infty, 0 < q \le \infty$, then there exist a sequence a^k of $(2, p, \infty)$ atoms and a real number sequence $\mu_k \in l_q$ such that

$$f_n = \sum_{k\in Z} \mu_k a_n^k, \forall n \in N$$

and

$$(\sum_{k\in Z}|\mu_k|^q)^{\frac{1}{q}}\leq c\|f\|_{Q_{p,q}}.$$

Conversely, if $0<q\leq 1, q\leq p<\infty$, and the martingale f has the above decomposition, then $f\in Q_{p,q}$ and

$$\|f\|_{Q_{p,q}}\leq c\inf(\sum_{k\in Z}|\mu_k|^q)^{\frac{1}{q}},$$

where the inf is taken all the above decompositions.

Proof Suppose that $f\in Q_{p,q}$. Let $\beta=(\beta_n)_{n\geq 0}$ is the optimal control of $S_n(f)$, i.e., $\beta\in\Lambda, S_n(f)\leq\beta_{n-1}, \|f\|_{Q_{p,q}}=\|\beta_\infty\|_{p,q}$. The stopping times τ_k are defined in this case by

$$\tau_k=\inf\{n\in N:\beta^n>2^k\}(\inf\phi=\infty).$$

Let a^k and $\mu_k (k\in Z)$ be define as in the proof of Theorem 3.1.1. Then for a fixed k, (a_n^k) is also a martingale. Since $S(f_n^{\tau_k})=S_{\tau_k}(f)\leq\beta_{\tau_{k-1}}\leq 2^k, S(f_n^{\tau_{k+1}})\leq 2^{k+1}$,

$$S(a_n^k)\leq\frac{S(f_n^{\tau_{k+1}})+S(f_n^{\tau_k})}{\mu_k}\leq P(\tau_k<\infty)^{-\frac{1}{p}}, \forall n\in N.$$

Similarly to Theorem 3.1.1, we can show a^k is really a $(2,p,\infty)$ atom. And

$$(\sum_{k\in Z}|\mu_k|^q)^{\frac{1}{q}}=3(\sum_{k\in Z}(2^kP(\tau_k<\infty)^{\frac{1}{p}})^q)^{\frac{1}{q}}=3(\sum_{k\in Z}(2^kP(\beta_\infty>2^k)^{\frac{1}{p}})^q)^{\frac{1}{q}}\leq c\|\beta_\infty\|_{p,q}=\|f\|_{Q_{p,q}}.$$

Conversely, if the a^k is $(2,p,\infty)$-atom, one can show $\|a^k\|^q_{H^S_{p,q}}\leq\frac{1}{q}$. The rest can be proved similarly to Theorem 3.1.1.

Theorem 3.1.3 If the martingale $f\in D_{p,q}, 0<p<\infty, 0<q\leq\infty$, then there exist a sequence a^k of $(3,p,\infty)$ atoms and a real number sequence $\mu_k\in l_q$ such that

$$f_n=\sum_{k\in Z}\mu_k a_n^k, \forall n\in N$$

and

$$(\sum_{k\in Z}|\mu_k|^q)^{\frac{1}{q}}\leq c\|f\|_{D_{p,q}}.$$

Conversely, if if $0<q\leq 1, q\leq p<\infty$, and the martingale f has the above decomposition, then $f\in D_{p,q}$ and

$$\|f\|_{D_{p,q}}\leq c\inf(\sum_{k\in Z}|\mu_k|^q)^{\frac{1}{q}},$$

where the inf is taken all the above decomposition.

The proof of Theorem 3.1.3 is similar to one of Theorem 3.1.2, so here omit it.

§ 3.2 Boundedness on sublinear operator

As one of applications of the atomic decompositions, we shall obtain a sufficient condition for a sublinear operator to be bounded from Lorentz martingale spaces to function Lorentz spaces. Applying the condition to Mf, Sf and sf, we deduce a series of inequalities on Lorentz martingale spaces.

An operator $T : X \to Y$ is called a sublinear operator if it satisfies

$$|T(f+g)| \le |Tf| + |Tg|, |T(\alpha f)| \le |\alpha||Tf|,$$

where X is a martingale space, Y is a measurable function space.

Theorem 3.2.1 Let $T : H_r^s \to L_r$ be a bounded sublinear operator for some $1 \le r < \infty$. If

$$P(|Ta| > 0) \le cP(\tau < \infty)$$

for all $(1, p, \infty)$-atoms a, where τ is the stopping time associate with a, then for $0 < p < r, 0 < q \le \infty$, we have

$$\|Tf\|_{p,q} \le c\|f\|_{H^s_{p,q}}, \qquad f \in H^s_{p,q}.$$

Proof Assume that $f \in H^s_{p,q}$. By Theorem 2.1, f can be decomposed into the sum of a sequence of $(1, p, \infty)-$atoms. For any fixed $y > 0$ choose $j \in Z$ such that $2^j \le y < 2^{j+1}$ and let

$$f = \sum_{k\in Z} \mu_k a^k = \sum_{k=-\infty}^{j-1} \mu_k a^k + \sum_{k=j}^{\infty} \mu_k a^k =: g + h.$$

Recall that $\mu_k = 2^k 3P^{1/p}(\tau_k < \infty)$ and $s(a^k) = 0$ on the set $\{\tau_k = \infty\}$. we have

$$\begin{aligned}
\|g\|_{H_r^s} &\le (\int_\Omega (\sum_{k=-\infty}^{j-1} \mu_k s(a^k))^r dP)^{1/r} \le \sum_{k=-\infty}^{j-1} \mu_k (\int_\Omega (s(a^k))^r dP)^{1/r} \\
&\le \sum_{k=-\infty}^{j-1} \mu_k P^{-\frac{1}{p}}(\tau_k < \infty) P^{\frac{1}{r}}(\tau_k < \infty) \\
&= \sum_{k=-\infty}^{j-1} 2^k P^{\frac{1}{r}}(\tau_k < \infty) \\
&= \sum_{k=-\infty}^{j-1} 2^k P^{\frac{1}{r}}(s(f) > 2^k)
\end{aligned}$$

It follows from the boundedness of T that

$$\begin{aligned}
P(|Tg|>y) &\leq y^{-r}E|Tg|^r \leq cy^{-r}\|g\|_{H_r^s}^r \\
&\leq cy^{-r}(\sum_{k=-\infty}^{j-1} 2^k P^{\frac{1}{r}}(s(f)>2^k))^r \\
&= y^{-r}(\sum_{k=-\infty}^{j-1} 2^{k(1-\frac{p}{r})}2^{k\frac{p}{r}}P^{\frac{1}{r}}(s(f)>2^k))^r \\
&\leq cy^{-r}(y^{\frac{p}{r}}P^{\frac{1}{r}}(s(f)>2^k))^r(\sum_{k=-\infty}^{j-1} 2^{k(1-\frac{p}{r})})^r \\
&\leq cP(s(f)>y)
\end{aligned}$$

On the other hand, since $|Th| \leq \sum_{k=j}^{\infty} \mu_k |Ta^k|$, we get

$$\begin{aligned}
P(|Th|>y) &\leq P(|Th|>0) \leq \sum_{k=j}^{\infty} P(|Ta^k|>0) \\
&\preceq \sum_{k=j}^{\infty} P(\tau_k<\infty) = \sum_{k=j}^{\infty} 2^{-kp}2^{kp}P(sf>2^k) \\
&\leq cyP(sf>y)\sum_{k=j}^{\infty} 2^{-kp} \\
&\leq cP(s(f)>y)
\end{aligned}$$

Since T is subliear, $(|Tf|)^*(t) \leq (|Tg|+|Th|)^*(t) \leq |Tg|^*(\frac{t}{2}) + |Th|^*(\frac{t}{2})$, thus

$$\|Tf\|_{p,q} \leq \|Tg\|_{p,q} + \|Th\|_{p,q} \leq \|f\|_{H_{p,q}^s}.$$

Similarly to the proof of Theorem 3.1, we can prove the following Theorems 3.2 and 3.3 by using Theorems 2.2 and 2.3, respectively. Here we only give the theorems and omit the proofs.

Theorem 3.2.2 Let $T : Q_r \to L_r$ be a bounded sublinear operator for some $1 \leq r < \infty$. If

$$P(|Ta|>0) \leq cP(\tau<\infty)$$

for all $(2,p,\infty)-$atoms a, where τ is the stopping time associate with a, then for $0<p<r, 0<q\leq\infty$, we have

$$\|Tf\|_{p,q} \preceq \|f\|_{Q_{p,q}}, \qquad f \in Q_{p,q}.$$

Theorem 3.2.3 Let $T : D_r \to L_r$ be a bounded sublinear operator for some $1 \leq r < \infty$. If

$$P(|Ta|>0) \leq cP(\tau<\infty)$$

for all $(3, p, \infty)$-atoms a, where τ is the stopping time associate with a, then for $0 < p < r, 0 < q \leq \infty$,

$$\|Tf\|_{p,q} \leq c\|f\|_{D_{p,q}}, \qquad f \in D_{p,q}.$$

Theorem 3.2.4 For all martingale $f = (f_n)_{n\geq 0}$ the following inequalities hold:

1)If $0 < p < 2, 0 < q \leq \infty$,

$$H^s_{p,q} \hookrightarrow H^*_{p,q}, \qquad H^s_{p,q} \hookrightarrow H^S_{p,q},$$

if $p > 2, 0 < q \leq \infty$,

$$H^*_{p,q} \hookrightarrow H^s_{p,q}, \qquad H^S_{p,q} \hookrightarrow H^s_{p,q},$$

2) If $0 < p < \infty, 0 < q \leq \infty$,

$$Q_{p,q} \hookrightarrow H^*_{p,q}, \quad Q_{p,q} \hookrightarrow H^S_{p,q}, \quad Q_{p,q} \hookrightarrow H^s_{p,q}$$

$$D_{p,q} \hookrightarrow H^*_{p,q}, \quad D_{p,q} \hookrightarrow H^S_{p,q}, \quad D_{p,q} \hookrightarrow H^s_{p,q}$$

Proof 1) The maximal operator $Tf = Mf$ is sublinear, and $\|Mf\|_2 \leq \|sf\|_2$. If a is any $(1, p, \infty)-$atom and τ is the corresponding stopping time, then $\{|Ta| > 0\} = \{|Ma| > 0\} \subset \{\tau < \infty\}$ and hence $P(|Ta| > 0) \leq cP(\tau < \infty)$. It follows from Theorem 3.1 that

$$\|Mf\|_{p,q} \leq c\|f\|_{H^s_{p,q}}, \qquad (0 < p < 2).$$

Similarly, consider the operator $Tf = Sf$ we get $\|Sf\|_{p,q} \preceq \|f\|_{H^s_{p,q}}$. Conversely, we use interpolation by considering the following operator to obtain the case $p > 2, 0 < q \leq \infty$. In fact, consider operator $Q : L_p(l_\infty) \to L_p$ by $Q(f) = s(f)$, then Q is bounded for all $p \geq 2$. So by interpolation, Q is bounded from $L_{p,q}(l_\infty)$ to $L_{p,q}$ for $p > 2, 0 < q \leq \infty$.

2) For all $0 < r < \infty$, $\|Mf\|_r, \|Sf\|_r, \|sf\|_r \preceq \|f\|_{Q_r}$ and $\|Mf\|_r, \|Sf\|_r, \|sf\|_r \preceq \|f\|_{D_r}$. Note that $a^k_n = 0$ on the set $\{n \leq \tau_k\}$, thus

$$\chi(n \leq \tau_k)E_{n-1}|\Delta_n a^k|^2 = E_{n-1}\chi(n \leq \tau_k)|\Delta_n a^k|^2 = 0.$$

Hence $s(a^k) = 0$ on the set $\{\tau_k = \infty\}$. By Theorem 3.2.2 and 3.2.3, we can complete the proofs.

Remark If put $p = q$ in the above embedding, Theorem 2.11 in [97] can be deduced; if put $q = \infty$, Theorem 7 and Theorem 8 in [39] can be concluded.

Remark We conjecture that for $1 \leq p < \infty, 0 < q \leq \infty$, $H^S_{p,q} = H^*_{p,q}$, however our method doesn't show these.

§ 3.3 Restricted weak interpolation

We say that a sublinear operator T is of restricted weak-type (p, q) if T maps $H^s_{p,1}$ to $L_{p,\infty}$. Then we have the next interpolation from one restricted weak-type estimate to another.

Theorem 3.3.1 Let T is of restricted weak-type (p_i, q_i) for $i = 0, 1$, and $1 < p_i, q_i < \infty$. Put

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad \forall 0 \le \theta \le 1.$$

Then T is also of restricted weak-type (p, q).

Proof Suppose that $f \in H^s_{p,1}$, from Theorem 2.1, $f = \sum_{k\in Z} \mu_k a^k$, a^k is $(1, p, \infty)$-atoms with respect to stopping time τ_k, and $\sum_{k\in Z} |\mu_k| \preceq \|f\|_{H^s_{p,1}}$. Now we can estimate $\|Ta^k\|_{q,\infty} \le c$. In fact

$$\begin{aligned}
\|Ta^k\|_{q,\infty} &= \sup_{t>0} t^{\frac{1}{p}}(Ta^k)^*(t) = \sup_{t>0}(t^{\frac{1}{q_0}}(Ta^k)^*(t))^{1-\theta}(t^{\frac{1}{q_1}}(Ta^k)^*(t))^{\theta} \\
&\le \|Ta^k\|^{1-\theta}_{q_0,\infty}\|Ta^k\|^{\theta}_{q_1,\infty} \\
&\le c\|sa^k\|^{1-\theta}_{p_0,1}\|sa^k\|^{\theta}_{p_1,1} \\
&\le c\|sa^k\|^{1-\theta}_{2p_0,2p_0}\|\chi_{\{\tau_k<\infty\}}\|^{1-\theta}_{2p_0,l}\|sa^k\|^{\theta}_{2p_1,2p_1}\|\chi_{\{\tau_k<\infty\}}\|^{\theta}_{2p_1,m} \\
&\le cP(\tau_k < \infty)^{-\frac{1}{p}}(P(\tau_k < \infty)^{\frac{1-\theta}{2p_0}}P(\tau_k < \infty)^{\frac{\theta}{2p_1}})^2 \\
&\le c,
\end{aligned}$$

where $l = \frac{2p_0}{2p_0-1}$ and $m = \frac{2p_1}{2p_1-1}$. Consequently,

$$\|Tf\|_{q,\infty} \le \sum_{k\in Z} |\mu_k|\|Ta^k\|_{q,\infty} \le c\sum_{k\in Z} |\mu_k| \preceq \|f\|_{H^s_{p,1}}.$$

The proof is finished.

Now we show how restricted weak-type estimate can be transferred to strong type. It is also the version of the classical Marcinkiewicz interpolation theorem in the martingale setting(see Theorem 4.13 in [5]).

Theorem 3.3.2 Let T is of restricted weak-type (p_i, q_i) for $i = 0, 1$, and $1 < p_i < \infty, 1 < q_i \le \infty, q_0 \ne q_1$. Put

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad \forall 0 \le \theta \le 1.$$

Then T is of type $(H^s_{p,r}, L_{q,r})$, for $0 < r < 1$ and $r \le q$.

Proof For $0 < r < 1$ and $r \le q$, we know $\|\cdot\|_{q,r}$ is equivalent to a $r-$norm, so it is enough to prove $\|Ta\|_{q,r} \le c$, for all $(1, p, \infty)$-atoms. Once it is proved, from

Theorem 3.2.1,

$$\|Tf\|_{q,r}^r \le \sum_{k\in Z} \mu_k^r \|Ta\|_{q,r}^r \le c\sum_{k\in Z}\mu_k^r \le c\|f\|_{H_{p,r}^s}^r.$$

Now we shall show $\|Ta\|_{q,r} \le c$. Consider the case $q_1, q_2 < \infty$. From the previous proof, it is easy to know

$$\|a\|_{H_{p_i,1}^s}^{p_i} \le cP(\tau<\infty)^{1-\frac{p_i}{p}}, \qquad i=0,1.$$

Thus, say $q_0 < q < q_1$, we get

$$\begin{aligned}
\frac{1}{q}\|Ta\|_{q,r}^q &= \int_0^\infty y^{r-1}P(|Ta|>y)^{\frac{r}{q}}dy \\
&\le \int_0^\delta y^{r-1}(\frac{1}{y}\|a\|_{H_{p_0,1}^s})^{\frac{q_0 r}{q}}dy + \int_\delta^\infty y^{r-1}(\frac{1}{y}\|a\|_{H_{p_1,1}^s})^{\frac{q_1 r}{q}}dy \\
&\le c(\delta^{\frac{r}{q}(q-q_0)}P(\tau<\infty)^{\frac{rq_0}{q}(\frac{1}{p_0}-\frac{1}{p})} + \delta^{\frac{r}{q}(q-q_1)}P(\tau<\infty)^{\frac{rq_1}{q}(\frac{1}{p_1}-\frac{1}{p})})
\end{aligned}$$

Taking $\delta = P(\tau<\infty)^\alpha$, with α satisfying

$$q\alpha = (\frac{1}{p_0}-\frac{1}{p})/(\frac{1}{q}-\frac{1}{q_0}) = (\frac{1}{p_1}-\frac{1}{p})/(\frac{1}{q}-\frac{1}{q_1})$$

In fact, from $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$ we can know $q\alpha = (\frac{1}{p_0}-\frac{1}{p_1})/(\frac{1}{q_1}-\frac{1}{q_0})$, and

$$\frac{r}{q}[\alpha(q-q_0)+q_0(\frac{1}{p_0}-\frac{1}{p})] = \frac{r}{q}[\alpha(q-q_1)+q_1(\frac{1}{p_1}-\frac{1}{p})] = 0.$$

Then $\|Ta\|_{q,r}^q \preceq c$.

When one of q_i is ∞, say $q_1=\infty$, the proof is unchanged. More precisely, we have

$$\|Ta\|_\infty \le c\|a\|_{H_{p_1,1}^s} \le cP(\tau<\infty)^{\frac{1}{p_1}-\frac{1}{p}}.$$

Thus, from $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $\frac{1}{q}=\frac{1-\theta}{q_0}$

$$\begin{aligned}
\frac{1}{q}\|Ta\|_{q,r}^q &= \int_0^{\|Ta\|_\infty} y^{r-1}P(|Ta|>y)^{\frac{r}{q}}dy \\
&\le \int_0^{\|Ta\|_\infty} y^{r-1}(\frac{1}{y}\|a\|_{H_{p_0,1}^s})^{\frac{q_0 r}{q}}dy \\
&\le cP(\tau<\infty)^{\frac{rq_0}{q}(\frac{1}{p_0}-\frac{1}{p})}P(\tau<\infty)^{\frac{r}{q}(q-q_0)(\frac{1}{p_1}-\frac{1}{p})} \\
&\le c
\end{aligned}$$

We complete the proof.

Remark From Theorems 3.2.2 and 3.2.3, we can conclude that the familiar results hold for $Q_{p,1}$ and $D_{p,1}$. We shall not state these explicitly. □

Chapter 4 Embeddings on vector-valued Lorentz martingale spaces

§ 4.1 Introductions and Notations

As we all know that Lorentz spaces are the extensions of Lebesgue spaces, and some important facts in Lebesgue spaces have been found to have their satisfactory counterparts in Lorentz spaces. Many papers have tried to reveal these results. It is also well known the validity of a classical (scalar-valued) result in the vector-valued setting, i.e., for functions or martingale with values in a Banach space, depends on the geometric properties of the underlying Banach space; the relevant properties are often the uniform convexity and smoothness. Let us recall Pisier's celebrated work [80]on martingale inequalities in uniformly convex spaces. Let $1 < q < \infty$. Then a Banach space X has an equivalent q-uniformly convex norm iff for one $1 < p < \infty$ (or equivalently, for every $1 < p < \infty$) there exists a positive constant C such that

$$\left\| \left(\|f_1\|^q + \sum_{n \geq 2} \|f_n - f_{n-1}\|^q \right)^{1/q} \right\|_p \leq C \sup_{n \geq 1} \|f_n\|_p \tag{4.1}$$

for all L_p-martingales f with values in X. The validity of the converse inequality amounts to saying that X has an equivalent q-uniformly smooth norm. Then the main goal of the present paper is to extend (4.1) to the Lorentz spaces case. More precisely, we obtain the following results. Let $1 < q < \infty$. Then a Banach space X has an equivalent q-uniformly convex norm iff for some $1 < r < \infty, 1 \leq s \leq \infty$ (or equivalently, for every $1 < r < \infty, 1 \leq s \leq \infty$) there exists a positive constant C such that

$$\left\| \left(\|f_1\|^q + \sum_{n \geq 2} \|f_n - f_{n-1}\|^q \right)^{1/q} \right\|_{r,s} \leq c \sup_{n \geq 1} \|f_n\|_{r,s} \tag{4.2}$$

for all $L_{r,s}$-martingales f with values in X. Again, the validity of the converse inequality amounts to saying that X has an equivalent q-uniformly smooth norm.

In the preceding papers, the main methods to deal with martingale theory are the stopping time, atomic decomposition, scalar-valued martingale transform, interpolation and so on, see for instance [35] [54] [59] [97]. It should be mentioned that in this chapter we employ operator-valued martingale transforms, which were introduced by Martinez and Torrea [68] in 2000. Replacing the scalar-valued multiplying sequences by operator-valued multiplying sequences, they generalized the Burkholder martingale transforms [10]. The key fact in order to get our desired results is to identify the p-variant operator $S^{(p)}(f)$ of a Banach-valued martingale f with the maximal operator of a ℓ^p-valued martingale transform. As we can see, under this point of view, it is so short and transparent to obtain our desired martingale inequalities. Finally we give a equivalent characterization of UMD Banach lattices in the Lorentz spaces setting.

Let X be a Banach space. For $1 \le p \le \infty$ the usual L_p-space of strongly p-integrable X-valued functions on $(\Omega, \mathcal{F}, P)$ will be denoted by $L_p(\Omega; X)$ or simply by $L_p(X)$. Let $\{\mathcal{F}_n\}_{n\ge1}$ be an increasing sequence of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F} = \bigvee \mathcal{F}_n$. By an X-valued martingale relative to $\{\mathcal{F}_n\}_{n\ge1}$ we mean a sequence $f = \{f_n\}_{n\ge1}$ in $L_1(X)$ such that $\mathbb{E}(f_{n+1}\big|\mathcal{F}_n) = f_n$ for every $n \ge 1$. Let $df_n = f_n - f_{n-1}$ with the convention that $f_0 = 0$. $\{df_n\}_{n\ge1}$ is the martingale difference sequence of f. We will use the following standard notations from martingale theory

$$M_n(f) = \sup_{1\le k\le n} \|f_k\|, \qquad M(f) = \sup_{n\ge1} \|f_n\|;$$

$$S_n^{(q)}(f) = \Big(\sum_{k=1}^{n} \|df_k\|^q\Big)^{1/q}, \qquad S^{(q)}(f) = \Big(\sum_{k=1}^{\infty} \|df_n\|^q\Big)^{1/q}.$$

For $1 < r < \infty, 1 \le s \le \infty$, we define the following martingale spaces:

$$H^*_{r,s} = \{f = (f_n)_{n\ge1} : \|M(f)\|_{r,s} < \infty\}$$

$${}_qH^S_{r,s} = \{f = (f_n)_{n\ge1} : \|S^{(q)}(f)\|_{r,s} < \infty\}$$

We refer to [35] [59] and [97]for more facts on scalar martingale theory, and [27] and [56] for vector-valued case.

Given a X-valued martingale $f = (f_n)_{n\ge1}$, we define

$$\|f\|_{p,q} = \sup_n \|f_n\|_{p,q}.$$

§ 4.2 Operator-valued martingale transform

Lemma 4.2.1 Let X_1 and X_2 be two Banach space, T a martingale transform operator as above. Then the following statements are equivalent:

(1) There exists a positive constant C such that

$$P(M(Tf) > \lambda) \leq C\|M(f)\|_1 \quad \forall \lambda > 0$$

(2) For any $1 < p < \infty, 1 \leq q \leq \infty$, there exists $C = C_{p,q} > 0$ such that

$$\|M(Tf)\|_{p,q} \leq C\|Mf\|_{p,q}$$

(3) For some $1 < p_0 < \infty, 1 \leq q_0 \leq \infty$, there exists $C = C_{p_0,q_0} > 0$ such that

$$\|M(Tf)\|_{p_0,q_0} \leq C\|Mf\|_{p_0,q_0}$$

Lemma 4.2.2 Let $(A_n)_{n\geq 1}$ be a nonnegative, increasing and adapted sequence, $Y \geq 0$. If $\mathbb{E}(A_\infty - A_{\tau-1}|\mathcal{F}_\tau) \leq \mathbb{E}(Y|\mathcal{F}_\tau)$ for any stopping time τ, then we have

$$\|A_\infty\|_{p,q} \leq C\|Y\|_{p,q}\,, \qquad 1 < p < \infty, 1 \leq q \leq \infty.$$

Proof For any fixed $\lambda > 0$, setting stopping time $\tau = \inf\{n : A_n > \lambda\}$, then $A_{\tau-1} \leq \lambda$, and $\{A_\infty > \lambda\} = \{\tau < \infty\} \in \mathcal{F}_\tau$, by the condition, we get

$$\int_{\{A_\infty > \lambda\}} (A_\infty - \lambda) \leq \int_{\{A_\infty > \lambda\}} Y.$$

For $t \in [0,1]$, setting $\lambda = A_\infty^*(t)$, the inequality above implies

$$\int_0^t A_\infty^*(s)ds \leq \int_0^t Y^*(s)ds + tA_\infty^*(t).$$

Observe that, for any $t \in [0,1]$ and fixed $t_0 \in (0,1)$ we have

$$\int_0^t A_\infty^*(s)ds \geq \int_0^{t_0 t} A_\infty^*(s)ds + (1-t_0)tA_\infty^*(t) = t_0\int_0^t A_\infty^*(t_0 s)ds + (1-t_0)tA_\infty^*(t)$$

The preceding inequalities yield

$$\int_0^t A_\infty^*(t_0 s)ds \leq \frac{1}{t_0}\int_0^t Y^*(s)ds + tA_\infty^*(t) \leq \int_0^t \big(A_\infty^*(s) + \frac{1}{t_0}Y^*(s)\big)ds,$$

which leads to

$$A_\infty^{**}(t_0 t) \leq A_\infty^{**}(t) + \frac{1}{t_0}Y^{**}(t).$$

A change of variable gives

$$(t_0)^{-1/p}\|A_\infty\|_{(p,q)} \leq \|A_\infty\|_{(p,q)} + (t_0)^{-1}\|Y\|_{(p,q)}.$$

Noting $t_0 \in (0,1)$, we finally have

$$\|A_\infty\|_{p,q} \leq C\|Y\|_{p,q}.$$

□

Remark In Lemma 2.3 it is sufficient to verify $\mathbb{E}(A_\infty - A_{\tau-1}|\mathcal{F}_\tau) \leq \mathbb{E}(Y|\mathcal{F}_\tau)$ for stopping times taking constant values n, see for example [59].

Lemma 4.2.3 Let $1 < p < \infty, 1 \leq q \leq \infty$. Then for any X-valued martingale $f = (f_n)_{n\geq 1} \in H^*_{p,q}$ we have $f = g + h$, where g and h are martingales satisfying the following conditions:

(1)$\|dg_n\| \leq 4M_{n-1}(df)$; (2)$\big\| \sum_{n=1}^{\infty} \|dh_n\| \big\|_{p,q} \leq C\|M(df)\|_{p,q}$

Proof Setting $F_i = df_i \mathbb{1}_{\{\|df_i\| \leq 2M_{i-1}(df)\}}$ and $G_i = df_i \mathbb{1}_{\{\|df_i\| > 2M_{i-1}(df)\}}$. Now we let

$$dg_i = F_i - \mathbb{E}(F_i|\mathcal{F}_{i-1}), \quad g_n = \sum_{i=1}^{n} dg_i\,; \qquad dh_i = G_i - \mathbb{E}(G_i|\mathcal{F}_{i-1}), \quad h_n = \sum_{i=1}^{n} dh_i$$

It is obvious that $g = (g_n)_{n\geq 1}$ and $h = (h_n)_{n\geq 1}$ are martingales, and $\|dg_n\| \leq 4M_{n-1}(df)$. Note that $\|G_i\| = 2\|G_i\| - \|G_i\| \leq 2M_i(df) - 2M_{i-1}(df)$, we get

$$\begin{aligned}
\sum_{i=n}^{\infty} \|dh_i\| &\leq 2\sum_{i=n}^{\infty} \big(M_i(df) - M_{i-1}(df)\big) + 2\sum_{i=n}^{\infty} \mathbb{E}\big(M_i(df) - M_{i-1}(df)\big|\mathcal{F}_{i-1}\big) \\
&\leq 2M(df) + 2\sum_{i=1}^{\infty} \mathbb{E}\big(M_i(df) - M_{i-1}(df)\big|\mathcal{F}_{i-1}\big) \\
&\leq 2M(df) + 2\gamma_\infty
\end{aligned}$$

where $\gamma_\infty = \sum_{i=1}^{\infty} \mathbb{E}\big(M_i(df) - M_{i-1}(df)\big|\mathcal{F}_{i-1}\big)$. Then by Corollary 2 in [?],

$$\|\gamma_\infty\|_{p,q} \leq C\|\sum_{i=1}^{\infty} M_i(df) - M_{i-1}(df)\|_{p,q} \leq C\|M(df)\|_{p,q}.$$

Thus

$$\mathbb{E}\big(\sum_{i=n}^{\infty} \|dh_i\|\big|\mathcal{F}_{i-1}\big) \leq \mathbb{E}\big(2M(df) + 2\gamma_\infty\big|\mathcal{F}_{i-1}\big).$$

It follows from Lemma 2.4 that

$$\big\|\sum_{n=1}^{\infty} \|dh_n\|\big\|_{p,q} \leq C\|M(df) + \gamma_\infty\|_{p,q} \leq C\|M(df)\|_{p,q} + C\|\gamma_\infty\|_{p,q} \leq C\|M(df)\|_{p,q}.$$

□

Now we can give the proof of Lemma 2.3.

Proof of Lemma 4.2.1 (1) $\Rightarrow$ (2) Considering a martingale $f \in H^*_{p,q}$, by Lemma 2.6 we can decompose f as $f = g + h$, then

$$\|M(Tf)\|_{p,q} \leq C\|M(Tg)\|_{p,q} + C\|M(Th)\|_{p,q} \tag{4.3}$$

Since the boundedness of the sequence (υ_k) we get

$$\begin{aligned}
\|M(Th)\|_{p,q} &= \big\|\sup_n \|\sum_{k=1}^{n} \upsilon_k dh_k\|_{X_2}\big\|_{p,q} \leq \big\|\sup_n \sum_{k=1}^{n} \|\upsilon_k\|\|dh_k\|_{X_1}\big\|_{p,q} \\
&\leq \big\|\sup_n \sum_{k=1}^{n} \|dh_k\|_{X_1}\big\|_{p,q} = \big\|\sum_{k=1}^{\infty} \|dh_k\|_{X_1}\big\|_{p,q} \\
&\leq C\|M(f)\|_{p,q}
\end{aligned}$$

Setting $W_n = 4M_{n-1}(df)$, then W_n is nondecreasing and $\mathcal{F}_{n-1}$-measurable. Fix $\lambda > 0$. For $\beta > 0, \delta > 0$ satisfying $\beta > \delta + 1$, define the stopping times:

$$\mu = \inf\{n : \|(Tg)_n\|_{X_2} > \lambda\}, \quad \nu = \inf\{n : \|(Tg)_n\|_{X_2} > \beta\lambda\}, \quad \sigma = \inf\{n : \|g_n\|_{X_1} \bigvee W_{n+1} > \delta\lambda\}.$$

Now we denote $u_n = \mathbb{1}_{\{\mu<n\leq\nu \bigwedge \sigma\}}$. Since $\{\mu < n \leq \nu \bigwedge \sigma\}$ is $\mathcal{F}_{n-1}$-measurable, we can consider the martingale $a_n = \sum_{k=1}^n u_k dg_k$ and its martingale transform $(Ta)_n = \sum_{k=1}^n \upsilon_k u_k dg_k$. Note that $\|dg_n\| \leq W_n$, by the definition of stopping time σ, we have $M(a) \leq 2\delta\lambda$ in the set $\{\mu < \infty\}$ and $M(a) = 0$ in $\{\mu = \infty\}$. Then

$$\|M(a)\|_1 \leq 2\delta\lambda P(\mu < \infty) = 2\delta\lambda P\big(M(Tg) > \lambda\big).$$

By the condition (1), we get

$$P\big(M(Ta) > (\beta-\delta-1)\lambda\big) \leq \frac{C\|M(a)\|_1}{(\beta-\delta-1)\lambda} \leq \frac{2C\delta}{\beta-\delta-1}P\big(M(Tg) > \lambda\big).$$

If $w \in \{\mu < n \leq \nu \bigwedge \sigma\}$, then $(Ta)_n = (Tg)_n$; it is easy to see

$$P\big(M(Tg) > \beta\lambda, M(W) \leq \delta\lambda\big) \leq P\big(M(Ta) > (\beta-\delta-1)\lambda\big).$$

Thus

$$\begin{aligned} P\big(M(Tg) > \beta\lambda\big) &\leq P\big(M(Tg) > \beta\lambda, M(W) \leq \delta\lambda\big) + P\big(M(W) > \delta\lambda\big) \\ &\leq \frac{2C\delta}{\beta-\delta-1}P\big(M(Tg) > \lambda\big) + P\big(M(W) > \delta\lambda\big) \end{aligned}$$

Denote $\rho = \frac{2C\delta}{\beta-\delta-1}$. By the equivalent definition of $L_{p,q}$-norm, we get

$$\begin{aligned} \beta^{-1}\|M(Tg)\|_{p,q} &\leq \rho^{1/p}\|M(Tg)\|_{p,q} + \delta^{-1}\|M(W)\|_{p,q} \\ &\leq \rho^{1/p}\|M(Tg)\|_{p,q} + \delta^{-1}\|4M(df)\|_{p,q} \\ &\leq \rho^{1/p}\|M(Tg)\|_{p,q} + 8\delta^{-1}\|M(f)\|_{p,q} \end{aligned}$$

Now we take δ to satisfy $1 - \beta\rho^{1/p} > 0$, then $\|M(Tg)\|_{p,q} \leq C\|M(f)\|_{p,q}$. Finally it follows from (6.10),

$$\|M(Tf)\|_{p,q} \leq C\|M(f)\|_{p,q}.$$

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) We shall use Gundy's decomposition, see [59]. Fix $\lambda > 0$ we can decompose $f = a + b + e$ with a, b, c being martingales and satisfying respectively:

$$\lambda P\big(\sup\|da_n\| \neq 0\big) \leq C\|f\|_1; \quad \int\sum_{k=1}^{\infty}\|db_k\|dP \leq C\|f\|_1; \quad \sup_n\|e_n\| \leq C\lambda \quad \text{and} \quad \|e\|_1 \leq C\|f\|_1.$$

Then

$$P\big(M(Tf) > \lambda\big) \le P\big(M(Ta) > \lambda/3\big) + P\big(M(Tb) > \lambda/3\big) + P\big(M(Te) > \lambda/3\big).$$

Moreover, we have

$$P\big(M(Ta) > \lambda/3\big) \le P\big(\sup \|da_n\| \ne 0\big) \le \frac{C}{\lambda}\|f\|_1 \le \frac{C}{\lambda}\|M(f)\|_1.$$

and

$$\begin{aligned} P\big(M(Tb) > \lambda/3\big) &\le \frac{3}{\lambda}\int \sup_n \|(Tb)_n\| dP = \frac{3}{\lambda}\int \sup_n \|\sum_{k=1}^n \upsilon_k db_k\| dP \\ &\le \frac{3}{\lambda}\int \|\sum_{k=1}^\infty db_k\| dP \le \frac{C}{\lambda}\|f\|_1 \le \frac{C}{\lambda}\|M(f)\|_1. \end{aligned}$$

It is clear that $L_{p_0,1} \hookrightarrow L_{p_0,q_0} \hookrightarrow L_{p_0,\infty}$ for $1 < p_0 < \infty, 1 \le q_0 \le \infty$. Using the hypothesis and noting $\sup_n \|e_n\| \le C\lambda$, we get

$$\begin{aligned} \|M(Te)\|_{p_0,\infty} &\le C\|M(Te)\|_{p_0,q_0} \le C\|M(e)\|_{p_0,q_0} \le C\|e\|_{p_0,q_0} \le C\|e\|_{p_0,1} \\ &= \sup_n \int_0^\infty P(\|e_n\| > t)^{1/p_0} dt \le C \sup_n \int_0^{C\lambda} \big(t^{-1}\|e_n\|_1\big)^{1/p_0} dt \\ &= C\lambda^{1-1/p_0}\|e\|_1^{1/p_0} \end{aligned}$$

which leads to

$$P\big(M(Te) > \lambda/3\big) \le \frac{C}{\lambda}\|e\|_1 \le \frac{C}{\lambda}\|f\|_1 \le \frac{C}{\lambda}\|M(f)\|_1.$$

□

§ 4.3 Embeddings

Regarding the the maximal operator and p-variant operator as two martingale transform operators respectively and applying Lemma 4.2.1, by handling the two concrete martingale transform operators, we easily obtain some embeddings between vector-valued Lorentz martingale spaces. As usual, the geometric properties of the underlying Banach space are important. The following two lemmas are due to Liu [56].

Lemma 4.3.1 Let X be a Banach space. For $2 \le q < \infty$, the following statements are equivalent:

(1) X is isomorphic to a q-uniformly convex space

(2) For any X-valued martingale f there exists a constant $C > 0$ such that

$$\lambda P(S^{(q)}(f) > \lambda) \leq C\|M(f)\|_1 , \quad \forall \lambda > 0.$$

Lemma 4.3.2 Let X be a Banach space. For $1 < q \leq 2$, the following statements are equivalent:

(1) is isomorphic to a q-uniformly smooth space

(2) For any X-valued martingale f there exists a constant $C > 0$ such that

$$\lambda P(M(f) > \lambda) \leq C\|S^{(q)}(f)\|_1 , \quad \forall \lambda > 0.$$

Theorem 4.3.1 Let X be a Banach space. For $2 \leq q < \infty$, the following statements are equivalent:

(1) X is isomorphic to a q-uniformly convex space

(2) For any $1 < r < \infty, 1 \leq s \leq \infty$ (or equivalently, for some)

$$H^*_{r,s} \hookrightarrow {}_qH^S_{r,s} , \quad \forall f \in H^*_{r,s}$$

Proof Considering the martingale transform operator T from the family of X-valued martingales to that of $\ell^q(X)$-valued martingales. Let $\upsilon_k \in L\big(X, \ell^q(X)\big)$ be the operator defined by $\upsilon_k x = \{x_j\}_{j=1}^{\infty}$ for $x \in X$, where $x_j = x$ if $j = k$ and $x_j = 0$ otherwise. T is the martingale transform associated to the multiplying sequence (υ_k) :

$$(Tf)_n = \sum_{k=1}^{n} \upsilon_k df_k = (df_1, df_2, ..., df_n, 0, ...)$$

Then

$$M(Tf) = \sup_n \|(Tf)_n\|_{\ell^q(X)} = S^{(q)}(f)$$

Since X is isomorphic to a q-uniformly convex space, by Lemma 4.3.1, the martingale transform operator T satisfies (1) in Lemma 4.2.1. Thus the equivalence is obtained immediately. □

Theorem 4.3.2 Let X be a Banach space. For $1 < q \leq 2$, the following statements are equivalent:

(1) X is isomorphic to a q-uniformly smooth space

(2) For any $1 < r < \infty, 1 \leq s \leq \infty$ (or equivalently, for some)

$${}_qH^S_{r,s} \hookrightarrow H^*_{r,s} , \quad \forall f \in {}_qH^S_{r,s}$$

Proof Let $\ell^q(X)$-valued martingale $F = (F_n)_{n\geq 1}$, $F_n = \sum_{k=1}^n D_k$, $D_k = (D_k^j)_{j\geq 1}$. Define the martingale transform operator R from the family of $\ell^q(X)$-valued martingales to that of X-valued martingales. Let $\upsilon_k \in L\big(\ell^q(X), X\big)$ be the operator defined by $\upsilon_k x = x^k$ for all $x = \{x^j\}_{j\geq 1} \in \ell^q(X)$. R is the martingale transform associated to the multiplying sequence (υ_k) :

$$(RF)_n = \sum_{k=1}^n \upsilon_k D_k = \sum_{k=1}^n D_k^k.$$

Now for any X-valued martingale f with $f_n = \sum_{k=1}^n df_k$, we can choose the $\ell^q(X)$-valued martingale $F = (F_n)_{n\geq 1}$ with $D_k^j = df_k$ if $j = k$ and $D_k^j = 0$ if $j \neq k$. Then

$$(RF)_n = \sum_{k=1}^n D_k^k = \sum_{k=1}^n df_k = f_n, \quad M(RF) = M(f)$$

and

$$\|F_n\|_{\ell^q(X)} = \|(df_1, df_2, ..., df_n, 0, ...)\|_{\ell^q(X)} = S_n^{(q)}(f), \quad M(F) = S^{(q)}(f).$$

Since X is isomorphic to a q-uniformly smooth space, by Lemma 4.2.1, the martingale transform operator R satisfies (1) in Lemma 4.3.2. Thus the equivalence is obvious. $\square$

Corollary 4.3.1 Let X be a Banach space.The following statements are equivalent:

(1) X is isomorphic to a Hilbert space

(2) For any $1 < r < \infty, 1 \leq s \leq \infty$ (or equivalently, for some)

$${}_2H_{r,s}^S = H_{r,s}^*$$

Proof It is well known that a space which is 2-uniformly smooth and 2-uniformly convex is isomorphic a Hilbert space. $\square$

Now we can summarize martingale inequalities on Lorentz martingale spaces.

Theorem 4.3.3 The following inequalities are true

$$\|f\|_{H_{p,q}^*} \leq c\|f\|_{H_{p,q}^s}, \quad \|f\|_{H_{p,q}^S} \leq c\|f\|_{H_{p,q}^s} \quad (0 < p < 2, 0 < q \leq \infty)$$
$$\|f\|_{H_{p,q}^s} \leq c\|f\|_{H_{p,q}^*}, \quad \|f\|_{H_{p,q}^s} \leq c\|f\|_{H_{p,q}^S} \quad (2 < p < \infty, 0 < q \leq \infty)$$
$$\|f\|_{H_{p,q}^*} \approx \|f\|_{H_{p,q}^S} \quad (1 \leq p < \infty, 1 \leq q \leq \infty)$$
$$\|f\|_{H_{p,q}^*} \leq c\|f\|_{P_{p,q}}, \quad \|f\|_{H_{p,q}^S} \leq c\|f\|_{Q_{p,q}} \quad (0 < p < \infty, 0 < q \leq \infty)$$
$$\|f\|_{H_{p,q}^*} \leq c\|f\|_{Q_{p,q}}, \quad \|f\|_{H_{p,q}^S} \leq c\|f\|_{P_{p,q}} \quad (0 < p < \infty, 0 < q \leq \infty)$$
$$\|f\|_{H_{p,q}^s} \leq c\|f\|_{P_{p,q}}, \quad \|f\|_{H_{p,q}^s} \leq c\|f\|_{Q_{p,q}} \quad (0 < p < \infty, 0 < q \leq \infty).$$

Chapter 5 Carleson measures and vector-valued BMO martingales

§ 5.1 Introductions and Preliminaries

This paper deals with vector-valued martingale inequalities. It is well known that the validity of a classical (scalar-valued) result in the vector-valued setting, i.e. for functions or martingales with values in a Banach space X, depends on the geometrical or topological properties of X. For instance, the a.s. convergence of bounded L_p-martingales ($1 < p < \infty$) with values in X amounts to saying that X has the Radon-Nikodym property (see [27]). On the other hand, the validity of a one-sided Burkholder-Gundy inequalities for X-valued martingales is equivalent to the uniform convexity (smoothness) of X (see [80]).

It is also well known that martingale theory is intimately related to harmonic analysis. It was exactly with this in mind that Xu [104] developed the vector-valued Littlewood-Paley theory, which was inspired by Pisier's celebrated work [80] on martingale inequalities in uniformly convex spaces. Very recently, Ouyang and Xu [76] studied the endpoint case of the main results of [70] and [104] by means of the classical relationship between BMO functions and Carleson measures. Let us recall the main results of [76]. For a cube $I \subset \mathbb{R}^n$ let $\widehat{I}$ denote the tent over I. Let $1 < q < \infty$ and X be a Banach space. Then X has an equivalent norm which is q-uniformly convex iff there exists a positive c such that

$$\sup_{I \text{ cube}} \frac{1}{|I|} \int_{\widehat{I}} (t\|\nabla f(x,t)\|)^q \frac{dxdt}{t} \le c^q \|f\|^q_{BMO(R^n;X)}, \quad \forall\, f \in BMO(\mathbb{R}^n; X), \quad (5.1)$$

where f also denotes the Poisson integral of f on $\mathbb{R}^{n+1}_+$, and where

$$\|\nabla f(x,t)\| = \Big\|\frac{\partial}{\partial t} f(x,t)\Big\| + \sum_{i=1}^{n} \Big\|\frac{\partial}{\partial x_i} f(x,t)\Big\|.$$

The validity of the converse inequality is equivalent to the existence of an equivalent q-uniformly smooth norm. Inequality (5.1) means that $(t\|\nabla f(x,t)\|)^q \frac{dxdt}{t}$ is a Carleson measure on $\mathbb{R}^{n+1}_+$ for every $f \in BMO(\mathbb{R}^n; X)$.

The main goal of the present paper is to give the martingale version of Ouyang-Xu's results. This can be considered as the endpoint case of Pisier's theorem quoted previously, which we now recall as follows. Let $1 < q < \infty$. Then a Banach space X has an equivalent q-uniformly convex norm iff for one $1 < p < \infty$ (or equivalently, for every $1 < p < \infty$) there exists a positive constant c such that

$$\Big\| \big(\|f_1\|^q + \sum_{n \geq 2} \|f_n - f_{n-1}\|^q \big)^{1/q} \Big\|_p \leq c \sup_{n \geq 1} \|f_n\|_p \tag{5.2}$$

for all finite L_p-martingales f with values in X. Again, the validity of the converse inequality amounts to saying that X has an equivalent q-uniformly smooth norm. Ouyang-Xu's arguments heavily rely on Calderon-Zygmund singular integral theory. In fact, the Lusin function S_q in [70] and [76] can be represented as a singular integral operator with a regular operator-valued kernel. Similarly, Our proofs depend on martingale transform theory. More precisely, we will use operator-valued martingale transform theory as developed by T. Martinez and J.L. Torrea in [68] and [69].

In the remainder of this section we give some preliminaries necessary to the whole paper. The main object of this paper is the BMO space given in the following

Definition 5.1.1 Let $1 \leq p < \infty$ and X be a Banach space. The space $BMO_p(X)$ consists of all functions $f \in L_1(\Omega; X)$ such that

$$\|f\|_{BMO_p(X)} = \sup_{n \geq 1} \Big\| \mathbb{E}(\|f - f_{n-1}\|^p \big| \mathcal{F}_n)^{1/p} \Big\|_\infty < \infty.$$

Remark The following facts are well known in the scalar-valued case (see [35], [59] and [102]). Their proofs go straightforward over the Banach-valued setting.

(1) The spaces $BMO_p(X)$ are independent of p and all corresponding norms are equivalent. This allows us to denote any of them by $BMO(X)$.

(2) $L_\infty(X) \subset BMO(X) \subset L_p(X)$ for $1 \leq p < \infty$.

(3) We have

$$\|f\|_{BMO(X)} = \sup_{\tau} P(\tau < \infty)^{-1/p} \|f - f_{\tau-1}\|_{L_p(X)}\,, \quad 1 \leq p < \infty, \tag{5.3}$$

where the supremum is taken over all stopping times τ. On the other hand a function $f \in L_p(X), 1 \leq p < \infty$, belongs to $BMO(X)$ iff there exists an adapted process $(\theta_n)_{n \geq 0}$ such that $\theta_0 = 0$ and

$$C_\theta = \sup_n \Big\| \mathbb{E}(\|f - \theta_{n-1}\|^p \big| \mathcal{F}_n)^{1/p} \Big\|_\infty < \infty\,.$$

In this case, $\|f\|_{BMO(X)} \approx \inf_\theta C_\theta$.

Lemma 5.1.1 With the assumptions above the following statements are equivalent:

(1) There exists a positive constant c such that

$$\|Tf\|_{BMO(X_2)} \leq c\,\|f\|_{BMO(X_1)}\,, \quad \forall\, f \in BMO(X_1).$$

(2) There exists a positive constant c such that

$$\|(Tf)^*\|_{BMO(X_2)} \leq c\,\|f\|_{BMO(X_1)}\,, \quad \forall\, f \in BMO(X_1).$$

(3) For some $1 \leq p < \infty$ (or equivalently, for every $1 \leq p < \infty$) there exists a positive constant c such that

$$\|Tf\|_p \leq c\,\|f^*\|_p\,, \quad \forall\, f \in L_p(X_1).$$

The classical notion of Carleson measures in harmonic analysis has the following martingale analogue.

Definition 5.1.2 Let μ be a nonnegative measure on $\Omega \times \mathbb{N}$, where $\mathbb{N}$ is equipped with the counting measure dm. μ is called a Carleson measure if

$$\|\mu\|_C =: \sup \frac{\mu(\widehat{\tau})}{P(\tau < \infty)} < \infty,$$

where the supremum runs over all stopping times τ and where $\widehat{\tau}$ denotes the "tent" over τ:

$$\widehat{\tau} = \big\{(w,k) \in \Omega \times \mathbb{N} \,:\, \tau(w) \leq k, \tau(w) < \infty\big\}.$$

Throughout the paper we will use $A \approx B$ to abbreviate $c^{-1}B \leq A \leq cB$ for some positive constant c. The letter c will denote a positive constant, which may depend on p but never on the martingales in consideration, and which may change from line to line.

§ 5.2 Main results

The following theorem is the main result of this section. Recall that $\widehat{\tau}$ denotes the tent over a stopping time τ.

Theorem 5.2.1 Let X be a Banach space and $2 \leq q < \infty$. Then the following statements are equivalent:

(1) There exists a positive constant c such that for any finite X-valued martingale

$$\sup_{\tau} \frac{1}{P(\tau < \infty)} \int_{\widehat{\tau}} \|df_k\|^q dP \otimes dm \leq c^q \|f\|_{BMO}^q\,. \tag{5.4}$$

(2) X has an equivalent norm which is $q-$uniformly convex.

Inequality (5.4) means that $\|df_k\|^q dP \otimes dm$ is a Carleson measure on $\Omega \times \mathbb{N}$ for every $f \in BMO(X)$.

Lemma 5.2.1 Let $1 \leq p < \infty$. Then

$$\|f\|_{BMO(X)} \approx \inf_{\theta} \sup_{\tau} P(\tau < \infty)^{-1/p} \|f - \theta_{\tau-1}\|_p,$$

where the supremum runs over all stopping times τ and the infimum over all adapted processes θ such that $\theta_\infty = f$.

Proof Assume that $f \in BMO(X)$. Let τ be a stopping time. Then by Remark (3)

$$\begin{aligned}
\|f - \theta_{\tau-1}\|_p^p &= \mathbb{E}\|f - \theta_{\tau-1}\|^p \chi_{\{\tau<\infty\}} \\
&= \mathbb{E}\Big(\mathbb{E}(\|f - \theta_{\tau-1}\|^p | \mathcal{F}_\tau)\chi_{\{\tau<\infty\}}\Big) \\
&\leq C_\theta^p P(\tau < \infty).
\end{aligned}$$

This implies

$$\inf_{\theta} \sup_{\tau} P(\tau < \infty)^{-\frac{1}{p}} \|f - \theta_{\tau-1}\|_p \leq \inf_{\theta} C_\theta \leq c\|f\|_{BMO(X)}.$$

Conversely, assume $\beta = \inf_\theta \sup_\tau P(\tau < \infty)^{-1/p}\|f - \theta_{\tau-1}\|_p < \infty$, τ is any stopping time, $\forall F \in \mathcal{F}_\tau$, $F \subset \{\tau < \infty\}$. By defining $\tau_F = \tau$, if $\omega \in F$; otherwise $\tau_F = \infty$, we get

$$\begin{aligned}
\frac{1}{P(F)} \int_F \|f - \theta_{\tau-1}\|^p dP &= P(\tau_F < \infty)^{-1} \int_F \|f - \theta_{\tau_F-1}\|^p dP \\
&= P(\tau_F < \infty)^{-1} \|f - \theta_{\tau_F-1}\|_p^p,
\end{aligned}$$

which leads to

$$\sup_{\tau} \|\mathbb{E}(\|f - \theta_{\tau-1}\|^p | \mathcal{F}_n)^{1/p}\|_\infty \leq P(\tau_F < \infty)^{-1/p} \|f - \theta_{\tau_F-1}\|_p.$$

Thus

$$\|f\|_{BMO(X)} \leq c \inf_{\theta} C_\theta \leq c \inf_{\theta} \sup_{\tau} P(\tau < \infty)^{-1/p} \|f - \theta_{\tau-1}\|_p.$$

$\square$

Proof of Theorem 2.1 (1)$\Longrightarrow$(2). Assume that (1) holds. We first claim that

$$\|S^{(q)}(f)\|_{BMO} \leq c\|f\|_{BMO(X)}, \quad \forall f \in BMO(X).$$

Indeed, by Lemma 5.2.1

$$\begin{aligned}
\|S^{(q)}(f)\|_{BMO} &\leq c\sup_{\tau} P(\tau<\infty)^{-\frac{1}{q}}\|S^{(q)}(f)-S^{(q)}_{\tau-1}(f)\|_q \\
&\leq c\sup_{\tau} P(\tau<\infty)^{-\frac{1}{q}}\Big(\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^q\chi_{\{\tau<\infty\}}\Big)^{\frac{1}{q}} \\
&= c\sup_{\tau} P(\tau<\infty)^{-\frac{1}{q}}\Big(\int_{\hat{\tau}}\|df_k\|^q dP\otimes dm\Big)^{\frac{1}{q}} \\
&\leq c\|f\|_{BMO(X)}.
\end{aligned}$$

We now consider a martingale transform operator Q from the family of $X-$valued martingales to that of $l_q(X)-$valued martingales. Let $\upsilon_k \in L(X, l_q(X))$ be the operator defined by $\upsilon_k x = \{x_j\}_{j=1}^{\infty}$ for $x \in X$, where $x_j = x$ if $j = k$ and $x_j = 0$ otherwise. Q is the martingale transform associated to the sequence (υ_k) :

$$(Qf)_n = \sum_{k=1}^{n}\upsilon_k df_k = (df_1, df_2, ..., df_n, 0, ...).$$

Then

$$(Qf)^* = \sup_n \|(Qf)_n\|_{l^q(X)} = S^{(q)}(f).$$

It is clear that by the claim above Q satisfies the statement (2) in Lemma 5.1.1. Therefore, Q is L^q-bounded. Namely

$$\|S^{(q)}(f)\|_{L_q} = \|(Qf)^*\|_{L_q} \leq c\|f\|_{L_q(X)}.$$

Thus by Pisier' theorem X has an equivalent $q-$uniformly convex norm.

(2)$\Longrightarrow$(1). Suppose that X has an equivalent $q-$uniformly convex norm. By Pisier' theorem, we find for any $1 \leq n \leq m$

$$\mathbb{E}(\sum_{i=n}^{m}\|df_i\|^q|\mathcal{F}_n) \leq c\mathbb{E}(\|f_m - f_{n-1}\|^q|\mathcal{F}_n) \leq c\mathbb{E}(\|f - f_{n-1}\|^q|\mathcal{F}_n) \leq c\|f\|^q_{BMO(X)}.$$

This implies

$$\mathbb{E}(\sum_{i=n}^{\infty}\|df_i\|^q|\mathcal{F}_n) \leq c\|f\|^q_{BMO(X)}.$$

Now let τ be a stopping time. We then have

$$\begin{aligned}
& P(\tau<\infty)^{-\frac{1}{q}}\Big(\int_{\hat{\tau}}\|df_k\|^q dP\otimes dm\Big)^{\frac{1}{q}} \\
= \;& P(\tau<\infty)^{-\frac{1}{q}}\Big(\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^q\chi_{\{\tau<\infty\}}\Big)^{\frac{1}{q}} \\
= \;& P(\tau<\infty)^{-\frac{1}{q}}\Big(\mathbb{E}\big(\mathbb{E}(\sum_{k=\tau}^{\infty}\|df_k\|^q|\mathcal{F}_\tau)\chi_{\{\tau<\infty\}}\big)\Big)^{\frac{1}{q}} \\
\leq \;& cP(\tau<\infty)^{-\frac{1}{q}}\Big(\mathbb{E}\|f\|^q_{BMO(X)}\chi_{\{\tau<\infty\}}\Big)^{\frac{1}{q}} \\
\leq \;& c\|f\|_{BMO(X)}.
\end{aligned}$$

Taking the supremum over all stopping times τ, we get the desired inequality.

Theorem 5.2.2 Let X be a Banach space and $1<p\leq 2$. Then the following statements are equivalent:

(1) There exists a positive constant c such that for any $X-$valued martingale

$$|f\|^p_{BMO(X)}\leq c^p\sup_\tau P(\tau<\infty)^{-1}\int_{\hat{\tau}}\|df_k\|^p dP\otimes dm. \qquad (5.5)$$

(2) X has an equivalent $p-$uniformly smooth norm.

Inequality (5.5) means $f\in BMO(X)$, if $\|df_k\|^p dP\otimes dm$ is a Carleson measure on $\Omega\times\mathbb{N}$.

Proof (1) $\Longrightarrow$ (2). Suppose that (1) holds, then for any $X-$valued martingale we have

$$(2.3)\quad \|f\|_{BMO(X)}\leq c\sup_\tau P(\tau<\infty)^{-\frac{1}{p}}\Big(\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^p\chi_{\{\tau<\infty\}}\Big)^{\frac{1}{p}}.$$

Let X^* be the dual space of X. It suffice to prove X^* has an equivalent $q-$uniformly smooth norm, where q is the conjugate index of p.

To this end, we intend to claim for any X^*-valued martingale g,

$$\|S^{(q)}(g)\|_1\leq c\|g^*\|_1=c\|g\|_{H_1(X^*)}.$$

Since $\Big(L_1\big(l_q(X^*)\big)\Big)^*=L_\infty\big(l_p(X)\big)$, for any martingale $g\in H_1(X^*)$,

$$\begin{aligned}
\|S^{(q)}(g)\|_1 &= \sup\Big\{|\sum<dg_k,a_k>|:\|(a_k)\|_{L_\infty\big(l_p(X)\big)}\leq 1\Big\} \\
&= \sup\Big\{|\sum<dg_k,\mathbb{E}(a_k)-\mathbb{E}_{k-1}(a_k)>|:\|(a_k)\|_{L_\infty\big(l_p(X)\big)}\leq 1\Big\}.
\end{aligned}$$

Setting $df_k = \mathbb{E}_k(a_k) - \mathbb{E}_{k-1}(a_k)$, $f = \sum df_k$; then f is a $X-$valued martingale. We have

$$|\sum < dg_k, a_k > | = |\sum < dg_k, df_k > | = | < g, f > | \leq \|g\|_{H_1(X^*)}\|f\|_{BMO(X)}.$$

Now we shall estimate $\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^p\chi_{\{\tau<\infty\}}$ under the condition of $\|(a_k)\|_{L_\infty\left(l_p(X)\right)} \leq 1$. Indeed,

$$\begin{aligned}
\mathbb{E}(\sum_{k=\tau}^{\infty}\|df_k\|^p\chi_{\{\tau<\infty\}}) &\leq 2^p\Big(\mathbb{E}\sum_{k=\tau}^{\infty}\|\mathbb{E}_k(a_k)\|^p\chi_{\{\tau<\infty\}} + \mathbb{E}\sum_{k=\tau}^{\infty}\|\mathbb{E}_{k-1}(a_k)\|^p\chi_{\{\tau<\infty\}}\Big) \\
&= 2^p(I + II)
\end{aligned}$$

We shall estimate I and II respectively.

$$\begin{aligned}
I &\leq \mathbb{E}\sum_{k=\tau}^{\infty}\mathbb{E}_k\|a_k\|^p\chi_{\{\tau<\infty\}} \leq \mathbb{E}\mathbb{E}_\tau\Big(\sum_{k=0}^{\infty}\mathbb{E}_k(\|a_k\|^p\chi_{\{\tau\leq k\}})\Big) \\
&= \mathbb{E}\Big(\sum_{k=0}^{\infty}\|a_k\|^p\chi_{\{\tau\leq k\}}\Big) = \mathbb{E}\Big(\sum_{k=\tau}^{\infty}\|a_k\|^p\chi_{\{\tau<\infty\}}\Big) \\
&\leq \mathbb{E}\Big(\Big\|\sum_{k=\tau}^{\infty}\|a_k\|^p\Big\|_\infty\chi_{\{\tau<\infty\}}\Big) \\
&\leq P(\tau<\infty).
\end{aligned}$$

Similarly to I, we get

$$\begin{aligned}
II &= \mathbb{E}\|\mathbb{E}_{\tau-1}a_\tau\|^p\chi_{\{\tau<\infty\}} + \mathbb{E}\sum_{k=\tau-1}^{\infty}\|\mathbb{E}_{k-1}(a_k)\|^p\chi_{\{\tau<\infty\}} \\
&\leq 2P(\tau<\infty).
\end{aligned}$$

From (2.3) we then have $\|f\|_{BMO(X)} \leq c$. Therefore, we finally obtain

$$\|S^{(q)}(g)\|_1 \leq c\|g\|_{H_1(X^*)}, \qquad \forall g \in H_1(X^*).$$

By Piser's theorem, we get X^* has an equivalent $q-$uniformly norm. Thus we complete the proof of (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1). By the Remark 1.2, we have

$$\|f\|_{BMO(X)} = \sup_\tau P(\tau<\infty)^{-\frac{1}{p}}\|f - f_{\tau-1}\|_{L_p(X)}, \qquad \forall 1 < p \leq 2.$$

Now we consider the new nondecreasing $\sigma-$field sequence $\{\mathcal{F}_{k\vee\tau}\}_{k\geq1}$ and the corresponding martingale $\tilde{f}$ generated by $f - f_\tau$. Then by Doob's stopping time theorem, we have

$$\tilde{f}_k = \mathbb{E}(f - f_\tau|\mathcal{F}_{k\vee\tau}) = \mathbb{E}(f|\mathcal{F}_{k\vee\tau}) - f_\tau = f_{k\vee\tau} - f_\tau.$$

From the condition (2), we have $\|\tilde{f}\|_p \leq c\|S^{(p)}(\tilde{f})\|_p$. Thus

$$\begin{aligned}\mathbb{E}\|f-f_\tau\|^p &= \mathbb{E}\|\tilde{f}\|^p \leq c\mathbb{E}\sum_{k=1}^{\infty}\|d\tilde{f}_k\|^p = c\mathbb{E}\sum_{k=1}^{\infty}\|f_{(k+1)\vee\tau}-f_{k\vee\tau}\|^p \\ &= c\mathbb{E}\sum_{k=\tau}^{\infty}\|f_{k+1}-f_k\|^p = c\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^p\chi_{\{\tau<\infty\}}\end{aligned}$$

Therefore,

$$\mathbb{E}\|f-f_{\tau-1}\|^p \leq 2^p\Big(\mathbb{E}\|f-f_\tau\|^p+\mathbb{E}\|f_\tau-f_{\tau-1}\|^p\Big) \leq 2^{p+1}c\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^p\chi_{\{\tau<\infty\}}.$$

Then we obtain

$$\|f\|_{BMO(X)} \leq c\sup_\tau P(\tau<\infty)^{-\frac{1}{p}}\Big(\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^p\chi_{\{\tau<\infty\}}\Big)^{\frac{1}{p}}.$$

So the theorem is proved.

Corollary 5.2.1 Let X be a Banach space. Then the following statements are equivalent:

(1) There exists a positive constant c such that for any finite $X-$valued martingale

$$c^{-2}\sup_\tau P(\tau<\infty)^{-1}\int_{\hat{\tau}}\|df_k\|^2dP\otimes dm \leq \|f\|_{BMO}^2 \leq c^2\sup_\tau P(\tau<\infty)^{-1}\int_{\hat{\tau}}\|df_k\|^2dP\otimes dm.$$

(2) X is isomorphic to a Hilbert space.

Proof It is well known that a space which is both 2-uniformly smooth and 2-uniformly convex is isomorphic to a Hilbert space.

§ 5.3 UMD Banach lattice

Definition 5.3.1 A Banach space X is said to satisfy UMD property if there exists a positive constant c such that for $1<p<\infty$,

$$\|\varepsilon_1 df_1+...+\varepsilon_n df_n\|_p \leq c\|df_1+...+df_n\|_p, \quad \forall n\geq 1$$

for all $X-$valued martingale difference sequences $(df_1, df_2, ...)$ and all $\varepsilon_k=\pm 1$.

This definition is due to Burkholder [11]. It is known that the existence of one p_0 satisfying the inequality is enough to assure the existence of the rest of p, $1<p<\infty$.

X will denote a Banach lattice in this section. Without loss of generality we assume that X is a Banach lattice of measurable functions on some measure space $(\Sigma, d\mu)$. The reader is referred to [58] for informations about Banach lattices. In the Banach lattice case, it is nature to consider the following variant of square function $S^{(2)}(f)$.

Definition 5.3.2 Let X be a Banach lattice and $f = \{f_n\}_{n\geq 1}$ a $X-$valued martingale, $f_n = \sum_{k=1}^{n} df_k$. We define the operators

$$\tilde{S}_n f(w) = (\sum_{k=1}^{n} |df_k(w)|^2)^{\frac{1}{2}}, \quad \tilde{S} f(w) = \sup_n \tilde{S}_n f(w).$$

On the one hand for every fixed $w \in \Omega$, $\tilde{S}f(w)$ can be regard as a function defined Σ; on the another hand $\|\tilde{S}f\|$ can be seen as the norm of the element $(df_1, df_2, ...)$ in the Banach space

$$X(l^2) = \big\{(a_1, a_2, ...) : \big\|\big(\sum_{k=1}^{\infty} |a_k(w)|^2\big)^{\frac{1}{2}}\big\| < \infty\big\}.$$

$X(l^2)$ is also a Banach lattice when X is a Banach lattice. The following lemma is well known; see [89].

Lemma 5.3.3 Given a Banach lattice X, the following statements are equivalent :

(1) X satisfies the UMD property.

(2) There exists p, $1 < p < \infty$, and a constant c such that

$$c^{-1}\|f\|_{L_p(X)} \leq \|\tilde{S}f\|_{L_p(X)} \leq c\|f\|_{L_p(X)},$$

for any $X-$valued martingale.

Now we can prove the following characterization of UMD Banach lattices.

Theorem 5.3.4 Given a Banach lattice X, the following statements are equivalent:

(1) X satisfies the UMD property.

(2) There exists a positive constant c such that for any $X-$value martingale,

$$c^{-1}\|f\|_{BMO(X)} \leq \sup_\tau P(\tau < \infty)^{-\frac{1}{2}} \Big(\mathbb{E}\|(\sum_{k=\tau}^{\infty} |df_k|^2)^{\frac{1}{2}}\|^2 \chi_{\{\tau<\infty\}}\Big)^{\frac{1}{2}} \leq c\|f\|_{BMO(X)}.$$

Proof (2) $\Longrightarrow$ (1). Assume that (2) holds. By

$$\|\sum_k df_k\|_{BMO(X)} \approx \sup_\tau P(\tau < \infty)^{-\frac{1}{2}} \Big(\mathbb{E}\|(\sum_{k=\tau}^{\infty} |df_k|^2)^{\frac{1}{2}}\|^2 \chi_{\{\tau<\infty\}}\Big)^{\frac{1}{2}},$$

we get

$$\|\sum_k \varepsilon_k df_k\|_{BMO(X)} \leq c\|f\|_{BMO(X)}, \quad \forall \varepsilon_k = \pm 1.$$

For any fixed ε_k and any X-valued martingale f, $(Qf)_n = \sum_{k=1}^n \varepsilon_k df_k$ is a martingale transform operator from X to itself. By Lemma 1.4, we get

$$\|Qf\|_{L_p(X)} \leq c\|f\|_{L_p(X)}, \quad \forall p > 1.$$

By the definition of UMD space, we get X satisfies UMD property.
(1) $\Longrightarrow$ (2). We now consider the X-valued martingale $\tilde{f}$ defined in the proof of Theorem 2.3. By X satisfying UMD property, we have

$$\|\tilde{f}\|_{L_2(X)} \approx \|\tilde{S}\tilde{f}\|_{L_2(X)}.$$

Then

$$\begin{aligned} \mathbb{E}\|f - f_\tau\|^2 &= \mathbb{E}\|\tilde{f}\|^2 \leq c\mathbb{E}\|(\sum_{k=1}^{\infty} |d\tilde{f}_k|^2)^{1/2}\|^2 = c\mathbb{E}\|(\sum_{k=1}^{\infty} |f_{(k+1)\vee\tau} - f_{k\vee\tau}|^2)^{1/2}\|^2 \\ &= c\mathbb{E}\|(\sum_{k=\tau}^{\infty} |df_k|^2)^{1/2}\|^2 \chi_{\{\tau<\infty\}} \end{aligned}$$

It is obvious that $\mathbb{E}\|f_\tau - f_{\tau-1}\|^2 \leq \mathbb{E}\|(\sum_{k=\tau}^{\infty} |df_k|^2)^{1/2}\|^2 \chi_{\{\tau<\infty\}}$. Then

$$\mathbb{E}\|f - f_{\tau-1}\|^2 \leq c\Big(\mathbb{E}\|f - f_\tau\|^2 + \mathbb{E}\|f_\tau - f_{\tau-1}\|^2\Big) \leq c\mathbb{E}\|(\sum_{k=\tau}^{\infty} |df_k|^2)^{1/2}\|^2 \chi_{\{\tau<\infty\}}.$$

Conversely,

$$\begin{aligned} \mathbb{E}\|(\sum_{k=\tau}^{\infty} |df_k|^2)^{1/2}\|^2 \chi_{\{\tau<\infty\}} &\leq c\mathbb{E}\|\tilde{f}\|^2 = \mathbb{E}\|f - f_\tau\|^2 \\ &\leq c\Big(\mathbb{E}\|f - f_{\tau-1}\|^2 + \mathbb{E}\|f_\tau - f_{\tau-1}\|^2\Big) \\ &\leq c\mathbb{E}\|f - f_{\tau-1}\|^2 \end{aligned}$$

Thus

$$\mathbb{E}\|f - f_{\tau-1}\|^2 \approx \mathbb{E}\|(\sum_{k=\tau}^{\infty} |df_k|^2)^{1/2}\|^2 \chi_{\{\tau<\infty\}}$$

Recalling $\|f\|_{BMO(X)} = \sup_\tau P(\tau < \infty)^{-\frac{1}{2}}\|f - f_{\tau-1}\|_{L_2(X)}$, we obtain the desired inequality. Thus the theorem is proved.

Chapter 6 Noncommutative Lorentz martingale spaces

§ 6.1 Introductions and Preliminaries

Martingale inequalities and sums of independent random variables are important tools in classical harmonic analysis. A fundamental result duo to Burkholder [9] and [12]can be stated as follows. Given a probability space $(\Omega, \mathscr{F}, P)$, let $\{\mathscr{F}_n\}_{n\geq 1}$ be a nondecreasing sequence of σ-fields of $\mathscr{F}$ such that $\mathscr{F} = \vee \mathscr{F}_n$ and $\mathbb{E}_n$ the conditional expectation operator relative to $\mathscr{F}_n$. Given $2 \leq p < \infty$ and an L^p-bounded martingale $f = (f_n)_{n\geq 1,}$ we have

$$\|f\|_{L^p} \approx \Big\|\Big(\sum_{k=1}^{\infty} \mathbb{E}_{k-1}(|df_k|^2)\Big)^{1/2}\Big\|_{L^p} + \Big\|\Big(\sum_{k=1}^{\infty} |df_k|^p\Big)^{1/p}\Big\|_{L^p}. \tag{6.1}$$

The first term on the right is called the conditioned square function of f, while the second is called the p-variation of f. Rosenthal's inequalities [88] can be regarded as the particular case while the sequence $df = (df_1, df_2, ...)$ is a family of independent mean-zero random variables $df_k = a_k$. In this case it is easy to reduce Rosenthal's inequalities to

$$\|\sum_{k=1}^{\infty} a_k\|_{L^p} \approx \Big(\sum_{k=1}^{\infty} \|a_k\|_2^2\Big)^{1/2} + \Big(\sum_{k=1}^{\infty} \|a_k\|_p^p\Big)^{1/p}. \tag{6.2}$$

The noncommutative analogues of the above inequalities were successfully obtained by Junge and Xu in [50] and [51]. They replaced conditioned expectations onto the σ-subfields by the conditioned expectations onto an increasing sequence of von Neumann subalgebras of a given von Neumann algebra. More precisely, for $2 \leq p < \infty$, and any finite noncommutative $L^p(\mathcal{M})$-martingale $x = (x_n)_{n\geq 1}$, (1.1) has the following noncommutative version,

$$\|x\|_{L^p(\mathcal{M})} \approx \max\Big\{\Big\|\Big(\sum_k |dx_k|^p\Big)^{1/p}\Big\|_{L^p(\mathcal{M})}, \big\|s_c(x)\big\|_{L^p(\mathcal{M})}, \big\|s_r(x)\big\|_{L^p(\mathcal{M})}\Big\}, \tag{6.3}$$

where $s_c(x)$ and $s_r(x)$ denote column and row versions of conditioned square function. Moreover, they obtained a simpler inequality for $1 < p \leq 2$ by duality.

Recently, Randrianantoanina [84] proved a weak-type inequality for conditioned square functions, which implies Junge-Xu's noncommutative Burkholder' inequalities by interpolation. This alternate approach yields better constants some of which are optimal.

Our original motivation comes from the classical extension for Lorentz spaces of Rosenthal's inequalities (6.2) by Carothers and Dilworth [20], i.e., for $2 < p < \infty, 0 < q \leq \infty$ and any independent mean zero random variables $f_1, f_2, ..., f_n$,

$$\|\sum_{k=1}^{n} f_k\|_{L^{p,q}(\Omega)} \approx \max\Big\{\big\|\sum_{k=1}^{n} f_k\big\|_{L^2(\Omega)}\|, \, \|\sum_{k=1}^{n} \oplus f_k\big\|_{L^{p,q}(0,\infty)}\Big\}, \tag{6.4}$$

where $\sum_{k=1}^{n} \oplus f_k$ denotes the disjoint sum of $f_1, f_2, ..., f_n$, which is a function on $(0, \infty)$ with $d_f(t) = \sum_{k=1}^{n} d_{f_k}(t)$.

Inspired by (6.3) and (6.4), in this paper we consider Burkholder' inequalities in noncommutative Lorentz spaces $L^{p,q}(\mathcal{M}), 1 < p < \infty, 1 \leq q \leq \infty$. And one of our main results can be stated as follows (see Theorem 3.1 for the detailed statement): for $2 < p < \infty, 1 \leq q \leq \infty$, and any finite $L^{p,q}(\mathcal{M})$-martingale x, we have

$$\|x\|_{L^{p,q}(\mathcal{M})} \approx \max\Big\{\|\sum_{k} dx_k \otimes e_k\big\|_{L^{p,q}(\mathcal{M}\otimes\ell^{\infty})}, \, \big\|s_c(x)\big\|_{L^{p,q}(\mathcal{M})}, \, \big\|s_r(x)\big\|_{L^{p,q}(\mathcal{M})}\Big\}. \tag{6.5}$$

Note that if $p = q$, we come back the inequalities (6.3). We also extend this inequalities to the case $1 < p < 2, 1 \leq q \leq \infty$. Our main results are contained in section 3. Note that the proofs of these inequalities for L^p-spaces in [81] and [51] use an iteration argument; however this iteration seems inefficient (or more complicated) for the case of Lorentz spaces. We will adopt a different approach based on Randrianatoanina' weak type (1,1) inequality.

Now we introduce the noncommutative Lorentz spaces. Let $(\mathcal{M}, \tau)$ be a tracial noncommutative probability space. Namely $\mathcal{M}$ is a von Neumann algebra with a normal faithful normalized trace τ. We refer to [13] for noncommutative integration and more historical references. We only briefly recall some elementary facts on noncommutative Lorentz spaces. Let $L_0(\mathcal{M})$ denote the topological $*$-algebra of all measurable operators with respect to $(\mathcal{M}, \tau)$. For $x \in L_0(\mathcal{M})$, define its generalized singular number by

$$\mu_t(x) = \inf\{\lambda > 0 : \tau\big(\mathbb{1}_{(\lambda,\infty)}(|x|)\big) \leq t\}, \quad t > 0.$$

Then for $0 < p < \infty$,

$$L^p(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \tau(|x|^p) < \infty\}$$

and

$$\|x\|_{L^p(\mathcal{M})}^p = \tau(|x|^p) = \int_0^\infty \big(\mu_t(t)\big)^p dt.$$

Of special interest in this paper is the noncommutative Lorentz spaces $L^{p,q}(\mathcal{M})$ associated with $(\mathcal{M}, \tau)$:

$$L^{p,q}(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \|x\|_{L^{p,q}(\mathcal{M})} < \infty\},$$

where

$$\|x\|_{L^{p,q}(\mathcal{M})} = \Big(\int_0^\infty \big(t^{1/p}\mu_t(x)\big)^q \frac{dt}{t}\Big)^{1/q}$$

for $0 < q < \infty$ and with the usual modification for $q = \infty$.

The noncommutative Lorentz spaces behave well with respect to the real interpolation. Let $0 < \theta < 1, 0 < p_k, q_k \leq \infty, k = 0, 1$ and $p_0 \neq p_1$. Then

$$L^{p,q}(\mathcal{M}) = [L^{p_0,q_0}(\mathcal{M}),\ L^{p_1,q_1}(\mathcal{M})]_{\theta,q},$$

where $1/p = (1-\theta)/p_0 + \theta/p_1, 0 < q \leq \infty$.

The usual Hölder inequality also extends to the noncommutative setting. Let $0 < p_k, q_k \leq \infty, k = 0, 1$ and $1/p = 1/p_0 + 1/p_1, 1/q = 1/q_0 + 1/q_1$. Then for any $x \in L^{p_0,q_0}(\mathcal{M})$, $y \in L^{p_1,q_1}(\mathcal{M})$,

$$\|xy\|_{L^{p,q}(\mathcal{M})} \leq C\|x\|_{L^{p_0,q_0}(\mathcal{M})}\|y\|_{L^{p_1,q_1}(\mathcal{M})} \tag{6.6}$$

In particular, if $p = q = 1$,

$$|\tau(xy)| \leq \|xy\|_{L^1(\mathcal{M})} \leq \|x\|_{L^{p_0,q_0}(\mathcal{M})}\|y\|_{L^{p_1,q_1}(\mathcal{M})}, \forall x \in L^{p_0,q_0}(\mathcal{M}), y \in L^{p_1,q_1}(\mathcal{M}).$$

For $1 < p < \infty, 1 \leq q < \infty$, this defines a natural duality:

$$\Big(L^{p,q}(\mathcal{M})\Big)^* = L^{p',q'}(\mathcal{M}),$$

where p', q' denote the conjugate index of p, q respectively, and $\langle x, y\rangle = \tau(xy)$.

Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing sequence of von Neumann subalgebra of $\mathcal{M}$ such that the union of $\mathcal{M}_n'$s is weak*-dense in $\mathcal{M}$. For each $n \geq 1$, it is well known that there is unique normal faithful conditional expectation $\mathscr{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$. Moreover, $\mathscr{E}_n$ extends to a bounded projection from $L^{p,q}(\mathcal{M})$ onto $L^{p,q}(\mathcal{M}_n)$ for $1 < p < \infty, 1 \leq q \leq \infty$ which we still denote by $\mathscr{E}_n$.

For $1 \leq p < \infty, 1 \leq q \leq \infty$, and a finite sequence $a = (a_n)_{n\geq 1}$ in $\mathcal{M}$, we define

$$\|a\|_{L^{p,q}(\mathcal{M};\ell_c^2)} = \Big\|\Big(\sum_n |a_n|^2\Big)^{1/2}\Big\|_{L^{p,q}(\mathcal{M})}, \quad \|a\|_{L^{p,q}(\mathcal{M};\ell_r^2)} = \Big\|\Big(\sum_n |a_n^*|^2\Big)^{1/2}\Big\|_{L^{p,q}(\mathcal{M})}$$

and

$$\|a\|_{L^{p,q}(\mathcal{M},\mathscr{E}_{n-1};\ell_c^2)} = \Big\| \Big(\sum_n \mathscr{E}_{n-1}|a_n|^2 \Big)^{1/2} \Big\|_{L^{p,q}(\mathcal{M})}, \quad \|a\|_{L^{p,q}(\mathcal{M},\mathscr{E}_{n-1};\ell_r^2)} = \Big\| \Big(\sum_n \mathscr{E}_{n-1}|a_n^*|^2 \Big)^{1/2} \Big\|_{L^{p,q}(\mathcal{M})}.$$

Now, any finite sequence $a = (a_n)$ in $L^{p,q}(\mathcal{M})$ can be regarded as an element in $L^{p,q}(\mathcal{M} \otimes B(\ell^2))$. Therefore, $\|\cdot\|_{L^{p,q}(\mathcal{M},\ell_c^2)}$ defines a quasi-norm on the family of all finite sequences in $L^{p,q}(\mathcal{M})$. The corresponding completion is a quasi-Banach space, denoted by $L^{p,q}(\mathcal{M},\ell_c^2)$(if $q=\infty$ the competition should be taken in a certain weak topology). It is shown in [47] that $\|\cdot\|_{L^p(\mathcal{M},\mathscr{E}_{n-1};\ell_c^2)}$ is a quasi-norm. Similarly, we can show $\|\cdot\|_{L^{p,q}(\mathcal{M},\mathscr{E}_{n-1};\ell_c^2)}$ defines a quasi-norm on the family of all finite sequences in $L^{p,q}(\mathcal{M})$. The corresponding completion is a quasi-Banach space, denoted by $L^{p,q}(\mathcal{M},\mathscr{E}_{n-1};\ell_c^2)$. There are same arguments for $L^{p,q}(\mathcal{M},\ell_r^2)$ and $L^{p,q}(\mathcal{M},\mathscr{E}_{n-1};\ell_r^2)$.

Recalled that a noncommutative martingale with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ in $L^1(\mathcal{M},\tau)$ such that

$$\mathscr{E}_n(x_{n+1}) = x_n, \quad \forall n \geq 1.$$

If additionally, $x \in L^{p,q}(\mathcal{M})$ for some $1 < p < \infty, 0 < q \leq \infty$, then x is called an $L^{p,q}(\mathcal{M})$-martingale. In this case, we set

$$\|x\|_{L^{p,q}(\mathcal{M})} = \sup_{n\geq 1} \|x_n\|_{L^{p,q}(\mathcal{M})}.$$

If $\|x\|_{L^{p,q}(\mathcal{M})} < \infty$, then x is called a bounded $L^{p,q}(\mathcal{M})$-martingale. The difference sequence $dx = (dx_n)_{n\geq 1}$ is defined by $dx_n = x_n - x_{n-1}$ with the usual convention that $x_0 = 0$. For concrete natural examples of noncommutative martingale, we refer to [105].

We describe the square functions and conditional square functions of noncommutative martingales. Following [81] and [47] , we will consider the following column and row versions of square function and conditional quare function: for a finite martingale $x = (x_n)$, set (recalling that $\mathscr{E}_0 = \mathscr{E}_1$)

$$S_c(x) = \Big(\sum_n |dx_n|^2 \Big)^{1/2}, \quad S_r(x) = \Big(\sum_n |dx_n^*|^2 \Big)^{1/2};$$

$$s_c(x) = \Big(\sum_n \mathscr{E}_{n-1}|dx_n|^2 \Big)^{1/2}, \quad s_r(x) = \Big(\sum_n \mathscr{E}_{n-1}|dx_n^*|^2 \Big)^{1/2}.$$

Observe that

$$\|S_c(x)\|_{L^{p,q}(\mathcal{M})} = \|dx\|_{L^{p,q}(\mathcal{M};\ell_c^2)}, \quad \|s_c(x)\|_{L^{p,q}(\mathcal{M})} = \|dx\|_{L^{p,q}(\mathcal{M},\mathscr{E}_{n-1};\ell_c^2)}.$$

Now we define Hardy spaces $\mathcal{H}_p(\mathcal{M})$. For $1 \leq p < 2$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M}),$$

with the norm

$$\|x\|_{\mathcal{H}_p} = \inf\{\|y\|_{\mathcal{H}_p^c} + \|z\|_{\mathcal{H}_p^r} : x = y + z, y \in \mathcal{H}_p^c(\mathcal{M}), z \in \mathcal{H}_p^r(\mathcal{M})\}$$

For $2 \leq p < \infty$,

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \bigcap \mathcal{H}_p^r(\mathcal{M}),$$

with the norm

$$\|x\|_{\mathcal{H}_p} = \max\{\|x\|_{\mathcal{H}_p^c}, \|x\|_{\mathcal{H}_p^r}\}.$$

§ 6.2 The Burkholder-Gundy inequality

We now extend the noncommutative Burkholder-Gundy's inequalities in the Lorentz spaces setting. One should note we cant directly obtain the Burkholder-Gundy inequalities from the results in [13]. We employ the recent results in [73], which play an important role in our proof. First we give a lemma.

Lemma 6.2.1 Let $1 < p < \infty, 0 < q < \infty$, and (ε_n) be Redermacher sequence. Then there is a positive constant C such that for all finite martingale $x \in L^{p,q}(\mathcal{M})$, we have

$$\|\sum_n dx_n \otimes \varepsilon_n\|_{L^{p,q}(\mathcal{M} \otimes L^\infty(\Omega))} \approx \|x\|_{L^{p,q}(\mathcal{M})}.$$

Proof Consider the operator

$$T : L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M} \otimes L^\infty(\Omega))$$

by

$$Tx = \sum_n dx_n \otimes \varepsilon_n, \quad \forall x \in L^p(\mathcal{M}) \quad \text{and} \quad x_n = \mathscr{E}_n(x).$$

By Theorem 2.1 in [81], it is easy to know

$$\|x\|_{L^p(\mathcal{M})} = \|\sum_n dx_n\|_{L^p(\mathcal{M})} \approx \|\sum_n dx_n \otimes \varepsilon_n\|_{L^p(\mathcal{M} \otimes L^\infty(\Omega))}.$$

Then T is bounded in $L^p(\mathcal{M})$ for all $1 < p < \infty$. Thus by interpolation we obtain

$$\|\sum_n dx_n \otimes \varepsilon_n\|_{L^{p,q}(\mathcal{M} \otimes L^\infty(\Omega))} \leq C\|x\|_{L^{p,q}(\mathcal{M})}.$$

In order to prove the inverse inequality, we consider the operator

$$S : L^p(\mathcal{M} \otimes L^\infty(\Omega)) \longrightarrow L^p(\mathcal{M})$$

by

$$S(\sum_n a_n \otimes \varepsilon_n) = \sum_n \mathscr{E}_n(a_n) - \mathscr{E}_{n-1}(a_n), \quad \forall (a_n) \in L^p(\mathcal{M})$$

Note that $(\mathscr{E}_n(a_n) - \mathscr{E}_{n-1}(a_n))$ is a martingale difference sequence, we have

$$\begin{aligned}
\|\sum_n \mathscr{E}_n(a_n) - \mathscr{E}_{n-1}(a_n)\|_{L^p(\mathcal{M})} &\leq C\|\sum_n (\mathscr{E}_n(a_n) - \mathscr{E}_{n-1}(a_n)) \otimes \varepsilon_n\|_{L^p(\mathcal{M})\otimes L^\infty(\Omega)} \\
&\leq \|\sum_n \mathscr{E}_n(a_n) \otimes \varepsilon_n\|_{L^p(\mathcal{M})\otimes L^\infty(\Omega)} \\
&+ \|\sum_n \mathscr{E}_{n-1}(a_n) \otimes \varepsilon_n\|_{L^p(\mathcal{M})\otimes L^\infty(\Omega)}.
\end{aligned}$$

By Khintchine inequality and Stein inequality, for $p \geq 2$,

$$\begin{aligned}
\|\sum_n \mathscr{E}_n(a_n) \otimes \varepsilon_n\|_{L^p(\mathcal{M})\otimes L^\infty(\Omega)} &\leq C\|(\sum_n \mathscr{E}_n(a_n)^*\mathscr{E}_n(a_n))^{1/2}\|_{L^p(\mathcal{M})} \vee \|(\sum_n \mathscr{E}_n(a_n)\mathscr{E}_n(a_n)^*)^{1/2}\|_{L^p(\mathcal{M})} \\
&\leq C\|(\sum_n a_n^* a_n)^{1/2}\|_{L^p(\mathcal{M})} \vee \|(\sum_n a_n a_n^*)^{1/2}\|_{L^p(\mathcal{M})} \\
&\leq C\|\sum_n a_n \otimes \varepsilon_n\|_{L^p(\mathcal{M}\otimes L^\infty(\Omega))}
\end{aligned}$$

If $1 < p < 2$, let $a_n = b_n + c_n$ with b_n and c_n in $L^p(\mathcal{M})$, then $\mathscr{E}_n(a_n) = \mathscr{E}_n(b_n) + \mathscr{E}_n(c_n)$

$$\begin{aligned}
\|\sum_n \mathscr{E}_n(a_n) \otimes \varepsilon_n\|_{L^p(\mathcal{M})\otimes L^\infty(\Omega)} &\leq C\|(\sum_n \mathscr{E}_n(b_n)^*\mathscr{E}_n(b_n))^{1/2}\|_{L^p(\mathcal{M})} + \|(\sum_n \mathscr{E}_n(c_n)\mathscr{E}_n(c_n)^*)^{1/2}\|_{L^p(\mathcal{M})} \\
&\leq C\|(\sum_n b_n^* b_n)^{1/2}\|_{L^p(\mathcal{M})} + \|(\sum_n c_n c_n^*)^{1/2}\|_{L^p(\mathcal{M})}
\end{aligned}$$

Taking the infimum over all decompositions $a_n = b_n + c_n$ with b_n and c_n in $L^p(\mathcal{M})$, then

$$\|\sum_n \mathscr{E}_n(a_n) \otimes \varepsilon_n\|_{L^p(\mathcal{M})\otimes L^\infty(\Omega)} \leq C\|\sum_n a_n \otimes \varepsilon_n\|_{L^p(\mathcal{M}\otimes L^\infty(\Omega))}$$

Similarly, $\|\sum_n \mathscr{E}_{n-1}(a_n) \otimes \varepsilon_n\|_{L^p(\mathcal{M})\otimes L^\infty(\Omega)} \leq C\|\sum_n a_n \otimes \varepsilon_n\|_{L^p(\mathcal{M}\otimes L^\infty(\Omega))}$. Thus for all $1 < p < \infty$, S is bounded from $L^p(\mathcal{M} \otimes L^\infty(\Omega))$ to $L^p(\mathcal{M})$. By interpolation again, we have

$$\|\sum_n \mathscr{E}_n(a_n) - \mathscr{E}_{n-1}(a_n)\|_{L^{p,q}(\mathcal{M})} \leq C\|\sum_n a_n \otimes \varepsilon_n\|_{L^{p,q}(\mathcal{M}\otimes L^\infty(\Omega))}.$$

Taking $(a_n) = (dx_n)$, we obtain the desired inequality. This complete the proof.

Theorem 6.2.1 Let $x = (x_n)_{n\geq 1}$ be any finite $L^{p,q}(\mathcal{M})$-martingale. Then x is bounded $L^{p,q}(\mathcal{M})$ iff $x \in \mathcal{H}^{p,q}(\mathcal{M})$; moreover, if this is the case, there is a positive constant C,

$$\|x\|_{L^{p,q}(\mathcal{M})} \approx \|x\|_{\mathcal{H}^{p,q}(\mathcal{M})}.$$

Proof By the recent results, Corollary 4.2 in [73], we have

$$\|\sum_n dx_n \otimes \varepsilon_n\|_{L^{p,q}(\mathcal{M}\otimes L^\infty(\Omega))} \approx \max\{\|S_c(x)\|_{L^{p,q}(\mathcal{M})}, \|S_r(x)\|_{L^{p,q}(\mathcal{M})}\}, \quad 2 < p < \infty, 0 < q \leq \infty,$$

and

$$\|\sum_n dx_n \otimes \varepsilon_n\|_{L^{p,q}(\mathcal{M}\otimes L^\infty(\Omega))} \approx \inf\{\|S_c(y)\|_{L^{p,q}(\mathcal{M})} + \|S_r(z)\|_{L^{p,q}(\mathcal{M})}\}, \quad 1 < p < 2, 0 < q \leq \infty.$$

Then by Lemma 3.1, we immediately obtain the desired equivalence.

Identifying bounded $L^{p,q}(\mathcal{M})$-martingales with their limits, we may reformulate Theorem 3.2 as follows.

Corollary 6.2.1 Let $1 < p < \infty, p \neq 2, 0 < q \leq \infty$. Then $L^{p,q}(\mathcal{M}) = \mathcal{H}^{p,q}(\mathcal{M})$ with equivalent norms.

§ 6.3 The Burkholder inequality

We now investigate the Burkholder inequality for noncommutative Lorentz spaces. The principal result of this section is the following

Theorem 6.3.1 Let $1 < p < \infty, 1 \leq q \leq \infty$ and $x = (x_n)_{n\geq 1}$ be a finite $L^{p,q}(\mathcal{M})$-martingale. Then

(1) for $2 < p < \infty$

$$\|x\|_{L^{p,q}(\mathcal{M})} \approx \max\left\{\|\sum_n dx_n \otimes e_n\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)}, \|s_c(x)\|_{L^{p,q}(\mathcal{M})}, \|s_r(x)\|_{L^{p,q}(\mathcal{M})}\right\}; \quad (6.7)$$

(2) for $1 < p < 2$

$$\|x\|_{L^{p,q}(\mathcal{M})} \approx \inf_{x=y+z+w}\left\{\|\sum_n dy_n \otimes e_n\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)} + \|s_c(z)\|_{L^{p,q}(\mathcal{M})} + \|s_r(w)\|_{L^{p,q}(\mathcal{M})}\right\}, \quad (6.8)$$

where the infimum runs over all decompositions $dx_n = dy_n + dz_n + dw_n$ with dy_n, dz_n and dw_n being martingale difference sequences.

We will employ the discrete version of the J-method. For $0 < \theta < 1$ and $1 \leq q \leq \infty$, we denote by $\lambda^{\theta,q}$ the space of all sequences $(\alpha_m)_{m=-\infty}^{\infty}$ for which

$$\|(\alpha_m)\|_{\lambda^{\theta,q}} = \Big(\sum_{m\in\mathbb{Z}}(2^{-m\theta}|\alpha_m|)^q\Big)^{1/q} < \infty.$$

Let (E_0, E_1) be a compatible couple and suppose that $0 < \theta < 1$ and $1 \leq q \leq \infty$. The interpolation space $(E_0, E_1)_{\theta,q;J}$ consists of elements $x \in E_0 + E_1$ which admit a representation

$$x = \sum_{m\in\mathbb{Z}} u_m \quad (convergence \quad in \quad E_0 + E_1) \tag{6.9}$$

with $u_m \in E_0 \bigcap E_1$ and such that

$$\|x\|_{\theta,q;J} = \inf\Big\{\big\|\{J(u_m, 2^m)\}\big\|_{\lambda^{\theta,q}}\Big\} < \infty,$$

where the infimum is taken over all representation of x as in (6.9) .

The following lemma from [84] is the key ingredient of our proof.

Lemma 6.3.1 Let $x = (x_n)_{1\leq n\leq N}$ be a finite L^2 martingale. Then there exist three adapted sequences $a = (a_n)_{1\leq n\leq N}, b = (b_n)_{1\leq n\leq N}$ and $c = (c_n)_{1\leq n\leq N}$ in $L^2(\mathcal{M}, \tau)$ such that:

(1) for every $1 \leq n \leq N$, we have the decomposition

$$dx_n = a_n + b_n + c_n;$$

(2) the L^2-norms satisfy

$$\|a\|_{L^2(\mathcal{M},\ell^2_c)} + \|b\|_{L^2(\mathcal{M},\ell^2_c)} + \|c\|_{L^2(\mathcal{M},\ell^2_r)} \leq K\|x\|_{L^2(\mathcal{M})};$$

(3) the conditional square functions satisfy the weak-type (1,1) inequality:

$$\begin{aligned}\|\sum_n a_n \otimes e_n\|_{L^{1,\infty}(\mathcal{M}\otimes\ell^\infty)} \quad &+ \quad \|(\sum_n \mathscr{E}_{n-1}|b_n|^2)^{1/2}\|_{L^{1,\infty}(\mathcal{M})} \\ &+ \quad \|(\sum_n \mathscr{E}_{n-1}|c_n^*|^2)^{1/2}\|_{L^{1,\infty}(\mathcal{M})} \leq K\|x\|_{L^1(\mathcal{M})}\end{aligned}$$

where (e_n) denotes the canonical unit of ℓ^∞ and K is an absolute constant.

Proof of Theorem 6.3.1

Step 1. We first combine Lemma 6.3.1 and the J-method to prove the low estimate of (6.8). Let $x = (x_n)_{1\leq n\leq N}$ be any finite $L^{p,q}$-martingale. For $1 < p < 2$, we choose θ satisfying $1/p = (1-\theta) + \theta/2$. Fix $(u_m)_{m=-\infty}^{\infty}$ in $L^2(\mathcal{M})$ such that

$$x_N = \sum_{m\in\mathbb{Z}} u_m$$

and

$$\big\|\{J(u_m,2^m)\}\big\|_{\lambda^{\theta,q}} \le 2\|x_N\|_{\theta,q;J},$$

where the J-functional and the interpolation are relative to the couple $(L^1(\mathcal{M}),L^2(\mathcal{M}))$. By Lemma 6.3.1 , for each fixed $m\in\mathbb{Z}$, we can find three finite adapted sequences a^m, b^m and c^m in $L^2(\mathcal{M})$, and an absolute constant $K>0$ such that:
(1) $\mathscr{E}_n(u_m)-\mathscr{E}_{n-1}(u_m)=a_n^m+b_n^m+c_n^m$, $1\le n\le N$;
(2) $J(\sum_{n\ge1}a_n^m\otimes e_n, t\,; L^{1,\infty}(\mathcal{M}\otimes\ell^\infty), L^2(\mathcal{M}\otimes\ell^\infty))\le KJ(u_m,t)$, $t>0$;
(3) $J\Big(\big(\sum_{n\ge1}\mathscr{E}_{n-1}|b_n^m|^2\big)^{1/2}, t\,; L^{1,\infty}(\mathcal{M}), L^2(\mathcal{M})\Big)\le KJ(u_m,t)$, $t>0$;
(4) $J\Big(\big(\sum_{n\ge1}\mathscr{E}_{n-1}|c_n^{m*}|^2\big)^{1/2}, t;\ L^{1,\infty}(\mathcal{M}), L^2(\mathcal{M})\Big)\le KJ(u_m,t)$, $t>0$, Then we deduce that

$$\Big\|\Big\{J\Big(\sum_{n\ge1}a_n^m\otimes e_n, 2^m\Big)\Big\}\Big\|_{\lambda^{\theta,q}}\le 2K\big(\|x_N\|_{\theta,q;J}\big), \tag{6.10}$$

$$\Big\|\Big\{J\Big(\big(\sum_{n\ge1}\mathscr{E}_{n-1}|b_n^m|^2\big)^{1/2}, 2^m\Big)\Big\}\Big\|_{\lambda^{\theta,q}}\le 2K\|x_N\|_{\theta,q;J} \tag{6.11}$$

and

$$\Big\|\Big\{J\Big(\big(\sum_{n\ge1}\mathscr{E}_{n-1}|c_n^{m*}|^2\big)^{1/2}, 2^m\Big)\Big\}\Big\|_{\lambda^{\theta,q}}\le 2K\|x_N\|_{\theta,q;J}. \tag{6.12}$$

From (6.10) and the definition of $\|\cdot\|_{\theta,q;J}$, we get that for any finite subset $S\subset\mathbb{Z}$

$$\Big\|\sum_{m\in S}\sum_{n\ge1}a_n^m\otimes e_n\Big\|_{\big[L^{1,\infty}(\mathcal{M}\otimes\ell^\infty),\,L^2(\mathcal{M}\otimes l^\infty)\big]_{\theta,q;J}}\le\Big\|\Big\{J\Big(\sum_{n\ge1}a_n^m\otimes e_n, 2^m\Big)\Big\}\Big\|_{\lambda^{\theta,q}}\le 2K\|x_N\|_{\theta,q;J}$$

For fixed $m\in S$, $a^m=\sum_{n\ge1}a_n^m\otimes e_n$ is an element of the Banach space $\big[L^{1,\infty}(\mathcal{M}\otimes\ell^\infty),\ L^2(\mathcal{M}\otimes\ell^\infty)\big]_{\theta,q;J}$. Then

$$\Big\|\sum_{m\in S}\pm a^m\Big\|_{\big[L^{1,\infty}(\mathcal{M}\otimes\ell^\infty),\,L^2(\mathcal{M}\otimes\ell^\infty)\big]_{\theta,q;J}}\le 4K\|x_N\|_{\theta,q;J}.$$

This means (since the constant C is independent of the finite subset of $\mathbb{Z}$) that the formal series $\sum_{m\in\mathbb{Z}}a^m$ is weak unconditionally Cauchy (see for instance Diestel [27] P.44 Theorem 6); but since the Banach space $\big[L^{1,\infty}(\mathcal{M}\otimes\ell^\infty),\ L^2(\mathcal{M}\otimes\ell^\infty)\big]_{\theta,q;J}$ contains no copy of c_0 (in fact it is reflexive), this implies that the series $\sum_{m\in\mathbb{Z}}a^m$ is (unconditionally) convergent. Hence if we set

$$a:=\sum_{m\in\mathbb{Z}}a^m$$

then the sequence $a = (a_n)_{n\geq 1}$ satisfies

$$\Big\| \sum_{n\geq 1} a_n \otimes e_n \Big\|_{\big[L^{1,\infty}(\mathcal{M}\otimes\ell^\infty),\, L^2(\mathcal{M}\otimes\ell^\infty)\big]_{\theta,q;J}} \leq 2K\|x_N\|_{\theta,q;J}. \tag{6.13}$$

Now we consider the space $L^2(\mathcal{M}, \mathscr{E}_{n-1}; \ell_c^2)$ as the column subspace of $L^2(\mathcal{M} \otimes B(\ell^2(\mathbb{N}^2)))$, and view the sequence $b^m = (b_n^m)_{n\geq 1}$ as a column vector with entries from $L^2(\mathcal{M} \otimes B(\ell^2(\mathbb{N}^2)))$ (see [47]for more details). Then for any fixed $m \in \mathbb{Z}$,

$$\begin{aligned} J\Big(\big(\sum_{n\geq 1} \mathscr{E}_{n-1}|b_n^m|^2\big)^{1/2}, 2^m\Big) &= \max\Big\{\Big\|\big(\sum_{n\geq 1} \mathscr{E}_{n-1}|b_n^m|^2\big)^{1/2}\Big\|_{L^{1,\infty}(\mathcal{M})}, 2^m\Big\|\big(\sum_{n\geq 1} \mathscr{E}_{n-1}|b_n^m|^2\big)^{1/2}\Big\|_{L^2(\mathcal{M})}\Big\} \\ &= J\Big(b^m,\, 2^m;\, L^{1,\infty}(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2))), L^2(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2)))\Big) \end{aligned}$$

Then (6.11) becomes

$$\Big\|\Big\{J\Big(b^m,\, 2^m;\, L^{1,\infty}(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2))), L^2(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2)))\Big)\Big\}\Big\|_{\lambda^{\theta,q}} \leq 2K\|x_N\|_{\theta,q;J}.$$

Similarly, if we set $b := \sum_{m\in\mathbb{Z}} b^m$, then $b = (b_n)_{n\geq 1}$ as a column vector, satisfies

$$\big\|b\big\|_{\big[L^{1,\infty}(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2))),L^2(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2)))\big]_{\theta,q;J}} \leq 2K\|x_N\|_{\theta,q;J}. \tag{6.14}$$

Again, if setting $c := \sum_{m\in\mathbb{Z}} c^m$, we have

$$\big\|c\big\|_{\big[L^{1,\infty}(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2))),L^2(\mathcal{M}\otimes B(\ell^2(\mathbb{N}^2)))\big]_{\theta,q;J}} \leq 2K\|x_N\|_{\theta,q;J}. \tag{6.15}$$

Note that a, b and c are adapted sequences. Moreover, it is clear from the construction that for $1 \leq n \leq N$,

$$dx_n = a_n + b_n + c_n.$$

Now we use the following well-known equalities, for $1/p = (1-\theta) + \theta/2, 1 \leq q \leq \infty$ and any semifinite von Neumann algebra $\mathcal{N}$,

$$\big[L^{1,\infty}(\mathcal{N}), L^2(\mathcal{N})\big]_{\theta,q;J} = L^{p,q}(\mathcal{N}) \quad \text{and} \quad \big[L^1(\mathcal{N}), L^2(\mathcal{N})\big]_{\theta,q;J} = L^{p,q}(\mathcal{N}).$$

Combining the previous inequalities, we conclude that there is positive constant $C > 0$ such that

$$\Big\|\sum_{n\geq 1} a_n\otimes e_n\Big\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)} + \Big\|\Big(\sum_{n\geq 1}\mathscr{E}_{n-1}|b_n|^2\Big)^{1/2}\Big\|_{L^{p,q}(\mathcal{M})} + \Big\|\Big(\sum_{n\geq 1}\mathscr{E}_{n-1}|c_n^*|^2\Big)^{1/2}\Big\|_{L^{p,q}(\mathcal{M})} \leq C\|x\|_{L^{p,q}(\mathcal{M})}.$$

To complete the proof, it is enough to set for $n \geq 1$,

$$dy_n = a_n - \mathscr{E}_{n-1}(a_n), \quad dz_n = b_n - \mathscr{E}_{n-1}(b_n), \quad dw_n = c_n - \mathscr{E}_{n-1}(c_n).$$

Then $(dy_n)_{\geq 1}$, $(dz_n)_{\geq 1}$ and $(dw_n)_{\geq 1}$ are martingale difference sequences with $dx_n = dy_n + dz_n + dw_n$. Note that $\mathscr{E}_{n-1}$ is bounded in $L^p(\mathcal{M})$, by interpolation we have for $1 < p < \infty, 1 \leq q \leq \infty$

$$\big\| \sum \mathscr{E}_{n-1}(a_n) \otimes e_n \big\|_{L^{p,q}(\mathcal{M})} \leq C \big\| \sum a_n \otimes e_n \big\|_{L^{p,q}(\mathcal{M})}$$

So

$$\big\| \sum_n dy_n \otimes e_n \big\|_{L^{p,q}(\mathcal{M}\otimes \ell^\infty)} \leq C \big\| \sum a_n \otimes e_n \big\|_{L^{p,q}(\mathcal{M})} \leq C \|x\|_{L^{p,q}(\mathcal{M})}.$$

Noting that $\mathscr{E}_{n-1}(b_n)^* \mathscr{E}_{n-1}(b_n) \leq \mathscr{E}_{n-1}(b_n^* b_n)$, we have

$$\mathscr{E}_{n-1}|dz_n|^2 = \mathscr{E}_{n-1}(b_n^* b_n) - \mathscr{E}_{n-1}(b_n)^* \mathscr{E}_{n-1}(b_n) \leq \mathscr{E}_{n-1}(b_n^* b_n)$$

Then we finally deduce

$$\big\| \sum_n dy_n \otimes e_n \big\|_{L^{p,q}(\mathcal{M}\otimes \ell^\infty)} + \|s_c(z)\|_{L^{p,q}(\mathcal{M})} + \|s_r(w)\|_{L^{p,q}(\mathcal{M})} \leq C \|x\|_{L^{p,q}(\mathcal{M})}.$$

Step 2. Applying the inequality established in step 1, by duality, we now prove the upper estimate of (6.7). Let $x = (x_n)_{n \geq 1}$ be any finite martingale, say, $x_n = x_N$ for all $n \geq N$. We first consider the case $2 < p < \infty$ and $1 \leq q < \infty$. Let $b_N \in L^{p',q'}(\mathcal{M}), 1 < p' < 2, 1 < q' \leq \infty$. Then b_N defines a finite martingale $b = (b_n)_{n\geq 1}$,

$$b_n = \mathscr{E}_n(b_N),\ n \geq 1.$$

Let $b_n = y_n + w_n + z_n$ be any decomposition of b satisfying the conditions in 2.2. We then obtain by Holder's inequality and Proposition 2.2,

$$\begin{aligned}
|\tau(x_N^* b_N)| &= |\tau(\sum_n dx_n^* db_n)| \\
&\leq |\tau(\sum_n dx_n^* dy_n)| + |\tau(\sum_n dx_n^* dw_n)| + |\tau(\sum_n dx_n^* dz_n)| \\
&= |\tau \otimes tr(\sum_n dx_n^* \otimes e_n dy_n \otimes e_n)| + |\tau(\sum_n \mathscr{E}_{n-1} dx_n^* dw_n)| + |\tau(\sum_n \mathscr{E}_{n-1} dx_n^* dz_n)| \\
&\leq \|\sum_n dx_n^* \otimes e_n\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)} \|\sum_n dy_n \otimes e_n\|_{L^{p',q'}(\mathcal{M}\otimes\ell^\infty)} \\
&+ \|s_c(x)\|_{L^{p,q}(\mathcal{M})} \|s_c(w)\|_{L^{p',q'}(\mathcal{M})} + \|s_r(x)\|_{L^{p,q}(\mathcal{M})} \|s_r(z)\|_{L^{p',q'}(\mathcal{M})} \\
&\leq C \max\Big\{ \big\| \sum_n dx_n \otimes e_n \big\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)}, \|s_c(x)\|_{L^{p,q}(\mathcal{M})}, \|s_r(x)\|_{L^{p,q}(\mathcal{M})} \Big\} \|b_N\|_{L^{p',q'}(\mathcal{M})}
\end{aligned}$$

Taking the supremum over all b_N such that $\|b_N\|_{L^{p',q'}(\mathcal{M})} \leq 1$, we deduce

$$\|x\|_{L^{p,q}(\mathcal{M})} \leq C \max\Big\{ \big\| \sum_n dx_n \otimes e_n \big\|_{L^{p,q}(\mathcal{M}\otimes l^\infty)}, \|s_c(x)\|_{L^{p,q}(\mathcal{M})}, \|s_r(x)\|_{L^{p,q}(\mathcal{M})} \Big\}.$$

If $2 < p < \infty, q = \infty$, considering the duality $L^{p,\infty}(\mathcal{M}) = \left(L^{p',1}(\mathcal{M})\right)^*$, we similarly obtain the desired result.

Step 3. Now we prove the low estimate of (6.7). First, we observe

$$\|\sum_n dx_n \otimes e_n\|_{L^2(\mathcal{M}\otimes\ell^\infty)} \leq \|x\|_{L^2(\mathcal{M})}, \quad \|\sum_n dx_n \otimes e_n\|_{L^\infty(\mathcal{M}\otimes\ell^\infty)} \leq \|x\|_{L^\infty(\mathcal{M})}$$

Then by interpolation , we get for $2 < p < \infty$, $0 < q \leq \infty$

$$\|\sum_n dx_n \otimes e_n\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)} \leq C\|x\|_{L^{p,q}(\mathcal{M})}.$$

Thus it remains to majorize $\|s_c(x)\|_{L^{p,q}(\mathcal{M})}$ and $\|s_r(x)\|_{L^{p,q}(\mathcal{M})}$. Again, we view $L^p(\mathcal{M};\mathscr{E}_{n-1},\ell^2_c)$ as a closed subspace of $L^p(\mathcal{M}\otimes B((\ell^2(\mathbb{N}^2))))$. Then there exists a linear operator T such that

$$\|s_c(x)\|_{L^p} = \|(dx_n)\|_{L^p(\mathcal{M};\mathscr{E}_{n-1},\ell^2_c)} = \|T(dx_n)\|_{L^p(\mathcal{M}\otimes B((\ell^2(\mathbb{N}^2))))}.$$

From Theorem 6.1 in [10], we know for any $2 \leq r < \infty$

$$\|T(dx_n)\|_{L^r(\mathcal{M}\otimes B((\ell^2(\mathbb{N}^2))))} \leq C\|x\|_{L^r(\mathcal{M})}.$$

By interpolation,

$$\|T(dx_n)\|_{L^{p,q}(\mathcal{M}\otimes B((\ell^2(\mathbb{N}^2))))} \leq C\|x\|_{L^{p,q}(\mathcal{M})}.$$

Thus we obtain

$$\|s_c(x)\|_{L^{p,q}} \leq C\|x\|_{L^{p,q}(\mathcal{M})}.$$

The same argument can also be applied to $\|s_r(x)\|_{L^{p,q}}$, we obtain the desired inequality.

Step 4. The low estimate of (6.8) is similar to step 2 by using the result in step 3, therefore omit it. Thus the proof of Theorem 3.1 is complete. □

As the commutative case, the noncommutative Rosenthal inequalities can be deduced from the Buekholder inequalities established in section 3. To state the noncommutative Rosenthal inequalities, we need to introduce a notion of independence in the noncommutative setting. The following definition is introduced in [51].

Definition 6.3.1 Let $(\mathcal{M},\tau)$ be a noncommutative probability space and $\mathcal{N}$ and $\mathcal{A}_k$ von Neumann subalgebras of $\mathcal{M}$ such that $\mathcal{N} \subset \mathcal{A}_k$ for every k. The sequence $\mathcal{A}_k$ may be finite.

(1) We say that $\mathcal{A}_k$ are independent over $\mathcal{N}$ or with respect to $\mathcal{E}_{\mathcal{N}}$ if for every k and for all $x \in \mathcal{A}_k$ and y in the von Neumann subalgebra generated by $(\mathcal{A}_j)_{j \neq k}$,

$$\mathscr{E}_{\mathcal{N}}(xy) = \mathscr{E}_{\mathcal{N}}(x)\mathscr{E}_{\mathcal{N}}(y).$$

(2) A sequence $(x_k) \subset L^{p,q}(\mathcal{M})$ is said to independent with respect to $\mathscr{E}_{\mathcal{N}}$ if there exist $\mathcal{A}_k$ such that $x_k \in L^{p,q}(\mathcal{A}_k)$ and $(\mathcal{A}_k)$ is independent with respect to $\mathscr{E}_{\mathcal{N}}$.

If $\mathcal{N} = \mathbb{C}$ then $\mathscr{E}_{\mathcal{N}} = \tau(\cdot)\mathbb{1}$, we say these notions are independent with respect to the state τ.

We refer to [51] and [105] for natural examples of independent sequences.

Now we investigate the Rosenthal inequalities in noncommutative $L^{p,q}(\mathcal{M})$. In this section we always assume that $\mathcal{N}$ and $(\mathcal{A}_n)$ are von Neumann subalgebras of $\mathcal{M}$ such that $(\mathcal{A}_n)$ is independent with respect to the conditional expectation $\mathscr{E} = \mathscr{E}_{\mathcal{N}}$.

Let $(\mathcal{A}_n)$ be an independent sequence of von Neumann subalgebras such that $(x_n) \subset L^{p,q}(\mathcal{M})$ with $\mathscr{E}_{\mathcal{N}}(x_n) = 0$. Let $\mathcal{M}_n$ be the von Neumann algebra generated by $(\mathcal{A}_1, ..., \mathcal{A}_n)$. Then $\mathcal{M}_n$ is an increasing filtration of subalgebras of $\mathcal{M}$. Let $\mathscr{E}_n$ be the associated conditional expectations. The independence assumption implies that for every $b \in \mathcal{M}_{n-1}$,

$$\mathscr{E}_{\mathcal{N}}(\mathscr{E}_{n-1}(x_n)b) = \mathscr{E}_{\mathcal{N}}(x_n b) = \mathscr{E}_{\mathcal{N}}(x_n)\mathscr{E}_{\mathcal{N}}(b) = 0.$$

Therefor,

$$\mathscr{E}_{n-1}(x_n) = 0.$$

Thus (x_n) is a martingale difference with respect to $(\mathcal{M}_n)$. Now we form a noncommutative martingale $y = (y_n)$ by setting $dy_n = x_n$. Applying once more the independence assumption, we get

$$\mathscr{E}_{n-1}(x_n x_n^*) = \mathscr{E}_{\mathcal{N}}(x_n x_n^*).$$

Thus we can directly deduce the following Rosenthal inequalities from the noncommutative Burkholder inequalities.

Theorem 6.3.2 Given $2 < p < \infty$ and $1 \leq q \leq \infty$. Let $(\mathcal{M}, \tau)$ be a noncommutative probability space, and $(x_n) \subset L^{p,q}(\mathcal{M})$ be any finite sequence independently with respect to $\mathscr{E}$ such that $\mathscr{E}(x_n) = 0$. Then

$$\|\sum_n x_n\|_{L^{p,q}(\mathcal{M})} \approx \max\Big\{ \big\|\sum_n x_n \otimes e_n\big\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)}, \|(x_n)\|_{L^{p,q}(\mathcal{M},\mathscr{E};\ell_c^2)}, \|(x_n)\|_{L^{p,q}(\mathcal{M},\mathscr{E};\ell_r^2)}\Big\}.$$

In the case $\mathcal{N} = \mathbb{C}$, the theorem above takes a simpler form. We can explicitly state as follows.

Corollary 6.3.1 Given $2 < p < \infty$ and $1 \leq q \leq \infty$. Let $(\mathcal{M}, \tau)$ be a noncommutative probability space, and $(x_n) \subset L^{p,q}$ be any finite sequence independently with respect to τ such that $\tau(x_n) = 0$. Then

$$\|\sum_n x_n\|_{L^{p,q}(\mathcal{M})} \approx \max\Big\{\big\|\sum_n x_n \otimes e_n\big\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)}, \Big(\sum_n \|x_n\|^2_{L^2(\mathcal{M})}\Big)^{1/2}\Big\}.$$

Now we can extend Theorem 6.3.2 to the case $1 < p < 2, 1 \leq q \leq \infty$. We start by considering the subspace $I^{p,q}(\mathcal{M} \otimes \ell^\infty)$ of $L^{p,q}(\mathcal{M} \otimes \ell^\infty)$ consisting of all sequences (x_n) such that $x_n \in L^{p,q}(\mathcal{A}_n)$ with $\mathscr{E}(x_n) = 0, 1 < p < \infty, 1 \leq q \leq \infty$. Alternately, $I^{p,q}(\mathcal{M} \otimes \ell^\infty)$ can be defined as the closure in $L^{p,q}(\mathcal{M} \otimes \ell^\infty)$. Similarly, we define the corresponding subspaces of $L^{p,q}(\mathcal{M}, \mathscr{E}; l^2_c)$ and $L^{p,q}(\mathcal{M}, \mathscr{E}; \ell^2_r)$, which are denoted respectively by $I^{p,q}(\mathcal{M}, \mathscr{E}; \ell^2_c)$ and $I^{p,q}(\mathcal{M}, \mathscr{E}; \ell^2_r)$.

Lemma 6.3.2 Let $1 < p < \infty, 1 \leq q \leq \infty$. Then $I^{p,q}(\mathcal{M} \otimes \ell^\infty)$ is complemented in $L^{p,q}(\mathcal{M} \otimes \ell^\infty)$. The similar statements hold for $I^{p,q}(\mathcal{M}, \mathscr{E}; \ell^2_c)$ and $I^{p,q}(\mathcal{M}, \mathscr{E}; \ell^2_r)$.

Proof Define the map $T : L^{p,q}(\mathcal{M}\otimes\ell^\infty) \longrightarrow L^{p,q}(\mathcal{M}\otimes\ell^\infty)$ by $T\big((x_k)\big) = \big(\mathscr{E}_{\mathcal{A}_k}(x_k)\big)$. Then

$$\begin{aligned} \|T\big((x_k)\big)\|^p_{L^p(\mathcal{M}\otimes\ell^\infty)} &= \|\big(\mathscr{E}_{\mathcal{A}_k}(x_k)\big)\|^p_{L^p(\mathcal{M}\otimes\ell^\infty)} = \sum \|\mathscr{E}_{\mathcal{A}_k}(x_k)\|^p_{L^p(\mathcal{M})} \\ &\leq \sum \|x_k\|^p_{L^p(\mathcal{M})} = \|(x_k)\|^p_{L^p(\mathcal{M}\otimes\ell^\infty)}. \end{aligned}$$

So T is a contraction on $L^p(\mathcal{M}\otimes\ell^\infty)$. By interpolation, T is bounded on $L^{p,q}(\mathcal{M}\otimes\ell^\infty)$ for $1 < p < \infty, 1 \leq q \leq \infty$. The same argument show $F\big((x_k)\big) = \big(\mathscr{E}(x_k)\big)$ is also bounded on $L^{p,q}(\mathcal{M} \otimes \ell^\infty)$. Then $(id - F)T$ is the desired projection from $L^{p,q}(\mathcal{M} \otimes \ell^\infty)$ onto $I^{p,q}(\mathcal{M} \otimes \ell^\infty)$. Similar arguments are true for $I^{p,q}(\mathcal{M}, \mathscr{E}; \ell^2_c)$ and $I^{p,q}(\mathcal{M}, \mathscr{E}; \ell^2_r)$, we omit the details.

we complete the proof of this Lemma. $\square$

Theorem 6.3.3 Given $1 < p < 2, 1 \leq q \leq \infty$. Let $(\mathcal{M}, \tau)$ be a noncommutative probability space, and $(x_n) \subset L^{p,q}$ be any finite sequence independently with respect to $\mathscr{E}$ such that $\mathscr{E}(x_n) = 0$. Then

$$\Big\|\sum_n x_n\Big\|_{L^{p,q}(\mathcal{M})} \approx \inf\Big\{\Big\|\sum_n y_n \otimes e_n\Big\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)} + \|(z_n)\|_{L^{p,q}(\mathcal{M},\mathscr{E};\ell^2_c)} + \|(w_n)\|_{L^{p,q}(\mathcal{M},\mathscr{E};\ell^2_r)}\Big\}.$$

where y_n, z_n and w_n are respectively independent with respect to $\mathscr{E}$ satisfying $\mathscr{E}(y_n) = \mathscr{E}(z_n) = \mathscr{E}(w_n) = 0$.

Proof Let $x_n = y_n + w_n + z_n$ be any decomposition satisfying the conditions above. Then

$$\Big\|\sum_n x_n\Big\|_{L^{p,q}(\mathcal{M})} \leq C\Big(\Big\|\sum_n y_n\Big\|_{L^{p,q}(\mathcal{M})} + \Big\|\sum_n w_n\Big\|_{L^{p,q}(\mathcal{M})} + \Big\|\sum_n z_n\Big\|_{L^{p,q}(\mathcal{M})}\Big).$$

For $1 < p < 2$,

$$\Big\|\sum_n y_n\Big\|_{L^p(\mathcal{M})} \leq C\mathbb{E}\Big(\Big\|\sum_n y_n\varepsilon_k\Big\|^2_{L^p(\mathcal{M})}\Big)^{1/2} \leq C\Big(\sum \|y_k\|^p_{L^p(\mathcal{M})}\Big)^{1/p},$$

where (ε_k) denote the Rademacher sequence. Noting Lemma 6.3.2 and by interpolation,

$$\|\sum_n y_n\|_{L^{p,q}(\mathcal{M})} \leq C\Big\|\sum_n y_n \otimes e_n\Big\|_{L^{p,q}(\mathcal{M}\otimes\ell^\infty)}.$$

Denoting $z = \sum_n z_n$, we have

$$\|z\|^2_{L^{p,q}(\mathcal{M})} = \|z^*z\|_{L^{p/2,q/2}(\mathcal{M})} \leq \|\mathscr{E}(z^*z)\|_{L^{p/2,q/2}(\mathcal{M})} = \Big\|\Big(\sum_n \mathscr{E}(z_n^* z_n)\Big)^{1/2}\Big\|.$$

Passing to adjoint, we get the same argument,

$$\Big\|\sum_n w_n\Big\|_{L^{p,q}(\mathcal{M})} \leq C\|(w_n)\|_{L^{p,q}(\mathcal{M},\mathscr{E};\ell^2_r)}.$$

To prove the converse inequality we use Theorem 6.3.2 and the duality again. Let $q \neq \infty$. Note that the infimum above is the norm of (x_n) in sum space $I^{p,q}(\mathcal{M}\otimes\ell^\infty) + I^{p,q}(\mathcal{M},\mathscr{E};\ell^2_c) + I^{p,q}(\mathcal{M},\mathscr{E};\ell^2_r)$. By the duality between sums and intersections and Lemma 6.3.2, we have

$$\Big(I^{p',q'}(\mathcal{M}\otimes\ell^\infty)\cap I^{p',q'}(\mathcal{M},\mathscr{E};\ell^2_c)\cap I^{p',q'}(\mathcal{M},\mathscr{E};\ell^2_r)\Big)^* = I^{p,q}(\mathcal{M}\otimes\ell^\infty)+I^{p,q}(\mathcal{M},\mathscr{E};\ell^2_c)+I^{p,q}(\mathcal{M},\mathscr{E};\ell^2_r)$$

Now let $(x'_n) \in I^{p',q'}(\mathcal{M}\otimes\ell^\infty) \cap I^{p',q'}(\mathcal{M},\mathscr{E};\ell^2_c) \cap I^{p',q'}(\mathcal{M},\mathscr{E};\ell^2_r)$ such that

$$\|(x'_n)\|_{I^{p',q'}(\mathcal{M}\otimes\ell^\infty)\cap I^{p',q'}(\mathcal{M},\mathscr{E};\ell^2_c)\cap I^{p',q'}(\mathcal{M},\mathscr{E};\ell^2_r)} \leq 1.$$

Then by Theorem 6.3.2,

$$\Big\|\sum x'_n\Big\|_{I^{p',q'}(\mathcal{M})} \leq C.$$

Thus by independence assumption and the Holder inequality

$$\Big|\tau\Big(\sum_n x_n^* x'_n\Big)\Big| = \Big|\tau\Big(\sum_n x_n^*\Big)\Big(\sum_n x'_n\Big)\Big| \leq C\Big\|\sum_n x_n\Big\|_{I^{p,q}(\mathcal{M})}.$$

If $q = \infty$, we can use $I^{p,\infty}(\mathcal{M}) = \big(I^{p',1}(\mathcal{M})\big)^*$. We also deduce the desired inequality.

We complete the proof. $\square$

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