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Runlian XIA

Espaces de Hardy locaux à valeurs opératorielles et Applications aux opérateurs pseudo-différentiels

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dirigée par Quanhua Xu

Rapporteurs: Tao Mei et Éric Ricard

Soutenue le 10 octobre 2017 devant le jury composé de :

M. Uwe Franz	Examinateur
M. Christian Le Merdy	Examinateur et Président du jury
M. Tao Mei	Rapporteur
M. Javier Parcet	Examinateur
M. Éric Ricard	Rapporteur
M. Quanhua Xu	Directeur

Laboratoire de Mathématiques de Besançon
16 route de Gray
25030 Besançon

École doctorale Carnot-Pasteur

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Résumé

Le but de cette thèse est d'étudier l'analyse sur les espaces $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, la version locale des espaces de Hardy à valeurs opératorielles construits par Tao Mei. Les espaces de Hardy locaux à valeurs opératorielles sont définis par les g-fonctions de Littlewood-Paley tronquées et les fonctions intégrables de Lusin tronquées associées au noyau de Poisson. Nous développons la théorie de Calderón-Zygmund sur $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$; nous étudions la dualité h_p^c-bmo_q^c et l'interpolation. D'après ces résultats, nous obtenons la caractérisation générale de $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ en remplaçant le noyau de Poisson par des fonctions tests raisonnables. Ceci joue un rôle important dans la décomposition atomique lisse de $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. En même temps, nous étudions aussi les espaces de Triebel-Lizorkin inhomogènes à valeurs opératorielles $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. Comme dans le cas classique, ces espaces sont connectés avec des espaces de Hardy locaux à valeurs opératorielles par les potentiels de Bessel. Grâce à l'aide de la théorie de Calderón-Zygmund, nous obtenons les caractérisations de type Littlewood-Paley et de type Lusin par des noyaux plus généraux. Ces caractérisations nous permettent d'étudier différentes propriétés de $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$, en particulier, la décomposition atomique lisse. Ceci est une extension et une amélioration de la décomposition atomique précédente de $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. Comme une application importante de cette décomposition atomique lisse, nous montrons la bornitude d'opérateurs pseudo-différentiels avec les symboles réguliers à valeurs opératorielles sur des espaces de Triebel-Lizorkin $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$, pour $\alpha\in\mathbb{R}$ et $1\le p\le\infty$. Finalement, grâce à la transférence, nous obtenons aussi la $F_p^{\alpha,c}$-bornitude d'opérateurs pseudo-différentiels sur les tores quantiques.

Mots-clefs

Espaces L_p non commutatifs, espaces de Hardy locaux, espaces BMO locaux, théorie de Calderón-Zygmund, dualité, caractérisations, espaces de Triebel-Lizorkin inhomogènes, interpolations, décompositions atomiques, opérateurs pseudo-différentiels, tores quantiques.

Abstract

This thesis is devoted to the study of the analysis on the spaces $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, the local version of operator-valued Hardy spaces studied by Tao Mei. The operator-valued local Hardy spaces are defined by the truncated Littlewood-Paley g-functions and the truncated Lusin square functions associated to the Poisson kernel. We develop the Calderón-Zygmund theory on $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, and study the h_p^c-bmo_q^c duality and the interpolation. Based on these results, we obtain general characterization of $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ which states that the Poisson kernel can be replaced by any reasonable test function. This characterization plays an important role in the smooth atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. We also investigate the operator-valued inhomogeneous Triebel-Lizorkin spaces $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. Like in the classical case, these spaces are connected with the operator-valued local Hardy spaces via Bessel potentials. Then by the aid of the Calderón-Zygmund theory, we obtain the Littlewood-Paley type and the Lusin type characterizations of $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ by more general kernels. These characterizations allow us to study various properties of $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$, in particular, the smooth atomic decomposition. This is an extension and an improvement of the previous atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. As an important application of this smooth atomic decomposition, we show the boundedness of pseudo-differential operators with regular operator-valued symbols on Triebel-Lizorkin spaces $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$, for $\alpha\in\mathbb{R}$ and $1\le p\le\infty$. Finally, by virtue of transference, we obtain the $F_p^{\alpha,c}$-boundedness of pseudo-differential operators on quantum tori.

Keywords

Noncommutative L_p-spaces, local Hardy spaces, local BMO spaces, Calderón-Zygmund theory, duality, characterizations, inhomogeneous Triebel-Lizorkin spaces, interpolations, atomic decompositions, pseudo-differential operators, quantum tori.

Contents

Introduction

0.1 Introduction

This thesis consists mainly of two topics: operator-valued local Hardy spaces and pseudo-differential operators. It follows the current line of investigation of noncommutative harmonic analysis. The latter field arose from the noncommutative integration theory developed by Murray and von Neumann, in order to provide a mathematical foundation for quantum mechanics. The object was to construct and study a linear functional on an operator algebra which plays the role of the classical integral. In [53], Pisier and Xu developed a pioneering work on noncommutative martingale theory; since then, many classical results have been successfully transferred to the noncommutative setting, see for instance, [28, 29, 31, 32, 55, 50, 56].

Inspired by the above mentioned developments and the Littlewood-Paley-Stein theory of quantum Markov semigroups (cf. [30, 35, 34]), Mei [42] studied operator-valued Hardy spaces, which were defined by the Littlewood-Paley g-function and Lusin area integral function associated to the Poisson kernel. These spaces are shown to be very useful for many aspects of noncommutative harmonic analysis. In [70], we obtain general characterizations of Mei's Hardy spaces, which state that the Poisson kernel can be replaced by any reasonable test function. This is done mainly by using the operator-valued Calderón-Zygmund theory.

In the classical setting, local Hardy spaces were first introduced by Goldberg [21]. Afterwards, many other inhomogeneous spaces have been studied. Our references for the classical theory are [21, 64, 18]. However, they have not been investigated so far in the operator-valued case. Motivated by [71, 70, 42], we provide a localization of Mei's operator-valued Hardy spaces on $\mathbb{R}^d$ in this thesis. The norms of these spaces are partly given by the truncated versions of the Littlewood-Paley g-function and Lusin area integral function. Some techniques that we use to deal with our local Hardy spaces are modelled after those of [70]; however, some highly not trivial modifications are needed. Since with the truncation, we only know the L_p-norms of the Poisson integrals of functions on the strip $\mathbb{R}^d \times (0,1)$, and lose information when the time is large than 1. This brings some substantial difficulties that the non-local case does not have, for example, the duality problem. Moreover, the noncommutative maximal function method is still unavailable in this setting, while in the classical case it is efficiently and frequently employed. However, based on tools developed recently, for instance, in [53, 28, 31, 55, 56, 30, 42, 43], we can overcome these difficulties. Parallel to Mei's Hardy spaces, we extend many results in [42, 70] to the inhomogeneous setting.

Goldberg's motivation of introducing the local Hardy spaces is the study of pseudo-differential operators on these spaces. Pseudo-differential operators were first explicitly defined by Kohn-Nirenberg [37] and Hörmander [25] to connect singular integrals and differential operators. One of the most important problems in pseudo-differential operator theory concerns the mapping properties of these operators on various function spaces, for instance, L_p-spaces, Sobolev, Besov and Triebel-Lizorkin spaces. However, it is known that pseudo-differential operators are not necessarily bounded on the classical Hardy space $\mathcal{H}_1(\mathbb{R}^d)$. This is the reason why local Hardy spaces were first introduced by Goldberg [21].

In the noncommutative setting, this line of research started with Connes' work [11] on pseudo-differential calculus for C^*-dynamical systems. But so far, the mapping properties are rarely studied. In this thesis, we consider the boundedness of noncommutative pseudo-differential operators on operator-valued Hardy spaces, or more generally, operator-valued Triebel-Lizorkin spaces. We then apply the outcome to the quantum torus case, and obtain a parallel theory in the latter case too.

Let us mention that independently and at the same time, González-Pérez, Junge and Parcet developed in [22] the pseudo-differential theory in quantum Euclidean spaces that are non compact analogues of quantum tori. Although there exists an overlap between them, the two works are very different in nature in regard to both results and arguments. Their results concern the boundedness of a pseudo-differential operator on the L_p-spaces with $1 < p < \infty$, while the ours deal with this boundedness on a column Triebel-Lizorkin spaces $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ with $\alpha \in \mathbb{R}$ and $1 \leq p \leq \infty$. Note that the mixture Triebel-Lizorkin space $F_p^{\alpha}(\mathbb{R}^d,\mathcal{M})$ coincide with $L_p(\mathcal{N})$ when $\alpha = 0$ and $1 < p < \infty$. On the other hand, the argument of [22] are based on a careful analysis of the L_2 and BMO cases, while our proof relies entirely on the atomic decomposition for $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ and a duality argument for the case $p = \infty$.

We now describe briefly our main results by gathering them together according to the three principal themes. All results below are stated only for the column spaces; but almost all of them admit row and mixture analogues.

Local Hardy spaces. The first family of results concerns the operator-valued local Hardy spaces $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$. The first major result of this part is the h_p^c-bmo_q^c duality for $1 \leq p < 2$, where q denotes the conjugate index of p. In particular, when $p = 1$, we obtain the operator-valued local analogue of the classical Fefferman-Stein theorem. The pattern of the proof of this theorem is similar to that of Mei's non-local case. We also show that $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M}) = \mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ for $2 < q < \infty$ like in the martingale and non-local settings. Thus the dual of $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ agrees with $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})$ when $1 < p < 2$.

The local Hardy spaces behave well with both complex and real interpolations. In particular, we have

$$\left(\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}), \mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\right)_{\frac{1}{p}} = \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}),$$

for $1 < p < \infty$. We reduce this interpolation problem to the corresponding one on the non-local Hardy spaces in order to use Mei's interpolation result in [42]. This proof is quite simple.

Like in [70], the Calderón-Zygmund theory plays a paramount role in this thesis. The usual $\mathcal{M}$-valued Calderón-Zygmund operators which satisfy the Hörmander condition are not bounded on inhomogeneous spaces. Thus in order to guarantee the boundedness of a Calderón-Zygmund operator on $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, we need to impose an extra decay at infinity to the kernel. Our treatment of this part is similar to [64]. Besides the local nature, there exists another difference: we also consider Hilbert space valued setting. This Hilbertian extension will be needed for general characterizations of operator-valued Triebel-Lizorkin spaces by the Lusin type square function.

The Calderón-Zygmund theory mentioned above will be applied to the general characterization of $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ with the Poisson kernel replaced by any reasonable test function. Additionally, to the characterization by the Littlewood Paley g-function, we also obtain the corresponding one by the Lusin square function. The latter will play an important role in the atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$.

We show that $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ admits an atomic decomposition as in the non-local case of [21]. However, for the study of pseudo-differential operators, we need a smooth atomic decomposition, that is, the atoms in consideration are required to be smooth and have size control on their derivatives too. We do this via tent spaces by using the Calderón reproducing identity. This is a quite technical part, some arguments are very lengthy and tedious.

Inhomogeneous Triebel-Lizorkin spaces. Local Hardy spaces are closely related to the inhomogeneous Triebel-Lizorkin spaces. Pursuing the investigation of Triebel-Lizorkin spaces on quantum tori carried out in [72], we consider operator-valued Triebel-Lizorkin spaces $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. The results in this part will be applied to the study of pseudo-differential operators with operator-valued kernels. We mention here two major results of this part. The first one gives a general characterization of $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ by any reasonable test function. This characterization can be realized either by the Littlewood Paley type g-function or by the Lusin type integral function. In the classical setting, such a characterization is achieved usually by virtue of maximal function techniques which are unfortunately no longer at our disposal in the noncommutative setting. As in [70], our arguments depend heavily on the Calderón-Zygmund theory mentioned in the previous part.

The second major result of this part is the atomic decomposition of the Triebel-Lizorkin spaces $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. This is an extension as well as an improvement of the previous atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$. Compared to the case of Hardy spaces, subatoms enter in the game; they will play a crucial role in the study of pseudo-differential operators. The proof of the atomic decomposition of $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ follows the same set-up as for $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$. Again, the Calderón reproducing identity, via tent spaces, is a key ingredient.

Pseudo-differential operators. Based on the smooth atomic decomposition, we obtain the boundedness of pseudo-differential operators in the class $S^0_{1,\delta}$ with $0 \leq \delta < 1$ on $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ for any $1 \leq p \leq \infty$ and $\alpha \in \mathbb{R}$. The main part concerns the case $p = 1$. As said before , the key ingredient of the proof for $p = 1$ is the smooth atomic decomposition of $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. Compared to the standard proof, via atomic decomposition, of the boundedness on $\mathcal{H}_1$ of a usual Calderón-Zygmund operator with a commutative or noncommutative kernel, the present proof is much subtler and more technical. We need a careful analysis of a pseudo-differential operator on subatoms.

By transference, our result yields the corresponding ones for the quantum torus case. The Euclidean space analogue of the latter case was studied by González-Pérez, Junge and Parcet in [22]. However, our approach is completely different from theirs.

In the remainder of this introduction, we will first introduce the necessary definitions of the function spaces in consideration and pseudo-differential operators, then describe the main results.

0.2 Definitions

0.2.1 Operator-valued local Hardy spaces

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal faithful semifinite trace τ. Let $\mathcal{N} = L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$ equipped with the tensor trace. For $1 \le p < \infty$, $L_p(\mathcal{M})$ denotes the noncommutative L_p-space associated to $(\mathcal{M}, \tau)$.

Let P be the Poisson kernel of $\mathbb{R}^d$:

$$\mathrm{P}(s) = c_d \, \frac{1}{(|s|^2+1)^{\frac{d+1}{2}}},$$

where c_d is the usual normalizing constant. Let

$$\mathrm{P}_\varepsilon(s) = \frac{1}{\varepsilon^d}\,\mathrm{P}(\frac{s}{\varepsilon}) = c_d\,\frac{\varepsilon}{(|s|^2+\varepsilon^2)^{\frac{d+1}{2}}}\,.$$

For any function f on $\mathbb{R}^d$ with values in $L_1(\mathcal{M})+\mathcal{M}$, its Poisson integral, whenever exists, will be denoted by $\mathrm{P}_\varepsilon(f)$:

$$\mathrm{P}_\varepsilon(f)(s) = \int_{\mathbb{R}^d} \mathrm{P}_\varepsilon(s-t)f(t)dt, \quad (s,\varepsilon) \in \mathbb{R}^{d+1}_+.$$

Let us denote the Hilbert space $L_2(\mathbb{R}^d, \frac{dt}{1+|t|^{d+1}})$ by R_d. Note that the Poisson integral of f exists if

$$f \in L_1(\mathcal{M}; \mathrm{R}^c_d) + L_\infty(\mathcal{M}; \mathrm{R}^c_d).$$

Now we define the local analogue of the Lusin area square function of f by

$$s^c(f)(s) = \Big(\int_{\widetilde{\Gamma}} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}}\Big)^{\frac{1}{2}}, \quad s \in \mathbb{R}^d,$$

where $\widetilde{\Gamma}$ is the truncated cone $\{(t,\varepsilon) \in \mathbb{R}^{d+1}_+ : |t| < \varepsilon < 1\}$. It is the intersection of the cone $\{(t,\varepsilon) \in \mathbb{R}^{d+1}_+ : |t| < \varepsilon\}$ and the strip $S \subset \mathbb{R}^{d+1}_+$ defined by:

$$S = \{(s,\varepsilon) : s \in \mathbb{R}^d, 0 < \varepsilon < 1\}.$$

For $1 \le p < \infty$ define the column local Hardy space $\mathrm{h}^c_p(\mathbb{R}^d, \mathcal{M})$ to be

$$\mathrm{h}^c_p(\mathbb{R}^d, \mathcal{M}) = \{f \in L_1(\mathcal{M}; \mathrm{R}^c_d) + L_\infty(\mathcal{M}; \mathrm{R}^c_d) : \|f\|_{\mathrm{h}^c_p} < \infty\},$$

where the $\mathrm{h}^c_p(\mathbb{R}^d, \mathcal{M})$-norm of f is defined by

$$\|f\|_{\mathrm{h}^c_p(\mathbb{R}^d,\mathcal{M})} = \|s^c(f)\|_{L_p(\mathcal{N})} + \|\mathrm{P} * f\|_{L_p(\mathcal{N})}.$$

The row local Hardy space $\mathrm{h}^r_p(\mathbb{R}^d, \mathcal{M})$ is the space of all f such that $f^* \in \mathrm{h}^c_p(\mathbb{R}^d, \mathcal{M})$, equipped with the norm $\|f\|_{\mathrm{h}^r_p} = \|f^*\|_{\mathrm{h}^c_p}$. Moreover, define the mixture space $\mathrm{h}_p(\mathbb{R}^d, \mathcal{M})$ as follows:

$$\mathrm{h}_p(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}^c_p(\mathbb{R}^d, \mathcal{M}) + \mathrm{h}^r_p(\mathbb{R}^d, \mathcal{M}) \text{ for } 1 \le p \le 2$$

equipped with the sum norm

$$\|f\|_{\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})} = \inf\{\|g\|_{\mathrm{h}^c_p} + \|h\|_{\mathrm{h}^r_p} : f = g + h, g \in \mathrm{h}^c_p(\mathbb{R}^d, \mathcal{M}), h \in \mathrm{h}^r_p(\mathbb{R}^d, \mathcal{M})\},$$

and

$$\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}) = \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}) \cap \mathrm{h}_p^r(\mathbb{R}^d,\mathcal{M}) \text{ for } 2 < p < \infty$$

equipped with the intersection norm

$$\|f\|_{\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})} = \max\{\|f\|_{\mathrm{h}_p^c}, \|f\|_{\mathrm{h}_p^r}\}.$$

For any cube $Q \subset \mathbb{R}^d$ with sides parallel to the axes, we will denote its center by c_Q, side length by $l(Q)$, and volume by $|Q|$. Let $f \in L_\infty(\mathcal{M};\mathrm{R}_d^c)$, the mean value of f over Q is denoted by $f_Q = \frac{1}{|Q|}\int_Q f(s)ds$. For $f \in L_\infty(\mathcal{M};\mathrm{R}_d^c)$, set

$$\|f\|_{\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})} = \max\Big\{ \sup_{|Q|<1} \|(\frac{1}{|Q|}\int_Q |f - f_Q|^2 dt)^{\frac12}\|_{\mathcal{M}}, \sup_{|Q|=1} \|(\int_Q |f|^2 dt)^{\frac12}\|_{\mathcal{M}} \Big\}.$$

Then we define

$$\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}) = \left\{ f \in L_\infty(\mathcal{M};\mathrm{R}_d^c) : \|f\|_{\mathrm{bmo}^c} < \infty \right\}.$$

Respectively, define $\mathrm{bmo}^r(\mathbb{R}^d,\mathcal{M})$ to be the space of all $f \in L^\infty(\mathcal{M};\mathrm{R}_d^r)$ such that

$$\|f^*\|_{\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})} < \infty$$

with the norm $\|f\|_{\mathrm{bmo}^r} = \|f^*\|_{\mathrm{bmo}^c}$. And $\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ is defined as the intersection of $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{bmo}^r(\mathbb{R}^d,\mathcal{M})$.

In order to describe the dual of $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ for $1 \le p < 2$, we will define a bmo type space called $\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ (with q the conjugate index of p).

Let $2 < q \le \infty$, we define $\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ to be the space of all $f \in L_q(\mathcal{M};\mathrm{R}_d^c)$ such that

$$\|f\|_{\mathrm{bmo}_q^c} = \Big(\Big\| {\sup_{\substack{s\in Q\subset\mathbb{R}^d\\ |Q|<1}}}^{+} \frac{1}{|Q|}\int_Q |f(t)-f_Q|^2 dt\Big\|_{L_{\frac q2}(\mathcal{N})}^{\frac q2} + \Big\| {\sup_{\substack{s\in Q\subset\mathbb{R}^d\\ |Q|=1}}}^{+} \frac{1}{|Q|}\int_Q |f(t)|^2 dt\Big\|_{L_{\frac q2}(\mathcal{N})}^{\frac q2}\Big)^{\frac1q} < \infty.$$

By this definition, $\mathrm{bmo}_\infty^c(\mathbb{R}^d,\mathcal{M})$ coincides with the space $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$ defined above.

0.2.2 Operator-valued inhomogeneous Triebel-Lizorkin spaces

Fix a Schwartz function φ on $\mathbb{R}^d$ satisfying the usual Littlewood-Paley decomposition property:

$$\begin{cases} \operatorname{supp}\varphi \subset \{\xi : \frac12 \le |\xi| \le 2\}. \\ \varphi > 0 \text{ on } \{\xi : \frac12 < |\xi| < 2\}, \\ \sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1, \xi \ne 0. \end{cases}$$

For each $k \in \mathbb{N}$, let φ_k be the function whose Fourier transform is equal to $\varphi(2^{-k}\cdot)$ and φ_0 be the function whose Fourier transform is equal to $1 - \sum_{k>0}\varphi(2^{-k}\cdot)$. Then $\{\varphi_k\}_{k\ge0}$ gives a Littlewood-Paley decomposition on $\mathbb{R}^d$, such that

$$\operatorname{supp}\widehat{\varphi}_k \subset \{\xi \in \mathbb{R}^d : 2^{k-1} \le |\xi| \le 2^{k+1}\},\ k \in \mathbb{N}, \quad \text{and} \ \operatorname{supp}\widehat{\varphi}_0 \subset \{\xi \in \mathbb{R}^d : |\xi| \le 2\}$$

and that

$$\sum_{k=0}^\infty \widehat{\varphi}_k(\xi) = 1 \quad \forall\, \xi \in \mathbb{R}^d.$$

Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. Denote by $\mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$ the $L_1(\mathcal{M}) + \mathcal{M}$-valued tempered distribution on $\mathbb{R}^d$. Then the column inhomogeneous Triebel-Lizorkin space $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ is defined by

$$F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}) = \{f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M}) : \|f\|_{F_p^{\alpha,c}} < \infty\},$$

where

$$\|f\|_{F_p^{\alpha,c}} = \big\|(\sum_{j\geq 0} 2^{2j\alpha}|\varphi_j * f(s)|^2)^{\frac{1}{2}}\big\|_p.$$

We then define the row and mixture inhomogeneous Triebel-Lizorkin spaces in the same way as for the local Hardy spaces.

0.2.3 Pseudo-differential operators

Let $n \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. Then $S^n_{\rho,\delta}$ denotes the collection of all infinitely differentiable functions σ defined on $\mathbb{R}^d \times \mathbb{R}^d$ and with values in $\mathcal{M}$, such that for each pair of multi-indices of nonnegative integers γ, β, there exists a constant $C_{\gamma,\beta}$ such that

$$\|D_s^\gamma D_\xi^\beta \sigma(s,\xi)\|_{\mathcal{M}} \leq C_{\gamma,\beta}(1+|\xi|)^{n+\delta|\gamma|_1-\rho|\beta|_1},$$

where $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}_0^d$, $|\gamma|_1 = \gamma_1 + \cdots + \gamma_d$ and $D_s^\gamma = \frac{\partial^{\gamma_1}}{\partial s_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_d}}{\partial s_d^{\gamma_d}}$.

Let $\sigma \in S^n_{\rho,\delta}$. For any function $f \in \mathcal{S}(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$, the (left) pseudo-differential operator is a mapping $f \mapsto T_\sigma^c f$ given by

$$T_\sigma^c f(s) = \int_{\mathbb{R}^d} \sigma(s,\xi)\widehat{f}(\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi.$$

σ is called the symbol of T_σ^c.

0.3 Properties

0.3.1 Duality

The first property of Hardy spaces is the duality theorem. We describe the dual of $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ $(1 \leq p < 2)$ as $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ (q being the conjugate index of p).

Theorem 0.1. *Let $1 \leq p < 2$ and q be its conjugate index. We have $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})^* = \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ with equivalent norms. More precisely, every $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ defines a continuous linear functional on $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ by*

$$\ell_g(f) = \tau \int f(s)g^*(s)ds, \quad \forall f \in L_p(\mathcal{M}; L_2^c(\mathbb{R}^d, (1+|t|^{d+1})dt)).$$

Conversely, every $\ell \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})^*$ can be written as above and is associated to some $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ with

$$\|\ell\|_{(\mathrm{h}_p^c)^*} \approx \|g\|_{\mathrm{bmo}_q^c}.$$

Denote by I^α the Riesz potential $(-(2\pi)^{-2}\Delta)^{\frac{\alpha}{2}}$. If $\alpha = 1$, we will abbreviate I^1 as I. For a tempered distribution f on $\mathbb{R}^d$ with values in $L_1(\mathcal{M}) + \mathcal{M}$, we have

$$I^\alpha f(s) = \int_{\mathbb{R}^d} \widehat{f}(\xi)|\xi|^\alpha e^{2\pi \mathrm{i} s\cdot\xi}d\xi, \quad \forall s \in \mathbb{R}^d.$$

The following equalities play an important role in study of the above duality problem:

$$\begin{aligned}\int_{\mathbb{R}^d} f(s)g^*(s)ds &= 4\int_{\mathbb{R}^d}\int_0^1 \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s)\varepsilon\, d\varepsilon ds\\ &\quad+\int_{\mathbb{R}^d}\mathrm{P}*f(s)(\mathrm{P}*g(s))^*ds+4\pi\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*ds.\\ &=\frac{4}{c_d}\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s+t)\frac{dtd\varepsilon}{\varepsilon^{d-1}}ds\\ &\quad+\int_{\mathbb{R}^d}\mathrm{P}*f(s)(\mathrm{P}*g(s))^*ds+4\pi\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*ds\end{aligned}$$

for nice $f,\ g\in L_1(\mathcal{M};\mathrm{R}_d^c)+L_\infty(\mathcal{M};\mathrm{R}_d^c)$. The presence of the terms $\mathrm{P}*g$ and $I(\mathrm{P})*g$ makes a main difference between the proofs of the non-local and local cases.

As in the classical theory, the dual of $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ is $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})$, which comes from the fact that the L_q-analogue of bmo space is actually equivalent to $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})$.

Theorem 0.2. *Let $2<q<\infty$. $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})=\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ with equivalent norms.*

The proof of the above theorem is modelled after that of [42, Theorem 4.7]. The main difference between them is that, due to the truncation, we have to deal with the terms $(I(\mathrm{P})*g,\, \mathrm{P}*g)$ concerning the properties of the Fourier transform of the considered function near the origin as explained above.

As a consequence of the previous two theorems, we obtain

Corollary 0.3. *For any $1<p<\infty$, $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})^*=\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})$ with equivalent norms.*

0.3.2 The relation between $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$

It should be pointed out that due to the noncommutativity, the column operator-valued local Hardy spaces $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and operator-valued Hardy spaces $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ defined by Mei [42] are not equivalent for $1<p<\infty$. We only have the inclusions:

$$\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})\subset\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})\quad\text{for}\quad 1\le p\le 2$$

and

$$\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})\subset\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})\quad\text{for}\quad 2<p<\infty.$$

However, if the Fourier transform of the function vanishes near the origin, we will have the reverse inclusion.

Theorem 0.4. *Let $\phi\in\mathcal{S}$ such that $\int_{\mathbb{R}^d}\phi(s)ds=1$.*

(1) *Let $1\le p\le 2$. For any $f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, we have $f-\phi*f\in\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\|f-\phi*f\|_{\mathcal{H}_p^c}\lesssim\|f\|_{\mathrm{h}_p^c}$.*

(2) *Let $2<p<\infty$. For any $f\in\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$, we have $f-\phi*f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\|f-\phi*f\|_{\mathrm{h}_p^c}\lesssim\|f\|_{\mathcal{H}_p^c}$.*

Moreover, if we consider the mixture versions, both $\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})$ and $\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})$ coincide with the space $L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})$:

Proposition 0.5. *For any $1<p<\infty$, $\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})=\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})=L_p(\mathcal{N})$ with equivalent norms.*

0.3.3 Lifting properties of $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$

We list some basic properties of $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. Much as in the classical case, the local Hardy spaces coincide with the inhomogeneous Triebel-Lizorkin spaces of order $\alpha=0$.

Proposition 0.6. *Let $1\le p<\infty$ and $\alpha\in\mathbb{R}$. Then*

(1) *$F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ is a Banach space.*

(2) *$F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})\subset F_p^{\beta,c}(\mathbb{R}^d,\mathcal{M})$ if $\alpha>\beta$.*

(3) *$F_p^{0,c}(\mathbb{R}^d,\mathcal{M})=\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ with equivalent norms.*

Given $a\in\mathbb{R}_+$, we define $D_{i,a}(\xi)=(2\pi \mathrm{i}\xi_i)^a$ for $\xi\in\mathbb{R}^d$, and D_i^a to be the Fourier multiplier with symbol $D_{i,a}(\xi)$ on Triebel-Lizorkin spaces $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. We set $D_a=D_{1,a_1}\cdots D_{d,a_d}$ and $D^a=D_1^{a_1}\cdots D_d^{a_d}$ for any $a=(a_1,\cdots,a_d)\in\mathbb{R}_+^d$. Note that if a is a positive integer, $D_i^a=\partial_i^a$ is the usual partial derivative, so there does not exist any conflict of notation. The operator D^a can be viewed as a fractional extension of partial derivatives. The following is the so-called reduction (or lifting) property of Triebel-Lizorkin spaces. For $\beta\in\mathbb{R}$, let J^β be the Bessel potential of order β, i.e. $J^\beta=(1-(2\pi)^{-2}\Delta)^{\frac{\beta}{2}}$. For a tempered distribution f on $\mathbb{R}^d$ with values in $L_1(\mathcal{M})+\mathcal{M}$, it is formulated as

$$J^\beta f(s)=\int_{\mathbb{R}^d}\widehat{f}(\xi)(1+|\xi|^2)^{\frac{\beta}{2}}e^{2\pi \mathrm{i}s\cdot\xi}d\xi\quad \forall s\in\mathbb{R}^d.$$

Proposition 0.7. *Let $1\le p<\infty$ and $\alpha\in\mathbb{R}$.*

(1) *For any $\beta\in\mathbb{R}$, J^β is an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ and $F_p^{\alpha-\beta,c}(\mathbb{R}^d,\mathcal{M})$. In particular, J^α is an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$.*

(2) *Let $\beta>0$. Then $f\in F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ if and only if $\varphi_0*f\in L_p(\mathcal{N})$ and $D_i^\beta f\in F_p^{\alpha-\beta,c}(\mathbb{R}^d,\mathcal{M})$ for all $i=1,\dots,d$. Moreover, in this case,*

$$\|f\|_{F_p^{\alpha,c}}\approx\|\varphi_0*f\|_p+\sum_{i=1}^d\|D_i^\beta f\|_{F_p^{\alpha-\beta,c}}.$$

We need to emphasize here that, different from the Triebel-Lizorkin spaces on quantum tori, the Riesz potential I^β is not an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ and $F_p^{\alpha-\beta,c}(\mathbb{R}^d,\mathcal{M})$.

0.3.4 Interpolation

As expected, we have the following complex and real interpolation results:

Theorem 0.8. *Let $1<p<\infty$. We have*

(1) $(\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M}))_{\frac{1}{p}}=\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$.

(2) $(\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M}))_{\frac{1}{p},p}=\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$.

The interpolation between any two local Hardy spaces $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{h}_{p'}^c(\mathbb{R}^d,\mathcal{M})$ with $1<p,p'<\infty$ is easy, since for any $1<p<\infty$, $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ is a complemented subspace of $L_p(\mathcal{N};L_2^c(\widetilde{\Gamma},\frac{dtd\varepsilon}{\varepsilon^{d+1}}))\oplus_p L_p(\mathcal{N})$. However, the interpolation problem with one end point being h_1-space is much subtler. The above interpolation equalities are obtained by transferring the problem to that of Hardy spaces in [42].

We also have the following corollary as the mixed version of the above theorem, which shows that $\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ are also good endpoints of $L_p(\mathcal{N})$.

Corollary 0.9. *Let $1 < p < \infty$. Then we have*

(1) $(X, Y)_{\frac{1}{p}} = L_p(\mathcal{N})$, *where* $X = \mathrm{bmo}(\mathbb{R}^d, \mathcal{M})$ *or* $L_\infty(\mathcal{N})$, *and* $Y = \mathrm{h}_1(\mathbb{R}^d, \mathcal{M})$ *or* $L_1(\mathcal{N})$.

(2) $(X, Y)_{\frac{1}{p},p} = L_p(\mathcal{N})$, *where* $X = \mathrm{bmo}(\mathbb{R}^d, \mathcal{M})$ *or* $L_\infty(\mathcal{N})$, *and* $Y = \mathrm{h}_1(\mathbb{R}^d, \mathcal{M})$ *or* $L_1(\mathcal{N})$.

The Bessel potentials J^α is an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$. In this way, the interpolation problem of Triebel Lizorkin spaces can be reduced to the corresponding problem of Hardy spaces. For the interpolation between two Triebel Lizorkin spaces with the same index α, the real and complex interpolations are simple corollary of Theorem 0.8. For different α, the real interpolation of two Triebel Lizorkin spaces will give Besov type spaces, which is not included in this thesis. For the complex interpolation, we need to use the complex order Bessel potentials and the result is:

Proposition 0.10. *Let $\alpha_0, \alpha_1 \in \mathbb{R}$ and $1 < p < \infty$. Then*

$$(F_\infty^{\alpha_0,c}(\mathbb{R}^d, \mathcal{M}), F_1^{\alpha_1,c}(\mathbb{R}^d, \mathcal{M}))_{\frac{1}{p}} = F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}), \quad \alpha = (1 - \frac{1}{p})\alpha_0 + \frac{\alpha_1}{p}.$$

0.4 Characterizations

In the classical theory, as far as we know, the existing proofs of the characterizations of local Hardy and inhomogeneous Triebel-Lizorkin spaces use maximal functions in a crucial way. As we mentioned earlier, this tool is no longer available in the noncommutative setting. Instead, we will use the Calderón-Zygmund theory. First, we begin with a general characterization of local Hardy spaces.

Consider a Schwartz function Φ on $\mathbb{R}^d$ of vanishing mean. We set $\Phi_\varepsilon(s) = \varepsilon^{-d}\Phi(\frac{s}{\varepsilon})$ for $\varepsilon > 0$. We will assume that Φ is nondegenerate in the following sense:

$$\forall \xi \in \mathbb{R}^d \setminus \{0\} \ \exists \, \varepsilon > 0 \ \text{ s.t. } \ \widehat{\Phi}(\varepsilon\xi) \neq 0.$$

Then there exists a Schwartz function Ψ of vanishing mean such that

$$\int_0^\infty \widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{d\varepsilon}{\varepsilon} = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}\,.$$

Next we can find two other functions ϕ, ψ such that $\widehat{\phi}, \widehat{\psi} \in H_2^\sigma(\mathbb{R}^d)$, $\widehat{\phi}(0), \widehat{\psi}(0) > 0$ and

$$\widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} = 1 - \int_0^1 \widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{d\varepsilon}{\varepsilon}.$$

For any $f \in L_1(\mathcal{M}; \mathrm{R}_d^c) + L_\infty(\mathcal{M}; \mathrm{R}_d^c)$, we define the local version of the conic and radial square functions of f associated to Φ by

$$s_\Phi^c(f)(s) = \Big(\iint_{\widetilde{\Gamma}} |\Phi_\varepsilon * f(s+t)|^2 \frac{dtd\varepsilon}{\varepsilon^{d+1}}\Big)^{\frac{1}{2}}, \, s \in \mathbb{R}^d,$$

$$g_\Phi^c(f)(s) = \Big(\int_0^1 |\Phi_\varepsilon * f(s)|^2 \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}, \, s \in \mathbb{R}^d.$$

where $\widetilde{\Gamma}$ is the truncated cone $\Big\{(t,\varepsilon) \in \mathbb{R}_+^{d+1} : |t| < \varepsilon < 1\Big\}$.

Theorem 0.11. *Let $1 \leq p < \infty$ and ϕ, Φ be as above. For any $f \in L_1(\mathcal{M}; \mathrm{R}_d^c) + L_\infty(\mathcal{M}; \mathrm{R}_d^c)$, $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ if and only if $s_\Phi^c(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$ if and only if $g_\Phi^c(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$. If this is the case, then*

$$\|f\|_{\mathrm{h}_p^c} \approx \|g_\Phi^c(f)\|_p + \|\phi * f\|_p \approx \|s_\Phi^c(f)\|_p + \|\phi * f\|_p$$

with the relevant constants depending only on d, p, Φ and ϕ.

One direction of the above norm equivalence can be deduced from the boundedness of the Hilbert-valued Calderón-Zygmund operator from $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ to $L_p(\mathcal{N}; H^c)$ for $1 \leq p \leq 2$ and the duality between $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{h}_q^c(\mathbb{R}^d, \mathcal{M})$. The other direction requires more complicated and technical computation, where the Carleson measure characterization of bmo_q^c is needed. In order to compare the square functions g_Φ^c and s_Φ^c, we need a sophisticated inequality, since we no longer have the harmonicity of the Poisson integral.

The above theorem admits a discrete version: The square functions s_Φ^c and g_Φ^c can be discretized as follows:

$$\begin{aligned} g_\Phi^{c,D}(f)(s) &= \Big(\sum_{j\geq 1} |\Phi_j * f(s)|^2\Big)^{\frac{1}{2}}, \\ s_\Phi^{c,D}(f)(s) &= \Big(\sum_{j\geq 1} 2^{dj} \int_{B(s,2^{-j})} |\Phi_j * f(t)|^2 dt\Big)^{\frac{1}{2}}. \end{aligned}$$

Here $B(s,r)$ denotes the ball of $\mathbb{R}^d$ with center s and radius r, and Φ_j is the inverse Fourier transform of $\Phi(2^{-j}\cdot)$. This time to get a resolvent of the unit on $\mathbb{R}^d$, we need to assume that Φ satisfies:

$$\forall\, \xi \in \mathbb{R}^d \setminus \{0\} \ \ \exists\, 0 < 2a \leq b < \infty \ \text{ s.t. } \ \Phi(\varepsilon\xi) \neq 0, \ \forall\, \varepsilon \in (a,\, b].$$

There exists another Schwartz function Ψ such that

$$\sum_{j=-\infty}^{+\infty} \widehat{\Phi}(2^{-j}\xi)\, \overline{\widehat{\Psi}(2^{-j}\xi)} = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}$$

and there exists two functions ϕ and ψ such that $\widehat{\phi}, \widehat{\psi} \in H_2^\sigma(\mathbb{R}^d)$ and

$$\sum_{j\geq 0} \widehat{\Phi}(2^{-j}\xi)\, \overline{\widehat{\Psi}(2^{-j}\xi)} + \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} = 1, \quad \forall \xi \in \mathbb{R}^d.$$

Theorem 0.12. *Let ϕ and Φ be test functions as above and $1 \leq p < \infty$. Then for any $f \in L_1(\mathcal{M}; \mathrm{R}_d^c) + L_\infty(\mathcal{M}; \mathrm{R}_d^c)$, $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ if and only if $s_\Phi^{c,D}(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$ if and only if $g_\Phi^{c,D}(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$. Moreover,*

$$\|f\|_{\mathrm{h}_p^c} \approx \|s_\Phi^{c,D}(f)\|_{L_p(\mathcal{N})} + \|\phi * f\|_p \approx \|g_\Phi^{c,D}(f)\|_p + \|\phi * f\|_p$$

with the relevant constants depending only on d, p, Φ and ϕ.

Since the noncommutative inhomogeneous Triebel-Lizorkin spaces are subspaces of Hilbert-valued noncommutative L_p-spaces, we develop Fourier multiplier theory for the later spaces. Then, with the aid of the above discrete characterization of local Hardy spaces, we obtain a more general characterization of $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ which states that the kernel which appears in the square function does not need to be a Schwartz function coming from the Littlewood-Paley decomposition.

Let $\Phi^{(0)}$ and Φ be two complex-valued infinitely differentiable functions defined respectively on $\mathbb{R}^d$ and $\mathbb{R}^d\backslash\{0\}$, which satisfy

$$\begin{cases} |\Phi^{(0)}(\xi)| > 0 \quad \text{if } |\xi| \leq 2, \\ \sup\limits_{k\in\mathbb{N}_0} 2^{-k\alpha_0}\|\Phi^{(0)}(2^k\cdot)\varphi\|_{H_2^\sigma} < \infty, \end{cases}$$

and

$$\begin{cases} |\Phi(\xi)| > 0 \quad \text{if } \frac{1}{2} \leq |\xi| \leq 2, \\ \sup\limits_{k\in\mathbb{N}_0} 2^{-k\alpha_0}\|\Phi(2^k\cdot)\varphi\|_{H_2^\sigma} < \infty, \\ \int_{\mathbb{R}^d}(1+|s|^2)^\sigma|\mathcal{F}^{-1}(\Phi\varphi^{(0)}I_{-\alpha_1})(s)|ds < \infty. \end{cases}$$

Recall that here $I_{-\alpha_1}(\xi)$ for $\xi \in \mathbb{R}^d$ is the symbol of the Fourier multiplier $I^{-\alpha_1}$, where $I^{-\alpha_1}$ is the Riesz potential $(-(2\pi)^{-2}\Delta)^{\frac{-\alpha_1}{2}}$.

Let $\Phi^{(j)} = \Phi(2^{-j}\cdot)$ for $j \geq 1$, and Φ_j be the function whose Fourier transform is equal to $\Phi^{(j)}$ for any $j \in \mathbb{N}_0$.

Theorem 0.13. *Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. Assume that $\alpha_0 < \alpha < \alpha_1$, $\alpha_1 \geq 0$ and $\Phi^{(0)}$, Φ satisfy conditions above. Then for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$, we have*

$$\|f\|_{F_p^{\alpha,c}} \approx \|(\sum_{j\geq 0} 2^{2j\alpha}|\Phi_j * f|^2)^{\frac{1}{2}}\|_p,$$

where the relevant constants are independent of f.

The continuous analogue of the above characterization holds as well. For any $L_1(\mathcal{M})+\mathcal{M}$-valued tempered distribution f on $\mathbb{R}^d$,

$$\|f\|_{F_p^{\alpha,c}} \approx \|\Phi_0 * f\|_p + \|(\int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon})^{\frac{1}{2}}\|_p.$$

.

Much as the local Hardy spaces, we also have the characterization via Lusin area square functions:

Theorem 0.14. *Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. Assume that $\alpha_0 < \alpha < \alpha_1$, $\alpha_1 \geq 0$ and $\Phi^{(0)}$, Φ satisfy the conditions above. Then for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$, we have*

$$\|f\|_{F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})} \approx \|\Phi_0 * f\|_p + \Big\|(\sum_{j\geq 1} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})}|\Phi_j * f(\cdot+t)|^2dt)^{\frac{1}{2}}\Big\|_p,$$

where the relevant constants are independent of f.

Since the local Hardy spaces can be seen as a special case of inhomogeneous Triebel-Lizorkin spaces, the above two theorems generalize Theorem 0.11 and Theorem 0.12 in this sense.

Note that the general characterization of Triebel-Lizorkin spaces on quantum tori has been studied in [72]. In that case, the Fourier transform of an operator x in $L_1(\mathbb{T}_\theta^d)$ ($\mathbb{T}_\theta^d$ being the d-dimensional quantum tori) is discrete, so we can always assume $\widehat{x}(0) = 0$ and omit the multiplier behaviour near the origin. However, for f in the inhomogeneous Triebel-Lizorkin spaces on $\mathbb{R}^d$, we need to deal with the properties of its Fourier transform near the origin. This makes our case more complicated than theirs.

0.5 Atomic decompositions

In the classical theory, atomic decomposition of Hardy or local Hardy spaces plays important roles in the analysis of these spaces. In this spirit, we also construct the noncommutative analogue of atoms.

We say that a function $a \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is an h_1^c-atom if

(1) a is supported in a bounded cube Q with $|Q| \le 1$;

(2) If $|Q| < 1$, $\int_Q a(s)ds = 0$;

(3) $\tau\big(\int_Q |a(s)|^2\, ds\big)^{\frac{1}{2}} \le |Q|^{-\frac{1}{2}}$.

Let $\mathrm{h}_{1,at}^c(\mathbb{R}^d, \mathcal{M})$ be the space of all f admitting a representation of the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where the a_j are h_1^c-atoms and $\lambda_j \in \mathbb{C}$ are such that $\sum_{j\in\mathbb{N}} |\lambda_j| < \infty$. We equip $\mathrm{h}_{1,at}^c(\mathbb{R}^d, \mathcal{M})$ with the following norm:

$$\|f\|_{\mathrm{h}_{1,at}^c} = \inf\Big\{ \sum_{j\in\mathbb{N}} |\lambda_j| : \ f = \sum_{j\in\mathbb{N}} \lambda_j a_j;\ a_j\text{'s are } h_1^c \text{ -atoms, } \lambda_j \in \mathbb{C} \Big\}.$$

By the atomic decomposition of Hardy spaces already studied in [42] and the duality between $L_1(\mathcal{M}; L_2^c(Q))$ and $L_\infty(\mathcal{M}; L_2^c(Q))$, we deduce the following atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$.

Theorem 0.15. $\mathrm{h}_{1,at}^c(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ *with equivalent norms.*

The smoothness of the atoms obtained above can be refined. The main idea is to find a smooth resolution of unit on the Euclidean space; atomic decompositions of tent spaces will also be of great service. Using the same strategy, we can get smooth atomic decompositions for $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$, which will be very useful when studying the pseudo-differential operators on Triebel-Lizorkin spaces.

In the classical theory, there exist several types of smooth atomic decompositions of Triebel-Lizorkin spaces. See, for instance [16, 47, 68]. However, not all of them can be transferred to the noncommutative setting by replacing L_∞-atoms with L_2-atoms. The idea of the following theorem comes from [68, Theorem 3.2.3], but many techniques used are different from that of [68, Theorem 3.2.3] due to the noncommutativity.

Definition 0.16. Let $\alpha \in \mathbb{R}$, and let K and L be two integers such that

$$K \ge ([\alpha]+1)_+ \quad \text{and} \quad L \ge \max\{[-\alpha], -1\}.$$

(1) A function $b \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is called an $(\alpha, 1)$-atom if

- $\operatorname{supp} b \subset 2Q_{0,k}$;
- $\tau(\int_{\mathbb{R}^d} |D^\gamma b(s)|^2 ds)^{\frac{1}{2}} \le 1,\ \ \forall \gamma \in \mathbb{N}_0^d\,,\ |\gamma|_1 \le K.$

(2) Let $Q = Q_{\mu,l}$, a function $a \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is called an (α, Q)-sub-atom if

- $\operatorname{supp} a \subset 2Q$;

- $\tau(\int_{\mathbb{R}^d} |D^\gamma a(s)|^2 ds)^{\frac{1}{2}} \leq |Q|^{\frac{\alpha}{d}-\frac{|\gamma|_1}{d}}, \ \forall \gamma \in \mathbb{N}_0^d, \ |\gamma|_1 \leq K;$
- $\int_{\mathbb{R}^d} s^\beta a(s) ds = 0, \ \forall \beta \in \mathbb{N}_0^d, \ |\beta|_1 \leq L.$

(3) A function $g \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is called an $(\alpha, Q_{k,m})$-atom if

$$\tau(\int_{\mathbb{R}^d} |J^\alpha g(s)|^2 ds)^{\frac{1}{2}} \leq |Q_{k,m}|^{-\frac{1}{2}} \quad \text{and} \quad g = \sum_{(\mu,l)\leq(k,m)} d_{\mu,l} a_{\mu,l},$$

for some $k \in \mathbb{N}_0$ and $m \in \mathbb{Z}^d$, where the $a_{\mu,l}$'s are $(\alpha, Q_{\mu,l})$-sub-atoms and the $d_{\mu,l}$'s are complex numbers such that

$$(\sum_{(\mu,l)\leq(k,m)} |d_{\mu,l}|^2)^{\frac{1}{2}} \leq |Q_{k,m}|^{-\frac{1}{2}}.$$

We refer the reader to chapter 8 for the precise definition of $D_{\mu,l}$.

Theorem 0.17. *Let $\alpha \in \mathbb{R}$ and K, L be two integers fixed above. Then any $f \in F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ can be represented as*

$$f = \sum_{j=1}^{\infty} (\mu_j b_j + \lambda_j g_j),$$

where the b_j's are $(\alpha, 1)$-atoms, the g_j's are (α, Q)-atoms, and μ_j, λ_j are complex numbers such that

$$\sum_{j=1}^{\infty} (|\mu_j| + |\lambda_j|) < \infty. \tag{0.1}$$

Moreover, the infimum of (0.1) with respect to all admissible representations is an equivalent norm in $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.

0.6 Pseudo-differential operators

In the classical theory, local Hardy spaces were first introduced in order to study the mapping properties of pseudo-differential operators. Now we have developed the theory of operator-valued local Hardy spaces, we can study the the boundedness of pseudo-differential operators in our noncommutative setting.

Based on the smooth atomic decompositions mentioned above, we prove that the image of an atom under the action of a pseudo-differential operator has bounded norm in $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. The case $1 < p < \infty$ is deduced from duality and interpolation.

Theorem 0.18. *Let $0 \leq \delta < 1$, $\sigma \in S_{1,\delta}^0$ and $\alpha \in \mathbb{R}$. Then T_σ^c is a bounded operator on $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ for every $1 \leq p \leq \infty$.*

The symbols in $\sigma \in S_{1,\delta}^0$ with $0 \leq \delta < 1$ are called regular symbols, which are bounded on $L_2(\mathcal{N})$, and behave well on symbolic calculus. When $\delta = 1$, we call symbols in $\sigma \in S_{1,1}^0$ forbidden symbols. Similarly to the classical case, they are not bounded on $L_2(\mathcal{N})$; alternatively, if $\alpha > 0$, we can prove their boundedness on $H_2^\alpha(\mathbb{R}^d; L^2(\mathcal{M}))$.

Theorem 0.19. *Let $\sigma \in S_{1,1}^0$ and $\alpha > 0$. Then T_σ^c is a bounded operator on $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.*

A very important application of the above theorem is the study of pseudo-differential operators over quantum tori through Neuwirth-Ricard's transference method. We refer the reader to [72] for details of the Triebel-Lizorkin spaces on quantum tori.

Let $0 \le \delta, \rho \le 1$, $n \in \mathbb{R}$ and $\gamma, \beta \in \mathbb{N}_0^d$ be multi-indices of nonnegative integers. Then the toroidal symbol class $S^n_{\mathbb{T}^d_\theta,\rho,\delta}(\mathbb{Z}^d)$ consists of those functions $\sigma : \mathbb{Z}^d \to \mathbb{T}^d_\theta$ which satisfy

$$\|D^\beta(\Delta_m^\gamma \sigma(m))\| \le C_{\beta,\gamma,m}(1+|m|)^{n-\rho|\gamma|_1+\delta|\beta|_1}, \quad \forall m \in \mathbb{Z}^d.$$

For any $x \in \mathbb{T}^d_\theta$, we define the corresponding toroidal pseudo-differential operator on $\mathbb{T}^d_\theta$ as follows:

$$T^c_\sigma x = \sum_{m \in \mathbb{Z}^d} \sigma(m)\widehat{x}(m)U^m.$$

Theorem 0.20. *Let $\sigma \in S^0_{\mathbb{T}^d_\theta,1,\delta}(\mathbb{Z}^d)$ and $\alpha \in \mathbb{R}$. Then*

- *If $0 \le \delta < 1$, then T^c_σ is a bounded operator on $F^{\alpha,c}_p(\mathbb{T}^d_\theta)$ for every $1 \le p \le \infty$.*
- *If $\delta = 1$ and $\alpha > 0$, then T^c_σ is a bounded operator on $F^{\alpha,c}_1(\mathbb{T}^d_\theta)$.*

Finally, it is worthwhile to mention the main results in [22] and compare them with the above two theorems. In [22], the authors proved that given appropriate assumptions, pseudo differential operators on quantum Euclidean spaces (denoted by $\mathbb{R}^d_\theta$) are bounded from $L_\infty(\mathbb{R}^d_\theta)$ to $\mathrm{BMO}^c(\mathbb{R}^d_\theta)$. With additional symmetric assumption on the pseudo differential operators, they can also get the boundedness from $L_\infty(\mathbb{R}^d_\theta)$ to $\mathrm{BMO}(\mathbb{R}^d_\theta)$, and then by duality and interpolation, L_p-boundedness follows. Compared to their method, our starting point is the local Hardy spaces $\mathrm{h}^c_1(\mathbb{R}^d, \mathcal{M})$ (or $F^{\alpha,c}_1(\mathbb{R}^d, \mathcal{M})$ more generally); by the smooth atomic decompositions, we prove directly the boundedness on $\mathrm{h}^c_1(\mathbb{R}^d, \mathcal{M})$, which also gives the boundedness on $\mathrm{h}^c_p(\mathbb{R}^d, \mathcal{M})$ for all $1 \le p < \infty$ by duality and interpolation. Our result states that T^c_σ is bounded on h^c_1, and also bounded on bmo^c, which is more precise than the L_∞ to BMO^c-boundedness in [22]. However, we do not have L_p-boundedness result, because no symmetric assumption is made. On the other hand, Theorems 0.18 and 0.19 do not apply to the quantum Euclidean case directly by transference, as our argument for Theorem 0.20. At this point, more work needs to be done for quantum Euclidean spaces.

Chapter 1

Preliminaries

1.1 Noncommutative L_p-spaces

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ and $S^+_{\mathcal{M}}$ be the set of all positive elements x in $\mathcal{M}$ with $\tau(s(x)) < \infty$, where $s(x)$ denotes the support of x, i.e., the smallest projection e such that $exe = x$. Let $S_{\mathcal{M}}$ be the linear span of $S^+_{\mathcal{M}}$. Then every $x \in S_{\mathcal{M}}$ has finite trace, and $S_{\mathcal{M}}$ is a w*-dense $*$-subalgebra of $\mathcal{M}$.

Let $1 \leq p < \infty$. For any $x \in S_{\mathcal{M}}$, the operator $|x|^p$ belongs to $S^+_{\mathcal{M}}$ (recalling $|x| = (x^*x)^{\frac{1}{2}}$). We define

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}.$$

One can prove that $\|\cdot\|_p$ is a norm on $S_{\mathcal{M}}$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L_p(\mathcal{M})$, which is the usual noncommutative L_p-space associated to $(\mathcal{M}, \tau)$. In this thesis, the norm of $L_p(\mathcal{M})$ will be often denoted simply by $\|\cdot\|_p$ if there is no confusion. But if different L_p-spaces appear in a same context, we will precise their norms in order to avoid possible ambiguity. We refer the reader to [73] and [54] for further information on noncommutative L_p-spaces.

Now we introduce noncommutative Hilbert space-valued L_p-spaces $L_p(\mathcal{M}; H^c)$ and $L_p(\mathcal{M}; H^r)$, which are studied at length in [30]. Let H be a Hilbert space and $v \in H$ with $\|v\| = 1$, and p_v be the orthogonal projection onto the one-dimensional subspace generated by v. Then define the following row and column noncommutative L_p-spaces:

$$L_p(\mathcal{M}; H^r) = (p_v \otimes 1_{\mathcal{M}})L_p(B(H)\overline{\otimes}\mathcal{M}) \text{ and } L_p(\mathcal{M}; H^c) = L_p(B(H)\overline{\otimes}\mathcal{M})(p_v \otimes 1_{\mathcal{M}}),$$

where the tensor product $B(H)\overline{\otimes}\mathcal{M}$ is equipped with the tensor trace while $B(H)$ is equipped with the usual trace, and where $1_{\mathcal{M}}$ denotes the unit of $\mathcal{M}$. For $f \in L_p(\mathcal{M}; H^c)$,

$$\|f\|_{L_p(\mathcal{M};H^c)} = \|(f^*f)^{\frac{1}{2}}\|_p.$$

A similar formula holds for the row space by passing to adjoint: $f \in L_p(\mathcal{M}; H^r)$ if and only if $f^* \in L_p(\mathcal{M}; H^c)$, and $\|f\|_{L_p(\mathcal{M};H^r)} = \|f^*\|_{L_p(\mathcal{M};H^c)}$. It is clear that $L_p(\mathcal{M}; H^c)$ and $L_p(\mathcal{M}; H^r)$ are 1-complemented subspaces of $L_p(B(H)\overline{\otimes}\mathcal{M})$ for any p.

1.2 Facts and notation

In this section, we collect some notation and facts which will be frequently used in this thesis. Throughout, we will use the notation $A \lesssim B$, which is an inequality up to a constant: $A \leq cB$ for some constant $c > 0$. The relevant constants in all such inequalities may depend on the dimension d, the test function Φ or p, etc, but never on the function f in consideration. The equivalence $A \approx B$ will mean $A \lesssim B$ and $B \lesssim A$ simultaneously.

Fix a Schwartz function φ on $\mathbb{R}^d$ satisfying the usual Littlewood-Paley decomposition property:

$$\begin{cases} \operatorname{supp} \varphi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}. \\ \varphi > 0 \text{ on } \{\xi : \frac{1}{2} < |\xi| < 2\}, \\ \sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1, \forall\, \xi \neq 0. \end{cases} \tag{1.1}$$

For each $k \in \mathbb{N}$ let φ_k be the function whose Fourier transform is equal to $\varphi(2^{-k}\cdot)$, and let φ_0 be the function whose Fourier transform is equal to $1 - \sum_{k>0} \varphi(2^{-k}\cdot)$. Then $\{\varphi_k\}_{k\geq 0}$ gives a Littlewood-Paley decomposition on $\mathbb{R}^d$ such that

$$\operatorname{supp} \widehat{\varphi}_k \subset \{\xi \in \mathbb{R}^d : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}, \quad \forall\, k \in \mathbb{N}, \quad \operatorname{supp} \widehat{\varphi}_0 \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2\} \tag{1.2}$$

and that

$$\sum_{k=0}^{\infty} \widehat{\varphi}_k(\xi) = 1\,, \quad \forall\, \xi \in \mathbb{R}^d. \tag{1.3}$$

The homogeneous counterpart of the above decomposition is given by $\{\dot{\varphi}_k\}_{k\in\mathbb{Z}}$. This time for every $k \in \mathbb{Z}$, these functions are given by $\widehat{\dot{\varphi}}_k(\xi) = \varphi(2^{-k}\xi)$. We have

$$\sum_{k\in\mathbb{Z}} \widehat{\dot{\varphi}}_k(\xi) = 1\,, \quad \forall\, \xi \neq 0. \tag{1.4}$$

The Bessel potential and the Riesz potential are $J^\alpha = (1 - (2\pi)^{-2}\Delta)^{\frac{\alpha}{2}}$ and $I^\alpha = (-(2\pi)^{-2}\Delta)^{\frac{\alpha}{2}}$, respectively. If $\alpha = 1$, we will abbreviate J^1 as J and I^1 as I. We denote also $J_\alpha(\xi) = (1 + |\xi|^2)^{\frac{\alpha}{2}}$ on $\mathbb{R}^d$ and $I_\alpha(\xi) = |\xi|^\alpha$ on $\mathbb{R}^d \setminus \{0\}$. Then J_α and I_α are the symbols of the Fourier multipliers J^α and I^α, respectively.

We denote by $H_2^\sigma(\mathbb{R}^d)$ the potential Sobolev space, consisting of all tempered distributions f such that $J^\sigma(f) \in L_2(\mathbb{R}^d)$.

Let $\mathcal{S}(\mathbb{R}^d; X)$ be the space of X-valued rapidly decreasing functions on $\mathbb{R}^d$ with the standard Fréchet topology, and $\mathcal{S}'(\mathbb{R}^d; X)$ be the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^d)$ to X. All operations on $\mathcal{S}(\mathbb{R}^d)$ such as derivations, convolution and Fourier transform transfer to $\mathcal{S}'(\mathbb{R}^d; X)$ in the usual way. On the other hand, $L_p(\mathbb{R}^d; X)$ naturally embeds into $\mathcal{S}'(\mathbb{R}^d; X)$ for $1 \leq p \leq \infty$, where $L_p(\mathbb{R}^d; X)$ stands for the space of strongly p-integrable functions from $\mathbb{R}^d$ to X. By this definition, Fourier multipliers on $\mathbb{R}^d$, in particular the Bessel and Riesz potentials, extend to vector valued tempered distributions in a natural way.

We will frequently use the following Cauchy-Schwarz type inequality for the operator square function,

$$\Big| \int_{\mathbb{R}^d} \phi(s) f(s) ds \Big|^2 \leq \int_{\mathbb{R}^d} |\phi(s)|^2 ds \int_{\mathbb{R}^d} |f(s)|^2 ds, \tag{1.5}$$

where $\phi : \mathbb{R}^d \to \mathbb{C}$ and $f : \mathbb{R}^d \to L_1(\mathcal{M}) + \mathcal{M}$ are functions such that all integrations of the above inequality make sense. We also require the operator-valued version of the Plancherel formula. For sufficiently nice functions $f : \mathbb{R}^d \to L_1(\mathcal{M}) + \mathcal{M}$, for example, for $f \in L_2(\mathbb{R}^d) \otimes L_2(\mathcal{M})$, we have

$$\int_{\mathbb{R}^d} |f(s)|^2 ds = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 d\xi. \tag{1.6}$$

Given two nice functions f and g, the polarized version of the above equality is

$$\int_{\mathbb{R}^d} f(s) g^*(s) ds = \int_{\mathbb{R}^d} \widehat{f}(\xi) \widehat{g}(\xi)^* d\xi. \tag{1.7}$$

1.3 Operator-valued Hardy spaces

Throughout the remainder of the thesis, unless explicitly stated otherwise, $(\mathcal{M},\tau)$ will be fixed as before and $\mathcal{N}=L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$, equipped with the tensor trace. In this section, we introduce Mei's operator-valued Hardy spaces. Contrary to the custom, we will use letters s,t to denote variables of $\mathbb{R}^d$ since letters x,y are reserved for operators in noncommutative L_p-spaces. Accordingly, a generic element of the upper half-space $\mathbb{R}^{d+1}_+$ will be denoted by (s,ε) with $\varepsilon>0$, where $\mathbb{R}^{d+1}_+=\{(s,\varepsilon): s\in\mathbb{R}^d,\ \varepsilon>0\}$.

Let P be the Poisson kernel on $\mathbb{R}^d$:

$$\mathrm{P}(s)=c_d\,\frac{1}{(|s|^2+1)^{\frac{d+1}{2}}}$$

with c_d the usual normalizing constant and $|s|$ the Euclidean norm of s. Let

$$\mathrm{P}_\varepsilon(s)=\frac{1}{\varepsilon^d}\,\mathrm{P}(\frac{s}{\varepsilon})=c_d\,\frac{\varepsilon}{(|s|^2+\varepsilon^2)^{\frac{d+1}{2}}}\,.$$

For any function f on $\mathbb{R}^d$ with values in $L_1(\mathcal{M})+\mathcal{M}$, its Poisson integral, whenever it exists, will be denoted by $\mathrm{P}_\varepsilon(f)$:

$$\mathrm{P}_\varepsilon(f)(s)=\int_{\mathbb{R}^d}\mathrm{P}_\varepsilon(s-t)f(t)dt,\quad (s,\varepsilon)\in\mathbb{R}^{d+1}_+.$$

Note that the Poisson integral of f exists if

$$f\in L_1(\mathcal{M};L_2^c(\mathbb{R}^d,\frac{dt}{1+|t|^{d+1}}))+L_\infty(\mathcal{M};L_2^c(\mathbb{R}^d,\frac{dt}{1+|t|^{d+1}})).$$

This space is the right space in which all functions considered in this thesis live as far as only column spaces are involved. As it will appear frequently later, to simplify notation, we will denote the Hilbert space $L_2(\mathbb{R}^d,\frac{dt}{1+|t|^{d+1}})$ by R_d:

$$\mathrm{R}_d=L_2(\mathbb{R}^d,\frac{dt}{1+|t|^{d+1}}).\tag{1.8}$$

The Lusin area square function of f is defined by

$$S^c(f)(s)=\Big(\int_\Gamma\big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\big|^2\,\frac{dt\,d\varepsilon}{\varepsilon^{d-1}}\Big)^{\frac12},\quad s\in\mathbb{R}^d,\tag{1.9}$$

where Γ is the cone $\{(t,\varepsilon)\in\mathbb{R}^{d+1}_+ : |t|<\varepsilon\}$. For $1\le p<\infty$ define the column Hardy space $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ to be

$$\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})=\{f:\|f\|_{\mathcal{H}_p^c}=\|S^c(f)\|_p<\infty\}.$$

Note that [42] uses the gradient of $\mathrm{P}_\varepsilon(f)$ instead of the sole radial derivative in the definition of S^c above, but this does not affect $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ (up to equivalent norms). At the same time, it is proved in [42] that $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ can be equally defined by the Littlewood-Paley g-function:

$$G^c(f)(s)=\Big(\int_0^\infty\varepsilon\,\big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\big|^2\,d\varepsilon\Big)^{\frac12},\quad s\in\mathbb{R}^d.\tag{1.10}$$

Thus

$$\|f\|_{\mathcal{H}_p^c}\approx\|G^c(f)\|_p,\quad f\in\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M}).$$

The row Hardy space $\mathcal{H}_p^r(\mathbb{R}^d,\mathcal{M})$ is the space of all f such that $f^* \in \mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$, equipped with the norm $\|f\|_{\mathcal{H}_p^r} = \|f^*\|_{\mathcal{H}_p^c}$. Finally, we define the mixture space $\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})$ as

$$\mathcal{H}_p(\mathbb{R}^d,\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M}) + \mathcal{H}_p^r(\mathbb{R}^d,\mathcal{M}) \quad \text{for} \quad 1 \leq p \leq 2$$

equipped with the sum norm

$$\|f\|_{\mathcal{H}_p} = \inf\left\{\|f_1\|_{\mathcal{H}_p^c} + \|f_2\|_{\mathcal{H}_p^r} : f = f_1 + f_2\right\},$$

and

$$\mathcal{H}_p(\mathbb{R}^d,\mathcal{M}) = \mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M}) \cap \mathcal{H}_p^r(\mathbb{R}^d,\mathcal{M}) \quad \text{for} \quad 2 < p < \infty$$

equipped with the intersection norm

$$\|f\|_{\mathcal{H}_p} = \max\left(\|f\|_{\mathcal{H}_p^c}, \|f\|_{\mathcal{H}_p^r}\right).$$

Observe that

$$\mathcal{H}_2^c(\mathbb{R}^d,\mathcal{M}) = \mathcal{H}_2^r(\mathbb{R}^d,\mathcal{M}) = L_2(\mathcal{N}) \quad \text{with equivalent norms.}$$

It is proved in [42] that for $1 < p < \infty$

$$\mathcal{H}_p(\mathbb{R}^d,\mathcal{M}) = L_p(\mathcal{N}) \quad \text{with equivalent norms.}$$

The operator-valued BMO spaces are also studied in [42]. Let Q be a cube in $\mathbb{R}^d$ (with sides parallel to the axes) and $|Q|$ its volume. For a function f with values in $\mathcal{M}$, f_Q denotes its mean over Q:

$$f_Q = \frac{1}{|Q|}\int_Q f(t)dt.$$

The column BMO norm of f is defined to be

$$\|f\|_{\mathrm{BMO}^c} = \sup_{Q\subset\mathbb{R}^d} \Big\| \frac{1}{|Q|}\int_Q |f(t) - f_Q|^2 dt \Big\|_{\mathcal{M}}^{\frac{1}{2}}. \tag{1.11}$$

Then

$$\mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M}) = \{f \in L_\infty(\mathcal{M}; \mathrm{R}_d^c) : \ \|f\|_{\mathrm{BMO}^c} < \infty\}.$$

Similarly, we define the row space $\mathrm{BMO}^r(\mathbb{R}^d,\mathcal{M})$ as the space of f such that f^* lies in $\mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M})$, and $\mathrm{BMO}(\mathbb{R}^d,\mathcal{M}) = \mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M}) \cap \mathrm{BMO}^r(\mathbb{R}^d,\mathcal{M})$ with the intersection norm.

In [42], it is showed that the dual of $\mathcal{H}_1^c(\mathbb{R}^d,\mathcal{M})$ can be naturally identified with $\mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M})$. This is the operator-valued analogue of the celebrated Fefferman H_1-BMO duality theorem.

On the other hand, one of the main results of [70] asserts that the Poisson kernel in the definition of Hardy spaces can be replaced by more general test functions.

Take any Schwartz function Φ with vanishing mean. We will assume that Φ is nondegenerate in the following sense:

$$\forall\, \xi \in \mathbb{R}^d \setminus \{0\},\ \exists\, \varepsilon > 0, \quad \text{s.t.} \ \ \widehat{\Phi}(\varepsilon\xi) \neq 0. \tag{1.12}$$

The radial and conic square functions of f associated to Φ are defined by replacing the partial derivative of the Poisson kernel P in $S^c(f)$ and $G^c(f)$ by Φ :

$$S_\Phi^c(f)(s) = \Big(\int_\Gamma |\Phi_\varepsilon * f(s+t)|^2 \frac{dt d\varepsilon}{\varepsilon^{d+1}}\Big)^{\frac{1}{2}}, \quad s \in \mathbb{R}^d \tag{1.13}$$

and

$$G^c_\Phi(f)(s) = \Big(\int_0^\infty |\Phi_\varepsilon * f(s)|^2 \frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}. \tag{1.14}$$

The following two lemmas are taken from [70]. The first one says that the two square functions above define equivalent norms in $\mathcal{H}^c_p(\mathbb{R}^d, \mathcal{M})$:

Lemma 1.1. *Let $1 \le p < \infty$ and $f \in L_1(\mathcal{M}; \mathrm{R}^c_d) + L_\infty(\mathcal{M}; \mathrm{R}^c_d)$. Then $f \in \mathcal{H}^c_p(\mathbb{R}^d, \mathcal{M})$ if and only if $G^c_\Phi(f) \in L_p(\mathcal{N})$ if and only if $S^c_\Phi(f) \in L_p(\mathcal{N})$. If this is the case, then*

$$\|G^c_\Phi(f)\|_p \approx \|S^c_\Phi(f)\|_p \approx \|f\|_{\mathcal{H}^c_p}$$

with the relevant constants depending only on p, d and Φ.

The above square functions G^c_Φ and S^c_Φ can be discretized as follows:

$$\begin{aligned} G^{c,D}_\Phi(f)(s) &= \Big(\sum_{j=-\infty}^{\infty} |\Phi_{2^{-j}} * f(s)|^2\Big)^{\frac12} \\ S^{c,D}_\Phi(f)(s) &= \Big(\sum_{j=-\infty}^{\infty} 2^{dj} \int_{B(s,2^{-j})} |\Phi_{2^{-j}} * f(t)|^2 dt\Big)^{\frac12}. \end{aligned} \tag{1.15}$$

Here $B(s,r)$ denotes the ball of $\mathbb{R}^d$ with center s and radius r. To prove that these discrete square functions also describe our Hardy spaces, we need to impose the following condition on the previous Schwartz function Φ, which is stronger than (1.12):

$$\forall\, \xi \in \mathbb{R}^d \setminus \{0\},\ \exists\, 0 < 2a \le b < \infty \ \text{ s.t. } \ \widehat{\Phi}(\varepsilon\xi) \neq 0,\ \forall\, \varepsilon \in (a,\, b]. \tag{1.16}$$

The following is the discrete version of Lemma 1.1:

Lemma 1.2. *Let $1 \le p < \infty$ and $f \in L_1(\mathcal{M}; \mathrm{R}^c_d) + L_\infty(\mathcal{M}; \mathrm{R}^c_d)$. Then $f \in \mathcal{H}^c_p(\mathbb{R}^d, \mathcal{M})$ if and only if $G^{c,D}_\Phi(f) \in L_p(\mathcal{N})$ if and only if $S^{c,D}_\Phi(f) \in L_p(\mathcal{N})$. Moreover,*

$$\|G^{c,D}_\Phi(f)\|_p \approx \|S^{c,D}_\Phi(f)\|_p \approx \|f\|_{\mathcal{H}^c_p}$$

with the relevant constants depending only on p, d and Φ.

Chapter 2

Operator-valued local Hardy spaces

2.1 Operator-valued local Hardy spaces

Let $f \in L_1(\mathcal{M}; \mathrm{R}_d^c) + L_\infty(\mathcal{M}; \mathrm{R}_d^c)$ (recalling that the Hilbert space R_d is defined by (1.8)). Then the Poisson integral of f is well-defined and takes values in $L_1(\mathcal{M}) + \mathcal{M}$. Now we define the local analogue of the Lusin area square function of f by

$$s^c(f)(s) = \Big(\int_{\widetilde{\Gamma}} \big|\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s+t)\big|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \Big)^{\frac{1}{2}},\ s \in \mathbb{R}^d,$$

where $\widetilde{\Gamma}$ is the truncated cone $\{(t,\varepsilon) \in \mathbb{R}_+^{d+1} : |t| < \varepsilon < 1\}$. It is the intersection of the cone $\{(t,\varepsilon) \in \mathbb{R}_+^{d+1} : |t| < \varepsilon\}$ and the strip $S \subset \mathbb{R}_+^{d+1}$ defined by:

$$S = \{(s,\varepsilon) : s \in \mathbb{R}^d, 0 < \varepsilon < 1\}.$$

For $1 \leq p < \infty$ define the column local Hardy space $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ to be

$$\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) = \{f \in L_1(\mathcal{M}; \mathrm{R}_d^c) + L_\infty(\mathcal{M}; \mathrm{R}_d^c) : \|f\|_{\mathrm{h}_p^c} < \infty\},$$

where the $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$-norm of f is defined by

$$\|f\|_{\mathrm{h}_p^c} = \|s^c(f)\|_{L_p(\mathcal{N})} + \|\mathrm{P} * f\|_{L_p(\mathcal{N})}.$$

The row local Hardy space $\mathrm{h}_p^r(\mathbb{R}^d, \mathcal{M})$ is the space of all f such that $f^* \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$, equipped with the norm $\|f\|_{\mathrm{h}_p^r} = \|f^*\|_{\mathrm{h}_p^c}$. Moreover, define the mixture space $\mathrm{h}_p(\mathbb{R}^d, \mathcal{M})$ as follows:

$$\mathrm{h}_p(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) + \mathrm{h}_p^r(\mathbb{R}^d, \mathcal{M}) \text{ for } 1 \leq p \leq 2$$

equipped with the sum norm

$$\|f\|_{\mathrm{h}_p} = \inf\{\|g\|_{\mathrm{h}_p^c} + \|h\|_{\mathrm{h}_p^r} :\ f = g + h, g \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}), h \in \mathrm{h}_p^r(\mathbb{R}^d, \mathcal{M})\},$$

and

$$\mathrm{h}_p(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) \cap \mathrm{h}_p^r(\mathbb{R}^d, \mathcal{M}) \text{ for } 2 < p < \infty$$

equipped with the intersection norm

$$\|f\|_{\mathrm{h}_p} = \max\{\|f\|_{\mathrm{h}_p^c}, \|f\|_{\mathrm{h}_p^r}\}.$$

The local analogue of the Littlewood-Paley g-function of f is defined by

$$g^c(f)(s) = \Big(\int_0^1 |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)|^2 \varepsilon d\varepsilon\Big)^{\frac{1}{2}},\ s \in \mathbb{R}^d.$$

We will see in chapter 6 that

$$\|s^c(f)\|_p + \|\mathrm{P} * f\|_p \approx \|g^c(f)\|_p + \|\mathrm{P} * f\|_p$$

for all $1 \le p < \infty$.

We close this section by some easy facts that will be frequently used later. Firstly, we have

$$\|s^c(f)\|_2^2 + \|\mathrm{P} * f\|_2^2 \approx \|f\|_2^2. \tag{2.1}$$

Indeed, by (1.6), we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\big|^2 ds &= \int_{\mathbb{R}^d} |\widehat{\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon}(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} 4\pi^2|\xi|^2|\widehat{f}(\xi)|^2 e^{-4\pi\varepsilon|\xi|} d\xi.\end{aligned}$$

Then

$$\int_{\mathbb{R}^d}\int_0^1 |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\big|^2 \varepsilon d\varepsilon ds = \frac{1}{4}\int_{\mathbb{R}^d}(1 - e^{-4\pi|\xi|} - 4\pi|\xi|e^{-4\pi|\xi|})|\widehat{f}(\xi)|^2 d\xi.$$

Therefore

$$\begin{aligned}\|s^c(f)\|_2^2 &= \tau\int_{\mathbb{R}^d}\int_{\widetilde{\Gamma}} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\big|^2 \frac{d\varepsilon dt}{\varepsilon^{d-1}} ds \\ &= \tau\int_{\mathbb{R}^d}\int_0^1\int_{B(s,\varepsilon)} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)\big|^2 \frac{d\varepsilon dt}{\varepsilon^{d-1}} ds \\ &= c_d\,\tau\int_{\mathbb{R}^d}\int_0^1 |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\big|^2 \varepsilon d\varepsilon ds \\ &= \frac{c_d}{4}\tau\int_{\mathbb{R}^d}(1 - e^{-4\pi|\xi|} - 4\pi|\xi|e^{-4\pi|\xi|})|\widehat{f}(\xi)|^2 d\xi,\end{aligned}$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$. Meanwhile,

$$\|\mathrm{P} * f\|_2^2 = \tau\int_{\mathbb{R}^d} e^{-4\pi|\xi|}|\widehat{f}(\xi)|^2 d\xi.$$

Then we deduce (2.1) from the equality

$$\frac{4}{c_d}\|s^c(f)\|_2^2 + \|\mathrm{P} * f\|_2^2 = \tau\int_{\mathbb{R}^d}(1 - 4\pi|\xi|e^{-4\pi|\xi|})|\widehat{f}(\xi)|^2 d\xi$$

and the fact that $0 \le 4\pi|\xi|e^{-4\pi|\xi|} \le \frac{1}{e}$ for every $\xi \in \mathbb{R}^d$. Passing to adjoint, (2.1) also tells us that $\|f\|_{\mathrm{h}_2^r(\mathbb{R}^d,\mathcal{M})} \approx \|f^*\|_2 = \|f\|_2$, whence

$$\mathrm{h}_2^c(\mathbb{R}^d,\mathcal{M}) = \mathrm{h}_2^r(\mathbb{R}^d,\mathcal{M}) = L_2(\mathcal{N}) \tag{2.2}$$

with equivalent norms.

Next, if we apply (1.7) instead of (1.6) in the above proof, we get the following polarized version of (2.1),

$$\begin{aligned}\int_{\mathbb{R}^d} f(s)g^*(s)ds &= 4\int_{\mathbb{R}^d}\int_0^1 \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s)\varepsilon\, d\varepsilon ds \\ &\quad + \int_{\mathbb{R}^d} \mathrm{P}*f(s)(\mathrm{P}*g(s))^*ds + 4\pi\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*ds. \\ &= \frac{4}{c_d}\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s+t)\frac{dtd\varepsilon}{\varepsilon^{d-1}}ds \\ &\quad + \int_{\mathbb{R}^d} \mathrm{P}*f(s)(\mathrm{P}*g(s))^*ds + 4\pi\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*ds\end{aligned} \tag{2.3}$$

for nice $f,\, g \in L_1(\mathcal{M};\mathrm{R}_d^c) + L_\infty(\mathcal{M};\mathrm{R}_d^c)$ (recalling that I is the Riesz potential).

2.2 Operator-valued bmo spaces

Now we introduce the noncommutative analogue of bmo spaces defined in [21]. For any cube $Q \subset \mathbb{R}^d$, in the whole thesis, we will denote its center by c_Q, its side length by $l(Q)$, and its volume by $|Q|$. Let $f \in L_\infty(\mathcal{M};\mathrm{R}_d^c)$. The mean value of f over Q is denoted by $f_Q := \frac{1}{|Q|}\int_Q f(s)ds$. We set

$$\|f\|_{\mathrm{bmo}^c} = \max\Big\{\sup_{|Q|<1}\|(\frac{1}{|Q|}\int_Q |f-f_Q|^2dt)^{\frac12}\|_{\mathcal{M}},\, \sup_{|Q|=1}\|(\int_Q|f|^2dt)^{\frac12}\|_{\mathcal{M}}\Big\}. \tag{2.4}$$

Then we define

$$\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}) = \{f\in L_\infty(\mathcal{M};\mathrm{R}_d^c) : \|f\|_{\mathrm{bmo}^c}<\infty\}.$$

Respectively, define $\mathrm{bmo}^r(\mathbb{R}^d,\mathcal{M})$ to be the space of all $f\in L^\infty(\mathcal{M};\mathrm{R}_d^r)$ such that

$$\|f^*\|_{\mathrm{bmo}^c}<\infty$$

with the norm $\|f\|_{\mathrm{bmo}^r} = \|f^*\|_{\mathrm{bmo}^c}$. And $\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ is defined as the intersection of these two spaces

$$\mathrm{bmo}(\mathbb{R}^d,\mathcal{M}) = \mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})\cap\mathrm{bmo}^r(\mathbb{R}^d,\mathcal{M})$$

equipped with the norm

$$\|f\|_{\mathrm{bmo}} = \max\{\|f\|_{\mathrm{bmo}^c},\|f\|_{\mathrm{bmo}^r}\}.$$

Remark 2.1. Let Q be a cube with volume $k^d \le |Q| < (k+1)^d$ for some positive integer k. Then Q can be covered by at most $(k+1)^d$ cubes with volume 1, say Q_j's. Evidently,

$$\frac{1}{|Q|}\int_Q|f|^2dt \le k^{-d}\int_Q|f|^2dt\le k^{-d}\sum_{j=1}^{(k+1)^d}\int_{Q_j}|f|^2dt.$$

Whence,

$$\sup_{|Q|\ge1}\|(\frac{1}{|Q|}\int_Q|f|^2dt)^{\frac12}\|_{\mathcal{M}}\le 2^{\frac d2}\sup_{|Q|=1}\|(\int_Q|f|^2dt)^{\frac12}\|_{\mathcal{M}}.$$

Thus, if we replace the second supremum in (2.4) over all cubes of volume one by that over all cubes of volume not less than one, we get an equivalent norm of $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$.

Proposition 2.2. *Let* $f \in \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$. *Then*

$$\|f\|_{L_\infty(\mathcal{M};\mathrm{R}_d^c)} \lesssim \|f\|_{\mathrm{bmo}^c}.$$

Moreover, $\mathrm{bmo}(\mathbb{R}^d, \mathcal{M})$, $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$ *and* $\mathrm{bmo}^r(\mathbb{R}^d, \mathcal{M})$ *are Banach spaces.*

Proof. Let Q_0 be the cube centered at the origin with side length 1 and $Q_m = Q_0 + m$ for each $m \in \mathbb{Z}^d$. For $f \in L_\infty(\mathcal{M}; \mathrm{R}_d^c)$,

$$\begin{aligned}
\|f\|^2_{L_\infty(\mathcal{M};\mathrm{R}_d^c)} = \Big\| \int_{\mathbb{R}^d} \frac{|f(t)|^2}{1+|t|^{d+1}} dt \Big\|_{\mathcal{M}} &\leq \sum_{m\in\mathbb{Z}^d} \Big\| \int_{Q_m} \frac{|f(t)|^2}{1+|t|^{d+1}} dt \Big\|_{\mathcal{M}} \\
&\lesssim \sum_{m\in\mathbb{Z}^d} \Big\| \frac{1}{1+|m|^{d+1}} \int_{Q_m} |f(t)|^2 dt \Big\|_{\mathcal{M}} \\
&\lesssim \|f\|^2_{\mathrm{bmo}^c} \sum_{m\in\mathbb{Z}^d} \frac{1}{1+|m|^{d+1}} \lesssim \|f\|^2_{\mathrm{bmo}^c}.
\end{aligned}$$

It is then easy to check that $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$ is a Banach space. □

Proposition 2.3. *We have the inclusion* $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{BMO}^c(\mathbb{R}^d, \mathcal{M})$. *More precisely, there exists a uniform constant C depending only on the dimension d, such that for any* $f \in \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$,

$$\|f\|_{\mathrm{BMO}^c} \leq C \|f\|_{\mathrm{bmo}^c}. \tag{2.5}$$

Proof. By virtue of Remark 2.1, it suffices to compare the term $\Big\|(\frac{1}{|Q|}\int_Q |f|^2 dt)^{\frac12}\Big\|_{\mathcal{M}}$ and the term $\Big\|(\frac{1}{|Q|}\int_Q |f-f_Q|^2 dt)^{\frac12}\Big\|_{\mathcal{M}}$ for $|Q| \geq 1$. By the triangle inequality and (1.5), we have

$$\begin{aligned}
\Big\|(\frac{1}{|Q|}\int_Q |f-f_Q|^2 dt)^{\frac12}\Big\|_{\mathcal{M}} &\leq \Big\|(\frac{1}{|Q|}\int_Q |f|^2 dt)^{\frac12}\Big\|_{\mathcal{M}} + \|f_Q\|_{\mathcal{M}} \\
&\leq 2\Big\|(\frac{1}{|Q|}\int_Q |f|^2 dt)^{\frac12}\Big\|_{\mathcal{M}},
\end{aligned}$$

which leads immediately to (2.5). □

Classically, BMO functions are related to Carleson measures (see [20]). A similar relation still holds in the present noncommutative local setting. We say that an $\mathcal{M}$-valued measure $d\lambda$ on the strip $S = \mathbb{R}^d \times (0,1)$ is a Carleson measure if

$$N(\lambda) = \sup_{|Q|\leq 1} \{ \frac{1}{|Q|} \Big\| \int_{T(Q)} d\lambda \Big\|_{\mathcal{M}} : Q \subset \mathbb{R}^d \text{ cube } \} < \infty,$$

where $T(Q) = Q \times (0, l(Q)]$.

Lemma 2.4. *Let* $g \in \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$. *Then* $d\lambda_g = |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)|^2 \varepsilon\, ds d\varepsilon$ *is an* $\mathcal{M}$*-valued Carleson measure on the strip S and*

$$\max\{N(\lambda_g)^{\frac12},\ \|\mathrm{P} * g\|_{L_\infty(\mathcal{N})}\} \lesssim \|g\|_{\mathrm{bmo}^c}.$$

Proof. Given a cube Q with $|Q| \leq 1$, we decompose $g = g_1 + g_2 + g_3$, where $g_1 = (g - g_{2Q})\mathbb{1}_{2Q}$ and $g_2 = (g - g_{2Q})\mathbb{1}_{\mathbb{R}^d \setminus 2Q}$. Since $\int \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s) ds = 0$ for any $\varepsilon > 0$, we have $\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g) = \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g_1) + \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g_2)$. By (1.5),

$$N(\lambda_g) \leq 2(N(\lambda_{g_1}) + N(\lambda_{g_2})).$$

We first deal with $N(\lambda_{g_1})$. By (1.6) and (2.5), we have

$$\begin{aligned}\int_{T(Q)} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g_1)(s)\Big|^2 \varepsilon ds d\varepsilon &\leq \int_{\mathbb{R}^d}\int_0^\infty \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g_1)(s)\Big|^2 \varepsilon ds d\varepsilon \\ &= \int_{\mathbb{R}^d}\int_0^\infty \Big|\widehat{\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon}(\xi)\Big|^2 |\widehat{g}_1(\xi)|^2 \varepsilon d\varepsilon ds \\ &\lesssim \int_{\mathbb{R}^d} |g_1(s)|^2 ds = \int_{2Q} |g-g_{2Q}|^2 ds \lesssim |Q|\,\|g\|^2_{\mathrm{bmo}^c}.\end{aligned}$$

Thus, $N(\lambda_{g_1}) \lesssim \|g\|^2_{\mathrm{bmo}^c}$. Since $|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s)| \lesssim \frac{1}{(\varepsilon+|s|)^{d+1}}$, applying (1.5), we obtain

$$\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g_2)(s)\Big|^2 \lesssim \frac{1}{\varepsilon}\int_{\mathbb{R}^d\setminus 2Q} \frac{|g(t)-g_{2Q}|^2}{(\varepsilon+|s-t|)^{d+1}}dt.$$

The integral on the right hand side of the above inequality can be treated by a standard argument as follows: for any $(s,\varepsilon)\in T(Q)$,

$$\begin{aligned}\int_{\mathbb{R}^d\setminus 2Q} \frac{|g(t)-g_{2Q}|^2}{(\varepsilon+|s-t|)^{d+1}}dt &\lesssim \int_{\mathbb{R}^d\setminus 2Q} \frac{|g(t)-g_{2Q}|^2}{|t-c_Q|^{d+1}}dt \\ &\lesssim \sum_{k\geq 1}\int_{2^{k+1}Q\setminus 2^kQ} \frac{|g(t)-g_{2Q}|^2}{|t-c_Q|^{d+1}}dt \\ &\lesssim \frac{1}{l(Q)}\sum_{k\geq 1} 2^{-k}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|g(t)-g_{2Q}|^2 dt \\ &\lesssim \frac{1}{l(Q)}\|f\|^2_{\mathrm{bmo}^c},\end{aligned}$$

where c_Q is the center of Q. Then, it follows that $N(\lambda_{g_2}) \lesssim \|g\|^2_{\mathrm{bmo}^c}$.

Now we deal with the term $\|\mathrm{P} * g(s)\|_{\mathcal{M}}$. Let $Q_m = Q_0 + m$ be the translate of the cube with volume one centered at the origin, so $\mathbb{R}^d = \cup_{m\in\mathbb{Z}^d} Q_m$. By (1.5), for any $s\in\mathbb{R}^d$, we have

$$\begin{aligned}\|\mathrm{P} * g(s)\|_{\mathcal{M}} &= \Big\|\sum_m \int_{Q_m} \mathrm{P}(t) g(s-t)dt\Big\|_{\mathcal{M}} \\ &\leq \sum_m \Big(\int_{Q_m}|\mathrm{P}(t)|^2 dt\Big)^{\frac12} \cdot \sup_{m\in\mathbb{Z}^d}\Big\|\Big(\int_{Q_m}|g(s-t)|^2 dt\Big)^{\frac12}\Big\|_{\mathcal{M}} \\ &\lesssim \sup_{|Q|=1}\Big\|\Big(\frac{1}{|Q|}\int_Q |g(t)|^2 dt\Big)^{\frac12}\Big\|_{\mathcal{M}} \\ &\lesssim \|g\|_{\mathrm{bmo}^c}.\end{aligned}$$

Thus, $\|\mathrm{P} * g\|_{L_\infty(\mathcal{N})} = \sup_{s\in\mathbb{R}^d}\|\mathrm{P} * g(s)\|_{\mathcal{M}} \lesssim \|g\|_{\mathrm{bmo}^c}$, which completes the proof. $\square$

Reexaming the last step of the above proof, we find that the only fact used for proving the inequality $\|\mathrm{P} * g\|_{L_\infty(\mathcal{N})} \lesssim \|g\|_{\mathrm{bmo}^c}$ is that

$$\sum_m \Big(\int_{Q_m}|\mathrm{P}(t)|^2 dt\Big)^{\frac12} < \infty.$$

Recall that $H_2^\sigma(\mathbb{R}^d)$ denotes the potential Sobolev space, consisting of distributions f such that $J^\sigma(f)\in L_2(\mathbb{R}^d)$. It is equipped with the norm $\|f\|_{H_2^\sigma(\mathbb{R}^d)} = \|J^\sigma f\|_{L_2(\mathbb{R}^d)}$. If ψ is a function on $\mathbb{R}^d$ such that $\widehat{\psi}\in H_2^\sigma(\mathbb{R}^d)$ for some $\sigma > \frac{d}{2}$, we have

$$\sum_m \Big(\int_{Q_m}|\psi(s)|^2 dt\Big)^{\frac12} \lesssim \Big(\sum_m \frac{1}{(1+|m|^2)^\sigma}\Big)^{\frac12}\Big(\int_{\mathbb{R}^d}(1+|s|^2)^\sigma|\psi(s)|^2 ds\Big)^{\frac12} \lesssim \|\widehat{\psi}\|_{H_2^\sigma}.$$

Then we have the following replacement of the above lemma:

Lemma 2.5. *Let $g \in \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$. If ψ is the (inverse) Fourier transform of a function in $H_2^\sigma(\mathbb{R}^d)$, we have*

$$\max\{N(\lambda_g)^{\frac{1}{2}},\ \|\psi * g\|_{L_\infty(\mathcal{N})}\} \lesssim \|g\|_{\mathrm{bmo}^c}. \tag{2.6}$$

In particular,

$$\max\{N(\lambda_g)^{\frac{1}{2}},\ \|J(\mathrm{P}) * g\|_{L_\infty(\mathcal{N})}\} \lesssim \|g\|_{\mathrm{bmo}^c}. \tag{2.7}$$

Proof. (2.6) follows from the above discussion; (2.7) is ensured by (2.6) and the fact that $(1+|\xi|^2)^{\frac{1}{2}} e^{-2\pi|\xi|} \in H_2^\sigma(\mathbb{R}^d)$, which can be checked by a direct computation. □

Remark 2.6. We will see in the next chapter that the converse inequality of (2.7) also holds.

Chapter 3

Dual spaces of h_p^c for $1 \le p < 2$

In this chapter, we will describe the dual of $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ for $1 \le p < 2$ as bmo type spaces. We will call these spaces $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ (with q the conjugate index of p). The argument used here is modelled on the one used in [21] when studying the duality between $\mathcal{H}_p^c(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{BMO}_q^c(\mathbb{R}^d, \mathcal{M})$.

3.1 Definition of bmo_q^c

Let $2 < q \le \infty$. We define $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ to be the space of all $f \in L_q(\mathcal{M}; \mathrm{R}_d^c)$ such that

$$\|f\|_{\mathrm{bmo}_q^c} = \Big(\Big\| \sup_{\substack{s \in Q \subset \mathbb{R}^d \\ |Q| < 1}}{}^{+} \frac{1}{|Q|} \int_Q |f(t) - f_Q|^2 dt \Big\|_{\frac{q}{2}}^{\frac{q}{2}} + \Big\| \sup_{\substack{s \in Q \subset \mathbb{R}^d \\ |Q| = 1}}{}^{+} \frac{1}{|Q|} \int_Q |f(t)|^2 dt \Big\|_{\frac{q}{2}}^{\frac{q}{2}} \Big)^{\frac{1}{q}} < \infty.$$

If $q = \infty$, $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ coincides with the space $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$ introduced in the previous chapter.

Note that the norm $\| \sup_i^+ a_i \|_{\frac{q}{2}}$ is just an intuitive notation since the pointwise supremum does not make any sense in the noncommutative setting. This is the norm of the Banach space $L_{\frac{q}{2}}(\mathcal{N}; \ell_\infty)$; we refer to [52, 28, 32] for more information.

If $1 \le p < \infty$ and $(a_i)_{i \in \mathbb{Z}}$ is a sequence of positive elements in $L_p(\mathcal{N})$, it has been proved by Junge (see [28], Remark 3.7) that

$$\| \sup_i{}^{+} a_i \|_p = \sup \{ \sum_{i \in \mathbb{Z}} \tau(a_i b_i) : \, b_i \in L_q(\mathcal{N}), \, b_i \ge 0, \, \| \sum_{i \in \mathbb{Z}} b_i \|_q \le 1 \}. \tag{3.1}$$

It is also known that a positive sequence $(x_i)_i$ belongs to $L_p(\mathcal{N}; \ell_\infty)$ if and only if there is an $a \in L_p(\mathcal{N})$ such that $x_i \le a$ for all i, and moreover,

$$\|(x_i)\|_{L_p(\mathcal{N}; \ell_\infty)} = \inf\{ \|a\|_p : \ \ a \in L_p(\mathcal{N}), x_i \le a, \forall i \}.$$

Then we get the following fact (which can be taken as an equivalent definition): $f \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ if and only if

$$\exists\, a \in L_{\frac{q}{2}}(\mathcal{N}) \text{ s.t. } \frac{1}{|Q|} \int_Q |f(t) - f_Q|^2 dt \le a(s), \ \forall\, s \in Q \text{ and } \forall\, Q \subset \mathbb{R}^d \text{ with } |Q| < 1 \tag{3.2}$$

and

$$\exists\, b \in L_{\frac{q}{2}}(\mathcal{N}) \text{ s.t. } \frac{1}{|Q|} \int_Q |f(t)|^2 dt \le b(s), \ \forall\, s \in Q \text{ and } \forall\, Q \subset \mathbb{R}^d \text{ with } |Q| = 1. \tag{3.3}$$

If this is the case, then

$$\|f\|_{\mathrm{bmo}_q^c} = \inf\{(\|a\|_{\frac{q}{2}}^{\frac{q}{2}} + \|b\|_{\frac{q}{2}}^{\frac{q}{2}})^{\frac{1}{q}} : a, b \text{ as in (3.2) and (3.3) respectively}\}.$$

Observe that the cubes considered in the definition of $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ can be reduced to cubes with dyadic lengths. Let Q_s^k denote the cube centered at s with side length 2^{-k}, $k \in \mathbb{Z}$. Set

$$f_k^{\#}(s) = \frac{1}{|Q_s^k|}\int_{Q_s^k} \big|f(t) - f_{Q_s^k}\big|^2 dt \quad \text{and} \quad f^{\#}(s) = \frac{1}{|Q_s^0|}\int_{Q_s^0} \big|f(t)\big|^2 dt.$$

Lemma 3.1. *If $q > 2$, then*

$$\Big(\|\sup_{k>0}{}^+ f_k^{\#}\|_{\frac{q}{2}}^{\frac{q}{2}} + \|f^{\#}\|_{\frac{q}{2}}^{\frac{q}{2}}\Big)^{\frac{1}{q}}$$

gives an equivalent norm in $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$.

Proof. It is obvious from the definition that

$$\|\sup_{k>0}{}^+ f_k^{\#}\|_{\frac{q}{2}}^{\frac{1}{2}} \le \|f\|_{\mathrm{bmo}_q^c} \quad \text{and} \quad \|f^{\#}\|_{\frac{q}{2}}^{\frac{1}{2}} \le \|f\|_{\mathrm{bmo}_q^c}.$$

We notice that for any cube Q with $|Q| < 1$ and $s \in Q$, there exists $k \ge -1$ such that $Q \subset Q_s^k$ and $|Q_s^k| \le 4^d|Q|$. Thus

$$\frac{1}{4^d}\Big\|\sup_{\substack{s\in Q\subset\mathbb{R}^d\\|Q|<1}}{}^+ \frac{1}{|Q|}\int_Q |f(t) - f_Q|^2 dt\Big\|_{\frac{q}{2}}^{\frac{1}{2}} \lesssim \|\sup_{k\ge -1}{}^+ f_k^{\#}\|_{\frac{q}{2}}^{\frac{1}{2}} \lesssim 2^d\|\sup_{k>0}{}^+ f_k^{\#}\|_{\frac{q}{2}}^{\frac{1}{2}}.$$

Similarly,

$$\frac{1}{4^d}\Big\|\sup_{\substack{s\in Q\subset\mathbb{R}^d\\|Q|=1}}{}^+ \frac{1}{|Q|}\int_Q |f(t)|^2 dt\Big\|_{\frac{q}{2}}^{\frac{1}{2}} \le 2^d\|f^{\#}\|_{\frac{q}{2}}^{\frac{1}{2}}.$$

Thus the lemma is proved. □

We can easily see that the analogues of Proposition 2.2 and Lemma 2.4 still hold in the present setting for the same reason. Thus we leave the proofs to the reader.

Proposition 3.2. *Let $q > 2$ and $f \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$. Then*

$$\|f\|_{L_q(\mathcal{M};\mathrm{R}_d^c)} \lesssim \|f\|_{\mathrm{bmo}_q^c}.$$

Lemma 3.3. *Let $f \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ and assume that the operators a and b satisfy* (3.2) *and* (3.3) *respectively. Then $d\lambda_g$ is a q-Carleson measure in the following sense:*

$$\frac{1}{|Q|}\int_{T(Q)} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon * f(t)|^2 \varepsilon dt d\varepsilon \lesssim a(s), \ \forall\, s \in Q \text{ and } \forall\, Q \subset \mathbb{R}^d \text{ with } |Q| < 1.$$

Moreover, $|\psi * f(s)|^2 \lesssim b(s)$ for any $s \in \mathbb{R}^d$, if ψ is the (inverse) Fourier transform of a function in $H_2^\sigma(\mathbb{R}^d)$.

3.2 A bounded map

In the sequel, we equip the truncated cone $\widetilde{\Gamma} = \{(s,\varepsilon) \in \mathbb{R}_+^{d+1} : |s| < \varepsilon < 1\}$ with the measure $\frac{dtd\varepsilon}{\varepsilon^{d+1}}$. For any $1 \leq p < \infty$, we will embed $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ into a larger space $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$. Here $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$ is the ℓ_p-direct sum of the Banach spaces $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma}))$ and $L_p(\mathcal{N})$, equipped with the norm

$$\|(f,g)\| = \Big(\|f\|^p_{L_p\left(\mathcal{N}; L_2^c(\widetilde{\Gamma})\right)} + \|g\|^p_{L_p(\mathcal{N})}\Big)^{\frac{1}{p}},$$

for $f \in L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma}))$ and $g \in L_p(\mathcal{N})$, with the usual modification for $p = \infty$.

Definition 3.4. We define a map F from $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ to $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$ by

$$F(f)(s,t,\varepsilon) = (\varepsilon \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s+t), \mathrm{P} * f(s)),$$

and a map E for sufficiently nice $h = (h', h'') \in L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$ by

$$E(h)(u) = \int_{\mathbb{R}^d} \Big[\frac{4}{c_d} \iint_{\widetilde{\Gamma}} h'(s,t,\varepsilon) \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(s+t-u) \frac{dtd\varepsilon}{\varepsilon^d} + h''(s)(\mathrm{P} + 4\pi I(\mathrm{P}))(s-u)\Big] ds\,.$$

By definition, the map F embeds $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ isometrically into $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$. The following results, Theorems 3.8 and 3.17 show that by identifying $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ as a subspace of $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$ via F, $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ is complemented in $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$ for every $1 < p < \infty$ by virtue of the map E.

Proposition 3.5. *Let $1 \leq p < \infty$. Then for any nice $f \in L_1(\mathcal{M}; \mathrm{R}_d^c) + L_\infty(\mathcal{M}; \mathrm{R}_d^c)$, we have*

$$E(F(f)) = f.$$

Proof. Applying (2.3), we get, for any nice function g,

$$\begin{aligned}
\int_{\mathbb{R}^d} E(F(f))(u)g(u)du &= \int_{\mathbb{R}^d} \Big[\frac{4}{c_d} \iint_{\widetilde{\Gamma}} \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s+t) \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(s+t-u) \frac{dtd\varepsilon}{\varepsilon^{d-1}} g(u)du \\
&\quad + \mathrm{P} * f(s) \int (\mathrm{P}(s-u) + 4\pi I(\mathrm{P})(s-u))g(u)du\Big] ds \\
&= \int_{\mathbb{R}^d} \Big[\frac{4}{c_d} \iint_{\widetilde{\Gamma}} \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s+t) \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(s+t) \frac{dtd\varepsilon}{\varepsilon^{d-1}} \\
&\quad + \mathrm{P} * f(s)(\mathrm{P} * g + 4\pi I(\mathrm{P}) * g)(s)\Big] ds \\
&= \int_{\mathbb{R}^d} f(u)g(u)du\,,
\end{aligned}$$

which completes the proof. □

The following dyadic covering lemma is known. Tao Mei ([42]) proved this lemma for the d-torus and also for the real line. For the case $\mathbb{R}^d$ with $d > 1$, we refer the interested readers to [10, 27] for more details. In the following, we will give a sketch of the way how we choose the dyadic covering.

Lemma 3.6. *There exist a constant $C > 0$, depending only on d, and $d+1$ dyadic increasing filtrations $\mathcal{D}^i = \{\mathcal{D}_j^i\}_{j \in \mathbb{Z}}$ of σ-algebras on $\mathbb{R}^d$ for $0 \leq i \leq d$, such that for any cube $Q \subset \mathbb{R}^d$, there is a cube $D_{m,j}^i$ satisfying $Q \subset D_{m,j}^i$ and $|D_{m,j}^i| \leq C|Q|$.*

Proof. Let $\{\alpha^i\}_{i=0}^d$ be a sequence in the interval $(0,1)$ such that

$$\min_{i\neq i'} |\alpha^i - \alpha^{i'}| > 0.$$

Then we define

$$\alpha_j^i = \begin{cases} \alpha^i, & j \geq 0, \\ \alpha^i + \frac{1}{3}(2^{-j} - 1), & j < 0 \text{ and } -j \text{ even}, \\ \alpha^i - \frac{1}{3}(2^{-j} + 1), & j < 0 \text{ and } -j \text{ odd}. \end{cases} \tag{3.4}$$

The σ-algebra $\mathcal{D}_j^i$ is generated by the cubes

$$D_{m,j}^i = (\alpha_j^i + m_1 2^{-j}, \alpha_j^i + (m_1+1)2^{-j}] \times \cdots \times (\alpha_j^i + m_d 2^{-j}, \alpha_j^i + (m_d+1)2^{-j}],$$

for all $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$.

For any cube $Q \subset \mathbb{R}^d$, there exist a constant C, depending only on $\{\alpha^i\}_{i=0}^d$ and d, and a dyadic cube $D_{m,j}^i$ such that $Q \subset D_{m,j}^i$ and $|D_{m,j}^i| \leq C|Q|$. □

To show the boundedness of the map E, we need the following assertion by Mei, see [42, Proposition 3.2]; we include a proof for this lemma, since the one in [42] is the one dimensional case. Let $1 \leq p < \infty$, and $f \in L_p(\mathcal{N})$ be a positive function. Let Q be a cube centered at the origin, and denote $Q_t = t + Q$. Then we define

$$f^Q(t) = \frac{1}{|Q|} \int_{Q_t} f(s) ds.$$

Lemma 3.7. *Let $1 \leq p < \infty$ and let $(f_k)_{k\in\mathbb{Z}}$ be a positive sequence in $L_p(\mathcal{N})$ and $(Q^k)_{k\in\mathbb{Z}}$ be a sequence of cubes centered at the origin. Then*

$$\| \sum_{k\in\mathbb{Z}} (f_k)^{Q^k} \|_p \lesssim \| \sum_{k\in\mathbb{Z}} f_k \|_p.$$

Proof. Similarly to the proof of [42, Proposition 3.2], we are going to apply [28, Theorem 0.1] for noncommutative martingales. By Lemma 3.6, we can cover every Q^k by some D_{m',j_k}^i, and thus by some D_{m,j_k-1}^i, which has twice the side length of D_{m',j_k}^i. Moreover, $|D_{m,j_k-1}^i| \leq C|Q^k|$. Obviously, $t + Q^k$ is still covered by $t + D_{m,j_k-1}^i$, but the later is not necessary a dyadic cube in $\mathcal{D}_{j_k-1}^i$. Let us adjust the translation vector $t = (t_1, ..., t_d)$ as follows. Write $Q^k = (-a, a] \times ... \times (-a, a]$ and $D_{m,j_k-1}^i = (b_1, b_2] \times ... \times (b_1, b_2]$, then either $b_2 - a \geq 2^{-j_k}$ or $-a - b_1 \geq 2^{-j_k}$. Without loss of generality, we can assume $b_2 - a \geq 2^{-j_k}$. Now set $\widetilde{t} = (\widetilde{t}_1, ..., \widetilde{t}_d)$ with $\widetilde{t}_j$ the largest real number in the set $2^{-j_k}\mathbb{Z}$ less than t_j. Then we can check that $t + Q^k$ is covered by $\widetilde{t} + D_{m,j_k-1}^i$ and that the later is a dyadic cube. Thus,

$$(f_k)^{Q^k} \leq C \sum_{0\leq i\leq d} E(f_k|\mathcal{D}_{j_k}^i),$$

where $E(\cdot|\mathcal{D}_j^i)$ denotes the conditional expectation with respect to $\mathcal{D}_j^i$. Then the lemma follows from [28, Theorem 0.1]. □

Theorem 3.8. *For $2 < p \leq \infty$, E extends to a bounded map from $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$ to $\mathrm{bmo}_p^c(\mathbb{R}^d, \mathcal{M})$.*

Proof. We have to show that for any $h = (h', h'') \in L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$,

$$\|E(h)\|_{\mathrm{bmo}_p^c} \lesssim \|h\|_{L_p\left(\mathcal{N}; L_2^c(\widetilde{\Gamma})\right) \oplus_p L_p(\mathcal{N})}.$$

Fix $h = (h', h'') \in L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$ and set $\varphi = E(h)$. For $v \in \mathbb{R}^d$ and $k \in \mathbb{N}$, denote by Q_v^k the cube centered at v with side length 2^{-k}, then we have $Q_v^k = v + Q_0^k$. We set

$$h_1'(s,t,\varepsilon) = h'(s,t,\varepsilon)\mathbb{1}_{Q_v^{k-1}}(s), \quad h_2'(s,t,\varepsilon) = h'(s,t,\varepsilon)\mathbb{1}_{(Q_v^{k-1})^c}(s)$$

and

$$\varphi_k^{\#}(v) = \frac{1}{|Q_v^k|}\int_{Q_v^k} \left|\varphi(u) - \varphi_{Q_v^k}\right|^2 du.$$

Let

$$B^{Q_0^k}(v) = \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} \left(\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon\right)^{Q_0^k}(s,t,v) h_2'(s,t,\varepsilon)\frac{dt d\varepsilon}{\varepsilon^d} ds$$

with $(\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon)^{Q_0^k}(s,t,v) = \frac{1}{|Q_v^k|}\int_{Q_v^k} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)du$. Then, we have

$$\begin{aligned}
\varphi_k^{\#}(v) &\lesssim \frac{1}{|Q_v^k|}\int_{Q_v^k} |\varphi(u) - B^{Q_0^k}(v)|^2 du \\
&\lesssim \frac{1}{|Q_v^k|}\int_{Q_v^k} \Big|\int_{(Q_v^{k-1})^c} \iint_{\widetilde{\Gamma}} h_2'(s,t,\varepsilon)\big[\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u) - (\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon)^{Q_0^k}(s,t,v)\big]\frac{dt d\varepsilon}{\varepsilon^d} ds\Big|^2 du \\
&\quad + \frac{1}{|Q_v^k|}\int_{Q_v^k} \Big|\int_{Q_v^{k-1}} \iint_{\widetilde{\Gamma}} h_1'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{dt d\varepsilon}{\varepsilon^d} ds\Big|^2 du \\
&\quad + \frac{1}{|Q_v^k|}\int_{Q_v^k} \Big|\int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u) + 4\pi I(\mathrm{P})(s-u)] ds\Big|^2 du.
\end{aligned}$$

When $s \in (Q_v^{k-1})^c$, $u \in Q_v^k$ and $(t,\varepsilon) \in \widetilde{\Gamma}$, we have $|s+t-u| + \varepsilon \approx |s-v| + \varepsilon$ with uniform constants. Then,

$$\begin{aligned}
&\iint_{\widetilde{\Gamma}} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u) - (\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon)^{Q_0^k}(s,t,v)\Big|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \\
&\lesssim \iint_{\widetilde{\Gamma}} \Big(\frac{2^{-k}}{(|s+t-u| + \varepsilon)^{d+2}}\Big)^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \lesssim \int_0^1 \int_{B(0,\varepsilon)} \frac{2^{-2k}}{(|s-v|^2 + \varepsilon^2)^{d+2}} dt \frac{d\varepsilon}{\varepsilon^{d-1}} \\
&= c_d \int_0^1 \frac{2^{-2k}\varepsilon}{(|s-v|^2 + \varepsilon^2)^{d+2}} d\varepsilon \lesssim \frac{2^{-2k}}{|s-v|^{2d+2}}.
\end{aligned}$$

Let $(a_k)_{k\in\mathbb{N}}$ be a positive sequence such that $\|\sum_{k\geq 1} a_k\|_{(\frac{p}{2})'} \leq 1$, where r' denotes the conjugate index of r. Let

$$\begin{aligned}
\mathrm{A} &= \sum_{k\geq 1} \tau \int_{\mathbb{R}^d} \int_{(Q_v^{k-1})^c} \frac{2^{-2k}}{|s-v|^{d+1}} ds \cdot \int_{(Q_v^{k-1})^c} \frac{1}{|s-v|^{d+1}} \iint_{\widetilde{\Gamma}} |h_2'(s,t,\varepsilon)|^2 \frac{dt d\varepsilon}{\varepsilon^{d+1}} ds \cdot a_k(v) dv \\
\mathrm{B} &= \sum_{k\geq 1} \tau \int_{\mathbb{R}^d} \frac{1}{|Q_v^k|}\int_{Q_v^k} \Big|\int_{Q_v^{k-1}} \iint_{\widetilde{\Gamma}} h_1'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{dt d\varepsilon}{\varepsilon^{d+1}} ds\Big|^2 du \cdot a_k(v) dv \\
\mathrm{C} &= \sum_{k\geq 1} \tau \int_{\mathbb{R}^d} \frac{1}{|Q_v^k|}\int_{Q_v^k} \Big|\int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u) + 4\pi I(\mathrm{P})(s-u)] ds\Big|^2 du \cdot a_k(v) dv.
\end{aligned}$$

Then,

$$\sum_{k\geq 1} \tau \int \varphi_k^{\#}(v) a_k(v) dv \lesssim \mathrm{A} + \mathrm{B} + \mathrm{C}.$$

First, we estimate the term A. Applying the Fubini theorem and the Hölder inequality, we arrive at

$$\begin{aligned}
\mathrm{A} &\lesssim \sum_{k\geq 1} \tau \int_{\mathbb{R}^d} 2^{-k} \int_{(Q_s^{k-1})^c} |v-s|^{-d-1} \iint_{\widetilde{\Gamma}} |h_2'(s,t,\varepsilon)|^2 \frac{dtd\varepsilon}{\varepsilon^{d+1}} ds\, a_k(v)dv \\
&\leq \Big\| \iint_{\widetilde{\Gamma}} |h_2'(\cdot,t,\varepsilon)|^2 \frac{dtd\varepsilon}{\varepsilon^{d+1}} \Big\|_{\frac{p}{2}} \cdot \Big\| \sum_{k\geq 1} 2^{-k} \int_{(Q_s^{k-1})^c} |v-s|^{-d-1} a_k(v)dv \Big\|_{(\frac{p}{2})'} \\
&\lesssim \|h'\|^2_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))} \cdot \Big\| \sum_{k\geq 1} 2^{-k} \sum_{j\leq k} \int_{Q_s^{j-2}\setminus Q_s^{j-1}} 2^{(j-1)(d+1)} a_k(v)dv \Big\|_{(\frac{p}{2})'}.
\end{aligned}$$

Here and in the context below, $\|\cdot\|_{(\frac{p}{2})'}$ is the norm of $L_{(\frac{p}{2})'}(\mathcal{N})$ with respect to the variable $s \in \mathbb{R}^d$. Now we apply Lemma 3.7 to estimate the second factor of the last term:

$$\begin{aligned}
&\Big\| \sum_{k\geq 1} \sum_{j\leq k} 2^{(j-1)d} \int_{Q_s^{j-2}\setminus Q_s^{j-1}} 2^{j-k-1} a_k(v)dv \Big\|_{(\frac{p}{2})'} \\
&\lesssim \Big\| \sum_{j\in\mathbb{Z}} \sum_{\substack{k\geq j\\ k\geq 1}} 2^{j-k-1} a_k \Big\|_{(\frac{p}{2})'} \lesssim \Big\| \sum_{k\geq 1} a_k \Big\|_{(\frac{p}{2})'} \leq 1.
\end{aligned}$$

Then we move to the estimate of B:

$$\begin{aligned}
\mathrm{B} &\leq \sum_{k\geq 1} \int_{\mathbb{R}^d} 2^{kd} \tau \int_{\mathbb{R}^d} \Big| \int_{Q_v^{k-1}} \iint_{\widetilde{\Gamma}} h_1'(s,t,\varepsilon) a_k^{\frac{1}{2}}(v) \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(s+t-u) \frac{dtd\varepsilon}{\varepsilon^d} ds \Big|^2 dudv \\
&\leq \sum_{k\geq 1} \int_{\mathbb{R}^d} 2^{kd} \sup_{\|f\|_2=1} \Big| \tau \int_{Q_v^{k-1}} \iint_{\widetilde{\Gamma}} h_1'(s,t,\varepsilon) a_k^{\frac{1}{2}}(v) \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(f)^*(s+t) \frac{dtd\varepsilon}{\varepsilon^d} ds \Big|^2 dv.
\end{aligned}$$

Since $\mathrm{h}_2^c(\mathbb{R}^d,\mathcal{M}) = L_2(\mathcal{N})$ with equivalent norms, by the Cauchy-Schwarz inequality and Lemma 3.7, we get

$$\begin{aligned}
\mathrm{B} &\leq \sum_{k\geq 1} \int_{\mathbb{R}^d} 2^{kd} \tau \int_{Q_v^{k-1}} \iint_{\widetilde{\Gamma}} |h_1'(s,t,\varepsilon)|^2 \frac{dtd\varepsilon}{\varepsilon^{d+1}} ds\, a_k(v)dv \cdot \|f\|_{\mathrm{h}_2^c} \\
&\lesssim \sum_{k\geq 1} \tau \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} |h_1'(s,t,\varepsilon)|^2 \frac{dtd\varepsilon}{\varepsilon^{d+1}} 2^{kd} \int_{Q_s^{k-1}} a_k(v)dvds \\
&\leq \|h'\|^2_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))} \Big\| \sum_{k\geq 1} 2^{kd} \int_{Q_s^{k-1}} a_k(v)dv \Big\|_{(\frac{p}{2})'} \\
&\leq 2^d \|h'\|^2_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))} \Big\| \sum_{k\geq 1} a_k \Big\|_{(\frac{p}{2})'} \leq 2^d \|h'\|^2_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))}.
\end{aligned}$$

The techniques used to estimate the term C are similar to that of B:

$$\begin{aligned}
\mathrm{C} &= \sum_{k\geq 1} \tau \int_{\mathbb{R}^d} 2^{kd} \int_{Q_v^{k-1}} \Big| \int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u) + 4\pi I(\mathrm{P})(s-u)]ds \Big|^2 a_k(v)dvdu \\
&\leq \Big\| \sum_{k\geq 1} 2^{kd} \int_{Q_s^{k-1}} a_k(v)dv \Big\|_{(\frac{p}{2})'} \Big\| \Big| \int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u) + 4\pi I(\mathrm{P})(s-u)]ds \Big|^2 \Big\|_{\frac{p}{2}} \\
&\lesssim \Big\| \Big| \int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u) + 4\pi I(\mathrm{P})(s-u)]ds \Big|^2 \Big\|_{\frac{p}{2}},
\end{aligned}$$

where the $\|\cdot\|_{\frac{p}{2}}$ is the norm of $L_{\frac{p}{2}}(\mathcal{N})$ with respect to the variable $u \in \mathbb{R}^d$. Take $f \in L_{p'}(\mathcal{N})$ with norm one such that

$$\Big\| \Big| \int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u) + 4\pi I(\mathrm{P})(s-u)]ds \Big|^2 \Big\|_{\frac{p}{2}} = \Big| \tau \int_{\mathbb{R}^d} h''(s)[\mathrm{P} * f(s) + 4\pi I(\mathrm{P}) * f(s)]ds \Big|^2.$$

Then

$$\Big|\tau\int_{\mathbb{R}^d} h''(s)[\mathrm{P}*f(s)+4\pi I(\mathrm{P})*f(s)]ds\Big|^2 \le \|h''\|_p^2\|\mathrm{P}*f+4\pi I(\mathrm{P})*f\|_{p'}^2$$
$$\lesssim \|h''\|_p^2\|f\|_{p'}^2 \le \|h''\|_p^2.$$

Combining the estimates of A,B and C with (3.1), we obtain

$$\|\sup_{k\ge 1}{}^{+}\varphi_k^{\#}\|_{\frac{p}{2}} \lesssim \|h\|^2_{L_p\left(\mathcal{N};L_2^c(\widetilde{\Gamma})\right)\oplus_p L_p(\mathcal{N})}.$$

It remains to establish the $L_{\frac{p}{2}}$-norm of $\varphi^{\#}(s)=\frac{1}{|Q_s^0|}\int_{Q_s^0}|\varphi(t)|^2dt$, which is relatively easy. For any positive operator a such that $\|a\|_{L_{(\frac{p}{2})'}(\mathcal{N})}\le 1$, we have

$$\begin{aligned}\tau\int\varphi^{\#}(v)a(v)dv &\lesssim \tau\int_{\mathbb{R}^d}\int_{Q_v^0}\Big|\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}} h'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{dtd\varepsilon}{\varepsilon^{d+1}}ds\Big|^2du\cdot a(v)dv\\ &\quad+\tau\int_{\mathbb{R}^d}\int_{Q_v^0}\Big|\int_{\mathbb{R}^d}h''(s)[\mathrm{P}(s-u)+4\pi I(\mathrm{P})(s-u)]ds\Big|^2du\cdot a(v)dv\\ &\overset{\mathrm{def}}{=}\mathrm{B}'+\mathrm{C}'.\end{aligned}$$

The terms B' and C' are treated in the same way as B and C respectively. The results are

$$\begin{aligned}\mathrm{B}' &\le \tau\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}|h'(s,t,\varepsilon)|^2\frac{dtd\varepsilon}{\varepsilon^{d+1}}\int_{Q_s^0}a(v)dvds \le \|h'\|^2_{L_p\left(\mathcal{N};L_2^c(\widetilde{\Gamma})\right)}\|a\|_{(\frac{p}{2})'}\\ \mathrm{C}' &\le \Big\|\int_{Q_s^0}a(v)dv\Big\|_{(\frac{p}{2})'}\Big\|\Big|\int_{\mathbb{R}^d}h''(s)[\mathrm{P}(s-\cdot)+4\pi I(\mathrm{P})(s-\cdot)]ds\Big|^2\Big\|_{\frac{p}{2}} \lesssim \|h''\|_p^2.\end{aligned}$$

So we obtain

$$\|\varphi^{\#}\|_{\frac{p}{2}} \lesssim \|h\|^2_{L_p\left(\mathcal{N};L_2^c(\widetilde{\Gamma})\right)\oplus_p L_p(\mathcal{N})}.$$

Thus, Lemma 3.1 ensures that

$$\|E(h)\|_{\mathrm{bmo}_p^c} \lesssim \|h\|_{L_p\left(\mathcal{N};L_2^c(\widetilde{\Gamma})\right)\oplus_p L_p(\mathcal{N})},$$

whence the theorem. □

Corollary 3.9. *Let $1\le p<2$. For any $f\in L_p\big(\mathcal{M};L_2^c(\mathbb{R}^d,(1+|t|^{d+1})dt)\big)$, we have*

$$\|f\|_{\mathrm{h}_p^c} \lesssim \|f\|_{L_p\left(\mathcal{M};L_2^c(\mathbb{R}^d,(1+|t|^{d+1})dt)\right)}.$$

Proof. To simplify the notation, we denote $L_2\big(\mathbb{R}^d,(1+|t|^{d+1})dt\big)$ by W_d. Let q be the conjugate index of p. By duality, we can choose $h=(h',h'')\in L_q(\mathcal{N};L_2^c)\oplus_q L_q(\mathcal{N})$ with norm one such that

$$\begin{aligned}&\|s^c(f)\|_p+\|\mathrm{P}*f\|_p\\ &=\Big|\tau\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)h'^*(s,t,\varepsilon)\frac{dtd\varepsilon}{\varepsilon^d}ds+\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)h''^*(s)ds\Big|\\ &=\Big|\tau\int f(u)\widetilde{E}(h)^*(u)du\Big|,\end{aligned}$$

where

$$\widetilde{E}(h)(u)=\int_{\mathbb{R}^d}\Big[\iint_{\widetilde{\Gamma}}h'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{dtd\varepsilon}{\varepsilon^d}+h''(s)\mathrm{P}(s-u)\Big]ds\,. \tag{3.5}$$

Following the proof of Theorem 3.8, we can easily check that $\widetilde{E}$ is also bounded from $L_q(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_q L_q(\mathcal{N})$ to $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$. They applying Proposition 3.2 and Theorem 3.8, we have

$$\begin{aligned}
&|\tau \int f(s)\widetilde{E}(h)^*(s)ds| \\
&\lesssim \sup_{\|\varphi\|_{\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})}\leq 1} |\tau \int f(s)\varphi^*(s)ds| \\
&\lesssim \sup_{\|\varphi\|_{L_q(\mathcal{M};\mathrm{R}_d^c)}\leq 1} \Big|\tau \int (1+|s|^{d+1}) f(s)\varphi^*(s)\frac{ds}{1+|s|^{d+1}}\Big| \\
&= \|(1+|s|^{d+1})f\|_{L_p(\mathcal{M};\mathrm{R}_d^c)} = \|f\|_{L_p(\mathcal{M};\mathrm{W}_d^c)}.
\end{aligned}$$

Thus we obtain the desired assertion. □

3.3 Duality

The following is the main theorem of this section.

Theorem 3.10. *Let $1 \leq p < 2$ and q be its conjugate index. We have $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})^* = \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ with equivalent norms. More precisely, every $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ defines a continuous linear functional on $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ by*

$$\ell_g(f) = \tau \int f(s)g^*(s)ds, \ \forall f \in L_p(\mathcal{M}; \mathrm{W}_d^c).$$

Conversely, every $\ell \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})^*$ can be written as above and is associated to some $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ with

$$\|\ell\|_{(\mathrm{h}_p^c)^*} \approx \|g\|_{\mathrm{bmo}_q^c}.$$

Proof. We first prove

$$|\ell_g(f)| \lesssim \|g\|_{\mathrm{bmo}_q^c}\|f\|_{\mathrm{h}_p^c} \tag{3.6}$$

for $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ compactly supported (relative to the variable of $\mathbb{R}^d$). We assume that f is sufficiently nice that all calculations below are legitimate. We need two auxiliary square functions. For $s \in \mathbb{R}^d$ and $\varepsilon \in [0, 1]$, we define

$$s^c(f)(s,\varepsilon) = \Big(\int_\varepsilon^1 \int_{B(s,r-\frac{\varepsilon}{2})} \big|\frac{\partial}{\partial r}\mathrm{P}_r(f)(t)\big|^2 \frac{dtdr}{r^{d-1}}\Big)^{\frac{1}{2}}, \tag{3.7}$$

$$\overline{s}^c(f)(s,\varepsilon) = \Big(\int_\varepsilon^1 \int_{B(s,\frac{r}{2})} \big|\frac{\partial}{\partial r}\mathrm{P}_r(f)(t)\big|^2 \frac{dtdr}{r^{d-1}}\Big)^{\frac{1}{2}}. \tag{3.8}$$

Both $\overline{s}^c(f)(s,\varepsilon)$ and $s^c(f)(s,\varepsilon)$ are decreasing in ε and $s^c(f)(s,0) = s^c(f)(s)$. In addition, it is clear that $\overline{s}^c(f)(s,\varepsilon) \leq s^c(f)(s,\varepsilon)$. Let $(e_i)_{i\in I}$ be an increasing family of τ-finite projections of $\mathcal{M}$ such that e_i converges to $1_\mathcal{M}$ in the strong operator topology. Then we can approximate $s^c(f)(s,\varepsilon)$ by $s^c(e_i f e_i)(s,\varepsilon)$. Thus we can assume that τ is finite; under this finiteness assumption, for any small $\delta > 0$ (which will tend to zero in the end of the proof), consider $s^c(f)(s,\varepsilon) + \delta 1_\mathcal{M}$ instead of $s^c(f)(s,\varepsilon)$, we can assume that $s^c(f)(s,\varepsilon)$ is invertible in $\mathcal{M}$ for every $(s,\varepsilon)\in S$. By (2.3) and the Fubini theorem, we have

$$\begin{aligned}\ell_g(f) &= \tau\int f(s)g^*(s)ds\\ &\lesssim \tau\int_{\mathbb{R}^d}\int_0^1 \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s)\varepsilon\, d\varepsilon ds\\ &\quad+\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(\mathrm{P}*g(s))^*ds+\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*ds\\ &= \frac{2^d}{c_d}\tau\int_{\mathbb{R}^d}\int_0^1\int_{B(s,\frac{\varepsilon}{2})}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(t)\frac{d\varepsilon dt}{\varepsilon^{d-1}}ds\\ &\quad+\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(\mathrm{P}*g(s))^*ds+\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*ds.\end{aligned}$$

Then,

$$\begin{aligned}|\ell_g(f)| &\lesssim \Big|\frac{2^d}{c_d}\tau\int_{\mathbb{R}^d}\int_0^1\int_{B(s,\frac{\varepsilon}{2})}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)s^c(f)(s,\varepsilon)^{\frac{p-2}{2}}s^c(f)(s,\varepsilon)^{\frac{2-p}{2}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(t)\frac{d\varepsilon dt}{\varepsilon^{d-1}}ds\Big|\\ &\quad+\Big(\Big|\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(\mathrm{P}*g(s))^*ds\Big|+\Big|\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*ds\Big|\Big)\\ &\stackrel{\mathrm{def}}{=} \mathrm{I}+\mathrm{II}.\end{aligned}$$

The term II is easy to deal with. By the Hölder inequality and (2.7), we get

$$\mathrm{II}\le \|\mathrm{P}*f\|_p\|\mathrm{P}*g\|_q+\|\mathrm{P}*f\|_p\|I(\mathrm{P})*g\|_q.$$

Then by [60, Proposition V.3 and Lemma V.3.2] we have

$$\|\mathrm{P}*g\|_q\lesssim\|J(\mathrm{P})*g\|_q,\ \text{ and }\ \|I(\mathrm{P})*g\|_q\lesssim\|J(\mathrm{P})*g\|_q.$$

Hence, by Lemma 3.3,

$$\mathrm{II}\lesssim\|g\|_{\mathrm{bmo}_q^c}\|f\|_{\mathrm{h}_p^c}.$$

Now we estimate the term I. By the Cauchy-Schwarz inequality

$$\begin{aligned}\frac{c_d^2}{4^d}\mathrm{I}^2 &\le \tau\int_{\mathbb{R}^d}\int_0^1\Big(\int_{B(s,\frac{\varepsilon}{2})}|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)|^2\frac{dt}{\varepsilon^{d-1}}\Big)s^c(f)(s,\varepsilon)^{p-2}d\varepsilon ds\\ &\quad\cdot\tau\int_{\mathbb{R}^d}\int_0^1\Big(\int_{B(s,\frac{\varepsilon}{2})}|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2\frac{dt}{\varepsilon^{d-1}}\Big)s^c(f)(s,\varepsilon)^{2-p}d\varepsilon ds\\ &\stackrel{\mathrm{def}}{=} A\cdot B.\end{aligned}$$

Note here that $s^c(f)(s,\varepsilon)$ is the function of two variables defined by (3.7), which is differentiable in the w* sense. We first deal with A. Using $\overline{s}^c(f)(s,\varepsilon)\le s^c(f)(s,\varepsilon)$, we have

$$\begin{aligned}A &\le \tau\int_{\mathbb{R}^d}\int_0^1\int_{B(s,\frac{\varepsilon}{2})}|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)|^2\overline{s}^c(f)(s,\varepsilon)^{p-2}\frac{d\varepsilon dt}{\varepsilon^{d-1}}ds\\ &= -\tau\int_{\mathbb{R}^d}\int_0^1(\frac{\partial}{\partial\varepsilon}\overline{s}^c(f)(s,\varepsilon)^2)\overline{s}^c(f)(s,\varepsilon)^{p-2}d\varepsilon ds\\ &= -2\tau\int_{\mathbb{R}^d}\int_0^1\overline{s}^c(f)(s,\varepsilon)^{p-1}\frac{\partial}{\partial\varepsilon}\overline{s}^c(f)(s,\varepsilon)d\varepsilon ds.\end{aligned}$$

Since $1 \le p < 2$ and $\overline{s}^c(f)(s,\varepsilon)$ is decreasing in ε, $\overline{s}^c(f)(s,\varepsilon)^{p-1} \le \overline{s}^c(f)(s,0)^{p-1}$. At the same time, $\frac{\partial}{\partial\varepsilon}\overline{s}^c(f)(s,\varepsilon) \le 0$. Therefore,

$$\begin{aligned} A &\lesssim -\tau \int_{\mathbb{R}^d} \overline{s}^c(f)(s,0)^{p-1} \int_0^1 \frac{\partial}{\partial\varepsilon}\overline{s}(f)^c(s,\varepsilon) d\varepsilon ds \\ &\lesssim \tau \int_{\mathbb{R}^d} s^c(f)(s,0)^p ds = \|f\|_{\mathrm{h}_p^c}^p. \end{aligned}$$

The estimate of B is harder. For any $j \in \mathbb{N}$, we need to create a square net partition in $\mathbb{R}^d$ as follows:

$$Q_{m,j} = (\frac{1}{\sqrt{d}}(m_1-1)2^{-j}, \frac{1}{\sqrt{d}}m_1 2^{-j}] \times \cdots \times (\frac{1}{\sqrt{d}}(m_d-1)2^{-j}, \frac{1}{\sqrt{d}}m_d 2^{-j}]$$

with $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$. Let $c_{m,j}$ denote the center of $Q_{m,j}$. Define

$$\mathbb{S}^c(f)(s,j) = \Big(\int_{2^{-j}}^1 \int_{B(c_{m,j},r)} |\frac{\partial}{\partial r}\mathrm{P}_r(f)(t)|^2 \frac{dtdr}{r^{d-1}}\Big)^{\frac12} \quad \text{if } s \in Q_{m,j}. \tag{3.9}$$

For any $s \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$ ($\mathbb{N}_0$ being the set of nonnegative integers), we define

$$d(s,k) = \mathbb{S}^c(f)(s,k)^{2-p} - \mathbb{S}^c(f)(s,k-1)^{2-p}.$$

Since $B(s, r-\frac{\varepsilon}{2}) \subset B(c_{m,j}, r)$ whenever $s \in Q_{m,j}$ and $\varepsilon \ge 2^{-j}$, we have

$$s^c(f)(s,\varepsilon) \le \mathbb{S}^c(f)(s,j),\ \forall s \in Q_{m,j}, \varepsilon \ge 2^{-j}.$$

It is clear that $\mathbb{S}^c(f)(s,j)$ is increasing in j, so $d(s,k) \ge 0$. At the same time, $d(s,k)$ is constant on $Q_{m,k}$ and $\sum_{k\le j} d(s,k) = \mathbb{S}^c(f)(s,j)^{2-p}$. Therefore,

$$\begin{aligned} B &\lesssim \tau \sum_{m\in\mathbb{Z}^d}\sum_{j\ge1}\int_{Q_{m,j}}\int_{2^{-j}}^{2^{-j+1}} \Big(\int_{B(s,\frac{\varepsilon}{2})} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \frac{dt}{\varepsilon^{d-1}}\Big)\mathbb{S}^c(f)(s,j)^{2-p} d\varepsilon ds \\ &= \tau \int_{\mathbb{R}^d}\sum_{j\ge1}\mathbb{S}^c(f)(s,j)^{2-p}\int_{2^{-j}}^{2^{-j+1}} \Big(\int_{B(s,\frac{\varepsilon}{2})} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \frac{dt}{\varepsilon^{d-1}}\Big) d\varepsilon ds \\ &= \tau \int_{\mathbb{R}^d}\sum_{j\ge1}\sum_{1\le k\le j} d(s,k)\int_{2^{-j}}^{2^{-j+1}} \Big(\int_{B(s,\frac{\varepsilon}{2})} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \frac{dt}{\varepsilon^{d-1}}\Big) d\varepsilon ds \\ &= \tau \int_{\mathbb{R}^d}\sum_{k\ge1} d(s,k)\sum_{j\ge k}\int_{2^{-j}}^{2^{-j+1}} \Big(\int_{B(s,\frac{\varepsilon}{2})} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \frac{dt}{\varepsilon^{d-1}}\Big) d\varepsilon ds \\ &= \tau \sum_{m}\sum_{k\ge1} d(s,k)\int_{Q_{m,k}}\int_0^{2^{-k+1}} \Big(\int_{B(s,\frac{\varepsilon}{2})} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \frac{dt}{\varepsilon^{d-1}}\Big) d\varepsilon ds\,. \end{aligned}$$

Since $g \in \mathrm{bmo}_q^c$, Lemma 3.3 ensures the existence of a positive operator $a \in L_{\frac{q}{2}}(\mathcal{N})$ such that $\|a\|_{\frac{q}{2}} \lesssim \|g\|_{\mathrm{bmo}_q^c}^2$ and

$$\frac{1}{|Q|}\int_{T(Q)} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \varepsilon dt d\varepsilon \le a(s) \ \text{ and for } s \in Q \text{ and for all cubes } Q \text{ with } |Q| < 1.$$

Let $\widetilde{Q}_{m,k}$ be the cube concentric with $Q_{m,k}$ and having side length 2^{-k+1}. By the Fubini theorem and Lemma 2.4, we have

$$\begin{aligned}\int_{Q_{m,k}}\int_0^{2^{-k+1}}\Big(\int_{B(s,\frac{\varepsilon}{2})}|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2\frac{dt}{\varepsilon^{d-1}}\Big)d\varepsilon ds &\leq 2^d\int_{\widetilde{Q}_{m,k}}\int_0^{2^{-k+1}}|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)|^2\varepsilon d\varepsilon ds\\ &= 2^d\int_{T(\widetilde{Q}_{m,k})}|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)|^2\varepsilon d\varepsilon ds\\ &\lesssim \int_{Q_{m,k}}a(s)ds.\end{aligned}$$

Then we deduce

$$\begin{aligned}B &\lesssim \tau\sum_m\sum_{k\geq 1}\int_{Q_{m,k}}d(s,k)a(s)ds\\ &= \tau\int_{\mathbb{R}^d}\sum_{k\geq 1}d(s,k)a(s)ds\\ &= \tau\int_{\mathbb{R}^d}\mathbb{S}^c(f)(s,+\infty)^{2-p}a(s)ds\\ &= \tau\int_{\mathbb{R}^d}S^c(f)(s)^{2-p}a(s)ds \leq \|S^c(f)\|_p^{2-p}\|a\|_{\frac{q}{2}}\\ &\leq \|f\|_{\mathrm{h}_p^c}^{2-p}\|a\|_{\frac{q}{2}} \lesssim \|f\|_{\mathrm{h}_p^c}^{2-p}\|g\|^2_{\mathrm{bmo}_q^c}.\end{aligned}$$

Combining the estimates of A and B, we complete the proof of inequality (3.6). With this in mind, a density argument then yields that ℓ_g extends to a continuous functional on $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ with norm

$$\|\ell_g\|_{(\mathrm{h}_p^c)^*}\lesssim\|g\|_{\mathrm{bmo}_q^c}.$$

Now we prove the converse. Suppose that $\ell\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})^*$. By the Hahn-Banach theorem, ℓ extends to a continuous functional on $L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})$ with the same norm. Thus, there exists $h=(h',h'')\in L_q(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_q L_q(\mathcal{N})$ such that

$$\begin{aligned}\ell(f) &= \tau\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)h'^*(s,t,\varepsilon)\frac{dtd\varepsilon}{\varepsilon^d}ds+\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)h''^*(s)ds,\\ &= \tau\int_{\mathbb{R}^d}f(u)\widetilde{E}(h)^*(u)du,\end{aligned}$$

where $\widetilde{E}$ is the map defined in (3.5), and that

$$\|h\|_{L_q\big(\mathcal{N};L_2^c(\widetilde{\Gamma})\big)\oplus_q L_q(\mathcal{N})}=\|\ell\|_{(\mathrm{h}_p^c)^*}\,.$$

Let $g=\widetilde{E}(h)$. Following the proof of Theorem 3.8, we have

$$\|g\|_{\mathrm{bmo}_q^c}\lesssim\|\ell\|_{(\mathrm{h}_p^c)^*}$$

and

$$\ell(f)=\tau\int_{\mathbb{R}^d}f(s)g^*(s)ds,\ \forall f\in L_p(\mathcal{M};\mathrm{W}_d^c).$$

Thus, we have accomplished the proof of the theorem. □

Corollary 3.11. *Let $2 < q \leq \infty$. Then $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ if and only if $d\lambda_g = |\frac{\partial}{\partial \varepsilon}\mathrm{P}_\varepsilon(g)(s)|^2 \varepsilon ds d\varepsilon$ is an $\mathcal{M}$-valued Carleson q-measure on S and $\|J(\mathrm{P}) * g\|_q < \infty$. Furthermore,*

$$\|g\|_{\mathrm{bmo}_q^c} \approx \Big\| \sup_{\substack{s \in Q \subset \mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_{T(Q)} |\frac{\partial}{\partial \varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \varepsilon dt d\varepsilon \Big\|_{\frac{q}{2}}^{\frac{1}{2}} + \|J(\mathrm{P}) * g\|_q.$$

Proof. From the proof of Theorem 3.10, we can see that if $d\lambda_g = |\frac{\partial}{\partial \varepsilon}\mathrm{P}_\varepsilon(g)(s)|^2 \varepsilon ds d\varepsilon$ is an $\mathcal{M}$-valued Carleson q-measure on S and $J(\mathrm{P}) * g \in L_q(\mathcal{N})$, then g defines a continuous functional on $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$:

$$\ell(f) = \tau \int_{\mathbb{R}^d} f(s) g^*(s) ds,$$

and

$$\|\ell\|_{(\mathrm{h}_p^c)^*} \lesssim \Big\| \sup_{\substack{s \in Q \subset \mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_{T(Q)} |\frac{\partial}{\partial \varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \varepsilon dt d\varepsilon \Big\|_{\frac{q}{2}}^{\frac{1}{2}} + \|J(\mathrm{P}) * g\|_q.$$

By Theorem 3.10 again, there exists a function $g' \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ such that

$$\|g'\|_{\mathrm{bmo}_q^c} \lesssim \Big\| \sup_{\substack{s \in Q \subset \mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_{T(Q)} |\frac{\partial}{\partial \varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \varepsilon dt d\varepsilon \Big\|_{\frac{q}{2}}^{\frac{1}{2}} + \|J(\mathrm{P}) * g\|_q$$

and that

$$\tau \int_{\mathbb{R}^d} f(s) g^*(s) ds = \tau \int_{\mathbb{R}^d} f(s) g'^*(s) ds,$$

for any $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$. Thus, $g = g'$ with

$$\|g\|_{\mathrm{bmo}_q^c} \lesssim \Big\| \sup_{\substack{s \in Q \subset \mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_{T(Q)} |\frac{\partial}{\partial \varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \varepsilon dt d\varepsilon \Big\|_{\frac{q}{2}}^{\frac{1}{2}} + \|J(\mathrm{P}) * g\|_q.$$

The inverse inequality is already contained in Lemmas 2.4 and Lemma 2.7. We obtain the desired assertion. □

3.4 The equivalence $\mathrm{h}_q = \mathrm{bmo}_q$

We begin with two lemmas concerning the comparison of $s^c(f)$ and $g^c(f)$. We require an auxiliary truncated square function. For $s \in \mathbb{R}^d$ and $\varepsilon \in [0, \frac{2}{3}]$, we define:

$$\widetilde{g}^c(f)(s, \varepsilon) = \Big(\int_\varepsilon^{\frac{2}{3}} |\frac{\partial}{\partial r}\mathrm{P}_r(f)(s)|^2 r dr \Big)^{\frac{1}{2}}. \tag{3.10}$$

Lemma 3.12. *We have*

$$\widetilde{g}^c(f)(s, \varepsilon) \lesssim s^c(f)(s, \frac{\varepsilon}{2}),$$

where the relevant constant depends only on the dimension d.

Proof. By translation, it suffices to prove this inequality for $s = 0$. Given $\varepsilon \in [0, \frac{2}{3}]$, for any r such that $\varepsilon \leq r \leq \frac{2}{3}$, let us denote the ball centered at $(0, r)$ and tangent to the boundary of the cone $\{(t,u) \in \mathbb{R}^{d+1}_+ : |t| < \frac{r-\frac{\varepsilon}{2}}{r} u\}$ by $\widetilde{B}_r$. We notice that the radius of $\widetilde{B}_r$ is greater than or equal to $\frac{r}{\sqrt{5}}$. By the harmonicity of $\frac{\partial}{\partial r}\mathrm{P}_r(f)$, we have

$$\frac{\partial}{\partial r}\mathrm{P}_r(f)(0) = \frac{1}{|\widetilde{B}_r|}\int_{\widetilde{B}_r} \frac{\partial}{\partial u}\mathrm{P}_u(f)(t)dt.$$

Then by (1.5), we arrive at

$$|\frac{\partial}{\partial r}\mathrm{P}_r(f)(0)|^2 \leq \frac{\sqrt{5}^{d+1}}{c_{d+1}r^{d+1}}\int_{\widetilde{B}_r} |\frac{\partial}{\partial u}\mathrm{P}_u(f)(t)|^2 dt,$$

where c_{d+1} is the volume of the unit ball of $\mathbb{R}^{d+1}$. Integrating the above inequality, we get

$$\int_\varepsilon^{\frac{2}{3}} |\frac{\partial}{\partial r}\mathrm{P}_r(f)(0)|^2 r dr \leq \int_\varepsilon^{\frac{2}{3}} \frac{\sqrt{5}^{d+1}}{c_{d+1}r^{d}}\int_{\widetilde{B}_r} |\frac{\partial}{\partial u}\mathrm{P}_u(f)(t)|^2 dtdudr. \tag{3.11}$$

Since $(t,u) \in \widetilde{B}_r$ implies $\frac{\sqrt{5}}{\sqrt{5}+1}u \leq r \leq \frac{\sqrt{5}}{\sqrt{5}-1}u$ and $\frac{\varepsilon}{2} \leq u \leq 1$, the right hand side of (3.11) can be majorized by

$$\frac{\sqrt{5}^{d+1}}{c_{d+1}}\int_{\frac{\varepsilon}{2}}^1 \int_{\widetilde{B}_r} |\frac{\partial}{\partial u}\mathrm{P}_u(f)(t)|^2 \int_{\frac{u}{2}}^{2u} \frac{1}{r^d} drdtdu \leq C|s^c(f)(0,\frac{\varepsilon}{2})|^2,$$

where C is a constant depending only on d. Therefore, $\widetilde{g}^c(f)(0,\varepsilon) \lesssim s^c(f)(0,\frac{\varepsilon}{2})$. □

Lemma 3.13. *Let $1 \leq p < \infty$. Then for any $f \in \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, we have*

$$\|s^c(f)\|_p + \|\mathrm{P} * f\|_p \lesssim \|g^c(f)\|_p + \|\mathrm{P} * f\|_p.$$

Proof. When $1 \leq p < 2$, let g be a function in $\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ (q being the conjugate index of p). Following a similar calculation as (2.3), we can easily check that

$$\begin{aligned}
&\tau\int_{\mathbb{R}^d} f(s)g^*(s)ds \\
&= 4\tau\int_{\mathbb{R}^d}\int_0^{\frac{2}{3}} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s)\varepsilon d\varepsilon ds \\
&\quad + \Big(\tau\int_{\mathbb{R}^d} \mathrm{P} * f(s)(\mathrm{P}_{\frac{1}{3}} * g(s))^* ds + \frac{8\pi}{3}\tau\int_{\mathbb{R}^d} \mathrm{P} * f(s)(I(\mathrm{P}_{\frac{1}{3}}) * g(s))^* ds\Big) \\
&\overset{\text{def}}{=} \mathrm{I} + \mathrm{II}.
\end{aligned}$$

The term II can be treated in the same way as in the proof of Theorem 3.10:

$$\mathrm{II} \lesssim \|\mathrm{P} * f\|_p \cdot \|J(\mathrm{P}_{\frac{1}{3}}) * f\|_p.$$

Applying Lemma 3.3, we have

$$\mathrm{II} \lesssim \|\mathrm{P} * f\|_p \cdot \|g\|_{\mathrm{bmo}_q^c}.$$

Concerning the term I, we have

$$\begin{aligned}
|\mathrm{I}|^2 &\lesssim \tau\int_{\mathbb{R}^d}\int_0^{\frac{2}{3}} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)|^2 \widetilde{g}^c(f)(s,\varepsilon)^{p-2}\varepsilon d\varepsilon ds \\
&\quad\cdot \tau\int_{\mathbb{R}^d}\int_0^{\frac{2}{3}} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)|^2 \widetilde{g}^c(f)(s,\varepsilon)^{2-p}\varepsilon d\varepsilon ds \\
&\overset{\text{def}}{=} A' \cdot B'.
\end{aligned}$$

The term A' is estimated exactly as A in the proof of Theorem 3.10, thus $A' \lesssim \|\widetilde{g}^c(f)\|_p^p$. To estimate B', by Lemma 3.12, we have

$$B' \leq \tau \int_{\mathbb{R}^d} \int_0^{\frac{2}{3}} |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(s)|^2 s^c(f)(s, \frac{\varepsilon}{2}) \varepsilon d\varepsilon ds.$$

Then we can apply almost the same argument as in the estimate of term B. There is only one minor difference: when $\varepsilon \geq 2^{-j}$ and $s \in Q_{m,j}$, we have $s^c(f)(s, \frac{\varepsilon}{2}) \leq \mathbb{S}^c(f)(s, j+1)$. Thus we conclude that

$$B' \lesssim \|g\|^2_{\mathrm{bmo}_q^c} \|s^c(f)\|_p^{2-p}.$$

Combining the estimates above, we get

$$\|s^c(f)\|_p + \|\mathrm{P} * f\|_p \lesssim \|\widetilde{g}^c(f)\|_p + \|\mathrm{P} * f\|_p \lesssim \|g^c(f)\|_p + \|\mathrm{P} * f\|_p.$$

The case $p = 2$ is obvious. For $p > 2$, choose a positive $g \in L_{(\frac{p}{2})'}(\mathcal{N})$ with norm one such that,

$$\begin{aligned} \|s^c(f)\|_p^2 &= \Big\| \iint_{\widetilde{\Gamma}} |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(\cdot + t)|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \Big\|_{\frac{p}{2}} \\ &= \tau \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s+t)|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} g(s) ds. \end{aligned}$$

Then by (3.1) and Lemma 3.7, we have

$$\tau \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s+t)|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} g(s) ds = \tau \int_{\mathbb{R}^d} \int_0^1 |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t)|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \int_{B(t,\varepsilon)} g(s) ds.$$

By the noncommutative Hardy-Littlewood maximal inequality (the one dimension $\mathbb{R}$ case is given by [42, Theorem 3.3], the case $\mathbb{R}^d$ is a simple corollary of (3.1) and Lemma 3.7), there exists a positive $a \in L_{(\frac{p}{2})'}(\mathcal{N})$ such that $\|a\|_{(\frac{p}{2})'} \leq 1$ and

$$\frac{1}{|B(t, 2^{-k})|} \int_{B(t,2^{-k})} g(s) ds \leq a(t), \quad \forall t \in \mathbb{R}^d, \, \forall \varepsilon > 0.$$

Thus,

$$\begin{aligned} \tau \int_{\mathbb{R}^d} \int_0^1 |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t)|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \int_{B(t,\varepsilon)} g(s) ds &\leq c_d \tau \int_{\mathbb{R}^d} \int_0^1 |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t)|^2 \varepsilon a(t) dt d\varepsilon \\ &\leq c_d \Big\| \int_0^1 |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t)|^2 \varepsilon dt d\varepsilon \Big\|_{\frac{p}{2}} \|a\|_{(\frac{p}{2})'} \\ &\leq c_d \|g^c(f)\|_p. \end{aligned}$$

Therefore, we obtain

$$\|s^c(f)\|_p \lesssim \|g^c(f)\|_p.$$

Thus, the assertion for the case $p > 2$ is also proved. □

To proceed further, we introduce the definition of tent spaces. In the noncommutative setting, these spaces were first defined and studied by Mei [43].

Definition 3.14. For any function defined on $\mathbb{R}^d \times (0, 1) = S$ with values in $L_1(\mathcal{M}) + \mathcal{M}$, whenever it exists, we define

$$A^c(f)(s) = \Big(\int_{\widetilde{\Gamma}} |f(t+s, \varepsilon)|^2 \frac{dt d\varepsilon}{\varepsilon^{d+1}} \Big)^{\frac{1}{2}}, s \in \mathbb{R}^d.$$

For $1 \leq p < \infty$, we define

$$T_p^c(\mathbb{R}^d, \mathcal{M}) = \{f : A^c(f) \in L_p(\mathcal{N})\}$$

equipped with the norm $\|f\|_{T_p^c(\mathbb{R}^d,\mathcal{M})} = \|A^c(f)\|_p$. For $p = \infty$, define the operator-valued column T_∞^c norm of f as

$$\|f\|_{T_\infty^c} = \sup_{|Q|\leq 1} \Big\| \Big(\frac{1}{|Q|} \int_{T(Q)} |f(s,\varepsilon)|^2 \frac{dsd\varepsilon}{\varepsilon} \Big)^{\frac{1}{2}} \Big\|_{\mathcal{M}},$$

and the corresponding space is

$$T_\infty^c(\mathbb{R}^d, \mathcal{M}) = \{f : \|f\|_{T_\infty^c} < \infty\}.$$

Remark 3.15. By the same idea used in the proof of Theorem 3.10, we can prove the duality that $T_p^c(\mathbb{R}^d,\mathcal{M})^* = T_q^c(\mathbb{R}^d,\mathcal{M})$ for $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For the case $p = 1$, it suffices to replace $\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)$ and $\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)$ in the proof of Theorem 3.10 by $f(s,\varepsilon)$ and $g(s,\varepsilon)$ respectively. A similar argument will give us the inclusion that $T_\infty^c(\mathbb{R}^d,\mathcal{M}) \subset T_1^c(\mathbb{R}^d,\mathcal{M})^*$. On the other hand, since $L_\infty(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \subset T_\infty^c(\mathbb{R}^d,\mathcal{M})$, we get the reverse inclusion. For $1 < p < \infty$, the tent space $T_p^c(\mathbb{R}^d,\mathcal{M})$ we define above is a complemented subspace of the column tent space defined in [42]. So by Remark 4.6 in [70], we obtain the duality that $T_p^c(\mathbb{R}^d,\mathcal{M})^* = T_q^c(\mathbb{R}^d,\mathcal{M})$.

Theorem 3.16. *For $2 < q < \infty$, $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M}) = \mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ with equivalent norms.*

Proof. First, we will show the inclusion $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$. By Theorem 3.10, it suffices to show that $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})^*$. By (2.3), for any $f \in \mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})$ and $g \in \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, we have

$$\begin{aligned}
\tau \int_{\mathbb{R}^d} g(s)f^*(s)ds &= \frac{4}{c_d} \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s+t) \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)^*(s+t) \frac{dtd\varepsilon}{\varepsilon^{d-1}} ds \\
&\quad + \int_{\mathbb{R}^d} \mathrm{P} * g(s)(\mathrm{P} * f(s))^* ds + 4\pi \int_{\mathbb{R}^d} \mathrm{P} * g(s)(I(\mathrm{P}) * f(s))^* ds \\
&= \frac{4}{c_d} \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s+t) \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)^*(s+t) \frac{dtd\varepsilon}{\varepsilon^{d-1}} ds \\
&\quad + \int_{\mathbb{R}^d} \mathrm{P} * g(s)(\mathrm{P} * f(s))^* ds + 4\pi \int_{\mathbb{R}^d} I(\mathrm{P}) * g(s)(\mathrm{P} * f(s))^* ds.
\end{aligned}$$

Then, by the Hölder inequality,

$$\begin{aligned}
|\tau \int_{\mathbb{R}^d} g(s)f^*(s)ds| &\leq \|\varepsilon \cdot \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)\|_{L_p(\mathcal{N};L_2(\widetilde{\Gamma}))} \|\varepsilon \cdot \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)\|_{L_q(\mathcal{N};L_2(\widetilde{\Gamma}))} \\
&\quad + \|(\mathrm{P} + I(\mathrm{P})) * g\|_p \cdot \|\mathrm{P} * f\|_q \\
&\lesssim \Big(\|s^c(g)\|_p + \|(\mathrm{P} + I(\mathrm{P})) * g\|_p \Big) \|f\|_{\mathrm{h}_q^c}.
\end{aligned}$$

Now, we will show that for any $1 \leq p < 2$ and $g \in \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, we have $\|(\mathrm{P}+I(\mathrm{P})) * g\|_p \lesssim \|g\|_{\mathrm{h}_p^c}$. Since $2 < q < \infty$, we have $1 < \frac{q}{2} < \infty$. Applying the noncommutative Hardy-Littlewood maximal inequality, we get

$$\|f\|_{\mathrm{bmo}_q^c} \lesssim \Big\| \sup_{s\in Q\subset\mathbb{R}^d}{}^+ \frac{1}{|Q|} \int_Q |f(t)|^2 dt \Big\|_{\frac{q}{2}}^{\frac{1}{2}} \lesssim \||f|^2\|_{\frac{q}{2}}^{\frac{1}{2}} = \|f\|_{\frac{q}{2}}.$$

This implies that $L_q(\mathcal{N}) \subset \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ for any $2 < q \le \infty$. Then by Theorem 3.10, we get $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) \subset L_p(\mathcal{N})$. Therefore we deduce that

$$\|(\mathrm{P} + I(\mathrm{P})) * g\|_p \lesssim \|g\|_p \lesssim \|g\|_{\mathrm{h}_p^c}. \tag{3.12}$$

Thus,

$$|\tau \int_{\mathbb{R}^d} g(s) f^*(s) ds| \lesssim \|f\|_{\mathrm{h}_q^c} \|g\|_{\mathrm{h}_p^c}.$$

We have proved $\mathrm{h}_q^c(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$.

Let us turn to the reverse inclusion $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{h}_q^c(\mathbb{R}^d, \mathcal{M})$. We need to make use of the tent spaces in Definition 3.14. We claim that for $q > 2$, any $f \in \mathrm{bmo}_q^c$ induces a linear functional on $T_p^c \oplus_p L_p(\mathcal{N})$. Indeed, for any $h = (h', h'') \in T_p^c \oplus_p L_p(\mathcal{N})$, we define

$$\begin{aligned} \ell_f(h) &= \tau \int_{\mathbb{R}^d} \int_0^1 h'(s, \varepsilon) \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)^*(s) d\varepsilon ds \\ &\quad + \tau \int_{\mathbb{R}^d} h''(s) [(\mathrm{P} * f)^*(s) + 4\pi (I(\mathrm{P}) * f)^*(s)] ds. \end{aligned} \tag{3.13}$$

Set

$$\begin{aligned} A^c(h')(s, \varepsilon) &= \int_\varepsilon^1 \int_{B(s, r - \frac{\varepsilon}{2})} |h'(s, \varepsilon)|^2 \frac{dt dr}{r^{d+1}}, \\ \overline{A}^c(h')(s, \varepsilon) &= \int_\varepsilon^1 \int_{B(s, \frac{r}{2})} |h'(s, \varepsilon)|^2 \frac{dt dr}{r^{d+1}}. \end{aligned}$$

Then by the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned} |\ell_f(h)| &\lesssim \Big(\tau \int_{\mathbb{R}^d} \int_0^1 \Big(\int_{B(s, \frac{\varepsilon}{2})} |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t)|^2 \frac{dt}{\varepsilon^{d-1}} \Big) \overline{A}^c(h')(s, \varepsilon)^{p-2} d\varepsilon ds \Big)^{\frac{1}{2}} \\ &\quad \cdot \Big(\tau \int_{\mathbb{R}^d} \int_0^1 \Big(\int_{B(s, \frac{\varepsilon}{2})} |\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t)|^2 \frac{dt}{\varepsilon^{d-1}} \Big) \overline{A}^c(h')(s, \varepsilon)^{2-p} d\varepsilon ds \Big)^{\frac{1}{2}} \\ &\quad + |\tau \int_{\mathbb{R}^d} h''(s) (\mathrm{P} * f(s))^* ds| + |\tau \int_{\mathbb{R}^d} h''(s) (I(\mathrm{P}) * f(s))^* ds|. \end{aligned}$$

Following a similar argument as in the proof of Theorem 3.10, we obtain that

$$|\ell_f(h)| \lesssim (\|h'\|_{T_p^c} + \|h''\|_{L_p}) \|f\|_{\mathrm{bmo}_q^c} \lesssim \|h\|_{T_p^c \oplus_p L_p} \cdot \|f\|_{\mathrm{bmo}_q^c},$$

which implies that $\|\ell_f\| \le c_q \|f\|_{\mathrm{bmo}_q^c}$. So the claim is proved.

Next we show that $\|f\|_{\mathrm{h}_q^c} \le C_q \|\ell_f\|$. By definition, we can regard T_p^c as a closed subspace of $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma}))$ in the natural way. Then, ℓ_f extends to a linear functional on $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$. Thus, there exists $g = (g', g'') \in L_q(\mathcal{N}; L_2^c(\widetilde{\Gamma}, \frac{dt d\varepsilon}{\varepsilon^{d+1}})) \oplus_q L_q(\mathcal{N})$ such that

$$\|g\|_{L_q(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_q L_q(\mathcal{N})} \le \|\ell_f\|$$

and for any $h = (h', h'') \in L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$,

$$\begin{aligned} \ell_f(h) &= \tau \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} h'(t, \varepsilon) g'^*(s, t, \varepsilon) \frac{dt d\varepsilon}{\varepsilon^{d+1}} ds + \tau \int_{\mathbb{R}^d} h''(s) g''^*(s) ds \\ &= \tau \int_{\mathbb{R}^d} \int_0^1 h'(s, \varepsilon) \int_{B(s, \varepsilon)} g'^*(s, t, \varepsilon) dt \frac{ds d\varepsilon}{\varepsilon^{d+1}} + \tau \int_{\mathbb{R}^d} h''(s) g''^*(s) ds. \end{aligned}$$

Comparing the equalities above with (3.13), we get

$$\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s) = \frac{1}{\varepsilon^{d+1}}\int_{B(s,\varepsilon)} g'(s,t,\varepsilon)dt$$

and

$$\mathrm{P} * f + 4\pi I(\mathrm{P}) * f = g''.$$

By Lemma 3.13, we have

$$\begin{aligned}\|f\|_{\mathrm{h}_q^c} &\lesssim \Big\|\Big(\int_0^1 |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)|^2\varepsilon d\varepsilon\Big)^{\frac12}\Big\|_q + \|\mathrm{P} * f\|_q\\ &\le c_d\Big\|\Big(\int_0^1 \frac{1}{\varepsilon^{d+1}}\int_{B(s,\varepsilon)}|g'(s,t,\varepsilon)|^2dtd\varepsilon\Big)^{\frac12}\Big\|_q + \|\mathrm{P} * f\|_q\\ &\lesssim \|g'\|_{L_q(\mathcal{N};L_2^c(\widetilde{\Gamma}))} + \|\mathrm{P} * f\|_q.\end{aligned}$$

Now let us majorize the second term $\|\mathrm{P} * f\|_q$ by $\|g''\|_q$. Indeed, consider the function

$$G(s) = 2\pi\int_0^\infty e^{-2\pi\varepsilon}\mathrm{P}_\varepsilon(s)d\varepsilon.$$

We can easily check that $G \in L_1(\mathbb{R}^d)$, $\|G\|_1 \le 1$ and $\widehat{G}(\xi) = (1+|\xi|)^{-1}$. This means that the operator $(1+I)^{-1}$ is a contractive Fourier multiplier on $L_q(\mathcal{N})$. Therefore,

$$\|\mathrm{P} * f\|_q \le \|(\mathrm{P} + I(\mathrm{P})) * f\|_q \le 4\pi\|g''\|_q.$$

Finally, we conclude that $\|f\|_{\mathrm{h}_q^c} \lesssim \|\ell_f\| \lesssim \|f\|_{\mathrm{bmo}_q^c}$ and thus $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M}) = \mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ with equivalent norms. □

The following theorem extends the content of Theorem 3.8.

Theorem 3.17.

(1) *The map E extends to a bounded map from $L_\infty(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_\infty L_\infty(\mathcal{N})$ into* $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$ *and*

$$\|E(h)\|_{\mathrm{bmo}^c} \lesssim \|h\|_{L_\infty(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_\infty L_\infty(\mathcal{N})}.$$

(2) *For $1 < p < \infty$, E extends to a bounded map from $L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})$ into $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and*

$$\|E(h)\|_{\mathrm{h}_p^c} \lesssim \|h\|_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})}.$$

Proof. (1) is already contained in Theorem 3.8. When $p > 2$, (2) follows from Theorem 3.10 and Theorem 3.16. The case $p = 2$ is trivial. For the case $1 < p < 2$, using the duality between $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$, we have

$$\|E(h)\|_{\mathrm{h}_p^c} \lesssim \sup_{\|f\|_{\mathrm{bmo}_q^c}\le 1}\Big|\tau\int_{\mathbb{R}^d} E(h)(s)f^*(s)ds\Big|.$$

Then, by Theorem 3.16 and (3.12), for $h = (h',h'') \in L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})$, we have

$$\begin{aligned}&\sup_{\|f\|_{\mathrm{bmo}_q^c}\le 1}\Big|\tau\int_{\mathbb{R}^d} E(h)(s)f^*(s)ds\Big|\\ &\lesssim \sup_{\|f\|_{\mathrm{h}_q^c}\le 1}\Big|\tau\int_{\mathbb{R}^d}\Big[\iint_{\widetilde{\Gamma}} h'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)^*(s+t)dtd\varepsilon + h''(s)([\mathrm{P}+4\pi I(\mathrm{P})] * f^*(s))\Big]ds\Big|\\ &\lesssim \|h\|_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})}.\end{aligned}$$

The desired inequality is proved. □

The above theorem shows that, for any $1 < p < \infty$, $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ is a complemented subspace of $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \oplus_p L_p(\mathcal{N})$. Thus, combined with Theorem 3.10, we deduce the following duality theorem:

Theorem 3.18. $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})^* = \mathrm{h}_q^c(\mathbb{R}^d, \mathcal{M})$ *with equivalent norms for any* $1 < p < \infty$.

Chapter 4

Interpolation

In this chapter we study the interpolation of local Hardy and bmo spaces by transferring the problem to that of the operator-valued Hardy and BMO spaces defined in [42]. We begin with an easy observation on the difference between bmo_q^c and BMO_q^c norms.

Lemma 4.1. *For $2 < q \leq \infty$, we have*

$$\|g\|_{\mathrm{bmo}_q^c} \approx \big(\|g\|_{\mathrm{BMO}_q^c}^q + \|J(\mathrm{P}) * g\|_q^q\big)^{\frac{1}{q}}.$$

Proof. Repeating the proof of Proposition 2.3 with $\|\cdot\|_{\mathcal{M}}$ replaced by $\|\cdot\|_{L_{\frac{q}{2}}(\mathcal{N};\ell_\infty)}$, we have $\|g\|_{\mathrm{BMO}_q^c} \lesssim \|g\|_{\mathrm{bmo}_q^c}$. By Lemma 3.3, it is also evident that $\|J(\mathrm{P}) * g\|_q \lesssim \|g\|_{\mathrm{bmo}_q^c}$. Then we obtain

$$\big(\|g\|_{\mathrm{BMO}_q^c}^q + \|J(\mathrm{P}) * g\|_q^q\big)^{\frac{1}{q}} \lesssim \|g\|_{\mathrm{bmo}_q^c}.$$

On the other hand, by Corollary 3.11, we have

$$\|g\|_{\mathrm{bmo}_q^c} \lesssim \Big\| {\sup_{\substack{s\in Q\subset\mathbb{R}^d\\ |Q|<1}}}^{+} \frac{1}{|Q|}\int_{T(Q)} |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)|^2 \varepsilon dt d\varepsilon \Big\|_{\frac{q}{2}}^{\frac{1}{2}} + \|J(\mathrm{P}) * g\|_q.$$

Clearly, the first term on the right side is estimated from above by $\|g\|_{\mathrm{BMO}_c^q}$ (see [70, Theorem 3.4]). Therefore,

$$\|g\|_{\mathrm{bmo}_q^c} \lesssim \|g\|_{\mathrm{BMO}_c^q} + \|J(\mathrm{P}) * g\|_q \approx \big(\|g\|_{\mathrm{BMO}_q^c}^q + \|J(\mathrm{P}) * g\|_q^q\big)^{\frac{1}{q}}.$$

Thus, the lemma is proved. $\square$

Define $F_q(\mathcal{N})$ to be the space of all $f \in L_q(\mathcal{M}; \mathrm{R}_d^c)$ such that $\|J(\mathrm{P}) * f\|_q < \infty$. From the above lemma, we see that $\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ and the space $\mathrm{BMO}_q^c(\mathbb{R}^d, \mathcal{M}) \oplus_q F_q(\mathcal{N})$ have equivalent norms. By the interpolation between $\mathrm{BMO}_q^c(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{BMO}^c(\mathbb{R}^d, \mathcal{M})$ (see [42] for more details), we deduce the following lemma:

Lemma 4.2. *Let $2 < q < \infty$ and $0 < \theta < 1$. Then*

$$\big(\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M}), \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})\big)_\theta \subset \mathrm{bmo}_\varrho^c(\mathbb{R}^d, \mathcal{M}) \quad \textit{with} \quad \varrho = \frac{q}{1-\theta}.$$

Proof. By Lemma 4.1, we can see that

$$\mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M}) = \mathrm{BMO}_q^c(\mathbb{R}^d, \mathcal{M}) \oplus_q F_q(\mathcal{N}).$$

with equivalent norms. Define a map

$$\begin{aligned} \Upsilon_q : F_q(\mathcal{N}) &\longrightarrow L_q(\mathcal{N}) \\ f &\longmapsto J(\mathrm{P}) * f. \end{aligned}$$

Thus, Υ_q defines an isometric embedding of $F_q(\mathcal{N})$ into $L_q(\mathcal{N})$. Then by the interpolation between $\mathrm{BMO}_q^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M})$, we get

$$\begin{aligned} \big(\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})\big)_\theta &= \big(\mathrm{BMO}_q^c(\mathbb{R}^d,\mathcal{M})\oplus_q F_q(\mathcal{N}),\mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M})\oplus_\infty F_\infty(\mathcal{N})\big)_\theta \\ &= \big(\mathrm{BMO}_q^c(\mathbb{R}^d,\mathcal{M}),\mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M})\big)_\theta \oplus_\varrho \big(F_q(\mathcal{N}),F_\infty(\mathcal{N})\big)_\theta \\ &\subset \mathrm{BMO}_\varrho^c(\mathbb{R}^d,\mathcal{M})\oplus_\varrho F_\varrho(\mathcal{N}) = \mathrm{bmo}_\varrho^c(\mathbb{R}^d,\mathcal{M}), \end{aligned}$$

which completes the proof. □

Theorem 4.3. *Let $1<p<\infty$. We have*

$$\big(\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\big)_{\frac{1}{p}} = \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}).$$

Proof. Let $1<p<2$ and $\frac{1}{p'}=\frac{1-\theta}{p}+\theta$. Since the map F in Definition 3.4 is an isometry from $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ to $L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})$, we have

$$\big(\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\big)_\theta \subset \mathrm{h}_{p'}^c(\mathbb{R}^d,\mathcal{M}). \tag{4.1}$$

By Theorem 3.18, h_p^c is a reflexive Banach space. Then applying [3, Corollary 4.5.2], we know that the dual of $\big(\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\big)_\theta$ is $\big(\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})\big)_\theta$. Therefore, if the inclusion (4.1) is proper, we will get the proper inclusion

$$\mathrm{bmo}_\varrho^c(\mathbb{R}^d,\mathcal{M}) \subsetneq \big(\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})\big)_\theta,$$

which is in contradiction with Lemma 4.2. Thus, we have

$$\big(\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\big)_\theta = \mathrm{h}_{p'}^c(\mathbb{R}^d,\mathcal{M}). \tag{4.2}$$

By duality and [3, Corollary 4.5.2] again, the above equality implies that for $q'=\frac{q}{1-\theta}$,

$$\big(\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})\big)_\theta = \mathrm{h}_{q'}^c(\mathbb{R}^d,\mathcal{M}). \tag{4.3}$$

For the case where $1<p_1,p_2<\infty$, the interpolation of $\mathrm{h}_{p_1}^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{h}_{p_2}^c(\mathbb{R}^d,\mathcal{M})$ is much easier to handle. Indeed, by Theorem 3.17, we have, for any $1<p<\infty$, $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ is a complemented subspace of $L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})$ via the maps F and E in Definition 3.4. This implies that, for any $1<p_1,p_2<\infty$,

$$\big(\mathrm{h}_{p_1}^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_{p_2}^c(\mathbb{R}^d,\mathcal{M})\big)_\theta = \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}),$$

with $\frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}$. Combining this equivalence with (4.2), (4.3), and applying Wolff's interpolation theorem (see [69]), we get the desired assertion. □

The following theorem is the mixed version of Theorem 4.3, which states that $\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ are also good endpoints of $L_p(\mathcal{N})$.

Theorem 4.4. *Let $1<p<\infty$. We have $(X,Y)_{\frac{1}{p}}=L_p(\mathcal{N})$ with equivalent norms, where $X=\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ or $L_\infty(\mathcal{N})$, and $Y=\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})$ or $L_1(\mathcal{N})$.*

Proof. By the same argument as in the proof of Theorem 4.3, we have the inclusion

$$\big(\mathrm{bmo}_q(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})\big)_\theta \subset \mathrm{bmo}_{q'}(\mathbb{R}^d,\mathcal{M}) \quad q'=\frac{q}{\theta},$$

which ensures by duality that

$$\big(\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})\big)_\theta \supset \mathrm{h}_{p'}(\mathbb{R}^d,\mathcal{M}) = L_{p'}(\mathcal{N})$$

for $\frac{1}{p'}=\frac{1-\theta}{p}+\theta$. Then by Proposition 6.15,

$$L_{p'}(\mathcal{N}) \subset \big(\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})\big)_\theta = \big(L_p(\mathcal{N}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})\big)_\theta.$$

Since $\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})\subset L_1(\mathcal{N})$, then

$$\big(\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})\big)_\theta \subset \big(L_p(\mathcal{N}),L_1(\mathcal{N})\big)_\theta = L_{p'}(\mathcal{N}).$$

Combining the estimates above, we have

$$\big(\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})\big)_\theta = L_{p'}(\mathcal{N}).$$

Again, using duality and Wolff's interpolation theorem, we can conclude the proof by the same trick as in the proof of Theorem 4.3. □

We end this chapter by some real interpolation results.

Corollary 4.5. *Let $1<p<\infty$. Then we have*

(1) $\big(\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\big)_{\frac{1}{p},p}=\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ *with equivalent norms.*

(2) $(X,Y)_{\frac{1}{p},p}=L_p(\mathcal{N})$ *with equivalent norms, where* $X=\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ *or* $L_\infty(\mathcal{N})$*, and* $Y=\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})$ *or* $L_1(\mathcal{N})$.

Proof. Both (1) and (2) follow from [3, Theorem 4.7.2]; we only prove (1). Let $1<p_1<p<p_2<\infty$ with $\frac{1}{p}=\frac{1-\eta}{p_1}+\frac{\eta}{p_2}$. By [3, Theorem 4.7.2], we write

$$\begin{aligned}&\big(\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\big)_{\frac{1}{p},p}\\ &=\Big(\big(\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\big)_{\frac{1}{p_1}},\big(\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\big)_{\frac{1}{p_2}}\Big)_{\eta,p}.\end{aligned} \tag{4.4}$$

Then the assertion (1) follows from Theorem 4.4 and the facts that $\big(L_{p_1}(\mathcal{N}),L_{p_2}(\mathcal{N})\big)_{\eta,p}=L_p(\mathcal{N})$ and that $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ is a complemented subspace of $L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})$. □

Chapter 5

Calderón-Zygmund theory

In this chapter, we will introduce the Calderón-Zygmund theory on operator-valued inhomogeneous function spaces and deduce some Fourier multiplier theorems. It is closely related to the similar results of [24], [36], [49] and [72]. The results in this chapter will be used in the next several chapters to investigate various square funtions that characterize local Hardy spaces and inhomogeneous Triebel-Lizorkin spaces.

5.1 Calderón-Zygmund theory on local Hardy spaces

Let $K \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$ coincide on $\mathbb{R}^d \setminus \{0\}$ with a locally integrable $L_1(\mathcal{M}) + \mathcal{M}$-valued function. We define the left singular integral operator K^c associated to K by

$$K^c(f)(s) = \int_{\mathbb{R}^d} K(s-t)f(t)dt,$$

and the right singular integral operator K^r associated to K by

$$K^r(f)(s) = \int_{\mathbb{R}^d} f(t)K(s-t)dt.$$

Both $K^c(f)$ and $K^r(f)$ are well-defined for sufficiently nice functions f with values in $L_1(\mathcal{M}) \cap \mathcal{M}$, for instance, for $f \in \mathcal{S} \otimes (L_1(\mathcal{M}) \cap \mathcal{M})$.

Let $\mathrm{bmo}_0^c(\mathbb{R}^d, \mathcal{M})$ denote the subspace of $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$ consisting of compactly supported functions. The following lemma is an analogue of Lemma 2.1 in [70] for inhomogeneous spaces. Notice that the usual Calderón-Zygmund operators (the operators satisfying the condition (1) and (3) in the following lemma) are not necessarily bounded on the local Hardy space $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$. Thus, we need to impose an extra decay at infinity on the kernel K.

Lemma 5.1. *Assume that*

(1) *the Fourier transform of K is bounded:* $\sup_{\xi \in \mathbb{R}^d} \|\widehat{K}(\xi)\|_{\mathcal{M}} < \infty$;

(2) *K satisfies the size estimate: there exist C_1 and $\rho > 0$ such that*

$$\|K(s)\|_{\mathcal{M}} \leq \frac{C_1}{|s|^{d+\rho}}, \ \forall |s| \geq 1;$$

(3) *K has the Lipschitz regularity: there exist C_2 and $\gamma > 0$ such that*

$$\|K(s-t) - K(s)\|_{\mathcal{M}} \leq C_2 \frac{|t|^\gamma}{|s-t|^{d+\gamma}}, \ \forall |s| > 2|t|.$$

Then K^c is bounded on $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ for $1\le p<\infty$ and from $\mathrm{bmo}_0^c(\mathbb{R}^d,\mathcal{M})$ to $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$.

A similar statement also holds for K^r and the corresponding row spaces.

Proof. First suppose that K^c maps constant functions to zero. This amounts to requiring that $K^c(\mathbb{1}_{\mathbb{R}^d})=0$. Let $Q\subset\mathbb{R}^d$ be a cube with $|Q|<1$. Since the assumption of Lemma 2.1 in [70] are included in the ones of this lemma, we get

$$\Big\|\big(\frac{1}{|Q|}\int_Q |K^c(f)-K^c(f)_Q|^2dt\big)^{\frac12}\Big\|_{\mathcal{M}}\lesssim \|f\|_{\mathrm{BMO}^c}\lesssim\|f\|_{\mathrm{bmo}^c}.$$

Now let us focus on the cubes with side length 1. Let Q be a cube with $|Q|=1$ and $\widetilde{Q}=2Q$ be the cube concentric with Q and with side length 2. Decompose f as $f=f_1+f_2$, where $f_1=\mathbb{1}_{\widetilde{Q}}f$ and $f_2=\mathbb{1}_{\mathbb{R}^d\setminus\widetilde{Q}}f$. Then $K^c(f)=K^c(f_1)+K^c(f_2)$. We have

$$\Big\|\frac{1}{|Q|}\int_Q|K^c(f)|^2ds\Big\|_{\mathcal{M}}\lesssim\Big\|\frac{1}{|Q|}\int_Q|K^c(f_1)|^2ds\Big\|_{\mathcal{M}}+\Big\|\frac{1}{|Q|}\int_Q|K^c(f_2)|^2ds\Big\|_{\mathcal{M}}.$$

The first term is easy to estimate. By assumption (1) and (1.5),

$$\begin{aligned}\Big\|\frac{1}{|Q|}\int_Q|K^c(f_1)|^2ds\Big\|_{\mathcal{M}}&\le\Big\|\frac{1}{|Q|}\int_{\mathbb{R}^d}|\widehat{K}(\xi)\widehat{f_1}(\xi)|^2d\xi\Big\|_{\mathcal{M}}\\&\lesssim\Big\|\frac{1}{|Q|}\int_{\mathbb{R}^d}|\widehat{f_1}(\xi)|^2d\xi\Big\|_{\mathcal{M}}\\&=\Big\|\frac{1}{|Q|}\int_{\widetilde{Q}}|f(s)|^2ds\Big\|_{\mathcal{M}}\\&\lesssim\sup_{|Q|=1}\Big\|\frac{1}{|Q|}\int_Q|f(s)|^2ds\Big\|_{\mathcal{M}}.\end{aligned}$$

To estimate the second term, using assumption (2) and (1.5) again, we have

$$\begin{aligned}|K^c(f_2)(s)|^2&=\Big|\int_{\mathbb{R}^d}K(s-t)f_2(t)dt\Big|^2=\Big|\int_{\mathbb{R}^d\setminus\widetilde{Q}}K(s-t)f(t)dt\Big|^2\\&\le\int_{\mathbb{R}^d\setminus\widetilde{Q}}\|K(s-t)\|_{\mathcal{M}}dt\cdot\int_{\mathbb{R}^d\setminus\widetilde{Q}}\|K(s-t)\|_{\mathcal{M}}^{-1}|K(s-t)f(t)|^2dt\\&\lesssim\int_{\mathbb{R}^d\setminus\widetilde{Q}}\|K(s-t)\|_{\mathcal{M}}|f(t)|^2dt\\&\lesssim\int_{\mathbb{R}^d\setminus\widetilde{Q}}\frac{1}{|s-t|^{d+\rho}}|f(t)|^2dt.\end{aligned}$$

Set $\widetilde{Q}_m=\widetilde{Q}+2m$ for every $m\in\mathbb{Z}^d$. Then $\mathbb{R}^d\setminus\widetilde{Q}=\cup_{m\neq0}\widetilde{Q}_m$. Continuing the estimate of $|K^c(f_2)(s)|^2$, for any $s\in Q$, we have

$$\begin{aligned}|K^c(f_2)(s)|^2&\le\sum_{m\neq0}\int_{\widetilde{Q}_m}\frac{1}{|s-t|^{d+\rho}}|f(t)|^2dt\\&\approx\sum_{m\neq0}\frac{1}{|m|^{d+\rho}}\int_{\widetilde{Q}_m}|f(t)|^2dt\lesssim\|f\|_{\mathrm{bmo}^c}.\end{aligned}$$

Combining the previous estimates, we deduce that K^c is bounded from $\mathrm{bmo}_0^c(\mathbb{R}^d,\mathcal{M})$ to $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$.

Now we illustrate that the additional requirement that $K^c(\mathbb{1}_{\mathbb{R}^d})=0$ is not needed. First, a similar argument as above ensures that for every compactly supported $f\in L_\infty(\mathcal{N})$,

$\|K^c(f)\|_{\mathrm{bmo}^c} \lesssim \|f\|_\infty$. Then we follow the argument of [19, Proposition II.5.15] to extend K^c on the whole $L_\infty(\mathcal{N})$, as

$$K^c(f)(s) = \lim_j \big[K^c(f\mathbb{1}_{B_j})(s) - \int_{1<|t|\le j} K(-t)f(t)dt\big], \quad \forall s\in\mathbb{R}^d,$$

where B_j is the ball centered at the origin with radius j. Let us show that the sequence on the right hand side converges pointwise in the norm $\|\cdot\|_{\mathcal{M}}$ and uniformly on compact sets $F\subset\mathbb{R}^d$. To this end, we denote by g_j the j-th term of this sequence. Let l be the first natural number such that $l\ge 2\sup_{s\in F}|s|$. Then for $s\in F$ and $j>l$, we have

$$g_j(s) = g_l(s) + \int_{l<|t|\le j}(K(s-t)-K(-t))f(t)dt.$$

By assumption (3), the integral on the right hand side is bounded by a bounded multiple of $\|f\|_\infty$, uniformly on $s\in F$. This ensures the convergence of g_j, so $K^c(f)$ is a well-defined function. Now we have to estimate the bmo^c-norm of $K^c(f)$. Taking any cube $Q\subset\mathbb{R}^d$, by the uniform convergence of g_j on Q in $\mathcal{M}$,

$$\|\Big(\int_Q |K^c(f)(s)-(K^c(f))_Q|^2 ds\Big)^{\frac12}\|_{\mathcal{M}} = \lim_j \|\Big(\int_Q |g_j(s)-(g_j)_Q|^2 ds\Big)^{\frac12}\|_{\mathcal{M}}.$$

Similarly,

$$\|\Big(\int_Q |K^c(f)(s)|^2 ds\Big)^{\frac12}\|_{\mathcal{M}} = \lim_j \|\Big(\int_Q |g_j(s)|^2 ds\Big)^{\frac12}\|_{\mathcal{M}}.$$

Hence, by the fact that g_j and $K^c(f\mathbb{1}_{B_j})$ differ by a constant, we obtain

$$\|K^c(f)\|_{\mathrm{bmo}^c} = \lim_j \|g_j\|_{\mathrm{bmo}^c} \lesssim \limsup_j \|K^c(f\mathbb{1}_{B_j})\|_{\mathrm{bmo}^c} + \|f\|_\infty \lesssim \|f\|_\infty.$$

Therefore, K^c defined above extends to a bounded operator from $L_\infty(\mathcal{N})$ to $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$. In particular, $K^c(\mathbb{1}_{\mathbb{R}^d})$ determines a function in $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$. Then for f and Q as above, we have $K^c(f) = K^c(f_1)+K^c(f_2)+K^c(\mathbb{1}_{\mathbb{R}^d})f_{\widetilde{Q}}$, so

$$\begin{aligned}\|K^c(f)\|_{\mathrm{bmo}^c} &\le \|K^c(f_1)\|_{\mathrm{bmo}^c} + \|K^c(f_2)\|_{\mathrm{bmo}^c} + \|K^c(\mathbb{1}_{\mathbb{R}^d})\|_{\mathrm{bmo}^c}\|f_{\widetilde{Q}}\|_{\mathcal{M}}\\ &\lesssim \|f\|_{\mathrm{bmo}^c} + \|f_{\widetilde{Q}}\|_{\mathcal{M}} \lesssim \|f\|_{\mathrm{bmo}^c}.\end{aligned}$$

Thus we have proved the bmo^c-boundedness of K^c in the general case.

By duality, the boundedness of K^c on $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$ is equivalent to that of its adjoint map $(K^c)^*$ on $\mathrm{bmo}_0^c(\mathbb{R}^d,\mathcal{M})$. It is easy to see that $(K^c)^*$ is also a singular integral operator:

$$(K^c)^*(g) = \int_{\mathbb{R}^d}\widetilde{K}(s-t)g(t)dt,$$

where $\widetilde{K}(s) = K^*(-s)$. Obviously, $\widetilde{K}$ also satisfies the same assumption as K, so $(K^c)^*$ is bounded on $\mathrm{bmo}_0^c(\mathbb{R}^d,\mathcal{M})$. Thus we get the boundedness of K^c on $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. Then, by the interpolation between $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$ in Theorem 4.3, we get the boundedness of K^c on $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ for $1<p<\infty$. The assertion is proved. □

Remark 5.2. Under the assumption of the above lemma, $K^c(\mathbb{1}_{\mathbb{R}^d})$ is a constant, so $K^c(\mathbb{1}_{\mathbb{R}^d})$ is zero as an element in $\mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M})$.

A special case of Lemma 5.1 concerns the Hilbert-valued kernel K. Let H be a Hilbert space and $\mathsf{k} : \mathbb{R}^d \to H$ be a H-valued kernel. We view the Hilbert space as the column matrices in $B(H)$ with respect to a fixed orthonormal basis. Put $K(s) = \mathsf{k}(s) \otimes 1_{\mathcal{M}} \in B(H)\overline{\otimes}\mathcal{M}$. For nice functions $f : \mathbb{R}^d \to L_1(\mathcal{M}) + \mathcal{M}$, $K^c(f)$ takes values in the column subspace of $L_1(B(H)\overline{\otimes}\mathcal{M}) + L_\infty(B(H)\overline{\otimes}\mathcal{M})$. Consequently,

$$\|K^c(f)\|_{L_p(B(H)\overline{\otimes}\mathcal{N})} = \|K^c(f)\|_{L_p(\mathcal{N};H^c)}.$$

Since $\mathsf{k}(s) \otimes 1_{\mathcal{M}}$ commutes with $\mathcal{M}$, $K^c(f) = K^r(f)$ for $f \in L_2(\mathcal{N})$. Let us denote this common operator by k^c. Here the superscript c refers to the previous convention that H is identified with the column matrices in $B(H)$. Thus, Lemma 5.1 implies the following

Corollary 5.3. *Assume that*

(1) $\sup_{\xi\in\mathbb{R}^d} \|\widehat{\mathsf{k}}(\xi)\|_H < \infty$;

(2) $\|\mathsf{k}(s)\|_H \lesssim \frac{1}{|s|^{d+\rho}}$, $\forall |s| \geq 1$, *for some* $\rho > 0$;

(3) $\|\mathsf{k}(s-t) - \mathsf{k}(s)\|_H \lesssim \frac{|t|^\gamma}{|s-t|^{d+\gamma}}$, $\forall |s| > 2|t|$, *for some* $\gamma > 0$.

Then the operator k^c *is bounded*

(1) *from* $\mathrm{bmo}_0^\alpha(\mathbb{R}^d, \mathcal{M})$ *to* $\mathrm{bmo}^\alpha(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$, *where* $\alpha = c$, $\alpha = r$ *or we leave out* α;

(2) *and from* $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ *to* $\mathrm{h}_p^c(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$ *for* $1 \leq p < \infty$.

Remark 5.4. Since $L_\infty(\mathcal{N}) \subseteq \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$, we have $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M}) \subseteq L_1(\mathcal{N})$. This ensures $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) \subseteq L_p(\mathcal{N})$ for $1 \leq p \leq 2$. Combined with Theorem 4.3, Corollary 5.3 and the fact that $\mathrm{h}_2^c(\mathbb{R}^d, \mathcal{M}) = L_2(\mathcal{N})$, we have

$$\|\mathsf{k}^c(f)\|_{L_p(\mathcal{N};H^c)} \lesssim \|\mathsf{k}^c(f)\|_{\mathrm{h}_p^c(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})} \lesssim \|f\|_{\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})}$$

for any $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$.

5.2 Multiplier theorems

We are going to develop two Fourier multiplier theorems in this section. They can be viewed as the special cases of Calderón-Zygmund theory. Our presentation follows closely the argument in section 4.1 of [72].

Recall again that φ is a fixed function satisfying (1.1), φ_0 is the inverse Fourier transform of $1 - \sum_{k>0} \varphi(2^{-k}\cdot)$ and φ_k is the inverse Fourier transform of $\varphi(2^{-k}\cdot)$ when $k > 0$. Moreover, we denote by $\varphi^{(k)}$ the Fourier transform of φ_k for every $k \in \mathbb{N}_0$, then they enjoy the properties in (1.2) and (1.3).

Firstly, let us state the following homogeneous version of [72, Theorem 4.1].

Theorem 5.5. *Let* $\sigma \in \mathbb{R}$ *with* $\sigma > \frac{d}{2}$. *Assume that* $(\phi_j)_{j\in\mathbb{Z}}$ *and* $(\rho_j)_{j\in\mathbb{Z}}$ *are two sequences of functions on* $\mathbb{R}^d\backslash\{0\}$ *such that*

$$\operatorname{supp} \phi_j\rho_j \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \;\; j \in \mathbb{Z}$$

and

$$\sup_{\substack{j\in\mathbb{Z} \\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d)} < \infty.$$

Let $1 < p < \infty$. Then for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$, we have

$$\big\|(\sum_{j\in\mathbb{Z}} 2^{2j\alpha}|\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_p \lesssim \sup_{\substack{j\in\mathbb{Z}\\ -2\le k\le 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma} \big\|(\sum_{j\geq K} 2^{2j\alpha}|\check{\rho}_j * f|^2)^{\frac12}\big\|_p,$$

where the constant depends on p, σ, d and φ.

Proof. Without loss of generality, we may take $\alpha = 0$. It suffices to show that for any integer K,

$$\big\|(\sum_{j\geq K} |\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_p \lesssim \sup_{\substack{j\in\mathbb{Z}\\ -2\le k\le 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma} \big\|(\sum_{j\in\mathbb{Z}} |\check{\rho}_j * f|^2)^{\frac12}\big\|_p, \tag{5.1}$$

with the relevant constant independent of $K \in \mathbb{Z}$. To this end, we set

$$\psi_{j-K} = \phi_j(2^K\cdot), \quad \eta_{j-K} = \rho_j(2^K\cdot), \text{ and } \widehat{g} = \widehat{f}(2^K\cdot).$$

By an easy computation, we have

$$\operatorname{supp}\psi_j\eta_j \subset \{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}, \quad \forall\, j \geq 0,$$

and

$$\check{\phi}_j * \check{\rho}_j * f = 2^{dK}\check{\psi}_{j-K} * \check{\rho}_{j-K} * g(2^K\cdot).$$

This ensures

$$\big\|(\sum_{j\geq K} |\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_p = 2^{\frac{(p-1)dK}{p}} \big\|(\sum_{j\geq 0} |\check{\psi}_j * \check{\eta}_j * g|^2)^{\frac12}\big\|_p. \tag{5.2}$$

Similarly,

$$\big\|(\sum_{j\geq K} |\check{\rho}_j * f|^2)^{\frac12}\big\|_p = 2^{\frac{(p-1)dK}{p}} \big\|(\sum_{j\geq 0} |\check{\eta}_j * g|^2)^{\frac12}\big\|_p. \tag{5.3}$$

Moreover, since $\psi_j(2^{j+k}\cdot) = \phi_{j+K}(2^{j+k+K}\cdot)$, we have

$$\begin{aligned} \sup_{\substack{j\geq 0\\ -2\le k\le 2}} \|\psi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma} &= \sup_{\substack{j\geq K\\ -2\le k\le 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma} \\ &\le \sup_{\substack{j\in\mathbb{Z}\\ -2\le k\le 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma}. \end{aligned} \tag{5.4}$$

Now applying [72, Theorem 4.1] to ψ_j, ρ_j and g defined above, we obtain

$$\|(\sum_{j\geq 0} |\check{\psi}_j * \check{\eta}_j * g|^2)^{\frac12}\|_p \lesssim \sup_{\substack{j\geq 0\\ -2\le k\le 2}} \|\psi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma(}\|(\sum_{j\geq 0} |\check{\eta}_j * g|^2)^{\frac12}\|_p.$$

Putting (5.2), (5.3) and (5.4) into this inequality, we then get (5.1), which yields Theorem 5.5 by approximation. $\square$

Theorem 5.5 is developed to deal with the multiplier problem of square functions, and also the multiplier problem of the Hardy spaces $\mathcal{H}_p^c(\mathbb{R}^d, \mathcal{M})$ by virtue of their characterizations (Lemma 1.2). In order to deal with relative problems on the local versions of square functions or Hardy spaces, we need the following version of Theorem 5.5. The main difference is that in the local case, we need to consider a Littlewood-Paley decomposition which covers the origin.

Theorem 5.6. *Let $1 < p < \infty, \alpha \in \mathbb{R}$ and $\sigma > \frac{d}{2}$. Assume that $(\phi_j)_{j\geq 0}$ and $(\rho_j)_{j\geq 0}$ are two sequences of functions on $\mathbb{R}^d$ such that*

$$\operatorname{supp}(\phi_j\rho_j) \subset \{\xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| \leq 2^{j+1}\},\quad j \in \mathbb{N},$$
$$\operatorname{supp}(\phi_0\rho_0) \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2\},$$

and that

$$\sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d)} < \infty \quad \text{and} \quad \|\phi_0(\varphi^{(0)} + \varphi^{(1)})\|_{H_2^\sigma(\mathbb{R}^d)} < \infty. \tag{5.5}$$

Then for any $L_1(\mathcal{M}) + \mathcal{M}$-valued distribution f,

$$\begin{aligned}\big\|(\sum_{j\geq 0} 2^{2j\alpha}|\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_p \lesssim \max\{ &\sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma}, \|\phi_0(\varphi^{(0)} + \varphi^{(1)})\|_{H_2^\sigma}\}\\ &\cdot \big\|(\sum_{j\geq 0} 2^{2j\alpha}|\check{\rho}_j * f|^2)^{\frac12}\big\|_p,\end{aligned}$$

where the constant depends on p, σ, d and φ.

Proof. This theorem follows easily from its homogeneous version, i.e., Theorem 5.5. Indeed, we can divide $\big\|(\sum_{j\geq 0} 2^{2j\alpha}|\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_p$ into two parts:

$$\big\|(\sum_{j\geq 0} 2^{2j\alpha}|\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_p \approx \big\|(\sum_{j\geq 1} 2^{2j\alpha}|\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_p + \|\check{\phi}_0 * \check{\rho}_0 * f\|_p$$

and treat them separately. Applying Theorem 5.5 to the sequences $(\phi_j)_{j\in\mathbb{Z}}$, $(\rho_j)_{j\in\mathbb{Z}}$ with $\phi_j = 0$ and $\rho_j = 0$ for $j \leq 0$, we get the estimate of the first term on the right hand side. The result is

$$\big\|(\sum_{j\geq 1} 2^{2j\alpha}|\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_p \lesssim \sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma} \big\|(\sum_{j\geq 1} |\check{\rho}_j * f|^2)^{\frac12}\big\|_p.$$

The second term $\|\check{\phi}_0 * \check{\rho}_0 * f\|_p$ is also easy to handle. By the support assumption on $\phi_0\rho_0$, we have

$$\check{\phi}_0 * \check{\rho}_0 * f = \mathcal{F}^{-1}\big(\phi_0(\varphi^{(0)} + \varphi^{(1)})\big) * \check{\rho}_0 * f.$$

Hence,

$$\|\check{\phi}_0 * \check{\rho}_0 * f\|_p \lesssim \|\phi_0(\varphi^{(0)} + \varphi^{(1)})\|_{H_2^\sigma} \|\check{\rho}_0 * f\|_p.$$

The assertion is proved. □

In the rest of this section, we will develop a more elaborated version of Theorem 5.6. Assume that we have a sequence of Hilbert spaces H_j for every $j \in \mathbb{N}_0$, and denote $H = \oplus_{j=0}^\infty H_j$. Then an element $f \in L_p(\mathcal{N}; H^c)$ has the form $f = (f_j)_{j\geq 0}$ with $f_j \in L_p(\mathcal{N}; H_j^c)$ for every j. In this case, it still makes sense to act on it by the Calderón-Zygmund operator $\mathsf{k} = (\check{\phi}_j)_{j\geq 0}$.

Since it will be frequently used in the following, we introduce an elementary inequality (see [72, Lemma 4.2]):

$$\|fg\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} \leq \|f\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} \int_{\mathbb{R}^d} (1 + |s|^2)^\sigma |\mathcal{F}^{-1}(g)(s)| ds, \tag{5.6}$$

where $\sigma > \frac{d}{2}$, and the functions $f : \mathbb{R}^d \to \ell_2$ and $g : \mathbb{R}^d \to \mathbb{C}$ satisfy

$$f \in H_2^\sigma(\mathbb{R}^d;\ell_2) \text{ and } \int_{\mathbb{R}^d}(1+|s|^2)^\sigma|\mathcal{F}^{-1}(g)(s)|ds < \infty.$$

Here $H_2^\sigma(\mathbb{R}^d;\ell_2)$ is the ℓ_2-valued Potential Sobolev space of order σ. Note also that ℓ_2 could be an ℓ_2-space on an arbitrary index set, depending on the problems in consideration.

The following lemma is an analogue of Lemma 4.3 in [72]. The main difference is that in order to get a Calderón-Zygmund operator which is bounded on local Hardy or bmo spaces, we need to use the test functions covering the origin.

Lemma 5.7. *Let $\phi = (\phi_j)_{j\geq 0}$ be a sequence of continuous functions on $\mathbb{R}^d$, viewed as a function from $\mathbb{R}^d$ to ℓ_2. For $\sigma > \frac{d}{2}$, we assume that*

$$\|\phi\|_{2,\sigma} \overset{\text{def}}{=} \max\big\{\sup_{k\geq 1}\|\phi(2^k\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)},\ \|\phi\varphi^{(0)}\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)}\big\} < \infty. \tag{5.7}$$

Let $\mathsf{k} = (\mathsf{k}_j)_{j\geq 0}$ with $\mathsf{k}_j = \mathcal{F}^{-1}(\phi_j)$. Then k is a Calderón-Zygmund kernel with values in ℓ_2, more precisely,

(1) $\|\widehat{\mathsf{k}}\|_{L_\infty(\mathbb{R}^d;\ell_2)} \lesssim \|\phi\|_{2,\sigma}$;

(2) $\int_{|s|\geq\frac{1}{2}}\|\mathsf{k}(s)\|_{\ell_2}ds \lesssim \|\phi\|_{2,\sigma}$.

(3) $\sup_{t\in\mathbb{R}^d}\int_{|s|>2|t|}\|\mathsf{k}(s-t)-\mathsf{k}(s)\|_{\ell_2}ds \lesssim \|\phi\|_{2,\sigma}$;

The relevant constants depend only on φ, σ and d.

Proof. For any $\xi \in \mathbb{R}^d$ and $k \geq 1$, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|\phi(2^k\xi)\varphi(\xi)\|_{\ell_2} &= \Big\|\int \mathcal{F}^{-1}(\phi(2^k\cdot)\varphi)(s)e^{-2\pi \mathrm{i} s\cdot\xi}ds\Big\|_{\ell_2}\\ &\leq \|\phi(2^k\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)}\Big(\int(1+|s|^2)^{-\sigma}ds\Big)^{\frac{1}{2}} \lesssim \|\phi\|_{2,\sigma}.\end{aligned}$$

In other words, we have $\|\phi\varphi(2^{-k}\cdot)\|_{L_\infty(\mathbb{R}^d;\ell_2)} \lesssim \|\phi\|_{2,\sigma}$. Likewise, $\|\phi\varphi^{(0)}\|_{L_\infty(\mathbb{R}^d;\ell_2)} \lesssim \|\phi\|_{2,\sigma}$ also holds. Thus, by (1.2) and (1.3), we easily deduce that $\|\widehat{\mathsf{k}}\|_{L_\infty(\mathbb{R}^d;\ell_2)} \lesssim \|\phi\|_{2,\sigma}$.

To show the third property of k, we decompose ϕ into

$$\phi = \sum_{k\geq 0}\phi\varphi^{(k)}.$$

The convergence of the above series can be proved by a limit procedure of its partial sums, which is quite formal. By (1.2) and (1.3), we write

$$\phi\varphi^{(k)} = \phi(\varphi^{(k-1)}+\varphi^{(k)}+\varphi^{(k+1)})\varphi^{(k)} \overset{\text{def}}{=} \phi_{(k)}\varphi^{(k)}, \quad k\geq 0.$$

Here we make the convention that $\varphi^{(k)} = 0$ if $k < 0$. Then for $s \in \mathbb{R}^d$,

$$\mathcal{F}^{-1}(\phi\varphi^{(k)})(s) = \mathcal{F}^{-1}(\phi_{(k)}) * \mathcal{F}^{-1}(\varphi^{(k)})(s) = 2^{kd}\mathcal{F}^{-1}(\phi_{(k)}(2^k\cdot)) * \mathcal{F}^{-1}(\varphi)(2^k s), \quad k\geq 0.$$

By (5.6), we have

$$\Big(\int_{\mathbb{R}^d}(1+|2^k s|^2)^\sigma\|\mathcal{F}^{-1}(\phi\varphi^{(k)})(s)\|_{\ell_2}^2ds\Big)^{\frac{1}{2}} \lesssim 2^{\frac{kd}{2}}\|\phi_{(k)}(2^k\cdot)\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)}.$$

Notice that if $k \geq 1$, we have $\varphi^{(k)}(2^k \cdot) = \varphi$. Thus, if $k \geq 2$,

$$\begin{aligned}\|\phi_{(k)}(2^k\cdot)\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} &\leq \sum_{j=-1}^{1} \|\phi(2^k\cdot)\varphi^{(k-j)}(2^k\cdot)\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} \\ &\lesssim \sum_{j=-1}^{1} \|\phi(2^{k-j}\cdot)\varphi^{(k-j)}(2^{k-j}\cdot)\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} \\ &= \sum_{j=-1}^{1} \|\phi(2^{k-j}\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} \leq 3\|\phi\|_{2,\sigma}.\end{aligned}$$

For $k = 0, 1$, we treat them in the same way,

$$\begin{aligned}\|\phi_{(1)}(2\cdot)\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} &\lesssim \|\phi\varphi^{(0)}\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} + \|\phi(2\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} + \|\phi(4\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)}; \\ \|\phi_{(0)}\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} &\lesssim \|\phi\varphi^{(0)}\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} + \|\phi(2\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)}.\end{aligned}$$

In summary, we obtain

$$(\int_{\mathbb{R}^d} (1 + |2^k s|^2)^\sigma \|\mathcal{F}^{-1}(\phi\varphi^{(k)})(s)\|_{\ell_2}^2 ds)^{\frac{1}{2}} \lesssim 2^{\frac{kd}{2}} \|\phi\|_{2,\sigma}.$$

Thus, by the Cauchy-Schwarz inequality, for any $t \in \mathbb{R}^d \setminus \{0\}$ and $k \geq 0$, we have

$$\begin{aligned}\int_{|s|>|t|} \|\mathcal{F}^{-1}(\phi\varphi^{(k)})(s)\|_{\ell_2} ds &\lesssim 2^{\frac{kd}{2}} \|\phi\|_{2,\sigma} (\int_{|s|>|t|} (1 + |2^k s|^2)^{-\sigma} ds)^{\frac{1}{2}} \\ &\lesssim (2^k|t|)^{\frac{d}{2}-\sigma} \|\phi\|_{2,\sigma}.\end{aligned} \tag{5.8}$$

Consequently,

$$\int_{|s|>2|t|} \|\mathcal{F}^{-1}(\phi\varphi^{(k)})(s) - \mathcal{F}^{-1}(\phi\varphi^{(k)})(s-t)\|_{\ell_2} ds \lesssim (2^k|t|)^{\frac{d}{2}-\sigma} \|\phi\|_{2,\sigma}.$$

We notice that $\frac{d}{2} - \sigma < 0$, so the estimate above is good only when $2^k|t| \geq 1$. Otherwise, we need another estimate (with $e_t(\xi) = e^{2\pi \mathrm{i}\xi\cdot t}$)

$$\begin{aligned}&\mathcal{F}^{-1}(\phi\varphi^{(k)})(s) - \mathcal{F}^{-1}(\phi\varphi^{(k)})(s-t) \\ &= \mathcal{F}^{-1}(\phi_{(k)}\varphi^{(k)}(1 - e_t))(s) \\ &= 2^{kd}\mathcal{F}^{-1}(\phi_{(k)}(2^k\cdot)) * [\mathcal{F}^{-1}(\varphi) - \mathcal{F}^{-1}(\varphi)(\cdot - 2^k t)](2^k s).\end{aligned}$$

Thus,

$$\begin{aligned}&(\int_{\mathbb{R}^d} (1 + |2^k s|^2)^\sigma \|\mathcal{F}^{-1}(\phi\varphi^{(k)})(s) - \mathcal{F}^{-1}(\phi\varphi^{(k)})(s-t)\|_{\ell_2}^2 ds)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{kd}{2}} \|\phi\|_{2,\sigma} 2^k|t| \int (1 + |s|^2)^\sigma |\mathcal{F}^{-1}(\varphi)(s - \theta 2^k t)| ds \\ &\lesssim 2^{\frac{kd}{2}} \|\phi\|_{2,\sigma} 2^k|t| (\int |J^\sigma[\varphi(s) e^{2\pi \mathrm{i} s\cdot\theta 2^k t}]|^2 ds)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{kd}{2}} \|\phi\|_{2,\sigma} 2^k|t|,\end{aligned}$$

where $\theta \in [0, 1]$. Then as before, for $2^k|t| < 1$, we have

$$\int_{|s|>2|t|} \|\mathcal{F}^{-1}(\phi\varphi^{(k)})(s) - \mathcal{F}^{-1}(\phi\varphi^{(k)})(s-t)\|_{\ell_2} ds \lesssim 2^k|t| \|\phi\|_{2,\sigma}.$$

Combining the previous estimates, we obtain

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^d}\int_{|s|>2|t|}\|\mathsf{k}(s-t)-\mathsf{k}(s)\|_{\ell_2}ds\\
&\le \sup_{t\in\mathbb{R}^d}\sum_{k\ge 0}\int_{|s|>2|t|}\|\mathcal{F}^{-1}(\phi\varphi^{(k)})(s)-\mathcal{F}^{-1}(\phi\varphi^{(k)})(s-t)\|_{\ell_2}ds\\
&\lesssim \|\phi\|_{2,\sigma}\sup_{t\in\mathbb{R}^d}\sum_{k\ge 0}\min(2^k|t|,(2^k|t|)^{\frac{d}{2}-\sigma})\lesssim\|\phi\|_{2,\sigma}.
\end{aligned}$$

Finally, the second estimate of k can be deduced from (5.8) by letting $|t|=\frac{1}{2}$:

$$\begin{aligned}
\int_{|s|\ge\frac{1}{2}}\|\mathsf{k}(s)\|_{\ell_2}ds &\le \sum_{k\ge 0}\int_{|s|\ge\frac{1}{2}}\|\mathcal{F}^{-1}(\phi\varphi^{(k)})(s)\|_{\ell_2}ds\\
&\le \sum_{k\ge 0}(2^{k-1})^{\frac{d}{2}-\sigma}\|\phi\|_{2,\sigma}\lesssim\|\phi\|_{2,\sigma}.
\end{aligned}$$

The proof is complete. □

We keep the notation $H=\oplus_{j=0}^{\infty}H_j$. By the above lemma, we can apply the Calderón-Zygmund theory developed earlier in this chapter, to deduce the following lemma:

Lemma 5.8. *Let $\phi=(\phi_j)_{j\ge 0}$ be a sequence of continuous functions on $\mathbb{R}^d$ satisfying* (5.7) *and $1<p<\infty$. For any $f=(f_j)_{j\ge 0}\in L_p(\mathcal{N};H^c)$, we have*

$$\|(\check{\phi}_j * f_j)_{j\ge 0}\|_{L_p(\mathcal{N};H^c)}\lesssim\|\phi\|_{2,\sigma}\|(f_j)_{j\ge 0}\|_{L_p(\mathcal{N};H^c)},$$

where the relevant constant depends only on φ, σ, p and d.

Proof. Consider k as a diagonal matrix with diagonal entries $(\mathsf{k}_j)_{j\ge 0}$ determined by $\widehat{\mathsf{k}}_j=\phi_j$ and $f=(f_j)_{j\ge 0}$ as a column matrix. The associated Calderón-Zygmund operator is defined on $L_p(B(H)\overline{\otimes}\mathcal{N})$ by

$$\mathsf{k}(f)(s)=\int_{\mathbb{R}^d}\mathsf{k}(s-t)f(t)dt.$$

Now it suffices to show that k is a bounded operator on $L_p(\mathcal{N};H^c)$.

We claim that k is bounded from $L_\infty(\mathcal{N};H^c)$ into $\mathrm{bmo}(\mathbb{R}^d,B(H)\overline{\otimes}\mathcal{M})$. For any $s\in\mathbb{R}^d$, put $K(s)=\mathsf{k}(s)\otimes 1_{\mathcal{M}}\in B(H)\overline{\otimes}\mathcal{M}$. Then we have $\|\mathsf{k}(s)\|_{\ell_2}\ge\|\mathsf{k}(s)\|_{\ell_\infty}=\|K(s)\|_{B(H)\otimes\mathcal{M}}$ and $\|f\|_{L_\infty(\mathcal{N};H^c)}=\|f\|_{B(H)\otimes\mathcal{N}}$. Thus, the claim is equivalent to saying that K is bounded from $L_\infty(\mathcal{N};H^c)$ into $\mathrm{bmo}(\mathbb{R}^d,B(H)\overline{\otimes}\mathcal{M})$, if we regard $L_\infty(\mathcal{N};H^c)$ as a subspace of $B(H)\overline{\otimes}\mathcal{N}$.

First, we show that K is bounded from $L_\infty(\mathcal{N};H^c)$ into $\mathrm{bmo}^c(\mathbb{R}^d,B(H)\overline{\otimes}\mathcal{M})$. Let Q be a cube in $\mathbb{R}^d$ centered at c. We decompose f as $f=g+h$ with $g=f\mathbb{1}_{\widetilde{Q}}$, where $\widetilde{Q}=2Q$ is the cube which has the same center as Q and twice the side length of Q. Set

$$a=\int_{\mathbb{R}^d\setminus\widetilde{Q}}K(c-t)f(t)dt.$$

Then

$$K(f)(s)-a=K(g)(s)+\int[K(s-t)-K(c-t)]h(t)dt.$$

Thus, for Q such that $|Q|<1$, we have

$$\frac{1}{|Q|}\int_Q|K(f)-a|^2ds\le 2(A+B),$$

where

$$A = \frac{1}{|Q|}\int_Q |K(g)|^2 ds,$$
$$B = \frac{1}{|Q|}\int_Q |\int [K(s-t) - K(c-t)]h(t)dt|^2 ds.$$

The term A is easy to estimate. By Lemma 5.7 and the Plancherel formula (1.6),

$$\begin{aligned}|Q|A &\leq \int |\widehat{K}(\xi)\widehat{g}(\xi)|^2 d\xi = \int \widehat{g}(\xi)^*\widehat{K}(\xi)^*\widehat{K}(\xi)\widehat{g}(\xi)d\xi \leq \int \|\widehat{K}(\xi)\|^2_{B(H)\overline{\otimes}\mathcal{M}}|\widehat{g}(\xi)|^2 d\xi \\ &\lesssim \int \|\widehat{k}(\xi)\|^2_{\ell_2}|\widehat{g}(\xi)|^2 d\xi \lesssim \|\phi\|^2_{2,\sigma}\int_{\widetilde{Q}} |f(s)|^2 ds \\ &\leq |\widetilde{Q}|\, \|\phi\|^2_{2,\sigma}\|f\|^2_{B(H)\overline{\otimes}\mathcal{N}} = |\widetilde{Q}|\, \|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)},\end{aligned}$$

whence

$$\|A\|_{B(H)\overline{\otimes}\mathcal{M}} \lesssim \|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)}.$$

To estimate B, writing $h = (h_j)_{j\geq 0}$, by Lemma 5.7, we get

$$\begin{aligned}&|\int [K(s-t) - K(c-t)]h(t)dt|^2 \\ &\lesssim \int_{\mathbb{R}^d\setminus\widetilde{Q}} \|K(s-t) - K(c-t)\|_{B(H)\overline{\otimes}\mathcal{M}}dt \int_{\mathbb{R}^d\setminus\widetilde{Q}} \|K(s-t) - K(c-t)\|_{B(H)\overline{\otimes}\mathcal{M}}|h(t)|^2 dt \\ &\lesssim \int_{\mathbb{R}^d\setminus\widetilde{Q}} \|\mathsf{k}(s-t) - \mathsf{k}(c-t)\|_{\ell_2}dt \int_{\mathbb{R}^d\setminus\widetilde{Q}} \|\mathsf{k}(s-t) - \mathsf{k}(c-t)\|_{\ell_2}|h(t)|^2 dt \\ &\lesssim \|\phi\|^2_{2,\sigma}\|f\|^2_{B(H)\overline{\otimes}\mathcal{N}} \lesssim \|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)}.\end{aligned}$$

Hence,

$$\|B\|_{B(H)\overline{\otimes}\mathcal{M}} \leq \frac{1}{|Q|}\int_Q \|\int [K(s-t) - K(c-t)]h(t)dt\|^2_{B(H)\overline{\otimes}\mathcal{M}} ds \lesssim \|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)}.$$

Combining the previous inequalities, we deduce that, for any $|Q| < 1$

$$\Big\|\big(\frac{1}{|Q|}\int_Q |K(f) - a|^2 ds\big)^{\frac{1}{2}}\Big\|_{B(H)\overline{\otimes}\mathcal{M}} \lesssim \|\phi\|_{2,\sigma}\|f\|_{L_\infty(\mathcal{N};H^c)}.$$

Now we consider the case when $|Q| = 1$. We have

$$\frac{1}{|Q|}\int_Q |K(f)|^2 ds \leq 2\frac{1}{|Q|}\int_Q |K(g)|^2 ds + 2\frac{1}{|Q|}\int_Q |K(h)|^2 ds.$$

The first term on the right hand side of the above inequality is equal to the term A, so it remains to estimate the second term. When $t \in \mathbb{R}^d\setminus\widetilde{Q}$, $s \in Q$ and $|Q| = 1$, we have $|s-t| \geq \frac{1}{2}$. Then by (2) in Lemma 5.7 and the Cauchy-Schwarz inequality, we easily deduce that

$$\begin{aligned}&|K(h)(s)|^2 = |\int |K(s-t)h(t)dt|^2 \\ &\leq \int_{\mathbb{R}^d\setminus\widetilde{Q}} \|K(s-t)\|_{B(H)\overline{\otimes}\mathcal{M}}dt \int_{\mathbb{R}^d\setminus\widetilde{Q}} \|K(s-t)\|_{B(H)\overline{\otimes}\mathcal{M}}|h(t)|^2 dt \\ &\lesssim \|f\|^2_{L_\infty(\mathcal{N};H^c)}\Big(\int_{\mathbb{R}^d\setminus\widetilde{Q}} \|\mathsf{k}(s-t)\|_{\ell_2}dt\Big)^2 \\ &\lesssim \|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)}.\end{aligned}$$

Thus, we have, for any $|Q|=1$,

$$\Big\|\Big(\frac{1}{|Q|}\int_Q |K(f)|^2 ds\Big)^{\frac{1}{2}}\Big\|_{B(H)\overline{\otimes}\mathcal{M}} \lesssim \|\phi\|_{2,\sigma}\|f\|_{L_\infty(\mathcal{N};H^c)}.$$

Therefore, K is bounded from $L_\infty(\mathcal{N};H^c)$ into $\mathrm{bmo}^c(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$.

Next we show that K is bounded from $L_\infty(\mathcal{N};H^c)$ into $\mathrm{bmo}^r(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$. We still use the same decomposition $f = g + h$, then we obtain

$$\frac{1}{|Q|}\int_Q |[K(f)-a]^*|^2 ds \le 2(A'+B'),$$

where

$$\begin{aligned} A' &= \frac{1}{|Q|}\int_Q |K(g)^*|^2 ds,\\ B' &= \frac{1}{|Q|}\int_Q \Big|\int [(K(s-t)-K(c-t))h(t)]^* dt\Big|^2 ds. \end{aligned}$$

The estimate of B' can be reduced to that of B. Indeed,

$$\begin{aligned} \|B'\|_{B(H)\overline{\otimes}\mathcal{M}} &\le \frac{1}{|Q|}\int_Q \Big\|\int [(K(s-t)-K(c-t))h(t)]^* dt\Big\|^2_{B(H)\overline{\otimes}\mathcal{M}} ds\\ &= \frac{1}{|Q|}\int_Q \Big\|\int [K(s-t)-K(c-t)]h(t)dt\Big\|^2_{B(H)\overline{\otimes}\mathcal{M}} ds\\ &\lesssim \|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)}. \end{aligned}$$

However, for A', we need a different argument. A' can be viewed as a bounded operator on $H\otimes L_2(\mathcal{M})$. So

$$\|A'\|_{B(\ell_2)\overline{\otimes}\mathcal{M}} = \sup_b\Big\{\frac{1}{|Q|}\int_Q \|\mathsf{k}(g)(s)\,b\|^2_{H\otimes L_2(\mathcal{M})} ds\Big\},$$

where the supremum runs over all b in the unit ball of $H\otimes L_2(\mathcal{M})$. By the Plancherel formula (1.6), we have

$$\begin{aligned} \int_Q \|\mathsf{k}(g)(s)\,b\|^2_{H\otimes L_2(\mathcal{M})} ds &= \int_Q \langle \mathsf{k}(g)(s)\,b, \mathsf{k}(g)(s)\,b\rangle_{H\otimes L_2(\mathcal{M})} ds\\ &\le \int \langle \widehat{\mathsf{k}}(\xi)\widehat{g}(\xi)\,b, \widehat{\mathsf{k}}(\xi)\widehat{g}(\xi)\,b\rangle_{H\otimes L_2(\mathcal{M})} d\xi. \end{aligned}$$

Let $\mathrm{diag}(f_j)_j$ be the diagonal matrix in $B(H)\overline{\otimes}\mathcal{N}$ with entries in $B(H_j)\overline{\otimes}\mathcal{N}$. By the Cauchy-Schwarz inequality, the Plancherel formula (1.6) and Lemma 5.7, we continue the estimate above as

$$\begin{aligned} \int \langle \widehat{\mathsf{k}}(\xi)\widehat{g}(\xi)\,b, \widehat{\mathsf{k}}(\xi)\widehat{g}(\xi)\,b\rangle_{H\otimes L_2(\mathcal{M})} d\xi &\le \sup_\xi \|\widehat{\mathsf{k}}(\xi)\|^2_{\ell_2}\int \langle \widehat{g}(\xi)\,b, \widehat{g}(\xi)\,b\rangle_{H\otimes L_2(\mathcal{M})} d\xi\\ &\lesssim \|\phi\|^2_{2,\sigma}\int_{\widetilde{Q}} \|\,\mathrm{diag}(f_j)_j(s)\,b\|^2_{H\otimes L_2(\mathcal{M})} ds\\ &\lesssim |Q|\|\phi\|^2_{2,\sigma}\|\,\mathrm{diag}(f_j)_j\|^2_{B(H)\overline{\otimes}\mathcal{N}}\|b\|^2_{H\otimes L_2(\mathcal{M})}\\ &\le |Q|\|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)}, \end{aligned}$$

whence,

$$\|A'\|_{B(\ell_2)\overline{\otimes}\mathcal{M}} \lesssim \|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)}.$$

Following the estimate of $\frac{1}{|Q|}\int_Q |K(f)(s)|^2 ds$, we get, when $|Q| = 1$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |K(f)^*|^2 ds &\le 2A' + 2\frac{1}{|Q|}\int_Q |K(h)^*|^2 ds\\ &\le 2A' + 2\frac{1}{|Q|}\int_Q \|K(h)^*\|^2_{B(H)\overline{\otimes}\mathcal{M}} ds\\ &= 2A' + 2\frac{1}{|Q|}\int_Q \|K(h)\|^2_{B(H)\overline{\otimes}\mathcal{M}} ds\\ &\lesssim \|\phi\|^2_{2,\sigma}\|f\|^2_{L_\infty(\mathcal{N};H^c)}.\end{aligned}$$

Therefore, K is bounded from $L_\infty(\mathcal{N};H^c)$ into $\mathrm{bmo}^r(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$. Then k is bounded from $L_\infty(\mathcal{N};H^c)$ into $\mathrm{bmo}(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$. It is clear that k is bounded from $L_2(\mathcal{N};H^c)$ into $L_2(B(H)\overline{\otimes}\mathcal{N})$, then by interpolation, k is bounded from $L_p(\mathcal{N};H^c)$ into $L_p(B(H)\overline{\otimes}\mathcal{N})$ for any $2 \le p < \infty$. The case $1 < p < 2$ is obtained by duality. $\square$

Note that when all H_j degenerate to one dimensional Hilbert spaces, then $H = \ell_2$, the above lemma gives a sufficient condition for $(\phi_j)_{j\ge 0}$ being a bounded Fourier multiplier on $L_p(\mathcal{N};\ell_2^c)$. So we can also use Lemmas 5.7 and 5.8 to prove Theorem 5.6 by an argument similar to the proof of [72, Theorem 4.1]; details are left to the reader. But here we intend to extend Theorem 5.6 to a more general setting.

Theorem 5.9. *Let p, α, σ, $(\phi_j)_{j\ge0}$ and $(\rho_j)_{j\ge0}$ be the same as in Theorem 5.6. Then, for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$,*

$$\begin{aligned}&\Big\|\big(\sum_{j\ge0} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})} |\check{\phi}_j * \check{\rho}_j * f(\cdot+t)|^2 dt\big)^{\frac12}\Big\|_p\\ &\lesssim \max\Big\{\sup_{\substack{j\ge1\\ -2\le k\le 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma}, \|\phi_0(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}\Big\}\\ &\quad\cdot\Big\|\big(\sum_{j\ge0} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})} |\check{\rho}_j * f(\cdot+t)|^2 dt\big)^{\frac12}\Big\|_p,\end{aligned}$$

where the constant depends only on p, σ, d and φ.

Proof. Set $H_j = L_2(B(0,2^{-j}), 2^{jd}dt)$ and $H = \oplus_{j=0}^\infty H_j$. So we have

$$\|(\sum_{j\ge0} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})} |\check{\phi}_j * \check{\rho}_j * f(\cdot+t)|^2 dt)^{\frac12}\|_p = \|(2^{j\alpha}\check{\phi}_j * \check{\rho}_j * f(\cdot+\cdot))_j\|_{L_p(\mathcal{N};H^c)}.$$

Let

$$\begin{aligned}&\zeta_j = \phi_j(\varphi^{(j-1)} + \varphi^{(j)} + \varphi^{(j+1)}),\ j\ge 2,\\ &\zeta_1 = \phi_1(\varphi + \varphi^{(1)} + \varphi^{(2)}),\\ &\zeta_0 = \phi_0(\varphi^{(0)} + \varphi) \quad \text{and} \quad \zeta_j = 0 \text{ if } j<0.\end{aligned}$$

By the support assumption on $\phi_j\rho_j$, we have that $\phi_j\rho_j = \zeta_j\rho_j$. So that for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$,

$$\check{\phi}_j * \check{\rho}_j * f = \check{\zeta}_j * \check{\rho}_j * f,\ j \in \mathbb{N}_0.$$

Now we show that $\zeta = (\zeta_j)_{j\ge0}$ satisfies (5.7) with ζ instead of ϕ. Indeed, by the support assumption of φ, the sequence $\zeta(2^k\cdot)\varphi = \big(\zeta_j(2^k\cdot)\varphi\big)_{j\ge0}$ has at most five nonzero terms of indices j with $k-2\le j\le k+2$. Thus for any $k\in\mathbb{N}_0$,

$$\|\zeta(2^k\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} \le \sum_{j=k-2}^{k+2} \|\zeta_j(2^k\cdot)\varphi\|_{H_2^\sigma}.$$

Moreover, by (5.6), we have

$$\|\zeta_j(2^k\cdot)\varphi\|_{H_2^\sigma} \lesssim \|\phi_j(2^k\cdot)\varphi\|_{H_2^\sigma}, \quad k-2 \leq j \leq k+2.$$

Therefore, the condition (5.5) yields

$$\sup_{k\geq 1} \|\zeta(2^k\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} \lesssim \sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma} + \|\phi_0(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma} < \infty,$$

where the relevant constant depends only on σ, φ and d. In a similar way, we have

$$\|\zeta\varphi^{(0)}\|_{H_2^\sigma(\mathbb{R}^d;\ell_2)} \leq \sum_{0\leq j\leq 2} \|\zeta_j\varphi^{(0)}\|_{H_2^\sigma} \lesssim \sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma} + \|\phi_0(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma} < \infty.$$

Now applying Lemma 5.8 with ζ_j instead of ϕ_j and $f_j = 2^{j\alpha}\check{\rho}_j * f(\cdot + t)$, we prove the theorem. □

The above theorem will be useful when we consider the conic square functions of local Hardy spaces and inhomogeneous Triebel-Lizorkin spaces in Chapter 8. Note that both Theorem 5.6 and Theorem 5.9 do not deal with the case $p = 1$. So we include the corresponding Fourier multiplier results for h_p^c with $1 \leq p \leq 2$ in the following. When the Hilbert space H degenerates to ℓ_2, we have

Corollary 5.10. *Let $\phi = (\phi_j)_{j\geq 0}$ be a sequence of continuous functions on $\mathbb{R}^d$ satisfying (5.7) and $1 \leq p \leq 2$. For $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$,*

$$\Big\|\big(\sum_{j\geq 0} |\check{\phi}_j * f|^2\big)^{\frac12}\Big\|_p \lesssim \|\phi\|_{2,\sigma}\|f\|_{\mathrm{h}_p^c}.$$

The relevant constant depends only on φ, σ and d.

Proof. Now we view $\mathsf{k} = (\mathsf{k}_j)_{j\geq 0}$ as a column matrix and the associated Calderón-Zygmund operator k is defined on $L_p(\mathcal{N})$:

$$\mathsf{k}(f)(s) = \int_{\mathbb{R}^d} \mathsf{k}(s-t)f(t)dt, \quad \forall s \in R.$$

Thus k maps function with values in $L_p(\mathcal{M})$ to sequence of functions. Then we have to show that k is bounded from $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ to $L_p(\mathcal{N}; \ell_2^c)$ for $1 \leq p \leq 2$. The case $p = 2$ is trivial, so by interpolation, it suffices to consider the case $p = 1$. To prove that k is bounded from $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ to $L_1(\mathcal{N}; \ell_2^c)$, passing to the dual spaces, it is equal to proving that the adjoint of k is bounded from $L_\infty(\mathcal{N}; \ell_2^c)$ to $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$. We keep all the notation in the proof of Lemma 5.8. For any finite sequence $f = (f_j)_{j\geq 0}$ (viewed as a column matrix), the adjoint of k is defined by

$$\mathsf{k}^*(f)(s) = \int_{\mathbb{R}^d} \sum_j \widetilde{\mathsf{k}}_j(s-t)f_j(t)dt,$$

where $\widetilde{\mathsf{k}}(s) = \mathsf{k}(-s)^*$ (so it is a row matrix). Put $\widetilde{K}(s) = \widetilde{\mathsf{k}}(s) \otimes 1_{\mathcal{M}}$. In this case, $\|\widetilde{K}(f)\|_{\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})} = \|\widetilde{K}(f)\|_{\mathrm{bmo}^c(\mathbb{R}^d,B(\ell_2)\otimes\mathcal{M})}$. Then we apply the estimates used in the previous lemma by replacing K with $\widetilde{K}$. It follows that k' is bounded from $L_\infty(\mathcal{N}; \ell_2^c)$ into $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$, the desired assertion is proved. □

In the setting where ℓ_2 is replaced by $H = \oplus_{j=0}^{\infty} H_j$ with $H_j = L_2(B(0, 2^{-j}), 2^{jd}dt)$, the counterpart of Corollary 5.10 is the following:

Corollary 5.11. *Let $\phi = (\phi_j)_{j\geq 0}$ be a sequence of continuous functions on $\mathbb{R}^d$ satisfying (5.7). Then for $1 \leq p \leq 2$ and any $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$,*

$$\|(\sum_{j\geq 0} 2^{dj} \int_{B(0,2^{-j})} |\check{\phi}_j * f(\cdot + t)|^2 dt)^{\frac{1}{2}}\|_p \lesssim \|\phi\|_{2,\sigma} \|f\|_{\mathrm{h}_p^c}.$$

The relevant constants depend only on φ, σ and d.

Proof. The proof of this corollary is similar to Corollary 5.10; let us point out the necessary change. Consider the H-valued Calderón-Zygmund operator k defined on $L_p(\mathcal{N})$ given by

$$\mathsf{k}(f)_j(\cdot + t) = \check{\phi}_j * f(\cdot + t).$$

The lemma is then reduced to showing that k is bounded from $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ to $L_p(\mathcal{N}; H^c)$ for $1 \leq p < 2$. Since each H_j is a normalized Hilbert space, such that the constant function 1 has Hilbert norm one, the kernel estimates of our k here are the same as the ones in Lemma 5.8. So we can repeat the proof in Lemma 5.8 and Corollary 5.10. The desired assertion follows. $\square$

Chapter 6

General characterizations of $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$

Applying the operator-valued Calderón-Zygmund theory developed in the last chapter, we will show in this chapter that the Poisson kernel in the square functions which are used to define $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ can be replaced by any reasonable test functions. As an application, we are able to compare the operator-valued local Hardy spaces $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ defined in this thesis with the operator-valued Hardy spaces $\mathcal{H}_p^c(\mathbb{R}^d, \mathcal{M})$ in [42].

6.1 General characterizations

Consider a Schwartz function Φ on $\mathbb{R}^d$ of vanishing mean. We set $\Phi_\varepsilon(s) = \varepsilon^{-d}\Phi(\frac{s}{\varepsilon})$ for $\varepsilon > 0$. We will assume that Φ is nondegenerate in the sense of (1.12). Then there exists a Schwartz function Ψ of vanishing mean such that

$$\int_0^\infty \widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{d\varepsilon}{\varepsilon} = 1, \forall \xi \in \mathbb{R}^d \setminus \{0\}. \tag{6.1}$$

This is a well-known elementary fact (ef. e.g. [61, p. 186]).

We will use multi-index notation. For $m = (m_1, \cdots, m_d) \in \mathbb{N}_0^d$ and $s = (s_1, \cdots, s_d) \in \mathbb{R}^d$, we set $s^m = s_1^{m_1} \cdots s_d^{m_d}$. Let $|m|_1 = m_1 + \cdots + m_d$ and $D^m = \frac{\partial^{m_1}}{\partial s_1^{m_1}} \cdots \frac{\partial^{m_d}}{\partial s_d^{m_d}}$.

Lemma 6.1. *$\int_0^1 \widehat{\Phi}(\varepsilon\cdot)\overline{\widehat{\Psi}(\varepsilon\cdot)}\frac{d\varepsilon}{\varepsilon}$ is an infinitely differentiable function on $\mathbb{R}^d$ if we define its value at the origin as* 0.

Proof. To prove the assertion, it suffices to show that $\int_0^1 \widehat{\Phi}(\varepsilon\cdot)\overline{\widehat{\Psi}(\varepsilon\cdot)}\frac{d\varepsilon}{\varepsilon}$ is infinitely differentiable at the origin. Given $\varepsilon \in (0, 1]$, we expand $\widehat{\Phi}(\varepsilon\cdot)$ in the Taylor series at the origin

$$\widehat{\Phi}(\varepsilon\xi) = \sum_{|\gamma|_1 \leq N} D^\gamma \widehat{\Phi}(0)\frac{\varepsilon^{|\gamma|_1}\xi^\gamma}{\gamma!} + \sum_{|\gamma|_1 = N+1} R_\gamma(\varepsilon\xi)\,\xi^\gamma,$$

with the remainder of integral form be

$$R_\gamma(\varepsilon\xi) = \frac{(N+1)\varepsilon^{N+1}}{\gamma!}\int_0^1 (1-\theta)^N D^\gamma \widehat{\Phi}(\theta\varepsilon\xi)d\theta\,.$$

Since $\widehat{\Phi}(0)=0$, the above Taylor series implies that

$$\widehat{\Phi}(\varepsilon\xi)=\sum_{1\leq|\gamma|_1\leq N} D^\gamma\widehat{\Phi}(0)\frac{\varepsilon^{|\gamma|_1}\xi^\gamma}{\gamma!}+\sum_{|\gamma|_1=N+1} R_\gamma(\varepsilon\xi)\,\xi^\gamma.$$

Similarly, we have

$$\widehat{\Psi}(\varepsilon\xi)=\sum_{1\leq|\beta|_1\leq N} D^\beta\widehat{\Psi}(0)\frac{\varepsilon^{|\beta|_1}\xi^\beta}{\beta!}+\sum_{|\beta|_1=N+1} R'_\beta(\varepsilon\xi)\,\xi^\beta,$$

where R'_β is the integral form remainder of $\widehat{\Psi}$. Thus, both $\widehat{\Phi}(\varepsilon\xi)$ and $\widehat{\Psi}(\varepsilon\xi)$ contain only powers of ε with order at least 1. Therefore, the integral $\int_0^1\widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{d\varepsilon}{\varepsilon}$ (and the integrals of arbitrary order derivatives of $\widehat{\Phi}(\varepsilon\xi)$ and $\widehat{\Psi}(\varepsilon\xi)$) converge uniformly for $\xi\in\mathbb{R}^d$ close to the origin. We then obtain that $\int_0^1\widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{d\varepsilon}{\varepsilon}$ is infinitely differentiable at the origin $\xi=0$. □

It follows immediately from Lemma 6.1 that $\int_1^\infty\widehat{\Phi}(\varepsilon\cdot)\overline{\widehat{\Psi}(\varepsilon\cdot)}\frac{d\varepsilon}{\varepsilon}$ is a Schwartz function if we define its value at the origin by 1. Then we can find two other functions ϕ, ψ such that $\widehat{\phi},\widehat{\psi}\in H_2^\sigma(\mathbb{R}^d)$, $\widehat{\phi}(0)>0,\widehat{\psi}(0)>0$ and

$$\widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)}=1-\int_0^1\widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{d\varepsilon}{\varepsilon},\quad\forall\xi\in\mathbb{R}^d.\tag{6.2}$$

Indeed, for $\beta>0$ large enough, the function $(1+|\cdot|^2)^{-\beta}$ belongs to $H_2^\sigma(\mathbb{R}^d)$. On the other hand, if $F\in\mathcal{S}(\mathbb{R}^d)$, the function $(1+|\cdot|^2)^\beta F$ is still in $H_2^\sigma(\mathbb{R}^d)$. Thus we obtain (6.2). The main target in this section is to use the test functions in (6.2) to characterize the space $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$.

For any $f\in L_1(\mathcal{M};\mathrm{R}_d^c)+L_\infty(\mathcal{M};\mathrm{R}_d^c)$, we define the local versions of the conic and radial square functions of f associated to Φ by

$$s_\Phi^c(f)(s)=\Big(\iint_{\widetilde{\Gamma}}|\Phi_\varepsilon*f(s+t)|^2\frac{dtd\varepsilon}{\varepsilon^{d+1}}\Big)^{\frac12},\ s\in\mathbb{R}^d,$$
$$g_\Phi^c(f)(s)=\Big(\int_0^1|\Phi_\varepsilon*f(s)|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12},\ s\in\mathbb{R}^d.$$

Fix the four test functions Φ,Ψ,ϕ,ψ as in (6.2). The following is one of our main results in this section.

Theorem 6.2. *Let $1\leq p<\infty$ and ϕ, Φ be as above. For any $f\in L_1(\mathcal{M};\mathrm{R}_d^c)+L_\infty(\mathcal{M};\mathrm{R}_d^c)$, $f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ if and only if $s_\Phi^c(f)\in L_p(\mathcal{N})$ and $\phi*f\in L_p(\mathcal{N})$ if and only if $g_\Phi^c(f)\in L_p(\mathcal{N})$ and $\phi*f\in L_p(\mathcal{N})$. If this is the case, then*

$$\|f\|_{\mathrm{h}_p^c}\approx\|g_\Phi^c(f)\|_p+\|\phi*f\|_p\approx\|s_\Phi^c(f)\|_p+\|\phi*f\|_p\tag{6.3}$$

with the relevant constants depending only on d,p,Φ and ϕ.

First, we deal with the case $1\leq p\leq 2$. We apply Corollary 5.3 to the square function operators s_Φ^c and g_Φ^c. Let $H=L_2((0,1),\frac{d\varepsilon}{\varepsilon})$. Define the kernel $\mathsf{k}:\mathbb{R}^d\to H$ by $\mathsf{k}(s)=\Phi_\cdot(s)$ with $\Phi_\cdot(s):\varepsilon\mapsto\Phi_\varepsilon(s)$. Then we can check that

$$\sup_{\xi\in\mathbb{R}^d}\|\widehat{\Phi}(\varepsilon\xi)\|_H<\infty,\quad\|\Phi_\varepsilon(s)\|_H\lesssim\frac{1}{|s|^{d+1}},\ \forall s\in\mathbb{R}^d\setminus\{0\}$$

and that

$$\|\nabla\Phi_\varepsilon(s)\|_H \lesssim \frac{1}{|s|^{d+1}}, \ \forall s \in \mathbb{R}^d \setminus \{0\}.$$

Thus, k satisfies the assumption of Corollary 5.3. By Remark 5.4, we have, for any $1 \leq p \leq 2$,

$$\|\Phi. * f\|_{L_p(\mathcal{N};H^c)} = \|g_\Phi^c(f)\|_p \lesssim \|f\|_{\mathrm{h}_p^c}.$$

The treatment of s_Φ^c is similar. In this case, we take the Hilbert space $H = L_2(\widetilde{\Gamma}, \frac{dtd\varepsilon}{\varepsilon^{d+1}})$. On the other hand, $\widehat{\phi} \in H_2^\sigma(\mathbb{R}^d)$ implies $\phi \in L_1(\mathbb{R}^d)$, then $\|\phi * f\|_{L_p(\mathcal{N})} \lesssim \|f\|_{L_p(\mathcal{N})} \lesssim \|f\|_{\mathrm{h}_p^c}$. Thus, combining the above estimates, we obtain one direction of (6.3) for the case $1 \leq p \leq 2$:

$$\|g_\Phi^c(f)\|_p + \|\phi * f\|_p \lesssim \|f\|_{\mathrm{h}_p^c}, \tag{6.4}$$

$$\|s_\Phi^c(f)\|_p + \|\phi * f\|_p \lesssim \|f\|_{\mathrm{h}_p^c}. \tag{6.5}$$

The proof of the other direction of (6.3) is long and technical. We follow the duality method used in [70], which involves unavoidably bmo spaces. Thus we need a Carleson measure characterization of bmo_q^c by general test functions, which is analogous to Lemma 3.3, in the more general setting.

Lemma 6.3. *Let $2 < q < \infty$, $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$ and $d\lambda_g = |\Psi_\varepsilon * g(s)|^2 \frac{dsd\varepsilon}{\varepsilon}$. Then $d\lambda_g$ is an $\mathcal{M}$-valued q-Carleson measure on the strip $\mathbb{R}^d \times (0,1)$. Furthermore, let ψ be any function on $\mathbb{R}^d$ such that*

$$\widehat{\psi} \in H_2^\sigma(\mathbb{R}^d) \quad \text{with} \quad \sigma > \frac{d}{2}. \tag{6.6}$$

We have

$$\max\Big\{\|\sup_{\substack{s\in Q\subset\mathbb{R}^d\\|Q|<1}}{}^{+} \frac{1}{|Q|}\int_{T(Q)} d\lambda_g\|_{\frac{q}{2}}^{\frac{1}{2}}, \, \|\psi * g\|_q\Big\} \lesssim \|g\|_{\mathrm{bmo}_q^c}.$$

Proof. Replacing $\varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon$ by Ψ_ε in Lemma 3.3, we have

$$\Big\|\sup_{\substack{s\in Q\subset\mathbb{R}^d\\|Q|<1}}{}^{+} \frac{1}{|Q|}\int_{T(Q)} d\lambda_g\Big\|_{\frac{q}{2}}^{\frac{1}{2}} \lesssim \|g\|_{\mathrm{bmo}_q^c}.$$

We obtain the desired conclusion. □

Lemma 6.4. *Let $1 \leq p \leq 2$ and q be its conjugate index. For $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) \cap L_2(\mathcal{N})$ and $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$,*

$$|\tau\int_{\mathbb{R}^d} f(s)g^*(s)ds| \lesssim (\|s_\Phi^c(f)\|_p + \|\phi * f\|_p)\|g\|_{\mathrm{bmo}_q^c}.$$

Proof. The proof of this Lemma is very similar to that of Theorem 3.10, we will just point out the necessary modifications to avoid duplication. We need two auxiliary square functions associated with Φ. For $s \in \mathbb{R}^d$, $\varepsilon \in [0,1]$, we define

$$s_\Phi^c(f)(s,\varepsilon) = \Big(\int_\varepsilon^1\int_{B(s,r-\frac{\varepsilon}{2})} |\Phi_r * f(t)|^2\frac{dtdr}{r^{d+1}}\Big)^{\frac{1}{2}}, \tag{6.7}$$

$$\overline{s}_\Phi^c(f)(s,\varepsilon) = \Big(\int_\varepsilon^1\int_{B(s,\frac{r}{2})} |\Phi_r * f(t)|^2\frac{dtdr}{r^{d+1}}\Big)^{\frac{1}{2}}. \tag{6.8}$$

By (6.2), we have

$$\begin{aligned}
&\tau\int_{\mathbb{R}^d} f(s)g^*(s)ds\\
&=\tau\int_{\mathbb{R}^d}\int_0^1 \Phi_\varepsilon * f(s)(\Psi_\varepsilon * g(s))^*\frac{dsd\varepsilon}{\varepsilon}+\tau\int_{\mathbb{R}^d}\phi * f(s)(\psi * g(s))^*ds\\
&=\frac{2^d}{c_d}\tau\int_{\mathbb{R}^d}\int_0^1\int_{B(s,\frac{\varepsilon}{2})}\Phi_\varepsilon * f(t)(\Psi_\varepsilon * g(t))^*\frac{dtd\varepsilon}{\varepsilon^{d+1}}ds+\tau\int_{\mathbb{R}^d}\phi * f(s)(\psi * g(s))^*ds\\
&=\frac{2^d}{c_d}\tau\int_{\mathbb{R}^d}\int_0^1\int_{B(s,\frac{\varepsilon}{2})}\Phi_\varepsilon * f(t)s_\Phi^c(f)(s,\varepsilon)^{\frac{p-2}{2}}s_\Phi^c(f)(s,\varepsilon)^{\frac{2-p}{2}}(\Psi_\varepsilon * g(t))^*\frac{dtd\varepsilon}{\varepsilon^{d+1}}ds\\
&\quad+\tau\int_{\mathbb{R}^d}\phi * f(s)(\psi * g(s))^*ds\\
&\overset{\mathrm{def}}{=}\mathrm{I}+\mathrm{II}.
\end{aligned}$$

Then by the Cauchy-Schwarz inequality,

$$\begin{aligned}
|\mathrm{I}|^2&\lesssim\tau\int_{\mathbb{R}^d}\int_0^1\Big(\int_{B(s,\frac{\varepsilon}{2})}|\Phi_\varepsilon * f(t)|^2\frac{dt}{\varepsilon^{d+1}}\Big)\overline{s}_\Phi^c(f)(s,\varepsilon)^{p-2}d\varepsilon ds\\
&\quad\cdot\tau\int_{\mathbb{R}^d}\int_0^1\Big(\int_{B(s,\frac{\varepsilon}{2})}|\Psi_\varepsilon * g(t)|^2\frac{dt}{\varepsilon^{d+1}}\Big)s_\Phi^c(f)(s,\varepsilon)^{2-p}d\varepsilon ds\\
&\overset{\mathrm{def}}{=}A\cdot B.
\end{aligned}$$

Replacing $\varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)$ and $\varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)$ in the proof of Theorem 3.10 by $\Phi_\varepsilon * f$ and $\Psi_\varepsilon * g$ respectively and applying Lemma 6.3, we get the estimates for the terms A and B that

$$A\lesssim\|s_\Phi^c(f)\|_p^p\quad\text{and}\quad B\lesssim\|g\|^2_{\mathrm{bmo}_q^c}\|s_\Phi^c(f)\|_p^{2-p}.$$

The term II is easy to deal with. By the Hölder inequality and Lemma 6.3 again, we get

$$\Big|\tau\int_{\mathbb{R}^d}\phi * f(s)(\psi * g(s))^*ds\Big|\le\|\phi * f\|_p\|\psi * g\|_q\lesssim\|\phi * f\|_p\|g\|_{\mathrm{bmo}_q^c}.$$

Combining the estimates for A, B and II, we finally get the desired inequality. □

We will also need the radial version of Lemma 6.4. To this end, we need to majorize the radial square function by the conic one. When we consider the Poisson kernel, this result follows from the harmonicity of the Poisson integral (see Lemma 3.12). However, in the general case, the harmonicity is no longer available. To overcome this difficulty, a more sophisticated inequality has been developped in [70] to compare non-local radial and conic functions. Observe that the result given in [70, Lemma 4.3] is a pointwise one, which also works for the local version of square functions if we consider integration over the interval $0<\varepsilon<1$. The following lemma is an obvious consequence of [70, Lemma 4.3].

Lemma 6.5. *Let $f\in L_1(\mathcal{M};\mathrm{R}_d^c)+L_\infty(\mathcal{M};\mathrm{R}_d^c)$. Then*

$$g_\Phi^c(f)(s)^2\lesssim\sum_{|m|_1\le d}s_{D^m\Phi}^c(f)(s)^2,\ \forall s\in\mathbb{R}^d.$$

Lemma 6.6. *Let $1\le p\le 2$. For $f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})\cap L_2(\mathcal{N})$ and $g\in\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$,*

$$\Big|\tau\int_{\mathbb{R}^d}f(s)g^*(s)ds\Big|\lesssim(\|g_\Phi^c(f)\|_p+\|\phi * f\|_p)^{\frac{p}{2}}\|f\|_{\mathrm{h}_p^c}^{1-\frac{p}{2}}\|g\|_{\mathrm{bmo}_q^c}.$$

Proof. This proof is similar to that of Lemma 6.4 and we keep the notation there. Let $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ with compact support (relative to the variable of $\mathbb{R}^d$). We assume that f is sufficiently nice so that all calculations below are legitimate. Now we need the radial version of $s_\Phi^c(f)(s,\varepsilon)$,

$$g_\Phi^c(f)(s,\varepsilon) = \Big(\int_\varepsilon^1 |\Phi_r * f(s)|^2 \frac{dr}{r}\Big)^{\frac{1}{2}}$$

for $s \in \mathbb{R}^d$ and $0 \le \varepsilon \le 1$. By approximation, we can assume that $g_\Phi^c(f)(s,\varepsilon)$ is invertible for every $(s,\varepsilon) \in S$. By (6.1), (6.2), (1.7) and the Fubini theorem, we have

$$\begin{aligned}
&\Big|\tau \int_{\mathbb{R}^d} f(s) g^*(s) ds\Big|^2 \\
&\lesssim \tau \int_{\mathbb{R}^d}\int_0^1 |\Phi_\varepsilon * f(s)|^2 g_\Phi^c(f)(s,\varepsilon)^{p-2} \frac{d\varepsilon ds}{\varepsilon} \cdot \tau \int_{\mathbb{R}^d}\int_0^1 |\Psi_\varepsilon * g(s)|^2 g_\Phi^c(f)(s,\varepsilon)^{2-p} \frac{d\varepsilon ds}{\varepsilon} \\
&\quad + \Big|\tau \int_{\mathbb{R}^d} \phi * f(s) (\psi * g(s))^* ds\Big|^2 \\
&\overset{\text{def}}{=} A'B' + \mathrm{II}'.
\end{aligned}$$

II' is treated exactly in the same way as before,

$$\mathrm{II}' \lesssim \|\phi * f\|_p^2 \|\psi * g\|_q^2 \lesssim \|\phi * f\|_p^p \|f\|_{\mathrm{h}_p^c}^{2-p} \|g\|_{\mathrm{bmo}_q^c}^2.$$

A' is also estimated similarly as in Lemma 6.4, we have $A' \lesssim \|g_\Phi^c(f)\|_p^p$.

To estimate B', we notice that the proof of [70, Lemma 1.3] also gives

$$g_\Phi^c(f)(s,\varepsilon)^2 \lesssim \sum_{|m|_1 \le d} s_{D^m\Phi}^c(f)(s,\varepsilon)^2,$$

where $s_{D^m\Phi}^c(f)(s,\varepsilon)$ is defined by (6.7) with $D^m\Phi$ instead of Φ. Then by the above inequality, Lemma 6.3 and inequality (6.5) with $D^m\Phi$ instead of Φ, we obtain

$$\begin{aligned}
B' &\lesssim \sum_{|m|_1 \le d} \tau \int_{\mathbb{R}^d}\int_0^1 |\Psi_\varepsilon * g(s)|^2 s_{D^m\Phi}^c(f)(s,\varepsilon)^{2-p} \frac{d\varepsilon ds}{\varepsilon} \\
&\lesssim \sum_{|m|_1 \le d} \|g\|_{\mathrm{bmo}_q^c}^2 \|s_{D^m\Phi}^c(f)\|_p^{2-p} \\
&\lesssim \|g\|_{\mathrm{bmo}_q^c}^2 \|f\|_{\mathrm{h}_p^c}^{2-p}.
\end{aligned}$$

Therefore,

$$|\tau \int_{\mathbb{R}^d} f(s) g^*(s) ds|^2 \lesssim (\|g_\Phi^c(f)\|_p + \|\phi * f\|_{L_p})^p \|f\|_{\mathrm{h}_p^c}^{2-p} \|g\|_{\mathrm{bmo}_q^c}^2,$$

which completes the proof. □

Proof of Theorem 6.2. From Lemmas 6.4, 6.6 and Theorem 3.10, we conclude that for $1 \le p \le 2$,

$$\|f\|_{\mathrm{h}_p^c} \lesssim \|s_\Phi^c(f)\|_p + \|\phi * f\|_p,$$
$$\|f\|_{\mathrm{h}_p^c} \lesssim \|g_\Phi^c(f)\|_p + \|\phi * f\|_p.$$

Therefore, combined with (6.4) and (6.5), we have proved the assertion for the case $1 \le p \le 2$.

Now we turn to the case $2<p<\infty$. By Theorem 3.18, we can choose $g\in \mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})$ (with q the conjugate index of p) with norm one such that

$$\|f\|_{\mathrm{h}_p^c}\approx \tau\int_{\mathbb{R}^d} f(s)g^*(s)ds=\tau\int_{\mathbb{R}^d}\int_0^1 \Phi_\varepsilon * f(s)\cdot(\Psi_\varepsilon * g(s))^*\frac{dsd\varepsilon}{\varepsilon}+\tau\int_{\mathbb{R}^d}\phi * f(s)(\psi * g(s))^* ds.$$

Then by the Hölder inequality and (6.4) (applied to g,Ψ and q),

$$\begin{aligned}\|f\|_{\mathrm{h}_p^c} &\lesssim \|g_\Phi^c(f)\|_p\|g_\Psi^c(g)\|_q+\|\phi * f\|_p\|\psi * g\|_q\\ &\lesssim (\|g_\Phi^c(f)\|_p+\|\phi * f\|_p)\|g\|_{\mathrm{h}_q^c}=\|g_\Phi^c(f)\|_p+\|\phi * f\|_p.\end{aligned}$$

Similarly,

$$\|f\|_{\mathrm{h}_p^c}\lesssim \|s_\Phi^c(f)\|_p+\|\phi * f\|_p.$$

It remains to show the two reverse inequalities for $2<p<\infty$. First, we define a map $E_{\Phi,\phi}$ for sufficiently nice $h=(h',h'')\in L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})$ by

$$E_{\Phi,\phi}(h)(u)=\int_{\mathbb{R}^d}\Big[\iint_{\widetilde{\Gamma}} h'(s,t,\varepsilon)\Phi_\varepsilon(s+t-u)\frac{dtd\varepsilon}{\varepsilon^{d+1}}+h''(s)\phi(s-u)\Big]ds\,.$$

This map can be seen as an analogue of the map E in Theorem 3.8. By modelling on the proof of Theorem 3.8 and Theorem 3.17, we can check that $E_{\Phi,\phi}$ is also a bounded map from $L_q(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_q L_q(\mathcal{N})$ to $\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})$ for $1<q<2$. Now, we estimate the sum $\|s_\Phi^c(f)\|_p+\|\phi * f\|_p$. Note that there exists a function $h_0=(h_0',h_0'')\in L_q(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_q L_q(\mathcal{N})$ with norm one such that

$$\begin{aligned}\|s_\Phi^c(f)\|_p+\|\phi * f\|_p&=\Big|\tau\int_{\mathbb{R}^d}\int_{\widetilde{\Gamma}}\Phi_\varepsilon * f(s+t)h_0'(s,t,\varepsilon)^*\frac{dtd\varepsilon}{\varepsilon^{d+1}}ds+\tau\int_{\mathbb{R}^d}\phi * f(s)h_0''(s)^*ds\Big|\\ &=\Big|\int_{\mathbb{R}^d} f(u)E_{\Phi,\phi}(h_0^*)(u)du\Big|\\ &\lesssim \|E_{\Phi,\phi}(h_0)\|_{\mathrm{h}_q^c}\|f\|_{\mathrm{h}_p^c}\lesssim \|f\|_{\mathrm{h}_p^c}.\end{aligned}$$

On the other hand, Lemma 6.5 gives the inequality

$$\|g_\Phi^c(f)\|_p+\|\phi * f\|_p\lesssim \|f\|_{\mathrm{h}_p^c},$$

which completes the proof. □

6.2 Discrete characterizations

The square functions s_Φ^c and g_Φ^c can be discretized as follows:

$$\begin{aligned}g_\Phi^{c,D}(f)(s)&=\Big(\sum_{j\geq 1}|\Phi_j * f(s)|^2\Big)^{\frac12},\\ s_\Phi^{c,D}(f)(s)&=\Big(\sum_{j\geq 1}2^{dj}\int_{B(s,2^{-j})}|\Phi_j * f(t)|^2dt\Big)^{\frac12}.\end{aligned}$$

Here Φ_j is the inverse Fourier transform of $\Phi(2^{-j}\cdot)$. This time, to get a resolvent of the unit on $\mathbb{R}^d$, we need to assume that Φ satisfies (1.16). Then adapting the proof of [61, Lemma V.6] , we can find a Schwartz function Ψ of vanishing mean such that

$$\sum_{j=-\infty}^{+\infty}\widehat{\Phi}(2^{-j}\xi)\,\overline{\widehat{\Psi}(2^{-j}\xi)}=1,\quad \forall\xi\in\mathbb{R}^d\setminus\{0\}. \tag{6.9}$$

There exist two other functions ϕ and ψ such that $\widehat{\varphi}, \widehat{\psi} \in H_2^\sigma(\mathbb{R}^d)$, $\widehat{\phi}(0) > 0, \widehat{\psi}(0) > 0$ and

$$\sum_{j=1}^{\infty} \widehat{\Phi}(2^{-j}\xi)\, \overline{\widehat{\Psi}(2^{-j}\xi)} + \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} = 1, \quad \forall \xi \in \mathbb{R}^d. \tag{6.10}$$

The following discrete version of Theorem 6.2 will play a crucial role in the study of operator-valued Triebel-Lizorkin spaces on $\mathbb{R}^d$ in chapter 8. Now we fix the pairs (Φ, Ψ) and (ϕ, ψ) satisfying (6.9) and (6.10).

Theorem 6.7. *Let ϕ and Φ be test functions as in* (6.10) *and $1 \le p < \infty$. Then for any $f \in L_1(\mathcal{M}; \mathrm{R}_d^c) + L_\infty(\mathcal{M}; \mathrm{R}_d^c)$, $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ if and only if $s_\Phi^{c,D}(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$ if and only if $g_\Phi^{c,D}(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$. Moreover,*

$$\|s_\Phi^{c,D}(f)\|_p + \|\phi * f\|_p \approx \|g_\Phi^{c,D}(f)\|_p + \|\phi * f\|_p \approx \|f\|_{\mathrm{h}_p^c}$$

with the relevant constants depending only on d, p, Φ and ϕ.

The following paragraphs are devoted to the proof of Theorem 6.7, which is similar to that of Theorem 6.2. We will just indicate the necessary modifications. We first prove the discrete counterparts of Lemmas 6.4 and 6.6.

Lemma 6.8. *Let $1 \le p < 2$ and q be the conjugate index of p. For any $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) \cap L_2(\mathcal{N})$ and $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$,*

$$\Big|\tau \int_{\mathbb{R}^d} f(s) g^*(s) ds\Big| \lesssim \big(\|s_\Phi^{c,D}(f)\|_p + \|\phi * f\|_p\big)\, \|g\|_{\mathrm{bmo}_q^c}.$$

Proof. First, note that by (6.10), we have

$$\tau \int_{\mathbb{R}^d} f(s) g^*(s) ds = \tau \int_{\mathbb{R}^d} \sum_{j\ge 1} \Phi_j * f(s)\big(\Psi_j * g(s)\big)^*\, ds + \tau \int_{\mathbb{R}^d} \phi * f(s)\big(\psi * g(s)\big)^* ds.$$

The second term on the right hand side of the above formula is exactly the same as the corresponding term II in the proof of Lemma 6.4. Now we need the discrete versions of s_Φ^c and $\overline{s}_\Phi^c$: For $j \ge 1$, $s \in \mathbb{R}^d$, let

$$\begin{aligned} s_\Phi^{c,D}(f)(s,j) &= \Big(\sum_{1\le k\le j} 2^{dk} \int_{B(s,2^{-k}-2^{-j-1})} |\Phi_j * f(t)|^2 dt\Big)^{\frac{1}{2}} \\ \overline{s}_\Phi^{c,D}(f)(s,j) &= \Big(\sum_{1\le k\le j} 2^{dk} \int_{B(s,2^{-k-1})} |\Phi_j * f(t)|^2 dt\Big)^{\frac{1}{2}}. \end{aligned}$$

Denote $s_\Phi^{c,D}(f)(s,j)$ and $\overline{s}_\Phi^{c,D}(f)(s,j)$ simply by $s(s,j)$ and $\overline{s}(s,j)$, respectively. By approximation, we may assume that $s(s,j)$ and $\overline{s}(s,j)$ are invertible for every $s \in \mathbb{R}^d$ and $j \ge 1$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} &\Big|\tau \int_{\mathbb{R}^d} \sum_{j\ge 1} \Phi_j * f(s)\big(\Psi_j * g(s)\big)^*\, ds\Big|^2 \\ &= \Big|\frac{2^d}{c_d}\, \tau \int_{\mathbb{R}^d} \sum_j 2^{dj} \int_{B(s,2^{-j-1})} \Phi_j * f(t)\big(\Psi_j * g(t)\big)^*\, dt\, ds\Big|^2 \\ &\lesssim \tau \int_{\mathbb{R}^d} \sum_j s(s,j)^{p-2} \Big(2^{dj} \int_{B(s,2^{-j-1}))} |\Phi_j * f(t)|^2\, dt\Big) ds \\ &\quad \cdot \tau \int_{\mathbb{R}^d} \sum_j s(s,j)^{2-p} \Big(2^{dj} \int_{B(s,2^{-j-1})} |\Psi_j * g(t)|^2\, dt\Big) ds \\ &\overset{\text{def}}{=} \mathrm{A} \cdot \mathrm{B}. \end{aligned}$$

The term A is less easy to estimate than the corresponding term A in the proof of Lemma 6.4. To deal with it we simply set $\overline{s}_j = \overline{s}(s,j)$ and $\overline{s} = \overline{s}(s,+\infty) \leq s^{c,D}(f)(s)$. Then

$$\begin{aligned} \mathrm{A} &= \tau \int_{\mathbb{R}^d} \sum_{j\geq 1} s_j^{p-2}(\overline{s}_j^2 - \overline{s}_{j-1}^2) ds \\ &\leq \tau \int_{\mathbb{R}^d} \sum_{j} \overline{s}_j^{p-2}(\overline{s}_j^2 - \overline{s}_{j-1}^2) ds \\ &= \tau \int_{\mathbb{R}^d} \sum_{j} (\overline{s}_j - \overline{s}_{j-1}) ds + \tau \int_{\mathbb{R}^d} \sum_{j} \overline{s}_j^{p-2} \overline{s}_{j-1}(\overline{s}_j - \overline{s}_{j-1}) ds, \end{aligned}$$

where $\overline{s}_0 = 0$. Since $1 \leq p < 2$, $\overline{s}_j^{p-1} \leq s^{p-1}$, we have

$$\tau \int_{\mathbb{R}^d} \sum_{j} \overline{s}_j^{p-1}(\overline{s}_j - \overline{s}_{j-1}) ds \lesssim \tau \int_{\mathbb{R}^d} \overline{s}^p ds \leq \|s^{c,D}(f)\|_p^p.$$

On the other hand,

$$\tau \int_{\mathbb{R}^d} \sum_{j} \overline{s}_j^{p-2} \overline{s}_{j-1}(\overline{s}_j - \overline{s}_{j-1}) ds = \tau \int_{\mathbb{R}^d} \sum_{j} \overline{s}^{\frac{1-p}{2}} \overline{s}_j^{p-2} \overline{s}_{j-1} \overline{s}^{\frac{1-p}{2}} \overline{s}^{\frac{p-1}{2}} (\overline{s}_j - \overline{s}_{j-1}) \overline{s}^{\frac{p-1}{2}} ds,$$

since $\overline{s}_j \geq \overline{s}_{j-1}$ for any $j \geq 1$, we have $\overline{s}^{\frac{1-p}{2}} \overline{s}_j^{p-2} \overline{s}_{j-1} \overline{s}^{\frac{1-p}{2}} \leq 1$. Thus, by the Hölder inequality,

$$\tau \int_{\mathbb{R}^d} \sum_{j} \overline{s}_j^{p-2} \overline{s}_{j-1}(\overline{s}_j - \overline{s}_{j-1}) ds \leq \tau \int_{\mathbb{R}^d} \sum_{j} \overline{s}^{\frac{p-1}{2}} (\overline{s}_j - \overline{s}_{j-1}) \overline{s}^{\frac{p-1}{2}} ds = \tau \int_{\mathbb{R}^d} \overline{s}^p ds \leq \|s_\Phi^{c,D}(f)\|_p^p.$$

Combining the preceding inequalities, we get the desired estimate of A:

$$\mathrm{A} \leq 2\|s_\Phi^{c,D}(f)\|_p^p.$$

The estimate of the term B is, however, almost identical to that of B in the proof of Lemma 6.4. There are only two minor differences. The first one concerns the square function $\mathbb{S}^c(f)(s,j)$ in (3.9): it is now replaced by

$$\mathbb{S}^c(f)(s,j) = \Big(\sum_{1\leq k\leq j} 2^{dk} \int_{B(c_{m,j},2^{-k})} |\Phi_j * f(t)|^2 \, dt \Big)^{\frac{1}{2}} \quad \text{if} \quad s \in Q_{m,j}.$$

Then we have $s(s,j) \leq \mathbb{S}^c(f)(s,j)$. The second difference is about the Carleson characterization of bmo_q^c; we now use its discrete analogue. Namely, for $g \in \mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$, define

$$d\lambda_D(g) = \sum_{j\geq 1} |\Psi_j * g(s)|^2 ds \times d\delta_{2^{-j}}(\varepsilon),$$

where $\delta_{2^{-j}}(\varepsilon)$ is the unit Dirac mass at the point 2^{-j}, considered as a measure on $(0,1)$. Then $d\lambda_D(g)$ is a Carleson measure on the strip S and

$$\Big\| {\sup_{\substack{s\in Q\subset\mathbb{R}^d \\ |Q|<1}}}^{+} \frac{1}{|Q|} \int_{T(Q)} \sum_{j\geq 1} |\Psi_j * g(s)|^2 ds \times d\delta_{2^{-j}}(\varepsilon) \Big\|_{\frac{q}{2}} \lesssim \|g\|_{\mathrm{bmo}_q^c}^2 .$$

The proof of this inequality is the same as that of Lemma 6.3. Apart from these two differences, the remainder of the argument is identical to that in the proof of Lemma 6.4. □

Lemma 6.9. *Let* $1 \le p < 2$ *and* $f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) \cap L_2(\mathcal{N})$, $g \in \mathrm{bmo}_q^c(\mathbb{R}^d, \mathcal{M})$. *Then*

$$\Big|\tau \int_{\mathbb{R}^d} f(s) g^*(s) ds\Big| \lesssim \Big(\|g_\Phi^{c,D}(f)\|_p + \|\phi * f\|_p\Big)^{\frac{p}{2}} \|f\|_{\mathrm{h}_p^c}^{1-\frac{p}{2}} \|g\|_{\mathrm{bmo}_q^c}.$$

Proof. We use the truncated version of $g_\Phi^{c,D}(f)$:

$$g_\Phi^{c,D}(f)(s,j) = \Big(\sum_{k \le j} |\Phi_k * f(s)|^2\Big)^{\frac{1}{2}}.$$

The proof of [70, Lemma 4.3] is easily adapted to the present setting to ensure

$$g_\Phi^{c,D}(f)(s,j)^2 \lesssim \sum_{|m|_1 \le d} s_{D^m\Phi}^{c,D}(f)(s,j)^2.$$

Then

$$\Big|\tau \int_{\mathbb{R}^d} f(s) g^*(s) ds\Big|^2 \le \mathrm{I}' \cdot \mathrm{II}' + \Big|\tau \int \phi * f(s) (\psi * g(s))^* ds\Big|,$$

where

$$\mathrm{I}' = \tau \int_{\mathbb{R}^d} \sum_j g_\Phi^{c,D}(f)(s,j)^{p-2} |\Phi_j * f(s)|^2 ds,$$

$$\mathrm{II}' = \tau \int_{\mathbb{R}^d} \sum_j g_\Phi^{c,D}(f)(s,j)^{2-p} |\Psi_j * g(s)|^2 ds.$$

Both terms I' and II' are estimated exactly as before, so we have

$$\mathrm{I}' \le 2\|g_\Phi^c(f)\|_p^p \quad \text{and} \quad \mathrm{II}' \lesssim \|f\|_{\mathrm{h}_p^c}^{2-p} \|g\|_{\mathrm{bmo}_q^c}^2.$$

This gives the announced assertion. □

Armed with the previous two lemmas and the Calderón-Zygmund theorem (see Corollary 5.3), we can prove Theorem 6.7 in the same way as Theorem 6.2.

Remark 6.10. We notice that the assumption $\Phi \in \mathcal{S}$ can be relaxed in both Theorems 6.2 and 6.7, similarly to the test functions used in [70]. So far, we have used the following properties of Φ to prove the characterizations:

(1) Every $D^m\Phi$ in Lemma 6.5 with $0 \le |m|_1 \le d$ makes $f \mapsto s_{D^m\Phi}^c f$ and $f \mapsto g_{D^m\Phi}^c f$ Calderón-Zygmund singular integral operators in Corollary 5.3.

(2) There exist functions Ψ, ψ and ϕ such that (6.2) (or (6.10) for discrete characterizations) holds.

(3) The above Ψ makes $d\mu(f) = |\Psi_\varepsilon * f(s)|^2 \frac{d\varepsilon ds}{\varepsilon}$ a q-Carleson measure, satisfying Lemma 6.3.

Even though the Poisson kernel and its potentials are not Schwartz functions, they can still be used to characterize $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$. Let us take $\Phi = -2\pi I(\mathrm{P})$ and $\phi = \mathrm{P}$ for example. A simple calculation shows that we can choose $\Psi = -8\pi I(\mathrm{P})$ and $\psi = 4\pi I(\mathrm{P}) + \mathrm{P}$ to fulfill (6.2). By the inverse Fourier transform formula, we have

$$\begin{aligned} -2\pi f * I(\mathrm{P})_\varepsilon(t) &= -2\pi \int e^{2\pi \mathrm{i} t\cdot\xi} \widehat{f}(\xi) |\varepsilon\xi| e^{-2\pi\varepsilon|\xi|} d\xi \\ &= \varepsilon \frac{\partial}{\partial\varepsilon} \int e^{2\pi \mathrm{i} t\cdot\xi} \widehat{f}(\xi) e^{-2\pi\varepsilon|\xi|} d\xi = \varepsilon \frac{\partial}{\partial\varepsilon} (\mathrm{P}_\varepsilon(f)(t)). \end{aligned}$$

So we return back to the original definition of $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$. Therefore, Theorem 6.2 implies that

$$\|f\|_{\mathrm{h}_p^c} \approx \|s_\Phi^c(f)\|_p + \|\phi * f\|_p \approx \|g_\Phi^c(f)\|_p + \|\phi * f\|_p.$$

In particular, we have the following equivalent norm of $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$.

Theorem 6.11. *Let $1 \leq p < \infty$. Then for any $f \in \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, we have*

$$\|f\|_{\mathrm{h}_p^c} \approx \|g^c(f)\|_p + \|\mathrm{P} * f\|_p.$$

6.3 The relation between $\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})$ and $\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})$

Due to the noncommutativity, for any $1 < p < \infty$ and $p \neq 2$, the column operator-valued local Hardy space $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and the column operator-valued Hardy space $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ are not equivalent. On the other hand, if we consider the mixture spaces $\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})$ and $\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})$, then we will have the same situation as in the classical case.

Since $\|\mathrm{P} * f\|_p \lesssim \|f\|_p \lesssim \|f\|_{\mathcal{H}_p^c}$ for any $1 \leq p \leq 2$, we deduce the inclusion

$$\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}) \quad \text{for} \quad 1 \leq p \leq 2. \tag{6.11}$$

Then by the duality obtained in Theorem 3.18, we have

$$\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M}) \subset \mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M}) \quad \text{for} \quad 2 < p < \infty. \tag{6.12}$$

However, we can see from the following proposition that we do not have the inverse inclusion of (6.11) or (6.12).

Proposition 6.12. *Let ϕ be a function on $\mathbb{R}^d$ such that $\widehat{\phi}(0) \geq 0$ and $\widehat{\phi} \in H_2^\sigma(\mathbb{R}^d)$ with $\sigma > \frac{d}{2}$. Let $2 < p < \infty$. If for any $f \in \mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$,*

$$\|\phi * f\|_p \lesssim \|f\|_{\mathcal{H}_p^c}, \tag{6.13}$$

then we must have $\widehat{\phi}(0) = 0$.

Proof. We will prove the assertion by contradiction. Suppose that there exists ϕ such that $\widehat{\phi}(0) > 0$, $\widehat{\phi} \in H_2^\sigma(\mathbb{R}^d)$ and (6.13) holds for any $f \in \mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$. Since both $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ and $L_p(\mathcal{N})$ are homogeneous spaces, we have, for any $\varepsilon > 0$,

$$\|\phi * f(\varepsilon\cdot)\|_p = \|(\phi_\varepsilon * f)(\varepsilon\cdot)\|_p = \varepsilon^{-\frac{d}{p}}\|\phi_\varepsilon * f\|_p \quad \text{and} \quad \|f(\varepsilon\cdot)\|_{\mathcal{H}_p^c} = \varepsilon^{-\frac{d}{p}}\|f\|_{\mathcal{H}_p^c}.$$

This implies that

$$\|\phi_\varepsilon * f\|_p \lesssim \|f\|_{\mathcal{H}_p^c}, \tag{6.14}$$

for any $\varepsilon > 0$ with the relevant constant independent of ε. Now we consider a function $f \in L_p(\mathcal{N})$ which takes values in $\mathcal{S}_\mathcal{M}^+$ and such that $\operatorname{supp}\widehat{f}$ is compact, i.e. there exists a positive real number N such that $\operatorname{supp}\widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leq N\}$. Since $\widehat{\phi}(0) > 0$, we can find $\varepsilon_0 > 0$ and $c > 0$ such that $\widehat{\phi}(\varepsilon_0\xi) \geq c$ whenever $|\xi| \leq N$. Thus, in this case, $\|\phi_\varepsilon * f\|_p \geq c\|f\|_p$. Then by (6.14), we have

$$\|f\|_p \lesssim \|f\|_{\mathcal{H}_p^c},$$

which leads to a contradiction when $p > 2$. Therefore, $\widehat{\phi}(0) = 0$. $\square$

By the definition of the h_p^c-norm and the duality in Theorem 3.18, we get the following result:

Corollary 6.13. *Let $1\leq p<\infty$ and $p\neq 2$. $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ are not equivalent.*

Although $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ do not coincide when $p\neq 2$, for those functions whose Fourier transforms vanish at the origin, their h_p^c-norms and $\mathcal{H}_p^c$-norms are still equivalent.

Theorem 6.14. *Let $\phi\in\mathcal{S}$ such that $\int_{\mathbb{R}^d}\phi(s)ds=1$.*

(1) *If $1\leq p\leq 2$ and $f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, then $f-\phi*f\in\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\|f-\phi*f\|_{\mathcal{H}_p^c}\lesssim\|f\|_{\mathrm{h}_p^c}$.*

(2) *If $2<p<\infty$ and $f\in\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$, then $f-\phi*f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and $\|f-\phi*f\|_{\mathrm{h}_p^c}\lesssim\|f\|_{\mathcal{H}_p^c}$.*

Proof. (1) Let $f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ and Φ be a nondegenerate Schwartz function with vanishing mean. By the general characterization of $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ in Lemma 1.1, $\|f-\phi*f\|_{\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})}\approx\|G_\Phi^c(f-\phi*f)\|_p$. Let us split $\|G_\Phi^c(f-\phi*f)\|_p$ into two parts:

$$
\begin{aligned}
&\|G_\Phi^c(f-\phi*f)\|_p\\
&\lesssim\Big\|\Big(\int_0^1|\Phi_\varepsilon*(f-\phi*f)|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\Big\|_p+\Big\|\Big(\int_1^\infty|\Phi_\varepsilon*(f-\phi*f)|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\Big\|_p\\
&=\Big\|\Big(\int_0^1|\Phi_\varepsilon*(f-\phi*f)|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\Big\|_p+\Big\|\Big(\int_1^\infty|(\Phi_\varepsilon-\Phi_\varepsilon*\phi)*f|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\Big\|_p.
\end{aligned}
$$

In order to estimate the first term in the last equality, we notice that $\phi*f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$, thus we have $f-\phi*f\in\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$. Then by Theorem 6.2, this term can be majorized from above by $\|f\|_{\mathrm{h}_p^c}$.

To deal with the second term, we express it as a Calderón-Zygmund operator with Hilbert-valued kernel. Let $H=L_2((1,+\infty),\frac{d\varepsilon}{\varepsilon})$ and define the kernel $\mathsf{k}:\mathbb{R}^d\to H$ by $\mathsf{k}(s)=\Phi_\cdot(s)-\Phi_\cdot*\phi(s)$ ($\Phi_\cdot(s)$ being the function $\varepsilon\mapsto\Phi_\varepsilon(s)$). Now we prove that k satisfies the hypotheses of Corollary 5.3. The condition (1) of that corollary is easy to verify. So we only check the conditions (2) and (3) there. By the fact that $\int_{\mathbb{R}^d}\phi(s)ds=1$ and the mean value theorem, we have

$$
\begin{aligned}
\big|(\Phi_\varepsilon-\Phi_\varepsilon*\phi)(s)\big|&=\Big|\int_{\mathbb{R}^d}[\Phi_\varepsilon(s)-\Phi_\varepsilon(s-t)]\,\phi(t)dt\Big|\\
&\leq\int_{\mathbb{R}^d}|t|\,\frac{1}{\varepsilon^{d+1}}\sup_{0<\theta<1}\big|\nabla\Phi(\frac{s-\theta t}{\varepsilon})\big|\,|\phi(t)|\,dt.
\end{aligned}
$$

Then we split the last integral into two parts:

$$
\begin{aligned}
\big\|(\Phi_\cdot-\Phi_\cdot*\phi)(s)\big\|_H&\lesssim\Big(\int_1^\infty\Big(\int_{|t|<\frac{|s|}{2}}|t|\,\frac{1}{\varepsilon^{d+1}}\sup_{0<\theta<1}\big|\nabla\Phi(\frac{s-\theta t}{\varepsilon})\big|\,|\phi(t)|\,dt\Big)^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\\
&\quad+\Big(\int_1^\infty\Big(\int_{|t|>\frac{|s|}{2}}|t|\,\frac{1}{\varepsilon^{d+1}}\sup_{0<\theta<1}\big|\nabla\Phi(\frac{s-\theta t}{\varepsilon})\big|\,|\phi(t)|\,dt\Big)^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\\
&\overset{\mathrm{def}}{=}\mathrm{I}+\mathrm{II}.
\end{aligned}
$$

If $|t|<\frac{|s|}{2}$, we have $|s-\theta t|\geq\frac{|s|}{2}$, thus $|\nabla\Phi(\frac{s-\theta t}{\varepsilon})|\lesssim\frac{\varepsilon^{d+\frac12}}{|s|^{d+\frac12}}$ for any $0\leq\theta\leq1$. Then

$$\mathrm{I}\lesssim\Big(\int_1^\infty\frac{1}{\varepsilon^2}d\varepsilon\Big)^{\frac12}\frac{1}{|s|^{d+\frac12}}\lesssim\frac{1}{|s|^{d+\frac12}}.$$

When $|t|>\frac{|s|}{2}$, since $\phi\in\mathcal{S}$, we have $\int_{|t|>\frac{|s|}{2}}|t|\,|\phi(t)|\,dt\lesssim\frac{1}{|s|^{d+\frac12}}$. Hence

$$\mathrm{II}\lesssim\Big(\int_1^\infty\frac{1}{\varepsilon^{2d+2}}\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\cdot\frac{1}{|s|^{d+\frac12}}\lesssim\frac{1}{|s|^{d+\frac12}}.$$

The estimates of I and II imply

$$\|(\Phi_\varepsilon-\Phi_\varepsilon*\phi)(s)\|_H\lesssim\frac{1}{|s|^{d+\frac12}}.$$

In a similar way, we obtain

$$\|\nabla(\Phi_\varepsilon-\Phi_\varepsilon*\phi)(s)\|_H\lesssim\frac{1}{|s|^{d+1}}.$$

Thus, it follows from Corollary 5.3 that $\Big\|\big(\int_1^\infty|(\Phi_\varepsilon-\Phi_\varepsilon*\phi)*f|^2\frac{d\varepsilon}{\varepsilon}\big)^{\frac12}\Big\|_p$ is also majorized from above by $\|f\|_{\mathrm{h}_p^c}$.

(2) The case $p>2$ can be deduced from the duality between h_p^c and h_q^c and that between $\mathcal{H}_p^c$ and $\mathcal{H}_q^c$ (q being the conjugate index of p). There exists $g\in\mathrm{h}_q^c(\mathbb{R}^d,\mathcal{M})$ with norm one such that

$$\begin{aligned}\|f-\phi*f\|_{\mathrm{h}_p^c}&=\Big|\tau\int_{\mathbb{R}^d}(f-\phi*f)(s)g^*(s)ds\Big|\\&=\Big|\tau\int_{\mathbb{R}^d}f(s)(g^*-\phi*g^*)(s)ds\Big|\\&\leq\|f\|_{\mathcal{H}_p^c}\|g-\overline{\phi}*g\|_{\mathcal{H}_q^c}\lesssim\|f\|_{\mathcal{H}_p^c}\|g\|_{\mathrm{h}_q^c}=\|f\|_{\mathcal{H}_p^c},\end{aligned}$$

which completes the proof. □

From the interpolation result of mixture local hardy spaces in Proposition 4.4, we can deduce the equivalence between mixture local Hardy spaces and L_p-spaces.

Proposition 6.15. *For any $1<p<\infty$, $\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})=\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})=L_p(\mathcal{N})$ with equivalent norms.*

Proof. It is known that $\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})=L_p(\mathcal{N})$ with equivalent norms. One can see [42, Corollary 5.4] for more details. One the other hand, since $L_\infty(\mathcal{N})\subset\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$, by duality, we get $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})\subset L_1(\mathcal{N})$. Combining (2.2) and the interpolation result in Theorem 4.3, we deduce that $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})\subset L_p(\mathcal{N})$ for any $1<p\leq2$ and $L_p(\mathcal{N})\subset\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ for any $2<p<\infty$. Similarly, we also have $\mathrm{h}_p^r(\mathbb{R}^d,\mathcal{M})\subset L_p(\mathcal{N})$ for any $1<p\leq2$ and $L_p(\mathcal{N})\subset\mathrm{h}_p^r(\mathbb{R}^d,\mathcal{M})$ for any $2<p<\infty$. Combined with (6.11) and (6.12), we get

$$\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})\subset\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})\subset L_p(\mathcal{N})\quad\text{for}\quad1<p\leq2,\tag{6.15}$$

and

$$L_p(\mathcal{N})\subset\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})\subset\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})\quad\text{for}\quad2<p<\infty.\tag{6.16}$$

Then (6.15), (6.16) and [42, Corollary 5.4] imply that

$$\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})=\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})=L_p(\mathcal{N})\quad\text{for}\quad1<p<\infty,$$

which completes the proof. □

Chapter 7

The atomic decomposition

We start this chapter by showing that $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$ admits an atomic decomposition as in the non-local case [42]. Then, we will give a smooth atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$, that is, the atoms in consideration are required to be smooth and admit size control on their derivatives too. This refinement will play an important role in the study of pseudo-differential operators in chapter 9.

7.1 The atomic decomposition

In this section, we will focus on the atomic decomposition of $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. The atomic decomposition of $\mathcal{H}_1^c(\mathbb{R}^d,\mathcal{M})$ studied in [42] will be very useful for us.

Definition 7.1. Let Q be a cube in $\mathbb{R}^d$ with $|Q|\leq 1$. If $|Q|=1$, an h_1^c-atom associated with Q is a function $a\in L_1(\mathcal{M};L_2^c(\mathbb{R}^d))$ such that

- $\operatorname{supp} a\subset Q$;
- $\tau\big(\int_Q |a(s)|^2ds\big)^{\frac12}\leq |Q|^{-\frac12}$.

If $|Q|<1$, we assume additionally:

- $\int_Q a(s)ds=0$.

Let $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d,\mathcal{M})$ be the space of all f admitting a representation of the form

$$f=\sum_{j=1}^{\infty}\lambda_j a_j,$$

where the a_j's are h_1^c-atoms and $\lambda_j\in\mathbb{C}$ such that $\sum_{j=1}^{\infty}|\lambda_j|<\infty$. The above series converges in the sense of distribution. We equip $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d,\mathcal{M})$ with the following norm:

$$\|f\|_{\mathrm{h}_{1,\mathrm{at}}^c}=\inf\{\sum_{j=1}^{\infty}|\lambda_j| : f=\sum_{j=1}^{\infty}\lambda_j a_j;\ a_j\text{'s are } \mathrm{h}_1^c \text{ -atoms, } \lambda_j\in\mathbb{C}\}.$$

Similarly, we define the row version $\mathrm{h}_{1,\mathrm{at}}^r(\mathbb{R}^d,\mathcal{M})$. Then we set

$$\mathrm{h}_{1,\mathrm{at}}(\mathbb{R}^d,\mathcal{M})=\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d,\mathcal{M})+\mathrm{h}_{1,\mathrm{at}}^r(\mathbb{R}^d,\mathcal{M}).$$

Theorem 7.2. *We have* $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d,\mathcal{M})=\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$ *with equivalent norms.*

Proof. First, we show the inclusion $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. To this end, it suffices to prove that for any atom a in Definition 7.1, we have

$$\|a\|_{\mathrm{h}_1^c} \lesssim 1. \tag{7.1}$$

Recall that the atomic decomposition of $\mathcal{H}_1^c(\mathbb{R}^d,\mathcal{M})$ has been considered in [42]. An $\mathcal{H}_1^c$-atom is a function $b \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ such that, for some cube Q,

- $\operatorname{supp} b \subset Q$;
- $\int_Q b(s)ds = 0$;
- $\tau\big(\int_Q |b(s)|^2 ds\big)^{\frac12} \leq |Q|^{-\frac12}$.

If a is supported in Q with $|Q|<1$, then a is also an $\mathcal{H}_1^c$-atom, whence $\|a\|_{\mathrm{h}_1^c} \lesssim \|a\|_{\mathcal{H}_1^c} \lesssim 1$. Now assume that the supporting cube Q of a is of side length one. We use the discrete characterization obtained in Theorem 6.7, i.e.

$$\|a\|_{\mathrm{h}_1^c} \approx \big\|\big(\sum_{j=1}^{\infty} |\Phi_j * a|^2\big)^{\frac12}\big\|_1 + \|\phi * a\|_1.$$

Apart from the assumption on Φ and ϕ in Theorem 6.2, we may take Φ and ϕ satisfying

$$\operatorname{supp}\Phi,\ \operatorname{supp}\phi \subset B_1 = \{s\in\mathbb{R}^d : |s|\leq 1\}.$$

Then

$$\operatorname{supp}\phi * a \subset 3Q \quad\text{and}\quad \operatorname{supp}\Phi_\varepsilon * a \subset 3Q \quad\text{for any}\quad 0<\varepsilon<1.$$

By the Cauchy-Schwarz inequality we have

$$\|\phi * a\|_1 \leq \int_{3Q} \big(\int_Q |\phi(t-s)|^2 ds\big)^{\frac12} \cdot \tau\big(\int |a(s)|^2 ds\big)^{\frac12}\, dt \lesssim 1.$$

Similarly,

$$\begin{aligned}
\big\|\big(\sum_{j=1}^{\infty} |\Phi_j * a|^2\big)^{\frac12}\big\|_1 &= \tau \int_{3Q} \big(\sum_{j=1}^{\infty} |\Phi_j * a(s)|^2\big)^{\frac12} ds \\
&\lesssim \tau\Big(\int_{3Q} \sum_{j=1}^{\infty} |\Phi_j * a(s)|^2 ds\Big)^{\frac12} \\
&= \tau\Big(\int_{\mathbb{R}^d} \sum_{j=1}^{\infty} |\widehat{\Phi}(2^{-j}\xi)\widehat{a}(\xi)|^2 d\xi\Big)^{\frac12} \\
&\leq \tau\big(\int |a(s)|^2 ds\big)^{\frac12} \leq 1.
\end{aligned}$$

Therefore, $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$.

Now we turn to proving the converse inclusion. Observe that $\mathcal{H}_1^c$-atoms are also h_1^c-atoms. Then by the atomic decomposition of $\mathcal{H}_1^c(\mathbb{R}^d,\mathcal{M})$ and the duality between $\mathcal{H}_1^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M})$, every continuous functional ℓ on $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d,\mathcal{M})$ corresponds to a function $g \in \mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M})$. Moreover, since for any cube Q with side length one, $L_1(\mathcal{M}; L_2^c(Q)) \subset \mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d,\mathcal{M})$, ℓ induces a continuous functional on $L_1(\mathcal{M}; L_2^c(Q))$ with norm less than or equal to $\|\ell\|_{(\mathrm{h}_{1,\mathrm{at}}^c)^*}$. Thus, the function g satisfies the condition that

$$g \in \mathrm{BMO}^c(\mathbb{R}^d,\mathcal{M}) \quad\text{and}\quad \sup_{\substack{Q\subset\mathbb{R}^d\\ |Q|=1}} \|g|_Q\|_{L_\infty(\mathcal{M}; L_2^c(Q))} \leq \|\ell\|_{(\mathrm{h}_{1,\mathrm{at}}^c)^*}. \tag{7.2}$$

Consequently, $g \in \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$ and

$$\ell(f) = \tau \int_{\mathbb{R}^d} f(s) g^*(s) ds, \ \forall f \in \mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M}).$$

Thus, $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M})^* \subset \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$. On the other hand, by the previous result, we have $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M})^*$. Thus, $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M})^* = \mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$ with equivalent norms. Since $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ and by the density of $\mathrm{h}_{1,\mathrm{at}}^c$ in h_1^c, we deduce that $\mathrm{h}_{1,\mathrm{at}}^c(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ with equivalent norms. $\square$

Remark 7.3. In Definition 7.1, we can replace the support Q of atoms by any bounded multiple of Q. For the convenience of the discussion of the smooth atoms in the rest of this chapter, we will replace Q by $2Q$.

7.2 Refinement on smoothness

The smoothness of the atom in $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ can be improved. In the classical theory, the smooth atoms have been widely studied and they play a crucial role when studying the mapping properties of pseudo-differential operators acting on local Hardy spaces, or more generally, on Triebel-Lizorkin spaces. Further details can be found in [8], [16] and [68]. In this section, we will show that in our operator-valued case, the atoms in Definition 7.1 can also be refined to be infinitely differentiable.

First, we introduce a lemma concerning the atomic decomposition of the tent space $T_1^c(\mathbb{R}^d, \mathcal{M})$ defined in Definition 3.14. A function $a \in L_1(\mathcal{M}; L_2(S, \frac{dsd\varepsilon}{\varepsilon}))$ is called a T_1^c-atom if

- supp $a \subset T(Q)$ for some cube Q in $\mathbb{R}^d$ with $|Q| \leq 1$;
- $\tau\Big(\int_{T(Q)} |a(s,\varepsilon)|^2 \frac{dsd\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \leq |Q|^{-\frac{1}{2}}$.

Let $T_{1,at}^c(\mathbb{R}^d, \mathcal{M})$ be the space of all $f : S \to L_1(\mathcal{M})$ admitting a representation of the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{7.3}$$

where the a_j's are T_1^c-atoms and $\lambda_j \in \mathbb{C}$ such that $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. We equip $T_{1,at}^c(\mathbb{R}^d, \mathcal{M})$ with the following norm

$$\|f\|_{T_{1,at}^c} = \inf\{\sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j;\ a_j\text{'s are } T_1^c\text{-atoms}, \lambda_j \in \mathbb{C}\}.$$

Lemma 7.4. *We have $T_{1,at}^c(\mathbb{R}^d, \mathcal{M}) = T_1^c(\mathbb{R}^d, \mathcal{M})$ with equivalent norms.*

Proof. In order to prove $T_{1,at}^c(\mathbb{R}^d, \mathcal{M}) \subset T_1^c(\mathbb{R}^d, \mathcal{M})$, it is enough to show that any T_1^c-atom a satisfies $\|a\|_{T_1^c} \lesssim 1$. By the support assumption of a, we have

$$\begin{aligned} \|a\|_{T_1^c} = \big\|A^c(a)\big\|_1 &= \tau \int_{\mathbb{R}^d} \Big(\int_0^1 \int_{B(t,\varepsilon)} |a(s,\varepsilon)|^2 \frac{dsd\varepsilon}{\varepsilon^{d+1}}\Big)^{\frac{1}{2}} dt \\ &\lesssim |Q|^{\frac{1}{2}} \tau\Big(\int_{\mathbb{R}^d} \int_0^1 \int_{B(t,\varepsilon)} |a(s,\varepsilon)|^2 \frac{dsd\varepsilon}{\varepsilon^{d+1}} dt\Big)^{\frac{1}{2}} \\ &= c_d^{\frac{1}{2}} |Q|^{\frac{1}{2}} \tau\Big(\int_{T(Q)} |a(t,\varepsilon)|^2 \frac{dtd\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \lesssim 1. \end{aligned}$$

Then by the duality $T_1^c(\mathbb{R}^d,\mathcal{M})^* = T_\infty^c(\mathbb{R}^d,\mathcal{M})$ mentioned in Remark 3.15, we have $T_\infty^c(\mathbb{R}^d,\mathcal{M}) \subset T_{1,at}^c(\mathbb{R}^d,\mathcal{M})^*$.

Now let Q be a cube in $\mathbb{R}^d$ with $|Q|\leq 1$. If $f\in L_1\big(\mathcal{M};L_2^c(T(Q),\frac{dsd\varepsilon}{\varepsilon})\big)$, then

$$a=|Q|^{-\frac{1}{2}}\|f\|^{-1}_{L_1\left(\mathcal{M};L_2^c(T(Q),\frac{dsd\varepsilon}{\varepsilon})\right)}f$$

is a T_1^c-atom supported in $T(Q)$. Therefore,

$$\|f\|_{T_{1,at}^c}\leq |Q|^{\frac{1}{2}}\|f\|_{L_1\left(\mathcal{M};L_2^c(T(Q),\frac{dsd\varepsilon}{\varepsilon})\right)}.$$

Thus, $L_1\big(\mathcal{M};L_2^c(T(Q),\frac{dsd\varepsilon}{\varepsilon})\big)\subset T_{1,at}^c(\mathbb{R}^d,\mathcal{M})$ for every cube Q. Therefore, every continuous functional ℓ on $T_{1,at}^c$ induces a continuous functional on $L_1\big(\mathcal{M};L_2^c(T(Q),\frac{dsd\varepsilon}{\varepsilon})\big)$ with norm smaller than or equal to $|Q|^{\frac{1}{2}}\|\ell\|_{(T_{1,at}^c)^*}$. Let Q_0 be the cube centered at the origin with side length 1 and $Q_m=Q_0+m$ for each $m\in\mathbb{Z}^d$. Then $\mathbb{R}^d=\cup_{m\in\mathbb{Z}^d}Q_m$. Consequently, we can choose a sequence of functions $(g_m)_{m\in\mathbb{Z}^d}$ such that

$$\ell(a)=\tau\int_{T(Q_m)}a(s,\varepsilon)g_m^*(s,\varepsilon)\frac{dsd\varepsilon}{\varepsilon},\quad \forall\, T_1^c\text{-atom } a \text{ with } \operatorname{supp} a\subset T(Q_m),$$

and

$$\|g_m\|_{L_\infty\left(\mathcal{M};L_2^c(T(Q_m),\frac{dsd\varepsilon}{\varepsilon})\right)}\leq \|\ell\|_{(T_{1,at}^c)^*}.$$

Let $g(s,\varepsilon)=g_m(s,\varepsilon)$ for $(s,\varepsilon)\in T(Q_m)$. Therefore, we have

$$\ell(a)=\tau\int_S a(s,\varepsilon)g^*(s,\varepsilon)\frac{dsd\varepsilon}{\varepsilon},\quad \forall\, T_1^c\text{-atom } a.$$

It follows that, for any cube Q with $|Q|\leq 1$, $g|_{T(Q)}\in L_\infty\big(\mathcal{M};L_2^c(T(Q),\frac{dsd\varepsilon}{\varepsilon})\big)$ and

$$\|g|_{T(Q)}\|_{L_\infty\left(\mathcal{M};L_2^c(T(Q_m),\frac{dsd\varepsilon}{\varepsilon})\right)}\leq |Q|^{\frac{1}{2}}\|\ell\|_{(T_{1,at}^c)^*},$$

which implies $g\in T_\infty^c(\mathbb{R}^d,\mathcal{M})$. Therefore, $T_{1,at}^c(\mathbb{R}^d,\mathcal{M})^*\subset T_\infty^c(\mathbb{R}^d,\mathcal{M})$. Thus, we have that $T_\infty^c(\mathbb{R}^d,\mathcal{M}) = T_{1,at}^c(\mathbb{R}^d,\mathcal{M})^*$ with equivalent norms. Finally, by the density of $T_{1,at}^c(\mathbb{R}^d,\mathcal{M})$ in $T_1^c(\mathbb{R}^d,\mathcal{M})$, we get the desired equivalence. $\square$

The following Lemma shows the connection between $T_p^c(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$. The proof is modelled on the classical argument of [8, Theorem 6].

Lemma 7.5. *Fix a Schwartz function Φ on $\mathbb{R}^d$ satisfying:*

$$\begin{cases}\Phi \textit{ is supported in the cube with side length 1 and centered at the origin;}\\ \int_{\mathbb{R}^d}\Phi(s)ds=0;\\ \Phi \textit{ is nondegenerate in the sense of } (1.12).\end{cases}\tag{7.4}$$

Let π_Φ be the map given by

$$\pi_\Phi(f)(s)=\int_0^1\int_{\mathbb{R}^d}\Phi_\varepsilon(s-t)f(t,\varepsilon)\frac{dtd\varepsilon}{\varepsilon},\quad s\in\mathbb{R}^d.$$

Then π_Φ is bounded from $T_p^c(\mathbb{R}^d,\mathcal{M})$ to $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$ for any $1\leq p<\infty$.

Proof. For any $1 < p < \infty$, let q be its conjugate index. By Theorem 3.18, it suffices to estimate $\tau \int \pi_\Phi(f)(s) g^*(s) ds$, for any $g \in \mathrm{h}_q^c(\mathbb{R}^d, \mathcal{M})$. Note that

$$\begin{aligned}\tau \int_{\mathbb{R}^d} \pi_\Phi(f)(s) g^*(s) ds &= \tau \int_{\mathbb{R}^d} \int_0^1 \Phi_\varepsilon(s-t) f(t,\varepsilon) \frac{dt d\varepsilon}{\varepsilon} g^*(s) ds \\ &= \tau \int_{\mathbb{R}^d} \int_0^1 f(t,\varepsilon)(\widetilde{\Phi}_\varepsilon * g)^*(t) \frac{d\varepsilon dt}{\varepsilon},\end{aligned}$$

where $\widetilde{\Phi}(s) = \overline{\Phi(-s)}$. Then by the Hölder inequality,

$$\begin{aligned}\Big|\tau \int_{\mathbb{R}^d} \pi_\Phi(f)(s) g^*(s) ds\Big| &= \frac{1}{c_d}\Big|\tau \int_{\mathbb{R}^d} \int_0^1 \int_{B(s,\varepsilon)} f(t,\varepsilon)(\widetilde{\Phi}_\varepsilon * g)^*(t) \frac{d\varepsilon dt}{\varepsilon^{d+1}} ds\Big| \\ &= \frac{1}{c_d}\Big|\tau \int_{\mathbb{R}^d} \int_{\widetilde{\Gamma}} f(s+t,\varepsilon)(\widetilde{\Phi}_\varepsilon * g)^*(s+t) \frac{d\varepsilon dt}{\varepsilon^{d+1}} ds\Big| \\ &\lesssim \|A^c(f)\|_p \|s^c_{\widetilde{\Phi}}(g)\|_q \\ &\leq \|f\|_{T_p^c} \|g\|_{\mathrm{h}_q^c}.\end{aligned}$$

Now we deal with the case $p = 1$. The arguement below is based on the atomic decompositions of $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ and $T_1^c(\mathbb{R}^d, \mathcal{M})$. By Lemma 7.4, it is enough to show that π_Φ maps a T_1^c-atom to a bounded multiple of an h_1^c-atom. Let a be an atom in T_1^c based on some cube Q with $|Q| \leq 1$. Since Φ is supported in the unit cube, it follows from the definition of π_Φ that $\pi_\Phi(a)$ is supported in $2Q$. Moreover, it satisfies the moment cancellation that $\int \pi_\Phi(a)(s) ds = 0$ since $\widehat{\Phi}(0) = 0$. So it remains to check that $\pi_\Phi(a)$ satisfies the size estimate. To this end, we use the Cauchy-Schwarz inequality and the Plancherel formula (1.6),

$$\begin{aligned}&\|\pi_\Phi(a)\|_{L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))} = \tau\Big(\int_{\mathbb{R}^d} |\widehat{\pi_\Phi(a)}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\ &= \tau\Big(\int_{\mathbb{R}^d} \Big|\int_0^1 \widehat{\Phi}(\varepsilon\xi)\widehat{a}(\xi,\varepsilon)\frac{d\varepsilon}{\varepsilon}\Big|^2 d\xi\Big)^{\frac{1}{2}} \\ &\leq \tau\Big(\int_{\mathbb{R}^d} \int_0^1 |\widehat{\Phi}(\varepsilon\xi)|^2 \frac{d\varepsilon}{\varepsilon} \int_0^1 |\widehat{a}(\xi,\varepsilon)|^2 \frac{d\varepsilon}{\varepsilon} d\xi\Big)^{\frac{1}{2}} \\ &\leq \tau\Big(\int_{T(Q)} |a(s,\varepsilon)|^2 \frac{ds d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \leq |Q|^{-\frac{1}{2}}.\end{aligned} \tag{7.5}$$

Therefore we obtain the boundedness of π_Φ from $T^c_{1,at}(\mathbb{R}^d, \mathcal{M})$ to $\mathrm{h}^c_{1,\mathrm{at}}(\mathbb{R}^d, \mathcal{M})$. □

Theorem 7.6. *For any $f \in L_1(\mathcal{M}; \mathrm{R}_d^c) + L_\infty(\mathcal{M}; \mathrm{R}_d^c)$, f belongs to $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ if and only if it can be represented as*

$$f = \sum_{j=1}^\infty (\mu_j b_j + \lambda_j g_j), \tag{7.6}$$

where

- *the b_j's are infinitely differentiable atoms supported in $2Q_{0,j}$ with $|Q_{0,j}| = 1$. For any multiple index $\gamma \in \mathbb{N}_0^d$, there exists a constant C_γ which depends on γ satisfying*

$$\tau\Big(\int_{2Q_{0,j}} |D^\gamma b_j(s)|^2 ds\Big)^{\frac{1}{2}} \leq C_\gamma; \tag{7.7}$$

- *the g_j's are infinitely differentiable atoms supported in $2Q_j$ with $|Q_j| < 1$, and such that*
$$\tau\Big(\int_{2Q_j} |g_j(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim |Q_j|^{-\frac{1}{2}} \quad \text{and} \quad \int_{2Q_j} g_j(s) ds = 0; \tag{7.8}$$
- *the coefficients μ_j and λ_j are complex numbers such that*
$$\sum_{j=1}^{\infty} (|\mu_j| + |\lambda_j|) < \infty. \tag{7.9}$$

Moreover, the infimum of (7.9) *with respect to all admissible representations gives rise to an equivalent norm on* $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$.

Proof. Since the b_j's and g_j's are atoms in $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$, it suffices to show that any $f \in \mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$ can be represented as in (7.6) and
$$\sum_{j=1}^{\infty} (|\mu_j| + |\lambda_j|) \lesssim \|f\|_{\mathrm{h}_1^c}.$$

Let κ be a radial, real and infinitely differentiable function on $\mathbb{R}^d$ which is supported in the unit cube centered at the origin. Moreover, we assume that $\widehat{\kappa}(\xi) > 0$ if $|\xi| < 1$. We take $\widehat{\Phi} = |\cdot|^2 \widehat{\kappa}$, which can be normalized as:
$$\int_0^{\infty} \widehat{\Phi}(\varepsilon\xi)^2 \frac{d\varepsilon}{\varepsilon} = 1, \quad \xi \in \mathbb{R}^d \backslash \{0\}.$$

And we define
$$\widehat{\phi}(\xi) = 1 - \int_0^1 \widehat{\Phi}(\varepsilon\xi)^2 \frac{d\varepsilon}{\varepsilon}, \quad \xi \in \mathbb{R}^d. \tag{7.10}$$

By the Paley-Wiener theorem, $\widehat{\Phi}$ can be extended to an analytic function $\widehat{\Phi}(z)$ of d complex variables $z = \{z_1, \dots, z_d\}$, and for any $\lambda > 0$, there exists a constant C_λ such that
$$|\widehat{\Phi}(z)| \le C_\lambda e^{(\frac{\lambda}{4} + \frac{\sqrt{d}}{2})|\xi_2|} (|\xi_1|^2 + |\xi_2|^2)$$

holds for any $z = \xi_1 + \mathrm{i}\xi_2$. Therefore,
$$\begin{aligned}
\int_0^1 |\widehat{\Phi}(\varepsilon z)|^2 \frac{d\varepsilon}{\varepsilon} &\le C_\lambda^2 \int_0^1 e^{\varepsilon(\frac{\lambda}{2} + \sqrt{d})|\xi_2|} \varepsilon^3 d\varepsilon \cdot (|\xi_1|^2 + |\xi_2|^2)^2 \\
&\le C_\lambda^2 \int_0^1 \varepsilon^3 d\varepsilon \cdot e^{(\frac{\lambda}{2} + \sqrt{d})|\xi_2|} (|\xi_1|^2 + |\xi_2|^2)^2 \\
&\le C_\lambda^2 e^{(\frac{\lambda}{2} + \sqrt{d})|\xi_2|} (1 + |\xi_1|^2)^2 (1 + |\xi_2|^2)^2 \\
&\le C_\lambda^2 e^{(\lambda + 2\sqrt{d})|\xi_2|} (1 + |\xi_1|)^4.
\end{aligned}$$

Now applying the Paley-Wiener-Schwartz theorem to distributions, we obtain that ϕ is a distribution with support in $\{s \in \mathbb{R}^d : |s| \le 2\sqrt{d}\}$. On the other hand, by Lemma 6.1, we know that ϕ is a Schwartz function, thus, $\operatorname{supp} \phi \subset \{s \in \mathbb{R}^d : |s| \le 2\sqrt{d}\}$. By (7.10), we arrive at the decomposition of f:
$$f = \phi * f + \int_0^1 \Phi_\varepsilon * \Phi_\varepsilon * f \frac{d\varepsilon}{\varepsilon}. \tag{7.11}$$

We first deal with $\phi * f$. By Theorem 7.2, we obtain an atomic decomposition of f:

$$f = \sum_j \widetilde{\mu}_j a_j, \tag{7.12}$$

where the a_j's are h_1^c-atoms and $\sum_j |\widetilde{\mu}_j| \lesssim \|f\|_{\mathrm{h}_1^c}$. Thus,

$$\phi * f = \sum_j \widetilde{\mu}_j \, \phi * a_j.$$

We now show that every $\phi * a_j$ can be decomposed into smooth atoms supported in cubes with side length two. Let $\mathcal{X}_0$ be a nonnegative infinitely differentiable function on $\mathbb{R}^d$ such that $\operatorname{supp} \mathcal{X}_0 \subset 2Q_0$ (with Q_0 the unit cube centered at the origin), and $\sum_{k\in\mathbb{Z}^d} \mathcal{X}_0(s-k) = 1$ for every $s \in \mathbb{R}^d$. See [61, Section VII.2.4] for the existence of such $\mathcal{X}_0$. Set $\mathcal{X}_k = \mathcal{X}_0(\cdot - k)$. Then each $\mathcal{X}_k$ is supported in the cube $2Q_k = k + 2Q_0$, and all $\mathcal{X}_k$'s form a smooth resolution of the unit:

$$1 = \sum_{k\in\mathbb{Z}^d} \mathcal{X}_k(s), \quad \forall\, s \in \mathbb{R}^d. \tag{7.13}$$

Take a to be one of the atoms in (7.12) supported in Q. Since ϕ has compact support, i.e. there exists a constant C such that $\operatorname{supp} \phi \subset CQ_0$, then $\phi * a$ is supported in $(C+1)Q_0$. Thus, we get the decomposition

$$\phi * a = \sum_{k=1}^{N} b_k \quad \text{with } b_k = \mathcal{X}_k \cdot (\phi * a),$$

where N is a positive integer depending only on the dimension d and C. For any $\beta, \gamma \in \mathbb{N}_0^d$, denote $\beta \le \gamma$ if $\beta_j \le \gamma_j$ for every $1 \le j \le d$. Then, by the Cauchy-Schwarz inequality, for any k,

$$\begin{aligned}
\tau\Big(\int_{\mathbb{R}^d} |D^\gamma b_k(s)|^2 ds\Big)^{\frac12} &\lesssim \sum_{\beta\le\gamma} \tau\Big(\int_{2Q_k} |D^\beta \phi * a(s) \cdot D^{\gamma-\beta}\mathcal{X}_k(s)|^2 ds\Big)^{\frac12}\\
&\lesssim \sum_{\beta\le\gamma} \tau\Big(\int_{2Q_k} \Big|\int_{\mathbb{R}^d} D^\beta\phi(s-t)a(t)dt\Big|^2 ds\Big)^{\frac12}\\
&\le \sum_{\beta\le\gamma} \Big(\int_Q\int_{2Q_k} |D^\beta\phi(s-t)|^2 dsdt\Big)^{\frac12} \cdot \tau\Big(\int_Q |a(t)|^2 dt\Big)^{\frac12}\\
&\lesssim |Q|^{\frac12} \tau\Big(\int_Q |a(t)|^2 dt\Big)^{\frac12} \le 1,
\end{aligned}$$

where the relevant constants depend on γ, ϕ and the $\mathcal{X}_k$'s. Thus, we have proved that $\phi * f$ can be decomposed as follows:

$$\phi * f = \sum_j \mu_j b_j,$$

with b_j as desired. Furthermore, $\sum_j |\mu_j| \lesssim \|f\|_{\mathrm{h}_1^c}$.

Now it remains to deal with the second term on the right hand side of (7.11). It follows from the definition of the tent space and Theorem 6.2 that $\Phi. * f \in T_1^c(\mathbb{R}^d, \mathcal{M})$ and

$$\|\phi * f\|_1 + \|\Phi. * f\|_{T_1^c} \lesssim \|f\|_{\mathrm{h}_1^c}.$$

By Lemma 7.4, we can decompose $\Phi_\varepsilon * f$ as follows:

$$\Phi_\varepsilon * f(s) = \sum_{j=1}^{\infty} \lambda_j \widetilde{a}_j(s,\varepsilon) \quad \text{with} \quad \sum_{j=1}^{\infty} |\lambda_j| \lesssim \|\Phi_\varepsilon * f\|_{T_1^c}, \tag{7.14}$$

where the $\widetilde{a}_j$'s are T_1^c-atoms based on cubes with side length less than or equal to 1. For each $\widetilde{a}_j(s,\varepsilon)$ based on Q_j in (7.14), we set

$$g_j(s) = \int_0^1 \Phi_\varepsilon * \widetilde{a}_j(s,\varepsilon)\frac{d\varepsilon}{\varepsilon} = \pi_\Phi \widetilde{a}_j(s), \quad \forall s \in \mathbb{R}^d. \tag{7.15}$$

We observe from the proof of Lemma 7.5 that g_j is a bounded multiple of an h_1^c-atom supported in $2Q_j$ with vanishing mean. Moreover, g_j is infinitely differentiable. Thus, g_j satisfies (7.8) with the relevant constant depending only on Φ. Combining (7.14) and (7.15), we obtain the decomposition

$$\int_0^1 \Phi_\varepsilon * \Phi_\varepsilon * f \frac{d\varepsilon}{\varepsilon} = \sum_{j=1}^{\infty} \lambda_j g_j,$$

with $\sum_{j=1}^{\infty} |\lambda_j| \lesssim \|f\|_{\mathrm{h}_1^c}$. The proof is complete. $\square$

Chapter 8

Operator-valued Triebel-Lizorkin spaces

In this chapter, we will focus on the study of the inhomogeneous Triebel-Lizorkin spaces. These spaces can be seen as the generalization of local Hardy spaces. We will see that the operator-valued inhomogeneous Triebel-Lizorkin spaces have two parameters p and α, in particular, if $\alpha = 0$, they will be equivalent to $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$. The first main task of this chapter is to give general characterizations by the Littlewood-Paley type g-function and by the Lusin type integral function. Since the maximal function techniques are no longer available in the noncommutative setting, as in [70], the Fourier multiplier theorem mentioned in chapter 5 will be an essential tool for us.

The second main task of this chapter is to give a subtle atomic decomposition of operator-valued inhomogeneous Triebel-Lizorkin spaces. This can be realized via tent spaces by using the Calderón-reproducing identity. We will see in the next chapter that this decomposition will be very useful in the study of continuity of pseudo-differential operators.

8.1 Definitions and basic properties

Recall that φ is a Schwartz function satisfying (1.1). For each $j \in \mathbb{N}$, φ_j is the function whose Fourier transform is equal to $\varphi(2^{-j}\cdot)$, and φ_0 is the function whose Fourier transform is equal to $1 - \sum_{j\geq 1} \varphi(2^{-j}\cdot)$. Moreover, the Fourier transform of φ_j is denoted by $\varphi^{(j)}$ for $j \in \mathbb{N}_0$.

Definition 8.1. Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$.

(1) The column Triebel-Lizorkin space $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ is defined by

$$F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}) = \{f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M}) : \|f\|_{F_p^{\alpha,c}} < \infty\},$$

where

$$\|f\|_{F_p^{\alpha,c}} = \Big\|\Big(\sum_{j\geq 0} 2^{2j\alpha}|\varphi_j * f|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

(2) The row space $F_p^{\alpha,r}(\mathbb{R}^d, \mathcal{M})$ consists of all f such that $f^* \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$, equipped with the norm $\|f\|_{F_p^{\alpha,r}} = \|f^*\|_{F_p^{\alpha,c}}$.

(3) The mixture space $F_p^\alpha(\mathbb{R}^d,\mathcal{M})$ is defined to be

$$F_p^\alpha(\mathbb{R}^d,\mathcal{M}) = \begin{cases} F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M}) + F_p^{\alpha,r}(\mathbb{R}^d,\mathcal{M}) & \text{if} \quad 1 \le p \le 2 \\ F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M}) \cap F_p^{\alpha,r}(\mathbb{R}^d,\mathcal{M}) & \text{if} \quad 2 < p < \infty, \end{cases}$$

equipped with

$$\|f\|_{F_p^\alpha} = \begin{cases} \inf\{\|g\|_{F_p^{\alpha,c}} + \|h\|_{F_p^{\alpha,r}} : f = g + h\} & \text{if} \quad 1 \le p \le 2 \\ \max\{\|f\|_{F_p^{\alpha,c}}, \|f\|_{F_p^{\alpha,r}}\} & \text{if} \quad 2 < p < \infty. \end{cases}$$

In the sequel, we will focus on the study of the column Triebel-Lizorkin spaces. All results obtained in this chapter also admit the row and mixture versions.

Before moving further, let us first include the following useful multiplier lemma for the case $p = 1$. This lemma is a complement of Theorem 5.6, which relies heavily on the characterization of $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$ given in Theorem 6.7.

Lemma 8.2. *We keep the assumption in Theorem 5.6. Assume additionally that for any $j \ge 1$, $\rho_j = \rho(2^{-j}\cdot)$ for some Schwartz function ρ with $\operatorname{supp}\rho \subset \{\xi : 2^{-1} \le |\xi| \le 2\}$ and $\rho(\xi) > 0$ for any $2^{-1} < |\xi| < 2$, and that $\operatorname{supp}\rho_0 \subset \{\xi : |\xi| \le 2\}$ and $\rho_0(\xi) > 0$ for any $|\xi| < 2$. Then for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$,*

$$\begin{aligned} \big\|(\sum_{j\ge 0} 2^{2j\alpha}|\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac12}\big\|_1 \lesssim \max\{ &\sup_{\substack{j\ge1 \\ -2\le k\le 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma}, \|\phi_0(\varphi^{(0)} + \varphi^{(1)})\|_{H_2^\sigma}\} \\ &\cdot \big\|(\sum_{j\ge0} 2^{2j\alpha}|\check{\rho}_j * f|^2)^{\frac12}\big\|_1. \end{aligned}$$

Proof. By the assumptions of ρ and ρ_0, we can select a Schwartz function $\widetilde{\rho}$ with the same properties as ρ and a Schwartz function $\widetilde{\rho}_0$ satisfying the same conditions as ρ_0, such that

$$\sum_{j=1}^\infty \rho(2^{-j}\xi)\overline{\widetilde{\rho}(2^{-j}\xi)} + \rho_0(\xi)\overline{\widetilde{\rho}_0(\xi)} = 1, \quad \forall \xi \in \mathbb{R}^d.$$

Let $\Psi_j = (I_{-\alpha}\rho)(2^{-j}\cdot)$, $\widetilde{\Psi}_j = (I_\alpha\rho)(2^{-j}\cdot)$ for $j \ge 1$ and $\Psi_0 = J_{-\alpha}\rho_0$, $\widetilde{\Psi}_0 = J_\alpha\rho_0$. We have

$$\sum_{j=1}^\infty \Psi_j(\xi)\overline{\widetilde{\Psi}_j(\xi)} + \Psi_0(\xi)\overline{\widetilde{\Psi}_0(\xi)} = 1, \quad \forall\, \xi \in \mathbb{R}^d.$$

Applying Theorem 6.7 to $g = J^\alpha f$ with the text functions in the above identity, we get

$$\|g\|_{\mathrm{h}_1^c} \approx \big\|(\sum_{j\ge0} |\check{\Psi}_j * g|^2)^{\frac12}\big\|_1.$$

Now let us show the following equivalence:

$$\big\|(\sum_{j\ge0} |\check{\Psi}_j * g|^2)^{\frac12}\big\|_1 \approx \big\|(\sum_{j\ge0} 2^{2j\alpha}|\check{\rho}_j * f|^2)^{\frac12}\big\|_1.$$

It is easy to see that $\check{\Psi}_0 * g = \check{\rho}_0 * f$ and $2^{j\alpha}\check{\rho}_j * f = \check{\Psi}_j * I^\alpha f$, so it suffices to prove

$$\big\|(\sum_{j\ge1} |\check{\Psi}_j * J^\alpha f|^2)^{\frac12}\big\|_1 \approx \big\|(\sum_{j\ge1} |\check{\Psi}_j * I^\alpha f|^2)^{\frac12}\big\|_1. \tag{8.1}$$

First, let us consider the case $\alpha \geq 0$. By [60, Lemma 3.2.2], there exists a finite measure μ_α on $\mathbb{R}^d$ such that

$$|\xi|^\alpha = \widehat{\mu}_\alpha(\xi)(1+|\xi|^2)^{\frac{\alpha}{2}}.$$

Thus, we have

$$\check{\Psi}_j * I^\alpha f = \mu_\alpha * \check{\Psi}_j * J^\alpha f, \quad \forall\, j \geq 1.$$

This implies that

$$\|(\sum_{j\geq 1} |\check{\Psi}_j * I^\alpha f|^2)^{\frac{1}{2}}\|_1 \lesssim \|(\sum_{j\geq 1} |\check{\Psi}_j * J^\alpha f|^2)^{\frac{1}{2}}\|_1.$$

Then, we move to the case $\alpha < 0$. Also by [60, Lemma 3.2.2], there exist two finite measures ν_α and λ_α on $\mathbb{R}^d$ such that

$$(1+|\xi|^2)^{-\frac{\alpha}{2}} = \widehat{\nu}_\alpha(\xi) + |\xi|^{-\alpha}\widehat{\lambda}_\alpha(\xi).$$

Let $(\dot{\varphi}_k)_{k\in\mathbb{Z}}$ be the homogeneous resolution of the unit defined in (1.4). It follows that

$$\frac{(1+|\xi|^2)^{-\frac{\alpha}{2}}}{|\xi|^{-\alpha}} \sum_{k\geq 0} \dot{\varphi}_k(\xi) = \frac{\widehat{\nu}_\alpha(\xi)}{|\xi|^{-\alpha}} \sum_{k\geq 0} \dot{\varphi}_k(\xi) + \widehat{\lambda}_\alpha(\xi) \sum_{k\geq 0} \dot{\varphi}_k(\xi).$$

Thus, by the support assumption of $\widehat{\rho}$, we have

$$\check{\Psi}_j * I^\alpha f = \omega_\alpha * \check{\Psi}_j * J^\alpha f,$$

with

$$\omega_\alpha = \nu_\alpha * \sum_{k\geq 0} \mathcal{F}^{-1}(I_\alpha \dot{\varphi}_k) + \lambda_\alpha * \mathcal{F}^{-1}(\sum_{k\geq 0} \dot{\varphi}_k).$$

Both $\mathcal{F}^{-1}(\sum_{k\geq 0} \dot{\varphi}_k)$ and $\sum_{k\geq 0} \mathcal{F}^{-1}(I_\alpha \dot{\varphi}_k)$ are finite measures. Since $\sum_{k\geq 0} \dot{\varphi}_k = 1 - \sum_{k<0} \dot{\varphi}_k$, and $\sum_{k<0} \dot{\varphi}_k$ is a Schwartz function, we know that $\mathcal{F}^{-1}(\sum_{k\geq 0} \dot{\varphi}_k) = \delta_0 - \mathcal{F}^{-1}(\sum_{k<0} \dot{\varphi}_k)$ is a finite measure, where δ_0 denotes the Dirac measure at the origin. Moreover, it is known in [72, Lemma 3.4] that $\|\mathcal{F}^{-1}(I_\alpha \dot{\varphi}_k)\|_1 \lesssim 2^{k\alpha}$. Then we have

$$\|\mathcal{F}^{-1}(\sum_{k\geq 0} I_\alpha \dot{\varphi}_k)\|_1 \lesssim \sum_{k\geq 0} 2^{k\alpha} < \infty.$$

Therefore, ω_α is a finite measure on $\mathbb{R}^d$. Thus,

$$\|(\sum_{j\geq 1} |\check{\Psi}_j * I^\alpha f|^2)^{\frac{1}{2}}\|_1 \lesssim \|(\sum_{j\geq 1} |\check{\Psi}_j * J^\alpha f|^2)^{\frac{1}{2}}\|_1.$$

Similarly, for $\alpha \in \mathbb{R}$, we can prove that

$$\|(\sum_{j\geq 1} |\check{\Psi}_j * J^\alpha f|^2)^{\frac{1}{2}}\|_1 \lesssim \|(\sum_{j\geq 1} |\check{\Psi}_j * I^\alpha f|^2)^{\frac{1}{2}}\|_1.$$

In summary, we have proved (8.1), which yields that

$$\|g\|_{\mathrm{h}_1^c} = \|J^\alpha f\|_{\mathrm{h}_1^c} \approx \big\|\big(\sum_{j\geq 0} 2^{2j\alpha} |\check{\rho}_j * f|^2\big)^{\frac{1}{2}}\big\|_1.$$

Now define a new sequence $\zeta = (\zeta_j)_{j\geq 0}$ by setting $\zeta_j = 2^{j\alpha} I_{-\alpha}\phi_j\rho_j$ for $j \geq 1$ and $\zeta_0 = J_{-\alpha}\phi_0\rho_0$. Then

$$\check{\zeta}_j * g = 2^{j\alpha}\check{\phi}_j * \check{\rho}_j * I^{-\alpha} g \quad \text{and} \quad \check{\zeta}_0 * g = \check{\phi}_0 * \check{\rho}_0 * f.$$

Repeating the argument for (8.1) with $\zeta = (\zeta_j)_{j\geq 0}$ instead of $\Psi = (\Psi_j)_{j\geq 0}$, we get

$$\big\|(\sum_{j\geq 0} 2^{2j\alpha}|\check{\phi}_j * \check{\rho}_j * f|^2)^{\frac{1}{2}}\big\|_1 = \big\|(\sum_{j\geq 0} |\check{\zeta}_j * I^\alpha f|^2)^{\frac{1}{2}}\big\|_1 \approx \big\|(\sum_{j\geq 0} |\check{\zeta}_j * g|^2)^{\frac{1}{2}}\big\|_1.$$

Then, we apply Corollary 5.10 to g with this new ζ instead of ϕ to get

$$\big\|(\sum_{j\geq 0} |\check{\zeta}_j * g|^2)^{\frac{1}{2}}\big\|_1 \lesssim \|\zeta\|_{2,\sigma}\|g\|_{\mathrm{h}_1^c} \approx \|\zeta\|_{2,\sigma}\big\|(\sum_{j\geq 0} 2^{2j\alpha}|\check{\rho}_j * f|^2)^{\frac{1}{2}}\big\|_1.$$

It suffices to estimate the term $\|\zeta\|_{2,\sigma}$. By the definition of $\zeta = (\zeta_j)_{j\geq}$, we have

$$\sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\zeta_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma} \lesssim \sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma},$$
$$\|\zeta_0(\varphi^{(0)} + \varphi^{(1)})\|_{H_2^\sigma} \lesssim \|\phi_0(\varphi^{(0)} + \varphi^{(1)})\|_{H_2^\sigma}.$$

So we can use the same argument at the end of the proof of Theorem 5.9, to get

$$\|\zeta\|_{2,\sigma} \lesssim \max\{\sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma}, \|\phi_0(\varphi^{(0)} + \varphi^{(1)})\|_{H_2^\sigma}\}.$$

Combining the above inequalities, we get the desired assertion. □

The above lemma will play a very important role in this section. Let us take a first view of its effect in the following proposition, which shows that $F_p^{\alpha,c}$-norm is independent of the choice of the function φ satisfying (1.1).

Proposition 8.3. *Let ψ be another Schwartz function satisfying the same condition* (1.1) *as φ. For each $j \in \mathbb{N}$, let ψ_j be the function whose Fourier transform is equal to $\psi(2^{-j}\cdot)$, and let ψ_0 be the function whose Fourier transform is equal to $1 - \sum_{j\geq 1}\psi(2^{-j}\cdot)$. Then*

$$\|f\|_{F_p^{\alpha,c}} \approx \big\|(\sum_{j\geq 0} 2^{2j\alpha}|\psi_j * f|^2)^{\frac{1}{2}}\big\|_p.$$

Proof. For any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$, by the support assumption of ψ and φ, we have, for any $j \geq 0$,

$$\psi_j * f = \sum_{k=-1}^{1} \psi_j * \varphi_{k+j} * f,$$

with the convention $\varphi_{-1} = 0$. Thus by Theorem 5.6 and Lemma 8.2,

$$\begin{aligned}
&\big\|(\sum_{j\geq 0} 2^{2j\alpha}|\psi_j * f|^2)^{\frac{1}{2}}\big\|_p\\
&\leq \sum_{k=-1}^{1} \big\|(\sum_{j\geq 0} 2^{2j\alpha}|\psi_j * \varphi_{k+j} * f|^2)^{\frac{1}{2}}\big\|_p\\
&\lesssim \max\{\sup_{-2\leq k\leq 2} \|\psi(2^k\cdot)\varphi\|_{H_2^\sigma}, \|\psi_0(\varphi^{(0)} + \varphi^{(1)})\|_{H_2^\sigma}\}\big\|(\sum_{j\geq 0} 2^{2j\alpha}|\varphi_j * f|^2)^{\frac{1}{2}}\big\|_p\\
&\lesssim \big\|(\sum_{j\geq 0} 2^{2j\alpha}|\varphi_j * f|^2)^{\frac{1}{2}}\big\|_p.
\end{aligned}$$

Changing the role of φ and ψ, we get the reverse inequality. □

Proposition 8.4. *Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. Then*

(1) *$F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ is a Banach space.*

(2) *$F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}) \subset F_p^{\beta,c}(\mathbb{R}^d, \mathcal{M})$ if $\alpha > \beta$.*

(3) *$F_p^{0,c}(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ with equivalent norms.*

Proof. (1) Let $\{f_i\}$ be a Cauchy sequence in $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. Then, the sequence $\{a_i\}$ with $a_i = (\varphi_0 * f_i, \ldots, 2^{j\alpha}\varphi_j * f_i, \ldots)$ is also a Cauchy sequence in $L_p(\mathcal{N}; \ell_2^c(\mathbb{N}_0))$. Thus, a_i converges to a function $f = (f^0, \ldots, f^j, \ldots)$ in $L_p(\mathcal{N}; \ell_2^c(\mathbb{N}_0))$. Formally we take

$$f = \sum_{j \geq 0} f^j. \tag{8.2}$$

Since for each $j \in \mathbb{N}$, $\operatorname{supp} \widehat{f^j} \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and $\operatorname{supp} \widehat{f^0} \subset \{\xi : |\xi| \leq 2\}$, the series (8.2) converges in $\mathcal{S}'(\mathbb{R}^d; L_p(\mathcal{M}))$. Let $\varphi_j = 0$ if $j < 0$. By the support assumption of φ, when $i \to \infty$, we get

$$\varphi_j * f_i = \sum_{k=j-1}^{j+1} \varphi_k * \varphi_j * f_i \to \sum_{k=j-1}^{j+1} \varphi_j * f^k = \varphi_j * f,$$

which implies that $f^j = 2^{j\alpha}\varphi_j * f$, for any $j \geq 0$. Thus, $f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ and $\{f_i\}$ converges to f in $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.

(2) is obvious.

(3) It is easy to see that for any φ satisfying (1.1) also satisfies (1.16). Then by the discrete characterization of $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ given in Theorem 6.7, we get the assertion. $\square$

Given $a \in \mathbb{R}_+$, we define $D_{i,a}(\xi) = (2\pi \mathrm{i}\xi_i)^a$ for $\xi \in \mathbb{R}^d$, and D_i^a to be the Fourier multiplier with symbol $D_{i,a}(\xi)$ on Triebel-Lizorkin spaces $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. We set $D_a = D_{1,a_1} \cdots D_{d,a_d}$ and $D^a = D_1^{a_1} \cdots D_d^{a_d}$ for any $a = (a_1, \cdots, a_d) \in \mathbb{R}_+^d$. Note that if a is a positive integer, $D_i^a = \partial_i^a$, so there does not exist any conflict of notation. The operator D^a can be viewed as a fractional extension of partial derivatives. The following is the so-called lifting (or reduction) property of Triebel-Lizorkin spaces.

Proposition 8.5. *Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$.*

(1) *For any $\beta \in \mathbb{R}$, J^β is an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ and $F_p^{\alpha-\beta,c}(\mathbb{R}^d, \mathcal{M})$. In particular, J^α is an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$.*

(2) *Let $\beta > 0$. Then $f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ if and only if $\varphi_0 * f \in L_p(\mathcal{N})$ and $D_i^\beta f \in F_p^{\alpha-\beta,c}(\mathbb{R}^d, \mathcal{M})$ for all $i = 1, \ldots, d$. Moreover, in this case,*

$$\|f\|_{F_p^{\alpha,c}} \approx \|\varphi_0 * f\|_p + \sum_{i=1}^d \|D_i^\beta f\|_{F_p^{\alpha-\beta,c}}.$$

Proof. (1) Let $f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. Applying Theorem 5.6 and Lemma 8.2 with $\rho = \varphi$, we obtain

$$\begin{aligned}\|J^\beta f\|_{F_p^{\alpha-\beta,c}} &= \big\|\big(\sum_{j\geq 0} 2^{2j(\alpha-\beta)}|\varphi_j * J^\beta f|^2\big)^{\frac12}\big\|_p \\ &\lesssim \max\{\sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} 2^{-j\beta}\|J_\beta(2^{j+k}\cdot)\varphi\|_{H_2^\sigma}, \|J_\beta(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}\} \\ &\quad \cdot \big\|\big(\sum_{j\geq 0} 2^{2j\alpha}|\varphi_j * f|^2\big)^{\frac12}\big\|_p.\end{aligned} \tag{8.3}$$

It is easy to check that all partial derivatives of $2^{-j\beta}J_\beta(2^{j+k}\cdot)\varphi$ of order less than or equal to $[\sigma]+1$ are bounded uniformly in $j \geq 0$ and $-2 \leq k \leq 2$, and that $J_\beta(\varphi^{(0)}+\varphi^{(1)}) \in H_2^\sigma(\mathbb{R}^d)$. Thus $\|J^\beta f\|_{F_p^{\alpha-\beta,c}} \lesssim \|f\|_{F_p^{\alpha,c}}$. So J^β is continuous from $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ to $F_p^{\alpha-\beta,c}(\mathbb{R}^d,\mathcal{M})$, and its inverse $J^{-\beta}$ is also continuous from $F_p^{\alpha-\beta,c}(\mathbb{R}^d,\mathcal{M})$ to $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$.

(2) If we take $\sigma \in (\frac{d}{2}, \beta+\frac{d}{2})$, then we have $\|D_{i,\beta}\varphi_0\|_{H_2^\sigma} < \infty$ and $\|D_{i,\beta}\varphi\|_{H_2^\sigma} < \infty$. Replacing J^β by D_i^β in (8.3), we obtain that, for any $i = 1,\dots,d$,

$$\|D_i^\beta f\|_{F_p^{\alpha-\beta,c}} \lesssim \|f\|_{F_p^{\alpha,c}},$$

which implies immediately that

$$\|\varphi_0 * f\|_p + \sum_{i=1}^d \|D_i^\beta f\|_{F_p^{\alpha-\beta,c}} \lesssim \|f\|_{F_p^{\alpha,c}}.$$

To show the reverse inequality, we choose a nonnegative infinitely differentiable function χ on $\mathbb{R}$ such that $\chi(s) = 0$ if $|s| < \frac{1}{2\sqrt{d}}$ and $\chi(s) = 1$ if $|s| \geq \frac{1}{\sqrt{d}}$. For $i = 1,\dots,d$, we define χ_i on $\mathbb{R}^d$ as follows:

$$\chi_i(\xi) = \frac{1}{\chi(\xi_1)|\xi_1|^\beta + \ldots + \chi(\xi_d)|\xi_d|^\beta}\frac{\chi(\xi_i)|\xi_i|^\beta}{(2\pi\mathrm{i}\xi_i)^\beta},$$

whenever the first denominator is positive, which is the case when $|\xi| \geq 1$. Then for any $j \geq 1$, $\chi_i\varphi_j$ is a well-defined infinitely differentiable function on $\mathbb{R}^d \backslash \{\xi : \xi_i = 0\}$ and

$$\varphi^{(j)} = \sum_{i=1}^d \chi_i D_{i,\beta}\varphi^{(j)}.$$

Then by Theorem 5.5, we have

$$\begin{aligned}\|f\|_{F_p^{\alpha,c}} &\leq \|\varphi_0 * f\|_p + \sum_{i=1}^d \big\|\big(\sum_{j\geq 1} 2^{2j\alpha}|\check{\chi}_i * \varphi_j * D_i^\beta f|^2\big)^{\frac12}\big\|_p \\ &\lesssim \|\varphi_0 * f\|_p + \sum_{i=1}^d \sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} 2^{j\beta}\|\chi_i(2^{j+k}\cdot)\varphi\|_{H_2^\sigma}\big\|\big(\sum_{j\geq 1} 2^{2j(\alpha-\beta)}|\varphi_j * D_i^\beta f|^2\big)^{\frac12}\big\|_p.\end{aligned}$$

However,

$$2^{j\beta}\|\chi_i(2^{j+k}\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d)} = \|\phi_i(2^k\cdot)\varphi\|_{H_2^\sigma(\mathbb{R}^d)},$$

where

$$\phi_i(\xi) = \frac{1}{\chi(2^j\xi_1)|\xi_1|^\beta + \ldots + \chi(2^j\xi_d)|\xi_d|^\beta}\frac{\chi(2^j\xi_i)|\xi_i|^\beta}{(2\pi\mathrm{i}\xi_i)^\beta}.$$

Since all partial derivatives of $\phi_i\varphi(2^k\cdot)$, of order less than a fixed integer, are bounded uniformly in j,k and i, and the norm of $\phi_i\varphi(2^k\cdot)$ in $H_2^\sigma(\mathbb{R}^d)$ are bounded from above by a constant independent of j,k and i. Then we deduce

$$\begin{aligned}\|f\|_{F_p^{\alpha,c}} &\lesssim \|\varphi_0 * f\|_p + \sum_{i=1}^d \big\|\big(\sum_{j\geq 1} 2^{2j(\alpha-\beta)}|\varphi_j * D_i^\beta f|^2\big)^{\frac12}\big\|_p \\ &\leq \|\varphi_0 * f\|_p + \sum_{i=1}^d \|D_i^\beta f\|_{F_p^{\alpha-\beta,c}}.\end{aligned}$$

The assertion is proved. $\square$

Definition 8.6. For $\alpha \in \mathbb{R}$, we define $F_\infty^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ as the space of all $f \in \mathcal{S}'(\mathbb{R}^d; \mathcal{M})$ such that

$$\|\varphi_0 * f\|_{\mathcal{N}} + \sup_{|Q|<1} \Big\| \frac{1}{|Q|} \int_Q \sum_{j \geq -\log_2(l(Q))} 2^{2j\alpha} |\varphi_j * f(s)|^2 ds \Big\|_{\mathcal{M}}^{\frac{1}{2}} < \infty.$$

We endow the space $F_\infty^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ with the norm:

$$\|f\|_{F_\infty^{\alpha,c}} = \|\varphi_0 * f\|_{\mathcal{N}} + \sup_{|Q|<1} \Big\| \frac{1}{|Q|} \int_Q \sum_{j \geq -\log_2(l(Q))} 2^{2j\alpha} |\varphi_j * f(s)|^2 ds \Big\|_{\mathcal{M}}^{\frac{1}{2}}.$$

Proposition 8.7. *Let $1 \leq p < \infty$, $\alpha \in \mathbb{R}$ and q be the conjugate index of p. Then the dual space of $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ coincides isomorphically with $F_q^{-\alpha,c}(\mathbb{R}^d, \mathcal{M})$.*

Proof. First, we show that J^α is an isomorphism between $F_\infty^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$. To this end, we now use the discrete Carleson characterization of $\mathrm{bmo}^c(\mathbb{R}^d, \mathcal{M})$ that for any Schwartz function Φ and $\phi \in H_2^\sigma(\mathbb{R}^d)$ satisfying (6.10),

$$\|f\|_{\mathrm{bmo}^c} \approx \|\phi * f\|_{\mathcal{N}} + \sup_{|Q|<1} \Big\| \frac{1}{|Q|} \int_Q \sum_{j \geq -\log_2(l(Q))} |\Phi_j * f(s)|^2 ds \Big\|_{\mathcal{M}}^{\frac{1}{2}}. \tag{8.4}$$

We can check the above equivalence by combining Lemma 6.8 and the argument of Corollary 3.11. By taking $\phi = \varphi_0$ and $\Phi = J^{-\alpha}\varphi$, we apply (8.4) to $J^\alpha f$:

$$\begin{aligned}
\|J^\alpha f\|_{\mathrm{bmo}^c} &\approx \|\varphi_0 * f\|_{\mathcal{N}} + \sup_{|Q|<1} \Big\| \frac{1}{|Q|} \int_Q \sum_{j \geq -\log_2(l(Q))} |(J^{-\alpha}\varphi)_j * (J^\alpha f)(s)|^2 ds \Big\|_{\mathcal{M}}^{\frac{1}{2}} \\
&= \|\varphi_0 * f\|_{\mathcal{N}} + \sup_{|Q|<1} \Big\| \frac{1}{|Q|} \int_Q \sum_{j \geq -\log_2(l(Q))} 2^{2j\alpha} |\varphi_j * f(s)|^2 ds \Big\|_{\mathcal{M}}^{\frac{1}{2}} \\
&= \|f\|_{F_\infty^{\alpha,c}}.
\end{aligned}$$

Since J^α is also an isomorphism between $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ for any $1 < p < \infty$, by the h_1^c-bmo^c duality in Theorem 3.10 and the h_p^c-h_q^c duality in Theorem 3.18, we obtain that $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})^* = F_q^{-\alpha,c}(\mathbb{R}^d, \mathcal{M})$ with equivalent norms. □

8.2 Interpolation

The main result in this section is the following complex interpolation of Triebel-Lizorkin spaces. It is deduced from the interpolation of local Hardy and bmo spaces in Theorem 4.3, and the boundedness of complex order Bessel potentials on them.

Proposition 8.8. *Let $\alpha_0, \alpha_1 \in \mathbb{R}$ and $1 < p < \infty$. Then*

$$\big(F_\infty^{\alpha_0,c}(\mathbb{R}^d, \mathcal{M}), F_1^{\alpha_1,c}(\mathbb{R}^d, \mathcal{M})\big)_{\frac{1}{p}} = F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}), \quad \alpha = (1 - \frac{1}{p})\alpha_0 + \frac{\alpha_1}{p}.$$

Proof. Let $f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. By Proposition 8.5, we have $J^\alpha f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$. Therefore, according to Theorem 4.3, there exists a continuous function on the strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Re} z \leq 1\}$, analytic in the interior, such that $J^\alpha f = F(\frac{1}{p}) \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ and such that

$$\sup_{t\in\mathbb{R}} \|F(\mathrm{i}t)\|_{\mathrm{bmo}^c} < \infty \quad \text{and} \quad \sup_{t\in\mathbb{R}} \|F(1+\mathrm{i}t)\|_{\mathrm{h}_1^c} < \infty.$$

We will use Bessel potentials of complex order. For $z \in \mathbb{C}$, define $J_z(\xi) = (1+|\xi|^2)^{\frac{z}{2}}$, and J^z to be the associated Fourier multiplier. We set

$$\widetilde{F}(z) = e^{(z-\frac{1}{p})^2} J^{-(1-z)\alpha_0 - z\alpha_1} F(z).$$

For any $t \in \mathbb{R}$,

$$\|\widetilde{F}(\mathrm{i}t)\|_{F_\infty^{\alpha_0,c}} \approx e^{-t^2+\frac{1}{p^2}} \|J^{\mathrm{i}t(\alpha_0-\alpha_1)} F(\mathrm{i}t)\|_{\mathrm{bmo}^c}$$

and

$$\|\widetilde{F}(1+\mathrm{i}t)\|_{F_1^{\alpha_1,c}} \approx e^{-t^2+(1-\frac{1}{p})^2} \|J^{\mathrm{i}t(\alpha_0-\alpha_1)} F(1+\mathrm{i}t)\|_{\mathrm{h}_1^c}.$$

We claim that J^{it} is a bounded Fourier multiplier on $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$, so by duality, it is bounded on $\mathrm{bmo}^c(\mathbb{R}^d,\mathcal{M})$ too. Therefore, we will have

$$\sup_{t\in\mathbb{R}} \|\widetilde{F}(\mathrm{i}t)\|_{F_\infty^{\alpha_0,c}} < \infty \quad \text{and} \quad \sup_{t\in\mathbb{R}} \|\widetilde{F}(1+\mathrm{i}t)\|_{F_1^{\alpha_1,c}} < \infty.$$

This will imply that $f = \widetilde{F}(\frac{1}{p}) \in \big(F_\infty^{\alpha_0,c}(\mathbb{R}^d,\mathcal{M}), F_1^{\alpha_1,c}(\mathbb{R}^d,\mathcal{M})\big)_{\frac{1}{p}}$. Hence,

$$F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M}) \subset \big(F_\infty^{\alpha_0,c}(\mathbb{R}^d,\mathcal{M}), F_1^{\alpha_1,c}(\mathbb{R}^d,\mathcal{M})\big)_{\frac{1}{p}}.$$

Then the reverse inclusion follows by duality.

Now, we prove the claim. First, we can easily check that J_{it} is d-times differentiable on $\mathbb{R}^d \setminus \{0\}$, and for any $m \in \mathbb{N}_0^d$ and $|m|_1 \le d$, we have

$$\sup\big\{|\xi|^{|m|_1}|D^m J_{\mathrm{it}}(\xi)| : \xi \neq 0\big\} \lesssim (1+|t|)^d.$$

Then, we can check that (with $J_{\mathrm{it}}(2^k\xi) = (1+|2^k\xi|^2)^{\frac{\mathrm{i}t}{2}}$),

$$\max_{-2\le k\le 2} \|J_{\mathrm{it}}(2^k\cdot)\varphi\|_{H_2^d} \lesssim (1+|t|)^d \quad \text{and} \quad \|J_{\mathrm{it}}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^d} \lesssim (1+|t|)^d.$$

By (3) in Proposition 8.4, if we take $(\varphi_j)_{j\ge 0}$ to be the Littlewood-Paley decomposition on $\mathbb{R}^d$ satisfying (1.2) and (1.3), we have

$$\|J^{\mathrm{it}} f\|_{\mathrm{h}_1^c} \approx \big\|\big(\sum_{j\ge 0} |\check{J}_{\mathrm{it}} * \varphi_j * f|^2\big)^{\frac{1}{2}}\big\|_1$$

and

$$\|f\|_{\mathrm{h}_1^c} \approx \big\|\big(\sum_{j\ge 0} |\varphi_j * f|^2\big)^{\frac{1}{2}}\big\|_1.$$

Then, we can apply Lemma 8.2 with $\rho_j = \varphi_j$, $\phi_j(2^j\cdot) = \check{J}_{\mathrm{it}}$, $\forall j \ge 0$ and $\alpha = 0$, $\sigma = d$,

$$\|J^{\mathrm{it}} f\|_{\mathrm{h}_1^c} \lesssim \max\big\{\max_{-2\le k\le 2} \|J_{\mathrm{it}}(2^k\cdot)\varphi\|_{H_2^d}, \|J_{\mathrm{it}}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^d}\big\}\|f\|_{\mathrm{h}_1^c} \lesssim (1+|t|)^d \|f\|_{\mathrm{h}_1^c}.$$

The desired assertion is proved. □

Remark 8.9. The real interpolation of the couple $\big(F_\infty^{\alpha,c}(\mathbb{R}^d,\mathcal{M}), F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})\big)$ follows easily from Proposition 8.5 and Corollary 4.5. But if $\alpha_1 \neq \alpha_2$, the real interpolation of $\big(F_\infty^{\alpha_1,c}(\mathbb{R}^d,\mathcal{M}), F_1^{\alpha_2,c}(\mathbb{R}^d,\mathcal{M})\big)$ will give Besov type spaces. We will not consider this problem in this thesis, and refer the reader to [72] for similar results on homogeneous Triebel-Lizorkin (and Besov) spaces.

8.3 Triebel-Lizorkin spaces with $\alpha > 0$

The following result shows that when $\alpha > 0$, the $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$-norm can be rewritten as the sum of two homogeneous norms. Recall that for a fixed Schwartz function φ in (1.1), the functions $\dot{\varphi}_j$'s determined by $\widehat{\dot{\varphi}}_j(\xi) = \varphi(2^{-j}\xi)$, $j \in \mathbb{Z}$ give a homogeneous Littlewood-Paley decomposition on $\mathbb{R}^d$ satisfying (1.4).

Proposition 8.10. *Let $\alpha > 0$. If $1 \le p < \infty$, then*

$$\|f\|_{F_p^{\alpha,c}} \approx \|\varphi_0 * f\|_p + \big\|\big(\sum_{j=-\infty}^{+\infty} 2^{2j\alpha}|\dot{\varphi}_j * f|^2\big)^{\frac{1}{2}}\big\|_p, \quad \forall\, f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}).$$

If $1 \le p \le 2$,

$$\|f\|_{F_p^{\alpha,c}} \approx \|f\|_p + \big\|\big(\sum_{j=-\infty}^{+\infty} 2^{2j\alpha}|\dot{\varphi}_j * f|^2\big)^{\frac{1}{2}}\big\|_p, \quad \forall\, f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}).$$

Proof. By the definition of the $F_p^{\alpha,c}$-norm, it is obvious that

$$\|f\|_{F_p^{\alpha,c}} \lesssim \|\varphi_0 * f\|_p + \big\|\big(\sum_{j=-\infty}^{+\infty} 2^{2j\alpha}|\dot{\varphi}_j * f|^2\big)^{\frac{1}{2}}\big\|_p.$$

To prove the reverse inequality, it suffices to show:

$$\big\|\big(\sum_{j=-\infty}^{0} 2^{2j\alpha}|\dot{\varphi}_j * f|^2\big)^{\frac{1}{2}}\big\|_p \lesssim \|\varphi_0 * f\|_p + \big\|\big(\sum_{j=1}^{+\infty} 2^{2j\alpha}|\dot{\varphi}_j * f|^2\big)^{\frac{1}{2}}\big\|_p.$$

By the support assumption of φ, we have $\varphi^{(0)} = 1$ for any $|\xi| \le 1$. Thus, when $j < 0$,

$$\varphi(2^j\cdot) = \varphi(2^j\cdot)\varphi^{(0)}.$$

Then

$$\dot{\varphi}_j * f = \dot{\varphi}_j * \varphi_0 * f. \tag{8.5}$$

By the triangle inequality, (8.5) and [71, Lemma 1.7], we obtain

$$\begin{aligned}
\big\|\big(\sum_{j=-\infty}^{0} 2^{j\alpha}|\dot{\varphi}_j * f|^2\big)^{\frac{1}{2}}\big\|_p &\lesssim \sum_{j=-\infty}^{-1} 2^{j\alpha}\|\dot{\varphi}_j * \varphi_0 * f\|_p + \|\dot{\varphi}_0 * f\|_p \\
&\lesssim \sum_{j=-\infty}^{-1} 2^{j\alpha}\|\dot{\varphi}_j\|_1\|\varphi_0 * f\|_p + \big\|\varphi(\varphi_0 + \varphi_1 + \varphi_2) * f\big\|_p \\
&\lesssim \sum_{j=-\infty}^{0} 2^{j\alpha}\|\varphi_0 * f\|_p + \big\|\big(\sum_{j=1}^{+\infty} 2^{2j\alpha}|\varphi_j * f|^2\big)^{\frac{1}{2}}\big\|_p \\
&\lesssim \|\varphi_0 * f\|_p + \big\|\big(\sum_{j=1}^{+\infty} 2^{2j\alpha}|\varphi_j * f|^2\big)^{\frac{1}{2}}\big\|_p.
\end{aligned}$$

Therefore, we have proved that $\|\varphi_0 * f\|_p + \big\|\big(\sum_{j=1}^{+\infty} 2^{2j\alpha}|\varphi_j * f|^2\big)^{\frac{1}{2}}\big\|_p$ gives rise to an equivalent norm on $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ when $\alpha > 0$.

For any $1 \leq p \leq 2$ and $\alpha > 0$, we have $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M}) \subset L_p(\mathcal{N})$. Therefore $\|f\|_p \lesssim \|f\|_{F_p^{\alpha,c}}$. Combined with the equivalence obtained above, we see that

$$\|f\|_p + \|(\sum_{j=-\infty}^{+\infty} 2^{2j\alpha}|\dot{\varphi}_j * f|^2)^{\frac{1}{2}}\|_p \lesssim \|f\|_{F_p^{\alpha,c}}.$$

The reverse inequality can be easily deduced by the fact that $\|\varphi_0 * f\|_p \leq \|\varphi_0\|_1\|f\|_p$. $\square$

We also have a continuous counterpart of Proposition 8.10. For any $\varepsilon \geq 0$, we define $\dot{\varphi}_\varepsilon = \mathcal{F}^{-1}(\varphi(\varepsilon\cdot))$.

Corollary 8.11. *Let $1 \leq p \leq 2$ and $\alpha > 0$. Then, for any $f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$,*

$$\|f\|_{F_p^{\alpha,c}} \approx \|f\|_p + \Big\|(\int_0^\infty \varepsilon^{-2\alpha}|\dot{\varphi}_\varepsilon * f|^2 \frac{d\varepsilon}{\varepsilon})^{\frac{1}{2}}\Big\|_p. \tag{8.6}$$

We notice that (8.6) is the sum of two homogeneous norms. For $\lambda > 0$, $\|f(\lambda\cdot)\|_p = \lambda^{-\frac{d}{p}}\|f\|_p$ and for the second term in (8.6), we have a corresponding assertion with $\alpha - \frac{d}{p}$ instead of $\frac{d}{p}$, since $[\dot{\varphi}_\varepsilon * f(\lambda\cdot)](s) = \dot{\varphi}_{\lambda\varepsilon} * f(\lambda s)$.

8.4 General characterizations

We have seen in section 8.1 that the definition of Triebel-Lizorkin spaces is independent of the choice of φ satisfying (1.1). In this section, we will show that this kernel is not necessarily a Schwartz function. Since the local Hardy spaces are included in the family of inhomogeneous Triebel-Lizorkin spaces, the following characterizations can be seen as generalizations of those in chapter 6. The multiplier in section 5.2 will be very useful in the following.

Let $\Phi^{(0)}$ and Φ be two complex-valued infinitely differentiable functions defined respectively on $\mathbb{R}^d$ and $\mathbb{R}^d\backslash\{0\}$, which satisfy

$$\begin{cases} |\Phi^{(0)}(\xi)| > 0 \quad \text{if } |\xi| \leq 2, \\ \sup_{k\in\mathbb{N}_0} 2^{-k\alpha_0}\|\Phi^{(0)}(2^k\cdot)\varphi\|_{H_2^\sigma} < \infty, \end{cases} \tag{8.7}$$

and

$$\begin{cases} |\Phi(\xi)| > 0 \quad \text{if } \frac{1}{2} \leq |\xi| \leq 2, \\ \sup_{k\in\mathbb{N}_0} 2^{-k\alpha_0}\|\Phi(2^k\cdot)\varphi\|_{H_2^\sigma} < \infty, \\ \int_{\mathbb{R}^d}(1+|s|^2)^\sigma|\mathcal{F}^{-1}(\Phi\varphi^{(0)}I_{-\alpha_1})(s)|ds < \infty. \end{cases} \tag{8.8}$$

Recall that here $I_{-\alpha_1}(\xi)$ for $\xi \in \mathbb{R}^d$ is the symbol of the Fourier multiplier $I^{-\alpha_1}$, where $I^{-\alpha_1}$ is the Riesz potential $(-(2\pi)^{-2}\Delta)^{\frac{-\alpha_1}{2}}$.

Let $\Phi^{(j)} = \Phi(2^{-j}\cdot)$, $j \geq 1$ and Φ_j be the function whose Fourier transform is equal to $\Phi^{(j)}$ for any $j \in \mathbb{N}_0$.

Theorem 8.12. *Let $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$. Assume that $\alpha_0 < \alpha < \alpha_1$, $\alpha_1 > 0$ and $\Phi^{(0)}$, Φ satisfy conditions* (8.7), (8.8)*. Then for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$, we have*

$$\|f\|_{F_p^{\alpha,c}} \approx \|(\sum_{j\geq 0} 2^{2j\alpha}|\Phi_j * f|^2)^{\frac{1}{2}}\|_p, \tag{8.9}$$

where the relevant constants are independent of f.

Proof. We follow the pattern of the proof of [72, Theorem 4.17]. Denote the norm on the right hand side of (8.9) by $\|f\|_{F^{\alpha,c}_{p,\Phi}}$.

Step 1. Let $\varphi_k = 0$ (and so is $\varphi^{(k)}$) if $k < 0$. Given a positive integer K, for any $j \in \mathbb{N}_0$, we write

$$\Phi^{(j)} = \sum_{k=-\infty}^{K-1} \Phi^{(j)}\varphi^{(j+k)} + \sum_{k=K}^{\infty} \Phi^{(j)}\varphi^{(j+k)}.$$

Then

$$\Phi_j * f = \sum_{k\leq K-1} \Phi_j * \varphi_{j+k} * f + \sum_{k\geq K} \Phi_j * \varphi_{j+k} * f. \tag{8.10}$$

Temporarily, we take for granted that the second series above is convergent not only in $\mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$ but also in $F^{\alpha,c}_p(\mathbb{R}^d, \mathcal{M})$, which is to be settled up in the last step. Then we obtain

$$\|f\|_{F^{\alpha,c}_{p,\Phi}} \leq \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned}
\mathrm{I} &= \sum_{k\leq K-1} \Big\|\big(\sum_{j\geq 1} 2^{2j\alpha}|\Phi_j * \varphi_{j+k} * f|^2\big)^{\frac12}\Big\|_p,\\
\mathrm{II} &= \sum_{k\leq K-1} \|\Phi_0 * \varphi_k * f\|_p,\\
\mathrm{III} &= \sum_{k\geq K} \Big\|\big(\sum_{j\geq 0} 2^{2j\alpha}|\Phi_j * \varphi_{j+k} * f|^2\big)^{\frac12}\Big\|_p.
\end{aligned}$$

The term II is easy to deal with. By [71, Lemma 1.7] and (8.7), we obtain

$$\begin{aligned}
\sum_{k=0}^{K-1} \|\Phi_0 * \varphi_k * f\|_p &= \sum_{k=0}^{K-1} \|\Phi_0 * (\varphi_{k-1} + \varphi_k + \varphi_{k+1}) * \varphi_k * f\|_p\\
&\lesssim \sum_{k=0}^{K-1} \|\varphi_k * f\|_p \|\Phi_0 * (\varphi_{k-1} + \varphi_k + \varphi_{k+1})\|_1\\
&\lesssim \sup_{k\in\mathbb{N}_0} 2^{-k\alpha_0}\|\Phi^{(0)}(2^k\cdot)\varphi\|_{H^\sigma_2} \sum_{k=0}^{K-1} 2^{k(\alpha_0-\alpha)}\|2^{k\alpha}\varphi_k * f\|_p\\
&\lesssim C_K\|f\|_{F^{\alpha,c}_p}.
\end{aligned}$$

Now let us treat the terms I and III separately. By the support assumption of $\varphi^{(k)}$ and the property that it is equal to 1 when $|\xi| \leq 1$, for $k \leq K-1$, we have

$$\begin{aligned}
\Phi(\xi)\varphi^{(k)}(\xi) &= \frac{\Phi(\xi)\varphi^{(0)}(2^{-K}\xi)}{|\xi|^{\alpha_1}}|\xi|^{\alpha_1}\varphi^{(k)}(\xi)\\
&= 2^{k\alpha_1}\eta(\xi)\rho^{(k)}(\xi),
\end{aligned} \tag{8.11}$$

where η, ρ are defined by

$$\eta(\xi) = \frac{\Phi(\xi)\varphi^{(0)}(2^{-K}\xi)}{|\xi|^{\alpha_1}} \quad \text{and} \quad \rho(\xi) = |\xi|^{\alpha_1}\varphi(\xi).$$

Let $\eta^{(j)} = \eta(2^{-j}\cdot)$, $j \in \mathbb{Z}$. For $j \geq 1$, define $\eta_j = \mathcal{F}^{-1}(\eta^{(j)})$. Then for any $j \geq 1$, we have

$$\Phi_j * \varphi_{j+k} * f = 2^{k\alpha_1}\eta_j * \rho_{j+k} * f.$$

Now we are ready to estimate I. Applying Theorem 5.6 and Lemma 8.2 twice, we get

$$
\begin{aligned}
\mathrm{I} &= \sum_{k\le K-1} 2^{k(\alpha_1-\alpha)} \big\|\big(\sum_{j\ge 1} 2^{2(j+k)\alpha}|\eta_j * \rho_{j+k} * f|^2\big)^{\frac12}\big\|_p \\
&= \sum_{k\le K-1} 2^{k(\alpha_1-\alpha)} \big\|\big(\sum_{j\ge k+1} 2^{2j\alpha}|\eta_{j-k} * \rho_j * f|^2\big)^{\frac12}\big\|_p \\
&\lesssim \sum_{k\le K-1} 2^{k(\alpha_1-\alpha)} \max\big\{\|\eta^{(-k)}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}, \max_{-2\le\ell\le 2}\|\eta^{(-k-\ell)}\varphi\|_{H_2^\sigma}\big\} \\
&\quad\cdot \big\|\big(\sum_{j\ge 0} 2^{2j\alpha}|\rho_j * f|^2\big)^{\frac12}\big\|_p \\
&\lesssim \sum_{k\le K-1} 2^{k(\alpha_1-\alpha)} \max\big\{\|\eta^{(-k)}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}, \max_{-2\le\ell\le 2}\|\eta^{(-k-\ell)}\varphi\|_{H_2^\sigma}\big\} \\
&\quad\cdot \max\big\{\|I_{\alpha_1}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}, \|I_{\alpha_1}\varphi\|_{H_2^\sigma}\big\}\big\|\big(\sum_{j\ge 0} 2^{2j\alpha}|\varphi_j * f|^2\big)^{\frac12}\big\|_p \\
&= \sum_{k\le K-1} 2^{k(\alpha_1-\alpha)} \max\big\{\|\eta^{(-k)}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}, \max_{-2\le\ell\le 2}\|\eta^{(-k-\ell)}\varphi\|_{H_2^\sigma}\big\} \\
&\quad\cdot \max\big\{\|I_{\alpha_1}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}, \|I_{\alpha_1}\varphi\|_{H_2^\sigma}\big\}\|f\|_{F_p^{\alpha,c}}.
\end{aligned}
\tag{8.12}
$$

First, it is obvious that $\|I_{\alpha_1}\varphi\|_{H_2^\sigma} < \infty$. Then we deal with the term $I_{\alpha_1}(\varphi^{(0)}+\varphi^{(1)})$, which can be reduced to $I_{\alpha_1}\varphi^{(0)}$ by dilation. Let N be a positive integer such that $\alpha_1 > \frac{1}{N}$. If the dimension d is odd, we consider the function $F(z) = e^{(z-\frac{N+2}{2N+2})^2}|\xi|^{\alpha_1-\frac12-\frac1N+(1+\frac1N)z}\varphi^{(0)}$, which is continuous on the strip $\{z\in\mathbb{C} : 0\le \mathrm{Re}(z)\le 1\}$, and analytic in the interior. A direct computation shows that $\sup_{t\in\mathbb{R}}\|F(\mathrm{i}t)\|_{H_2^{\frac d2-\frac12}} < \infty$ and $\sup_{t\in\mathbb{R}}\|F(1+\mathrm{i}t)\|_{H_2^{\frac d2+\frac12}} < \infty$. Then for $\theta = \frac{\frac1N+\frac12}{\frac1N+1} > \frac12$, we have

$$F(\theta) = I_{\alpha_1}\varphi^{(0)} \in H_2^\sigma(\mathbb{R}^d) = \big(H_2^{\frac d2-\frac12}(\mathbb{R}^d), H_2^{\frac d2+\frac12}(\mathbb{R}^d)\big)_\theta,$$

for some $\sigma > \frac d2$. On the other hand, if d is even, set $F(z) = e^{(z-\frac{1}{2N})^2}|\xi|^{N\alpha_1 z+\frac{\alpha_1}{2}}\varphi^{(0)}$. We can also check that $\sup_{t\in\mathbb{R}}\|F(\mathrm{i}t)\|_{H_2^{\frac d2}} < \infty$, and that $\sup_{t\in\mathbb{R}}\|F(1+\mathrm{i}t)\|_{H_2^{\frac d2+1}} < \infty$. Then for $\theta = \frac{1}{2N}$, we have

$$F(\theta) = I_{\alpha_1}\varphi^{(0)} \in H_2^{\frac d2+\frac{1}{2N}}(\mathbb{R}^d) = \big(H_2^{\frac d2}(\mathbb{R}^d), H_2^{\frac d2+1}(\mathbb{R}^d)\big)_\theta.$$

Thus, for any $\alpha_1 > 0$, we can always choose a positive $\sigma > \frac d2$ such that $I_{\alpha_1}\varphi^{(0)} \in H_2^\sigma(\mathbb{R}^d)$. Next, we have to estimate $\|\eta^{(-k)}\varphi\|_{H_2^\sigma}$ and $\|\eta^{(-k)}\varphi^{(0)}\|_{H_2^\sigma}$ uniformly in k, which will yield the convergence of the last sum in (8.12) by dilation again. To this end, note that by (8.8), $\check\eta$ is integrable on $\mathbb{R}^d$, then we use the Cauchy-Schwarz inequality in the following way:

$$
\begin{aligned}
|\mathcal{F}^{-1}(\eta^{(-k)}\varphi)(s)|^2 &= \Big|\int_{\mathbb{R}^d}\check\eta(t)\mathcal{F}^{-1}(\varphi)(s-2^k t)dt\Big|^2 \\
&\le \|\check\eta\|_1\int_{\mathbb{R}^d}|\check\eta(t)|\cdot|\mathcal{F}^{-1}(\varphi)(s-2^k t)|^2 dt.
\end{aligned}
$$

For $k \le K-1$, we have

$$\begin{aligned}
\|\eta^{(-k)}\varphi\|_{H_2^\sigma}^2 &= \int_{\mathbb{R}^d}(1+|s|^2)^\sigma|\mathcal{F}^{-1}(\eta^{(-k)}\varphi)(s)|^2 ds \\
&\le \|\check{\eta}\|_1 \int_{\mathbb{R}^d}(1+|s|^2)^\sigma \int_{\mathbb{R}^d}|\check{\eta}(t)|\cdot|\mathcal{F}^{-1}(\varphi)(s-2^k t)|^2 dtds \\
&\lesssim \|\check{\eta}\|_1 \int_{\mathbb{R}^d}(1+|2^k t|^2)^\sigma|\check{\eta}(t)| \int_{\mathbb{R}^d}(1+|s-2^k t|^2)^\sigma|\mathcal{F}^{-1}(\varphi)(s-2^k t)|^2 dsdt \qquad (8.13) \\
&\le 2^{K\sigma}\|\check{\eta}\|_1 \int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\check{\eta}(t)|dt \int_{\mathbb{R}^d}(1+|s|^2)^\sigma|\mathcal{F}^{-1}(\varphi)(s)|^2 ds \\
&\le C_{\varphi_0,\sigma,K}\big(\int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\check{\eta}(t)|dt\big)^2.
\end{aligned}$$

The other term is dealt with in the same way. Combining the previous inequalities, we obtain

$$\mathrm{I} \lesssim C_{\Phi,\varphi^{(0)},\alpha_1,\alpha,K}\int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\check{\eta}(t)|dt\,\|f\|_{F_p^{\alpha,c}}. \qquad (8.14)$$

In order to return from η back to φ_0, we write

$$\eta = I_{-\alpha_1}\Phi[\varphi^{(0)}(2^{-K}\cdot)-\varphi^{(0)}] + I_{-\alpha_1}\Phi\varphi^{(0)}.$$

Since $I_{-\alpha_1}\Phi(\varphi^{(0)}(2^{-K}\cdot)-\varphi^{(0)})$ is an infinitely differentiable function with compact support, we have

$$\int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\mathcal{F}^{-1}(I_{-\alpha_1}\Phi(\varphi^{(0)}(2^{-K}\cdot)-\varphi^{(0)}))(t)|dt = C'_{\Phi,\varphi^{(0)},\alpha_1,\alpha,K} < \infty.$$

Then (8.8) implies that

$$\int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\check{\eta}(t)|dt \lesssim C'_{\Phi,\varphi^{(0)},\alpha_1,\alpha,K} + \int_{\mathbb{R}^d}(1+|s|^2)^\sigma|\mathcal{F}^{-1}(I_{-\alpha_1}\Phi\varphi^{(0)})(s)|ds < \infty.$$

Therefore,

$$\mathrm{I} \lesssim \|f\|_{F_p^{\alpha,c}}.$$

Step 2. Now it remains to estimate the third term III. Let H be a Schwartz function such that

$$\operatorname{supp} H \subset \{\xi\in\mathbb{R}^d : \frac{1}{4}\le|\xi|\le 4\} \quad \text{and} \quad H(\xi)=1 \ \text{if} \ \frac{1}{2}\le|\xi|\le 2. \qquad (8.15)$$

Let $H^{(k)} = H(2^{-k}\cdot)$. For $k\ge K$, we have

$$\Phi(\xi)\varphi^{(k)}(\xi) = \frac{\Phi(\xi)}{|\xi|^{\alpha_0}}H^{(k)}(\xi)\varphi^{(k)}(\xi)|\xi|^{\alpha_0}, \qquad (8.16)$$

and

$$\Phi^{(0)}(\xi)\varphi^{(k)}(\xi) = \frac{\Phi^{(0)}(\xi)}{|\xi|^{\alpha_0}}H^{(k)}(\xi)\varphi^{(k)}(\xi)|\xi|^{\alpha_0}. \qquad (8.17)$$

For any $j\in\mathbb{N}_0$, we keep using the notation $\Phi_j = \mathcal{F}^{-1}(\Phi^{(j)})$ and $H_j = \mathcal{F}^{-1}(H^{(j)})$. Thus, we have

$$\Phi_j * \varphi_{j+k} * f = 2^{k\alpha_0}(I_{-\alpha_0}\Phi)_j * H_{j+k} * (I_{\alpha_0}\varphi)_{j+k} * f.$$

Therefore,

$$\begin{aligned}\mathrm{III} &= \sum_{k\geq K} 2^{k(\alpha_0-\alpha)} \big\| \big(\sum_{j\geq 0} 2^{2(j+k)\alpha} |(I_{-\alpha_0}\Phi)_j * H_{j+k} * (I_{\alpha_0}\varphi)_{j+k} * f|^2\big)^{\frac{1}{2}} \big\|_p \\ &= \sum_{k\geq K} 2^{k(\alpha_0-\alpha)} \big\| \big(\sum_{j\geq k} 2^{2j\alpha} |(I_{-\alpha_0}\Phi)_{j-k} * H_j * (I_{\alpha_0}\varphi)_j * f|^2\big)^{\frac{1}{2}} \big\|_p.\end{aligned}$$

Since both H and φ vanish near the origin, by Theorem 5.6 and Lemma 8.2, we obtain

$$\begin{aligned}&\sum_{k\geq K} 2^{k(\alpha_0-\alpha)} \big\| \big(\sum_{j\geq k} 2^{2j\alpha} |(I_{-\alpha_0}\Phi)_{j-k} * H_j * (I_{\alpha_0}\varphi)_j * f|^2\big)^{\frac{1}{2}} \big\|_p \\ &\lesssim \sup_{k\in\mathbb{N}_0} 2^{-k\alpha_0} \max\big\{ \max_{-2\leq \ell\leq 2} \|I_{-\alpha_0}\Phi(2^{k+\ell}\cdot)H(2^\ell\cdot)\varphi\|_{H_2^\sigma}, \|I_{-\alpha_0}\Phi^{(0)}(2^k\cdot)H(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}\big\} \\ &\quad \cdot \sum_{k\geq K} 2^{k(\alpha_0-\alpha)} \|f\|_{F_p^{\alpha,c}}.\end{aligned}$$

Then by (5.6), (8.7) and (8.8), we have, for any $-2\leq \ell\leq 2$,

$$\begin{aligned}&2^{-k\alpha_0}\|I_{-\alpha_0}\Phi(2^{k+\ell}\cdot)H(2^\ell\cdot)\varphi\|_{H_2^\sigma} \\ &\leq 2^{-k\alpha_0}\|\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma} \int_{\mathbb{R}^d} (1+|t|^2)^\sigma |\mathcal{F}^{-1}(I_{-\alpha_0}H(2^\ell\cdot))(t)|dt \\ &\lesssim 2^{-k\alpha_0}\|\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma} \leq \sup_{k\in\mathbb{N}_0} 2^{-k\alpha_0}\|\Phi(2^k\cdot)\varphi\|_{H_2^\sigma} < \infty,\end{aligned} \tag{8.18}$$

and

$$\begin{aligned}&2^{-k\alpha_0}\|I_{-\alpha_0}\Phi^{(0)}(2^k\cdot)H(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma} \\ &= 2^{-k\alpha_0}\|I_{-\alpha_0}\Phi^{(0)}(2^k\cdot)H\sum_{\ell'=-2}^{1}\varphi(2^{-\ell'}\cdot)\|_{H_2^\sigma} \\ &\lesssim 2^{-k\alpha_0}\sum_{\ell'=-2}^{1}\|I_{-\alpha_0}\Phi^{(0)}(2^{k+\ell'}\cdot)H(2^{\ell'}\cdot)\varphi\|_{H_2^\sigma} \\ &\leq 2^{-k\alpha_0}\sum_{\ell'=-2}^{1}\|\Phi^{(0)}(2^{k+\ell'}\cdot)\varphi\|_{H_2^\sigma}\int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\mathcal{F}^{-1}(I_{-\alpha_0}H(2^\ell\cdot))(t)|dt \\ &\lesssim \sup_{k\in\mathbb{N}_0} 2^{-k\alpha_0}\|\Phi^{(0)}(2^k\cdot)\varphi\|_{H_2^\sigma} < \infty.\end{aligned} \tag{8.19}$$

Then we get that

$$\mathrm{III} \leq C_{\Phi,\alpha_0,\alpha,K}\|f\|_{F_p^{\alpha,c}}.$$

Combining this estimate with those of I and II, we finally get

$$\|f\|_{F_{p,\Phi}^{\alpha,c}} \lesssim \|f\|_{F_p^{\alpha,c}}.$$

Step 3. We turn to the reverse inequality. Note that $\varphi^{(0)}(\xi)=1$ when $|\xi|\leq 1$, then by (8.7) and (8.8), for any $j\in\mathbb{N}$, we write

$$\varphi^{(j)}(\xi) = \varphi^{(j)}(\xi)\,\varphi^{(0)}(2^{-j-M}\xi) = \frac{\varphi^{(j)}(\xi)}{\Phi^{(j)}(\xi)}\varphi^{(0)}(2^{-j-M}\xi)\Phi^{(j)}(\xi), \quad j\in\mathbb{N}_0, \tag{8.20}$$

where M is a positive integer which will be chosen later. By Theorem 5.6 and Lemma 8.2,

$$\begin{aligned}
\|f\|_{F_p^{\alpha,c}} &= \Big\|\Big(\sum_{j\geq 0} 2^{2j\alpha}|\varphi_j * f|^2\Big)^{\frac{1}{2}}\Big\|_p \\
&\lesssim \max\Big\{\max_{-2\leq \ell\leq 2}\|\Phi^{-1}(2^{\ell}\cdot)\varphi(2^{\ell}\cdot)\varphi\|_{H_2^\sigma}, \|(\Phi^{(0)})^{-1}\varphi^{(0)}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}\Big\} \\
&\quad\cdot \Big\|\Big(\sum_{j\geq 0} 2^{2j\alpha}|(\varphi_0)_{j+M} * \Phi_j * f|^2\Big)^{\frac{1}{2}}\Big\|_p \\
&\lesssim \Big\|\Big(\sum_{j\geq 0} 2^{2j\alpha}|(\varphi_0)_{j+M} * \Phi_j * f|^2\Big)^{\frac{1}{2}}\Big\|_p,
\end{aligned}$$

where $(\varphi_0)_{j+M}$ is the Fourier inverse transform of $\varphi^{(0)}(2^{-j-M}\cdot)$. Let $h = 1-\varphi^{(0)}$. Write $\varphi^{(0)}(2^{-j-M}\xi)\Phi^{(j)}(\xi) = \Phi^{(j)}(\xi) - h^{(j+M)}(\xi)\Phi^{(j)}(\xi)$. Then, we have

$$\|f\|_{F_p^{\alpha,c}} \lesssim \|f\|_{F_{p,\Phi}^{\alpha,c}} + \Big\|\Big(\sum_{j\geq 0} 2^{2j\alpha}|h_{j+M} * \Phi_j * f|^2\Big)^{\frac{1}{2}}\Big\|_p,$$

where the relevant constant depends on p, σ, d and $\varphi^{(0)}$. Applying the arguments in the estimate of III, (8.16) with $h^{(M)}\Phi$ in place of Φ and (8.17) with $h^{(M)}\Phi^{(0)}$ in place of $\Phi^{(0)}$, we deduce

$$\begin{aligned}
&\Big\|\Big(\sum_{j\geq 0} 2^{2j\alpha}|h_{j+M} * \Phi_j * f|^2\Big)^{\frac{1}{2}}\Big\|_p \\
&\leq C_1 \sup_{k\geq M} 2^{-k\alpha_0}\max\Big\{\max_{-2\leq\ell\leq 2}\|h(2^{k-M+\ell}\cdot)\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma}, \|h(2^{k-M}\cdot)\Phi^{(0)}(2^{k}\cdot)\varphi\|_{H_2^\sigma}\Big\} \\
&\quad\cdot\sum_{k\geq M} 2^{k(\alpha_0-\alpha)}\|f\|_{F_p^{\alpha,c}} \\
&= \sup_{k\geq M} 2^{-k\alpha_0}\max\Big\{\max_{-2\leq\ell\leq 2}\|h(2^{k-M+\ell}\cdot)\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma}, \|h(2^{k-M}\cdot)\Phi^{(0)}(2^{k}\cdot)\varphi\|_{H_2^\sigma}\Big\} \\
&\quad\cdot C_1\frac{2^{M(\alpha_0-\alpha)}}{1-2^{\alpha_0-\alpha}}\|f\|_{F_p^{\alpha,c}},
\end{aligned}$$

where C_1 is a constant which depends on p, σ, d, H and α_0. Now we replace h in the above Sobolev norm by $1-\varphi^{(0)}$:

$$\|h(2^{k-M+\ell}\cdot)\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma} \leq \|\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma} + \|\varphi^{(0)}(2^{k-M+\ell}\cdot)\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma}.$$

The support assumptions of $\varphi^{(0)}$ and φ imply that when $k\geq M$, $\varphi^{(0)}(2^{k-M+\ell}\cdot)\varphi\neq 0$ if and only if $k+\ell = M$ or $k+\ell = M+1$. Then by (5.6), we have

$$\|\varphi^{(0)}(2^{k-M+\ell}\cdot)\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma} \leq C_2\|\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma},$$

where C_2 depends on $\varphi^{(0)}$, σ and d. Thus,

$$\|h(2^{k-M+\ell}\cdot)\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma} \leq (1+C_2)\|\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma}.$$

Similarly, we have

$$\|h(2^{k-M}\cdot)\Phi^{(0)}(2^{k}\cdot)\varphi\|_{H_2^\sigma} \leq (1+C_2)\|\Phi^{(0)}(2^{k}\cdot)\varphi\|_{H_2^\sigma}.$$

Putting all the estimates that we have obtained so far together, we get

$$
\begin{aligned}
&\|f\|_{F_p^{\alpha,c}} \\
&\leq C_3\Big(C_1(1+C_2)\frac{2^{M(\alpha_0-\alpha)}}{1-2^{\alpha_0-\alpha}}\sup_{k\geq M} 2^{-k\alpha_0}\max\{\|\Phi(2^{k+\ell}\cdot)\varphi\|_{H_2^\sigma}, \|\Phi^{(0)}(2^k\cdot)\varphi\|_{H_2^\sigma}\}\|f\|_{F_p^{\alpha,c}} \\
&\qquad + \|f\|_{F_{p,\Phi}^{\alpha,c}}\Big),
\end{aligned}
$$

where the three constants C_1, C_2, C_3 are independent of M, so we could take M large enough to make sure the multiple of $\|f\|_{F_p^{\alpha,c}}$ above is less than $\frac{1}{2}$, so that we have

$$\|f\|_{F_p^{\alpha,c}} \lesssim \|f\|_{F_{p,\Phi}^{\alpha,c}}.$$

Step 4. We now settle the convergence issue of the second series in (8.10). For every $j \geq 0$, $\Phi_j * \varphi_{j+k} * f$ is an $L_1(\mathcal{M}) + \mathcal{M}$-valued tempered distribution on $\mathbb{R}^d$. We now show that the series converges in $\mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$. By (8.18) and (8.19), for any $L > K$, we have

$$
\begin{aligned}
&2^{j\alpha}\sum_{k=K}^{L}\|\Phi_j * \varphi_{j+k} * f\|_p \\
&\lesssim \|I_{\alpha_0}\varphi\|_{H_2^\sigma}\sum_{k\geq K} 2^{k(\alpha_0-\alpha)}\sup_{k\in\mathbb{N}_0}\max\{2^{-k\alpha_0}\|\Phi(2^k\cdot)\varphi\|_{H_2^\sigma}, 2^{-k\alpha_0}\|\Phi^{(0)}(2^k\cdot)\varphi\|_{H_2^\sigma}\}\|f\|_{F_p^{\alpha,c}} \\
&\lesssim \|f\|_{F_p^{\alpha,c}}.
\end{aligned}
$$

Therefore, for any $j \geq 0$, $\sum_{k\geq K+1}\Phi_j * \varphi_{j+k} * f$ converges in $L_p(\mathcal{N})$, so in $\mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$ too. In the same way, we can show that the series also converges in $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$, which completes the proof. □

The following is the continuous analogue of Theorem 8.12. We will use similar notation for continuous parameters: given $\varepsilon > 0$, Φ_ε denotes the function whose Fourier transform is $\Phi^{(\varepsilon)} = \Phi(\varepsilon\cdot)$.

Theorem 8.13. *Keep the assumption of the previous theorem. For $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$, we have*

$$\|f\|_{F_p^{\alpha,c}} \approx \|\Phi_0 * f\|_p + \Big\|\Big(\int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}\Big\|_p. \tag{8.21}$$

Proof. This proof is very similar to the previous one. We keep the notation there and only point out the necessary modifications. First, we need to discretize the integral on the right hand side of (8.21). There exist two constants C_1, C_2 such that

$$C_1\sum_{j=0}^{\infty} 2^{2j\alpha}\int_{2^{-j-1}}^{2^{-j}}|\Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon} \leq \int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon} \leq C_2\sum_{j=0}^{\infty} 2^{2j\alpha}\int_{2^{-j-1}}^{2^{-j}}|\Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon}.$$

By approximation, we can assume that f is good enough so that each integral over the interval $(2^{-j-1}, 2^{-j})$ can be approximated uniformly by discrete sums. Instead of $\Phi^{(j)}(\xi) = \Phi(2^{-j}\xi)$, we have now $\Phi^{(\varepsilon)}(\xi) = \Phi(\varepsilon\xi)$ with $2^{-j-1} < \varepsilon \leq 2^{-j}$. We transfer the splitting (8.11) into:

$$\Phi^{(\varepsilon)}(\xi)\varphi_{j+k}(\xi) = \frac{\Phi(2^{-j}\cdot 2^j\varepsilon\xi)\varphi^{(0)}(2^{-K}\xi)}{|2^{-j}\xi|^{\alpha_1}}|2^{-j}\xi|^{\alpha_1}\varphi_{j+k}(\xi).$$

Thus,

$$\Phi_\varepsilon * \varphi_{j+k} * f = 2^{k\alpha_1}\eta_j * \rho_{j+k} * f$$

with

$$\eta(\xi) = \frac{\Phi(2^j\varepsilon\xi)\varphi^{(0)}(2^{-K}\xi)}{|\xi|^{\alpha_1}} \quad \text{and} \quad \rho(\xi) = |\xi|^{\alpha_1}\varphi(\xi).$$

We proceed as in step 1 of the previous theorem, where we can transfer (8.13) to the present setting:

$$\begin{aligned}\|\eta^{(-k)}\varphi\|_{H_2^\sigma} &\lesssim C_{\varphi^{(0)},\sigma,k}\int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\check{\eta}(t)|dt\\ &= C_{\varphi^{(0)},\sigma,k}\int_{\mathbb{R}^d}(1+|t|^2)^\sigma\big|\mathcal{F}^{-1}\big(I_{-\alpha_1}\Phi(\delta_j\cdot)\varphi^{(0)}(2^{-K}\cdot)\big)(t)\big|dt\\ &\le C_{\varphi^{(0)},\sigma,k}\delta_j^{\alpha_1}\int_{\mathbb{R}^d}(1+|t|^2)^\sigma\big|\mathcal{F}^{-1}\big(I_{-\alpha_1}\Phi\varphi^{(0)}(\delta_j^{-1}2^{-K}\cdot)\big)(t)\big|dt,\end{aligned}$$

where $\delta_j = 2^j\varepsilon$ and $\frac{1}{2} < \delta_j \le 1$. The last integral is estimated as follows:

$$\begin{aligned}&\int_{\mathbb{R}^d}(1+|t|^2)^\sigma\big|\mathcal{F}^{-1}\big(I_{-\alpha_1}\Phi\varphi^{(0)}(\delta_j^{-1}2^{-K}\cdot)\big)(t)\big|dt\\ &\le \int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\mathcal{F}^{-1}(I_{-\alpha_1}\Phi\varphi^{(0)})(t)|dt\\ &\quad + \int_{\mathbb{R}^d}(1+|t|^2)^\sigma\big|\mathcal{F}^{-1}\big(I_{-\alpha_1}\Phi[\varphi^{(0)}-\varphi^{(0)}(\delta_j^{-1}2^{-K}\cdot)]\big)(t)\big|dt\\ &\le \int_{\mathbb{R}^d}(1+|t|^2)^\sigma|\mathcal{F}^{-1}(I_{-\alpha_1}\Phi\varphi^{(0)})(t)|dt\\ &\quad + \sup_{\frac{1}{2}<\delta\le 1}\int_{\mathbb{R}^d}(1+|t|^2)^\sigma\big|\mathcal{F}^{-1}\big(I_{-\alpha_1}\Phi[\varphi^{(0)}-\varphi^{(0)}(\delta^{-1}2^{-K}\cdot)]\big)(t)\big|dt.\end{aligned}$$

Note that the above supremum is finite since $I_{-\alpha_1}\Phi[\varphi^{(0)}-\varphi^{(0)}(\delta^{-1}2^{-K}\cdot)]$ is a compactly supported and infinitely differentiable function and its inverse Fourier transform depends continuously on δ. Then it follows that for $2^{-j-1} \le \varepsilon \le 2^{-j}$,

$$\sum_{k\le K-1}\Big\|\Big(\int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}\Big\|_p \lesssim \sum_{k\le K-1}2^{k(\alpha_1-\alpha)}\|f\|_{F_p^{\alpha,c}} \lesssim \|f\|_{F_p^{\alpha,c}}.$$

We can make similar modifications in step 2 of the previous theorem and then establish the third part. Moreover, by the previous theorem, $\|\Phi_0 * f\|_p \lesssim \|f\|_{F_p^{\alpha,c}}$. Thus, we have proved

$$\|\Phi_0 * f\|_p + \Big\|\Big(\int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}\Big\|_p \lesssim \|f\|_{F_p^{\alpha,c}}.$$

For the reverse inequality, we follow the argument in step 3 in the previous proof. By (8.8), there exists $2 < a \le 2\sqrt{2}$ such that $\Phi(\xi) > 0$ on $\{\xi : a^{-1} \le |\xi| \le a\}$. Then for $j \ge 1$, $R_j = \{\varepsilon : a^{-1}2^{-j+1} < \varepsilon \le a2^{-j-1}\}$ are disjoint sub intervals on $(0,1]$, and $\frac{\varphi^{(j)}}{\Phi^{(\varepsilon)}}$ is well-defined for any $\varepsilon \in R_j$. We slightly modify (8.20) as follows: for any $\varepsilon \in R_j$, we have

$$\varphi^{(j)}(\xi) = \varphi^{(j)}\varphi^{(0)}(2^{-j-K}\xi) = \frac{\varphi^{(j)}(\xi)}{\Phi^{(\varepsilon)}(\xi)}\varphi^{(0)}(2^{-j-K}\xi)\Phi^{(\varepsilon)}(\xi), \quad j \in \mathbb{N}_0.$$

Since for any $-2 \le \ell \le 2$,

$$\|\Phi^{-1}(2^{-j}\varepsilon^{-1}2^\ell\cdot)\varphi(2^\ell\cdot)\varphi\|_{H_2^\sigma} \le \sup_{2a^{-1}\le\delta\le\frac{a}{2}}\|\Phi^{-1}(\delta 2^\ell\cdot)\varphi(2^\ell\cdot)\varphi\|_{H_2^\sigma} < \infty$$

and

$$\|(\Phi^{(0)})^{-1}(2^{-j}\varepsilon^{-1}\cdot)\varphi^{(0)}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma} \leq \sup_{2a^{-1}\leq\delta\leq\frac{a}{2}} \|(\Phi^{(0)})^{-1}(\delta\cdot)\varphi^{(0)}(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma} < \infty,$$

we follow the argument in step 3 in the previous theorem to get

$$\begin{aligned}\|f\|_{F_p^{\alpha,c}} &\lesssim \|(\sum_{j\geq 0} 2^{2j\alpha}\int_{R_j} |(\varphi_0)_{j+k} * \Phi_\varepsilon * f|^2)^{\frac12}\|_p \\ &\lesssim \|\Phi_0 * f\|_p + \Big\|(\int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon})^{\frac12}\Big\|_p + \|\Big(\sum_{j\geq 0} 2^{2j\alpha}\int_{R_j}|h_{j+k} * \Phi_\varepsilon * f|^2\frac{d\varepsilon}{\varepsilon}\Big)^{\frac12}\Big\|_p.\end{aligned}$$

The remaining of the proof follows step 3 with necessary modifications. □

8.5 Characterizations via Lusin functions

We are going to give some characterizations for Triebel-Lizorkin spaces via Lusin square functions. As what we did in the last section, we will keep using Fourier multiplier theorems as our main tool. For the case $p > 1$, we already have Theorem 5.9. For $p = 1$, we need the following lemma. By virtue of Corollary 5.11, its proof is similar to that of Lemma 8.2, and is left to the reader.

Lemma 8.14. *Keep the assumption in Theorem 5.9 and assume additionally that for any $j \geq 1$, $\rho_j = \rho(2^{-j}\cdot)$ for some Schwartz function with $\operatorname{supp}\rho \subset \{\xi : 2^{-1} \leq |\xi| \leq 2\}$ and $\rho(\xi) > 0$ for any $2^{-1} < |\xi| < 2$, and that $\operatorname{supp}\rho_0 \subset \{\xi : |\xi| \leq 2\}$ and $\rho_0(\xi) > 0$ for any $|\xi| < 2$. Then for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$,*

$$\begin{aligned}&\Big\|(\sum_{j\geq 0} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})} |\check{\phi}_j * \check{\rho}_j * f(\cdot+t)|^2 dt)^{\frac12}\Big\|_1 \\ &\lesssim \max\{\sup_{\substack{j\geq 1\\ -2\leq k\leq 2}} \|\phi_j(2^{j+k}\cdot)\varphi\|_{H_2^\sigma}, \|\phi_0(\varphi^{(0)}+\varphi^{(1)})\|_{H_2^\sigma}\} \\ &\quad\cdot \Big\|(\sum_{j\geq 0} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})} |\check{\rho}_j * f(\cdot+t)|^2 dt)^{\frac12}\Big\|_1.\end{aligned}$$

Combining the above Lemma with Theorem 5.9, we obtain the following characterization via Lusin square functions associated to φ given by the condition (1.1). We keep using the notation φ_j being the function whose Fourier transform is equal to $\varphi(2^{-j}\cdot)$ for $j \in \mathbb{N}$, and φ_0 being the function whose Fourier transform is equal to $1 - \sum_{j\geq 1}\varphi(2^{-j}\cdot)$.

Proposition 8.15. *For $1 \leq p < \infty$ and $f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$, we have*

$$\|f\|_{F_p^{\alpha,c}} \approx \|\varphi_0 * f\|_p + \Big\|(\sum_{j\geq 1} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})} |\varphi_j * f(\cdot+t)|^2 dt)^{\frac12}\Big\|_p. \tag{8.22}$$

Proof. For any $f \in F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$, by the lifting property in Proposition 8.5, we have $J^\alpha f \in \mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$. Then, we apply the discrete characterization in Theorem 6.7 with $\phi = J^{-\alpha}\varphi_0$ and $\Phi = I^{-\alpha}\varphi$ to $J^\alpha f$,

$$\|f\|_{F_p^{\alpha,c}} \approx \|J^\alpha f\|_{\mathrm{h}_p^c} \approx \|\varphi_0 * f\|_p + \|s_{I^{-\alpha}\varphi}^{c,D}(J^\alpha f)\|_p.$$

Following the argument in the proof of (8.1), we can prove

$$\big\|s^{c,D}_{I^{-\alpha}\varphi}(J^\alpha f)\big\|_p \approx \big\|s^{c,D}_{I^{-\alpha}\varphi}(I^\alpha f)\big\|_p.$$

Moreover, we can easily check that

$$\big\|s^{c,D}_{I^{-\alpha}\varphi}(I^\alpha f)\big\|_p = \Big\|(\sum_{j\geq 1} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})}|\varphi_j * f(\cdot+t)|^2 dt)^{\frac12}\Big\|_p.$$

Therefore, we conclude

$$\|f\|_{F^{\alpha,c}_p} \approx \|\varphi_0 * f\|_p + \Big\|(\sum_{j\geq 1} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})}|\varphi_j * f(\cdot+t)|^2 dt)^{\frac12}\Big\|_p.$$

The assertion is proved. □

Theorem 8.16. *Let $1\leq p<\infty$ and $\alpha\in\mathbb{R}$. Assume that $\alpha_0<\alpha<\alpha_1$, $\alpha_1>0$ and $\Phi^{(0)}$, Φ satisfy conditions* (8.7)*,* (8.8)*. Then for any $f\in\mathcal{S}'(\mathbb{R}^d;L_1(\mathcal{M})+\mathcal{M})$, we have*

$$\|f\|_{F^{\alpha,c}_p(\mathbb{R}^d,\mathcal{M})} \approx \|\Phi_0 * f\|_p + \Big\|(\sum_{j\geq 1} 2^{j(2\alpha+d)}\int_{B(0,2^{-j})}|\Phi_j * f(\cdot+t)|^2 dt)^{\frac12}\Big\|_p,$$

where the equivalent constant is independent of f.

Proof. This proof is very similar to that of Theorem 8.12. The main target is to replace the standard test functions φ and φ_0 in Proposition 8.15 with Φ and Φ_0 satisfying (8.7) and (8.8). This time we need to use the Lusin type multiplier theorem i.e. Theorem 5.9, instead of Theorem 5.6. For the special case $p=1$, we apply Lemma 8.14 instead of Lemma 8.2. □

Using a similar argument as in Theorem 8.13, we also have the following continuous analogue of the above theorem. This is the general characterization of Triebel-Lizorkin spaces by Lusin square functions.

Theorem 8.17. *Keep the assumption in the previous theorem. Then for any $L_1(\mathcal{M})+\mathcal{M}$-valued tempered distribution f on $\mathbb{R}^d$, we have*

$$\|f\|_{F^{\alpha,c}_p(\mathbb{R}^d,\mathcal{M})} \approx \|\Phi_0 * f\|_p + \Big\|(\int_{\widetilde{\Gamma}}\varepsilon^{-2\alpha}|\Phi_\varepsilon * f(\cdot+t)|^2\frac{dtd\varepsilon}{\varepsilon^{d+1}})^{\frac12}\Big\|_p.$$

8.6 Atomic decomposition

For every $l=(l_1,\cdots,l_d)\in\mathbb{Z}^d$, $\mu\in\mathbb{N}_0$, we define $Q_{\mu,l}$ in $\mathbb{R}^d$ to be the cubes centered at $2^{-\mu}l$, and with side length $2^{-\mu}$.

Let $\mathbb{D}_d$ be the collection of all the cubes $Q_{\mu,l}$ defined above. We write $(\mu,l)\leq(\mu',l')$ if

$$\mu\geq\mu' \quad\text{and}\quad Q_{\mu,l}\subset 2Q_{\mu',l'}.$$

For $a\in\mathbb{R}$, let $a_+=\max\{a,0\}$ and $[a]$ the largest integer less than or equal to a. Recall that $|\gamma|_1=\gamma_1+\cdots+\gamma_d$ for $\gamma\in\mathbb{N}_0^d$, $s^\beta=s_1^{\beta_1}\cdots s_d^{\beta_d}$ for $s\in\mathbb{R}^d$, $\beta\in\mathbb{N}_0^d$ and J^α is the Bessel potential of order α.

Definition 8.18. Let $\alpha\in\mathbb{R}$, and let K and L be two integers such that

$$K\geq([\alpha]+1)_+ \quad\text{and}\quad L\geq\max\{[-\alpha],-1\}.$$

(1) A function $b \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is called an $(\alpha, 1)$-atom if

- $\operatorname{supp} b \subset 2Q_{0,k}$;
- $\tau\big(\int_{\mathbb{R}^d} |D^\gamma b(s)|^2 ds\big)^{\frac{1}{2}} \leq 1, \ \forall \gamma \in \mathbb{N}_0^d, \ |\gamma|_1 \leq K$.

(2) Let $Q = Q_{\mu,l} \in \mathbb{D}_d$, a function $a \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is called an (α, Q)-subatom if

- $\operatorname{supp} a \subset 2Q$;
- $\tau\big(\int_{\mathbb{R}^d} |D^\gamma a(s)|^2 ds\big)^{\frac{1}{2}} \leq |Q|^{\frac{\alpha}{d} - \frac{|\gamma|_1}{d}}, \ \forall \gamma \in \mathbb{N}_0^d, \ |\gamma|_1 \leq K$;
- $\int_{\mathbb{R}^d} s^\beta a(s) ds = 0, \ \forall \beta \in \mathbb{N}_0^d, \ |\beta|_1 \leq L$.

(3) A function $g \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is called an $(\alpha, Q_{k,m})$-atom if

$$\tau\big(\int_{\mathbb{R}^d} |J^\alpha g(s)|^2 ds\big)^{\frac{1}{2}} \lesssim |Q_{k,m}|^{-\frac{1}{2}} \quad \text{and} \quad g = \sum_{(\mu,l)\leq(k,m)} d_{\mu,l} a_{\mu,l}, \tag{8.23}$$

for some $k \in \mathbb{N}_0$ and $m \in \mathbb{Z}^d$, where the $a_{\mu,l}$'s are $(\alpha, Q_{\mu,l})$-subatoms and the $d_{\mu,l}$'s are complex numbers such that

$$\big(\sum_{(\mu,l)\leq(k,m)} |d_{\mu,l}|^2\big)^{\frac{1}{2}} \leq |Q_{k,m}|^{-\frac{1}{2}}.$$

Remark 8.19. If $L < 0$, the third assumption of an (α, Q)-subatom means that no moment cancellation is required. In the second assumption of an $(\alpha, 1)$-atom b and that of an (α, Q)-subatom a, it is tacitly assumed that b and a have derivatives up to order K. For such a, we can define a norm by

$$\|a\|_* = \sup_{|\gamma|_1 \leq K} \|D^\gamma a\|_{L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))}.$$

Then the convergence in (8.23) is understood in this norm, and we will see that the atom g in (8.23) belongs to $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.

Remark 8.20. In the classical case, the first size estimate in (8.23) is not necessary. In other words, if $g = \sum_{(\mu,l)\leq(k,m)} d_{\mu,l} a_{\mu,l}$ with the subatoms $a_{\mu,l}$'s and the complex numbers $d_{\mu,l}$'s such that $\big(\sum_{(\mu,l)\leq(k,m)} |d_{\mu,l}|^2\big)^{\frac{1}{2}} \leq |Q_{k,m}|^{-\frac{1}{2}}$, then g satisfies that estimate in (8.23) automatically. We refer the readers to [68] for more details. Unfortunately, in the current setting, we are not able to prove this estimate, so we just add it in (8.23) for safety.

The following is our main result on the atomic decomposition of $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. The idea comes from [68, Theorem 3.2.3], but many techniques used are different from those of [68, Theorem 3.2.3] due to the noncommutativity.

Theorem 8.21. *Let $\alpha \in \mathbb{R}$ and K, L be two integers fixed as in Definition 8.18. Then any $f \in F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ can be represented as*

$$f = \sum_{j=1}^{\infty} (\mu_j b_j + \lambda_j g_j), \tag{8.24}$$

where the b_j's are $(\alpha, 1)$-atoms, the g_j's are (α, Q)-atoms, and μ_j, λ_j are complex numbers with

$$\sum_{j=1}^{\infty} (|\mu_j| + |\lambda_j|) < \infty. \tag{8.25}$$

Moreover, the infimum of (8.25) with respect to all admissible representations is an equivalent norm in $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.

Proof. Step 1. First, we show that any $f \in F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ admits the representation (8.24) and

$$\sum_{j=1}^{\infty}(|\mu_j|+|\lambda_j|) \lesssim \|f\|_{F_1^{\alpha,c}}.$$

The proof of this part is similar to the proof of Theorem 7.6. Let κ be the Schwartz function defined in the proof of Theorem 7.6. We take $\widehat{\Phi} = |\cdot|^N \widehat{\kappa}$ with N a positive even integer such that $N \geq \max\{L,\alpha\}$, then Φ can be normalized as follows:

$$\int_0^\infty \widehat{\Phi}(\varepsilon\xi)^2 \frac{d\varepsilon}{\varepsilon} = 1, \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\}.$$

Since $-\alpha + N \geq 0$, we have

$$\sum_{j=-\infty}^{\infty} (J_{-\alpha}\widehat{\Phi})(2^{-j}\xi)^2 = \mathit{\Phi}(\xi), \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\}, \tag{8.26}$$

and

$$\sum_{j=-\infty}^{\infty} (I_{-\alpha}\widehat{\Phi})(2^{-j}\xi)^2 = \mathit{\Phi}'(\xi), \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\}, \tag{8.27}$$

where $\mathit{\Phi}$ and $\mathit{\Phi}'$ are two functions which are rapidly decreasing and infinitely differentiable on $\mathbb{R}^d \setminus \{0\}$. Applying the Paley-Wiener-Schwartz theorem, we get a compactly supported function $\Phi_0 \in \mathcal{S}$ such that

$$\widehat{\Phi}_0(\xi) = 1 - \int_0^1 \widehat{\Phi}(\varepsilon\xi)^2 \frac{d\varepsilon}{\varepsilon}.$$

Denote by Φ_ε the Fourier inverse transform of $\Phi(\varepsilon\cdot)$. For any $f \in F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$, we have

$$f = \Phi_0 * f + \int_0^1 \Phi_\varepsilon * \Phi_\varepsilon * f \frac{d\varepsilon}{\varepsilon}. \tag{8.28}$$

Let us deal with the two terms on the right hand side of (8.28) separately.

The term $\Phi_0 * f$ is easy to treat. If $\alpha \geq 0$, Proposition 8.4 ensures that $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. Then we can repeat the first part of the proof of Theorem 7.6: for any $f \in F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$, $\Phi_0 * f$ admits the decomposition

$$\Phi_0 * f = \sum_j \mu_j b_j,$$

with

$$\sum_j |\mu_j| \lesssim \|f\|_{\mathrm{h}_1^c} \lesssim \|f\|_{F_1^{\alpha,c}},$$

where the b_j's, together with their derivatives $D^\gamma b_j$'s, satisfy (7.7) with some constants C_γ depending on γ. When K is fixed, we can normalize the b_j's by $\max_{|\gamma|_1 \leq K} |C_\gamma|$, then the new b_j's are $(\alpha,1)$-atoms. If $\alpha < 0$, by Propositions 8.4 and 8.5, we have $J^{[\alpha]} f \in F_1^{\alpha-[\alpha],c} \subset \mathrm{h}_1^c$. Then $J^{[\alpha]}\Phi_0 * f$ admits the decomposition

$$J^{[\alpha]}\Phi_0 * f = \sum_j \mu_j b_j,$$

with $\sum_j |\mu_j| \lesssim \|J^{[\alpha]} f\|_{\mathrm{h}_1^c} \lesssim \|f\|_{F_1^{\alpha,c}}$. Then

$$\Phi_0 * f = \sum_j \mu_j J^{-[\alpha]} b_j.$$

If $-[\alpha]$ is even, it is obvious that $\operatorname{supp} J^{-[\alpha]}b_j \subset \operatorname{supp} b_j$. Moreover, for any $\gamma \in \mathbb{N}_0^d$ such that $|\gamma|_1 \leq K$,

$$\tau\Big(\int_{\mathbb{R}^d} |D^\gamma J^{-[\alpha]} b_j(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim \sum_{|\gamma'|_1 \leq K-2[\alpha]} \tau\Big(\int_{\mathbb{R}^d} |D^{\gamma'} b_j(s)|^2 ds\Big)^{\frac{1}{2}} \leq C_K.$$

We normalize $J^{-[\alpha]}b_j$ by this constant C_K depending on K, then we can make it an $(\alpha,1)$-atom. Now we deal with the case when $-[\alpha]$ is odd. Since $-[\alpha]+1$ is even, it suffices to replace $[\alpha]$ in the above argument by $[\alpha]-1$, and then we get the desired decomposition.

Step 2. Now we turn to the second term on the right hand side of (8.28). It follows from Theorem 8.17 and the definition of the tent space that $\varepsilon^{-\alpha}\Phi_\varepsilon * f \in T_1^c(\mathbb{R}^d, \mathcal{M})$ and

$$\|\varepsilon^{-\alpha}\Phi_\varepsilon * f\|_{T_1^c} \lesssim \|f\|_{F_1^{\alpha,c}}.$$

By Lemma 7.4, we have

$$\varepsilon^{-\alpha}\Phi_\varepsilon * f(s) = \sum_{j=1}^\infty \lambda_j b_j(s,\varepsilon), \tag{8.29}$$

where the b_j's are T_1^c-atoms based on the cubes Q_j's with $|Q_j| \leq 1$. Then, if we set $a_j(s,\varepsilon) = \varepsilon^\alpha b_j(s,\varepsilon)$, we obtain

$$\Phi_\varepsilon * f(s) = \sum_{j=1}^\infty \lambda_j a_j(s,\varepsilon)$$

and

$$\sum_{j=1}^\infty |\lambda_j| \lesssim \|\varepsilon^{-\alpha}\Phi_\varepsilon * f\|_{T_1^c} \lesssim \|f\|_{F_1^{\alpha,c}}. \tag{8.30}$$

In particular,

$$\operatorname{supp} a_j \subset T(Q_j) \quad \text{and} \quad \tau\Big(\int_{T(Q_j)} \varepsilon^{-2\alpha}|a_j(s,\varepsilon)|^2 \frac{ds d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \leq |Q_j|^{-\frac{1}{2}}. \tag{8.31}$$

For every a_j, we set

$$g_j(s) = \pi_\Phi(a_j)(s) = \int_0^1 \Phi_\varepsilon * a_j(s,\varepsilon)\frac{d\varepsilon}{\varepsilon}. \tag{8.32}$$

Then $\operatorname{supp} g_j \subset 2Q_j$. We arrive at the decomposition

$$\int_0^1 \Phi_\varepsilon * \Phi_\varepsilon * f \frac{d\varepsilon}{\varepsilon} = \sum_{j=1}^\infty \lambda_j g_j.$$

Now we show that every g_j is an (α, Q_{k_j,m_j})-atom. Firstly, for any Q_j, there exist $k_j \in \mathbb{N}_0$ and $s \in \mathbb{R}^d$ such that

$$2^{-k_j-1} \leq l(Q_j) \leq 2^{-k_j} \quad \text{and} \quad c_{Q_j} = l(Q_j)s.$$

Take $m_j = [s] \in \mathbb{Z}^d$, where $[s] = ([s_1], \cdots, [s_d])$. Then, we can check that

$$Q_j \subset 2Q_{k_j,m_j}, \quad Q_{k_j,m_j} \in \mathbb{D}_d. \tag{8.33}$$

Next, by the argument similar to that in (7.5) and by (8.31), we have

$$\tau\Big(\int_{\mathbb{R}^d} |I^\alpha \pi_\Phi(a_j)(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim \tau\Big(\int_{T(Q_j)} \varepsilon^{-2\alpha}|a_j(t,\varepsilon)|^2 \frac{dt d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \leq |Q_j|^{-\frac{1}{2}} \lesssim |Q_{k_j,m_j}|^{-\frac{1}{2}}.$$

If $\alpha \leq 0$, it is clear that

$$\tau\Big(\int_{\mathbb{R}^d} |J^\alpha \pi_\Phi(a_j)(s)|^2 ds\Big)^{\frac{1}{2}} \leq \tau\Big(\int_{\mathbb{R}^d} |I^\alpha \pi_\Phi(a_j)(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim |Q_j|^{-\frac{1}{2}} \lesssim |Q_{k_j,m_j}|^{-\frac{1}{2}}.$$

If $\alpha > 0$, we have

$$\begin{aligned}\tau\Big(\int_{\mathbb{R}^d} |J^\alpha \pi_\Phi(a_j)(s)|^2 ds\Big)^{\frac{1}{2}} &\lesssim \tau\Big(\int_{\mathbb{R}^d} |\pi_\Phi(a_j)(s)|^2 ds\Big)^{\frac{1}{2}} + \tau\Big(\int_{\mathbb{R}^d} |I^\alpha \pi_\Phi(a_j)(s)|^2 ds\Big)^{\frac{1}{2}} \\ &\lesssim \tau\Big(\int_{T(Q_j)} |a_j(t,\varepsilon)|^2 \frac{dt d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} + |Q_j|^{-\frac{1}{2}} \\ &\lesssim \tau\Big(\int_{T(Q_j)} \varepsilon^{-2\alpha} |a_j(t,\varepsilon)|^2 \frac{dt d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} + |Q_j|^{-\frac{1}{2}} \\ &\leq 2|Q_j|^{-\frac{1}{2}} \lesssim |Q_{k_j,m_j}|^{-\frac{1}{2}}.\end{aligned}$$

Then we get that, for any $\alpha \in \mathbb{R}$,

$$\tau\Big(\int_{\mathbb{R}^d} |J^\alpha g_j(s)|^2 ds\Big)^{\frac{1}{2}} = \tau\Big(\int_{\mathbb{R}^d} |J^\alpha \pi_\Phi(a_j)(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim |Q_{k_j,m_j}|^{-\frac{1}{2}}. \tag{8.34}$$

Finally, we decompose the slice $T(Q_j) \cap \{2^{-\mu-1} \leq \varepsilon \leq 2^{-\mu}\}$ into $(d+1)$-dimensional dyadic cubes whose projections on $\mathbb{R}^d$ belong to $\mathbb{D}_d$, and with side length $2^{-\mu}$, $\mu \in \mathbb{N}_0$. Let $\widehat{Q}$ be one of those dyadic cubes with side length $2^{-\mu}$ and Q be its projection on $\mathbb{R}^d$. Let

$$a(s) = \int_{\widehat{Q}} \Phi_\varepsilon(s-t) a_j(t,\varepsilon) \frac{dt d\varepsilon}{\varepsilon}.$$

By the support assumption of Φ, it follows that

$$\operatorname{supp} a \subset 2Q, \qquad \operatorname{supp} a \subset 2Q_j \subset 4Q_{k_j,m_j}.$$

Then

$$\widehat{a}(\xi) = \int_{2^{-\mu}}^{2^{-\mu+1}} \widehat{\Phi}(\varepsilon\xi) \widehat{a_j(\cdot,\varepsilon)\mathbb{1}_Q}(\xi) \frac{d\varepsilon}{\varepsilon}.$$

Since $D^\beta \widehat{\Phi}(0) = 0$ for any $|\beta|_1 \leq N$, we obtain

$$\int_{\mathbb{R}^d} (-2\pi \mathrm{i} s)^\beta a(s) ds = D^\beta \widehat{a}(0) = 0, \quad \forall\, |\beta|_1 \leq L.$$

Furthermore, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\tau\Big(\int |a(s)|^2 ds\Big)^{\frac{1}{2}} &= \tau\Big(\int_{5Q} \Big|\int_{2^{-\mu}}^{2^{-\mu+1}} \int_Q \Phi_\varepsilon(s-t) a_j(t,\varepsilon) \frac{dt d\varepsilon}{\varepsilon}\Big|^2 ds\Big)^{\frac{1}{2}} \\ &\lesssim |Q|^{\frac{1}{2}} \Big(\int_{2^{-\mu}}^{2^{-\mu+1}} \int_Q \varepsilon^{-2d} \frac{dt d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \cdot \tau\Big(\int_{2^{-\mu}}^{2^{-\mu+1}} \int_Q |a_j(t,\varepsilon)|^2 \frac{dt d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \\ &\lesssim \tau\Big(\int_{2^{-\mu}}^{2^{-\mu+1}} \int_Q |a_j(s,\varepsilon)|^2 \frac{ds d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \\ &\lesssim |Q|^{\frac{\alpha}{d}} \tau\Big(\int_{2^{-\mu}}^{2^{-\mu+1}} \int_Q \varepsilon^{-2\alpha} |a_j(s,\varepsilon)|^2 \frac{ds d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}.\end{aligned}$$

Similarly, we have

$$\tau\Big(\int |D^\gamma a(s)|^2 ds\Big)^{\frac{1}{2}} \leq C'_\gamma |Q|^{\frac{\alpha}{d} - \frac{|\gamma|_1}{d}} \tau\Big(\int_{2^{-\mu}}^{2^{-\mu+1}} \int_Q \varepsilon^{-2\alpha} |a_j(s,\varepsilon)|^2 \frac{ds d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}.$$

The above discussion gives

$$g_j = \sum_{(\mu,l)\le(k_j,m_j)} d^j_{\mu,l} a^j_{\mu,l}, \tag{8.35}$$

where each $a^j_{\mu,l}$ is an $(\alpha, Q_{\mu,l})$-subatom. The normalizing factor is given by

$$d^j_{\mu,l} = \max_{|\gamma|_1\le K}\{C'_\gamma\}\tau\Big(\int_{2^{-\mu}}^{2^{-\mu+1}}\int_{Q_{\mu,l}} \varepsilon^{-2\alpha}|a_j(s,\varepsilon)|^2\frac{dsd\varepsilon}{\varepsilon}\Big)^{\frac12}.$$

By the elementary fact that $\ell_2(L_1(\mathcal{M})) \supset L_1(\mathcal{M};\ell_2^c)$, we get

$$\Big(\sum_{(\mu,l)\le(k_j,m_j)} |d^j_{\mu,l}|^2\Big)^{\frac12} \le C\tau\Big(\int_{T(Q_j)} \varepsilon^{-2\alpha}|a_j(s,\varepsilon)|^2\frac{dsd\varepsilon}{\varepsilon}\Big)^{\frac12} \le C|Q_{k_j,m_j}|^{-\frac12}, \tag{8.36}$$

where C is independent of f. We may assume $C = 1$, otherwise, we can put C in (8.29) in the numbers λ_j, which does not change (8.30). In summary, (8.33), (8.34), (8.35) and (8.36) ensure that g_j is an (α, Q_{k_j,m_j})-atom.

Step 3. Now we show the reverse assertion that if f is given by (8.24), then $f \in F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ and

$$\|f\|_{F_1^{\alpha,c}} \lesssim \sum_{j=1}^{\infty}(|\mu_j| + |\lambda_j|).$$

To this end, it is enough to show that every $(\alpha,1)$-atom b and every (α,Q)-atom g belong to $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ and

$$\|b\|_{F_1^{\alpha,c}} \lesssim 1 \quad \text{and} \quad \|g\|_{F_1^{\alpha,c}} \lesssim 1.$$

Let b be an $(\alpha,1)$-atom in $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. We observe that b is also an atom in $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. For $\alpha \le 0$, by Proposition 8.4, $\mathrm{h}_1^c \subset F_1^{\alpha,c}$. Then, we have $\|b\|_{F_1^{\alpha,c}} \lesssim \|b\|_{\mathrm{h}_1^c} \lesssim 1$. If $\alpha > 0$, by Proposition 8.5, we have

$$\|b\|_{F_1^{\alpha,c}} \approx \|\varphi_0 * b\|_1 + \sum_{i=1}^{d}\|D_i^K b\|_{F_1^{\alpha-K,c}}.$$

Note that for any $1 \le i \le d$, $D_i^K b$ is an atom in $\mathrm{h}_1^c(\mathbb{R}^d,\mathcal{M})$. Since $\alpha - K < 0$, by Proposition 8.4, we have

$$\|b\|_{F_1^{\alpha,c}} \lesssim \|\varphi_0 * b\|_1 + \sum_{i=1}^{d}\|D_i^K b\|_{\mathrm{h}_1^c} \lesssim 1.$$

On the other hand, let g be an $(\alpha, Q_{k,m})$-atom in the sense of Definition 8.18. We may use the discrete general characterization of $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ given in Theorem 8.12, i.e.

$$\|g\|_{F_1^{\alpha,c}} \approx \Big\|\Big(\sum_{j=0}^{\infty} 2^{2j\alpha}|\Phi_j * g|^2\Big)^{\frac12}\Big\|_1.$$

We split $\sum_{j=0}^{\infty}$ into two parts $\sum_{j=0}^{k-1}$ and $\sum_{j=k}^{\infty}$. When $j \ge k$, by the support assumption of Φ, we have $\operatorname{supp}\Phi_j * g \subset 5Q_{k,m}$. If $\alpha \ge 0$, by (8.27), (8.23) and the Plancherel formula, we obtain

$$\begin{aligned}
\tau\Big(\int_{5Q_{k,m}}\sum_{j=k}^{\infty}2^{2j\alpha}|\Phi_j*g(s)|^2ds\Big)^{\frac12} &= \tau\Big(\int_{5Q_{k,m}}\sum_{j=k}^{\infty}|(I^{-\alpha}\Phi)_j*I^{\alpha}g(s)|^2ds\Big)^{\frac12}\\
&\le \tau\Big(\int_{\mathbb{R}^d}\sum_{j=k}^{\infty}|(I_{-\alpha}\widehat{\Phi})(2^{-j}\xi)|^2|I_\alpha\widehat{g}(\xi)|^2d\xi\Big)^{\frac12}\\
&\lesssim \tau\Big(\int_{\mathbb{R}^d}|I_\alpha\widehat{g}(\xi)|^2d\xi\Big)^{\frac12} = \tau\Big(\int_{\mathbb{R}^d}|I^{\alpha}g(s)|^2ds\Big)^{\frac12}\\
&\le \tau\Big(\int_{\mathbb{R}^d}|J^{\alpha}g(s)|^2ds\Big)^{\frac12} \le |Q_{m,k}|^{-\frac12}.
\end{aligned}$$

If $\alpha<0$, by (8.26), (8.23) and the Plancherel formula again, we have

$$\begin{aligned}
\tau\Big(\int_{5Q_{k,m}}\sum_{j=k}^{\infty}2^{2j\alpha}|\Phi_j*g(s)|^2ds\Big)^{\frac12} &\le \tau\Big(\int_{5Q_{k,m}}\sum_{j=k}^{\infty}2^{2j\alpha}|J^{-\alpha}\Phi_j*J^{\alpha}g(s)|^2ds\Big)^{\frac12}\\
&\le \tau\Big(\int_{\mathbb{R}^d}\sum_{j=k}^{\infty}|(J_{-\alpha}\widehat{\Phi})(2^{-j}\xi)|^2|J_\alpha\widehat{g}(\xi)|^2d\xi\Big)^{\frac12}\\
&\lesssim \tau\Big(\int_{\mathbb{R}^d}|J_\alpha\widehat{g}(\xi)|^2d\xi\Big)^{\frac12} = \tau\Big(\int_{\mathbb{R}^d}|J^{\alpha}g(s)|^2ds\Big)^{\frac12}\\
&\le |Q_{m,k}|^{-\frac12}.
\end{aligned}$$

It follows that

$$\big\|\Big(\sum_{j=k}^{\infty}2^{2j\alpha}|\Phi_j*g|^2\Big)^{\frac12}\big\|_1\lesssim 1.$$

In order to estimate the sum $\sum_{j=0}^{k-1}$, we begin with a technical modification of g. Let

$$\widetilde{g}=2^{k(\alpha-d)}g(2^{-k}\cdot).$$

Then it is easy to see that $\widetilde{g}$ is an $(\alpha,Q_{0,m})$-atom. Moreover, we have

$$\Phi_j*g=2^{k(d-\alpha)}\Phi_{j-k}*\widetilde{g}(2^k\cdot),$$

which implies that

$$\big\|\Big(\sum_{j=0}^{k-1}2^{2j\alpha}|\Phi_j*g|^2\Big)^{\frac12}\big\|_1\le\big\|\Big(\sum_{j=-\infty}^{-1}2^{2j\alpha}|\Phi_j*\widetilde{g}|^2\Big)^{\frac12}\big\|_1+2^{-k\alpha}\|(\Phi_0)_{-k}*\widetilde{g}\|_1. \tag{8.37}$$

In other words, we can assume, by translation, that the atom g is based on a cube Q with side length 1 and centered at the origin. Then, let us estimate the right hand side of (8.37) with g instead of $\widetilde{g}$.

By the triangle inequality, we have

$$\begin{aligned}
\big\|\Big(\sum_{j=-\infty}^{-1}2^{2j\alpha}|\Phi_j*g|^2\Big)^{\frac12}\big\|_1 &\le \sum_{j=-\infty}^{-1}2^{j\alpha}\tau\int_{\mathbb{R}^d}|\Phi_j*g(s)|ds\\
&\le \sum_{j=-\infty}^{-1}\sum_{(\mu,l)\le(0,0)}|d_{\mu,l}|\,2^{j\alpha}\tau\int_{\mathbb{R}^d}|\Phi_j*a_{\mu,l}(s)|ds.
\end{aligned}$$

Now we estimate $2^{j\alpha}\tau\int_{\mathbb{R}^d}|\Phi_j * a_{\mu,l}(s)|ds$ for every $(\mu,l)\le(0,0)$. Let $M=[-\alpha]+1$. Then $M+\alpha>0$ and $L\ge M-1$. By the moment cancellation of $a_{\mu,l}$, we have

$$\begin{aligned}
&\Phi_j * a_{\mu,l}(s)\\
&=2^{jd}\int_{2Q_{\mu,l}}\big[\Phi(2^js-2^jt)-\Phi(2^js-2^j2^{-\mu}l)\big]a_{\mu,l}(t)dt\\
&=2^{j(d+M)}\\
&\quad\cdot\sum_{|\beta|_1=M}\frac{M+1}{\beta!}\int_{2Q_{\mu,l}}(2^{-\mu}l-t)^{\beta}\int_0^1(1-\theta)^M D^{\beta}\Phi(2^js-2^j(\theta t+(1-\theta)2^{-\mu}l))a_{\mu,l}(t)d\theta dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\Phi_j * a_{\mu,l}(s)|^2\lesssim&\sum_{|\beta|_1=M}2^{2j(d+M)}\int_{2Q_{\mu,l}}\int_0^1(1-\theta)^{2M}|D^{\beta}\Phi(2^js-2^j(\theta t+(1-\theta)2^{-\mu}l))|^2d\theta dt\\
&\cdot\int_{2Q_{\mu,l}}|t-2^{-\mu}l|^{2M}|a_{\mu,l}(t)|^2dt.
\end{aligned}$$

If $\Phi_j * a_{\mu,l}(s)\neq 0$, then we have $|2^js-2^jt|\le 1$ for some $t\in 2Q_{\mu,l}$. Hence, $\Phi_j * a_{\mu,l}(s)=0$ if $|s-2^{-\mu}l|>3\cdot 2^{-j-1}\sqrt{d}$. Consequently,

$$\begin{aligned}
&\sum_{j=-\infty}^{-1}2^{j\alpha}\tau\int_{\mathbb{R}^d}|\Phi_j * a_{\mu,l}(s)|ds\\
&\lesssim\sum_{j=-\infty}^{-1}2^{j(d+M+\alpha)}\tau\Big(\int_{2Q_{\mu,l}}|t-2^{-\mu}l|^{2M}|a_{\mu,l}(t)|^2dt\Big)^{\frac12}\\
&\quad\cdot\sum_{|\beta|_1=M}\int_{|s-2^{-\mu}l|\le\frac{3\sqrt{d}}{2^{j+1}}}\Big(\int_{2Q_{\mu,l}}\int_0^1(1-\theta)^{2M}|D^{\beta}\Phi(2^js-2^j(\theta t+(1-\theta)2^{-\mu}l))|^2d\theta dt\Big)^{\frac12}ds\\
&\lesssim\sum_{j=-\infty}^{-1}2^{j(d+M+\alpha)}\cdot 2^{-\mu M}|Q_{\mu,l}|^{\frac12}\tau\Big(\int_{2Q_{\mu,l}}|a_{\mu,l}(t)|^2dt\Big)^{\frac12}\int_{|s-2^{-\mu}l|\le\frac{3\sqrt{d}}{2^{j+1}}}ds\\
&\lesssim\sum_{j=-\infty}^{-1}2^{j(d+M+\alpha)}\cdot 2^{-jd}\cdot 2^{-\mu(\alpha+M)}|Q_{\mu,l}|^{\frac12}\\
&=2^{-\mu(\alpha+M)}\sum_{j=-\infty}^{-1}2^{j(M+\alpha)}|Q_{\mu,l}|^{\frac12}\lesssim 2^{-\mu(\alpha+M)}|Q_{\mu,l}|^{\frac12}.
\end{aligned}$$

Similarly, we also have

$$2^{-k\alpha}\tau\int_{\mathbb{R}^d}|(\Phi_0)_{-k} * a_{\mu,l}(s)|ds\lesssim 2^{-k(M+\alpha)}2^{-\mu(\alpha+M)}|Q_{\mu,l}|^{\frac12}\le 2^{-\mu(\alpha+M)}|Q_{\mu,l}|^{\frac12}.$$

Therefore, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\Big\|\Big(\sum_{j=-\infty}^{-1}2^{2j\alpha}|\Phi_j * g|^2\Big)^{\frac12}\Big\|_1&\le\sum_{j=-\infty}^{-1}2^{2j\alpha}\tau\int_{\mathbb{R}^d}|\Phi_j * g(s)|ds\\
&\lesssim\sum_{\mu=0}^{\infty}2^{-\mu(\alpha+M)}\Big(\sum_l|d_{\mu,l}|^2\Big)^{\frac12}\Big(\sum_l|Q_{\mu,l}|\Big)^{\frac12}\\
&\lesssim\sum_{\mu=0}^{\infty}2^{-\mu(\alpha+M)}<\infty,
\end{aligned}$$

and

$$2^{-k\alpha}\|(\Phi_0)_{-k} * g\|_1 \lesssim \sum_{\mu=0}^{\infty} 2^{-\mu(\alpha+M)} < \infty.$$

Therefore, $\|g\|_{F_1^{\alpha,c}} \lesssim 1$. The proof is complete. □

We close this chapter by a very useful result of pointwise multipliers, which can be deduced from the above atomic decomposition. Let $k \in \mathbb{N}$ and $\mathcal{L}^k(\mathbb{R}^d, \mathcal{M})$ be the collection of all $\mathcal{M}$-valued functions on $\mathbb{R}^d$ such that $D^\gamma h \in L_\infty(\mathcal{N})$ for all γ with $0 \le |\gamma|_1 \le k$.

Corollary 8.22. *Let $\alpha \in \mathbb{R}$ and let $k \in \mathbb{N}$ be sufficiently large and $h \in \mathcal{L}^k(\mathbb{R}^d, \mathcal{M})$. Then the map $f \mapsto hf$ is bounded on $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$*

Proof. First, consider the case $\alpha > 0$. We apply the atomic decomposition in Theorem 8.21 with $K = k$ and $L = -1$. In this case, no moment cancellation of subatoms is required. We can easily check that, multiplying every (sub)atom in Definition 8.18 by h, we get another (sub)atom. Moreover,

$$\|hf\|_{F_1^{\alpha,c}} \le \sum_{|\gamma|\le k} \sup_{s\in\mathbb{R}^d} \|D^\gamma h(s)\|_{\mathcal{M}} \cdot \|f\|_{F_1^{\alpha,c}}. \tag{8.38}$$

The case $\alpha \le 0$ can be deduced by induction. Assume that (8.38) is true for $\alpha > N \in \mathbb{Z}$. Let $\alpha > N - 1$. Any $f \in F_1^{\alpha,c}$ can be represented as $f = J^2 g = (1 - (2\pi)^{-2}\Delta)g$ with $g \in F_1^{\alpha+2,c}$ and $\|f\|_{F_1^{\alpha,c}} \approx \|g\|_{F_1^{\alpha+2,c}}$. Since

$$hf = (1 - (2\pi)^{-2}\Delta)(hg) + ((2\pi)^{-2}\Delta h)g + (2\pi)^{-2}\nabla h \cdot \nabla g,$$

we deduce

$$\begin{aligned}\|hf\|_{F_1^{\alpha,c}} &\lesssim \|(1 - (2\pi)^{-2}\Delta)(hg)\|_{F_1^{\alpha,c}} + \|(\Delta h)g\|_{F_1^{\alpha,c}} + \sum_{i=1}^{d} \|\partial_i h \cdot \partial_i g\|_{F_1^{\alpha,c}} \\ &\lesssim \|g\|_{F_1^{\alpha+2,c}} + \|(\Delta h)g\|_{F_1^{\alpha+2,c}} + \sum_{i=1}^{d} \|\partial_i h \cdot \partial_i g\|_{F_1^{\alpha+1,c}}.\end{aligned} \tag{8.39}$$

If $k \in \mathbb{N}$ is sufficiently large, we have

$$\|(\Delta h)g\|_{F_1^{\alpha+2,c}} \lesssim \|g\|_{F_1^{\alpha+2,c}}, \quad \|\partial_i h \cdot \partial_i g\|_{F_1^{\alpha+1,c}} \lesssim \|\partial_i g\|_{F_1^{\alpha+1,c}}.$$

Continuing the estimate in (8.39), we obtain

$$\|hf\|_{F_1^{\alpha,c}} \lesssim \|g\|_{F_1^{\alpha+2,c}} + \sum_i \|\partial_i g\|_{F_1^{\alpha+1,c}} \lesssim \|g\|_{F_1^{\alpha+2,c}} \lesssim \|f\|_{F_1^{\alpha,c}},$$

which completes the induction procedure. □

Chapter 9

Pseudo-differential operators on Triebel-Lizorkin spaces

In this chapter, we study the continuity of pseudo-differential operators in the context of operator-valued Triebel-Lizorkin spaces. We start by introducing some basic definitions and properties. The symbols of pseudo-differential operators defined in the first section are $B(X)$-valued, where X is a Banach space. However, in the later sections of this chapter, we will only consider those symbols with values in $\mathcal{M}$.

9.1 Definitions and basic properties

Let X be a Banach space, $n \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$. Then $S^n_{\rho,\delta}$ denotes the collection of all infinitely differentiable functions σ defined on $\mathbb{R}^d \times \mathbb{R}^d$ and with values in $B(X)$, such that for each pair of multi-indices of nonnegative integers γ, β, there exists a constant $C_{\gamma,\beta}$ such that

$$\|D_s^\gamma D_\xi^\beta \sigma(s,\xi)\|_{B(X)} \leq C_{\gamma,\beta}(1+|\xi|)^{n+\delta|\gamma|_1-\rho|\beta|_1},$$

where $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}_0^d$, $|\gamma|_1 = \gamma_1 + \cdots + \gamma_d$ and $D_s^\gamma = \frac{\partial^{\gamma_1}}{\partial s_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_d}}{\partial s_d^{\gamma_d}}$.

Definition 9.1. Let $\sigma \in S^n_{\rho,\delta}$. For any function $f \in \mathcal{S}(\mathbb{R}^d; X)$, the pseudo-differential operator T_σ is a mapping $f \mapsto T_\sigma f$ given by

$$T_\sigma f(s) = \int_{\mathbb{R}^d} \sigma(s,\xi)\widehat{f}(\xi)e^{2\pi \mathrm{i} s\cdot\xi} d\xi. \tag{9.1}$$

We call σ the symbol of T_σ.

Proposition 9.2. *Let $0 \leq \delta, \rho \leq 1$ and $n \in \mathbb{R}$. For any $\sigma \in S^n_{\rho,\delta}$, T_σ is continuous on $\mathcal{S}(\mathbb{R}^d; X)$.*

Proof. By integration by parts, for any $s \in \mathbb{R}^d$ and $\gamma \in \mathbb{N}_0^d$, we have

$$\begin{aligned} \|(2\pi \mathrm{i} s)^\gamma T_\sigma f\|_X &= \Big\|(2\pi \mathrm{i} s)^\gamma \int_{\mathbb{R}^d} \sigma(s,\xi)\widehat{f}(\xi)e^{2\pi \mathrm{i} s\cdot\xi} d\xi\Big\|_X \\ &= \Big\|\int_{\mathbb{R}^d} \sigma(s,\xi)\widehat{g}(\xi) D_\xi^\gamma(e^{2\pi \mathrm{i} s\cdot\xi}) d\xi\Big\|_X \\ &= \Big\|\int_{\mathbb{R}^d} D_\xi^\gamma[\sigma(s,\xi)\widehat{g}(\xi)]e^{2\pi \mathrm{i} s\cdot\xi} d\xi\Big\|_X < \infty. \end{aligned}$$

Thus, $T_\sigma f$ is rapidly decreasing. A similar argument works for the partial derivatives of $T_\sigma f$, then we easily check that $T_\sigma f$ maps $\mathcal{S}(\mathbb{R}^d; X)$ continuously to itself. $\square$

Another way to write (9.1) is as a double integral:

$$T_\sigma f(s) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(s,\xi) f(t) e^{2\pi i (s-t)\cdot\xi} dt d\xi. \tag{9.2}$$

However, the above ξ-integral does not necessarily converge absolutely, even for $f \in \mathcal{S}(\mathbb{R}^d; X)$. To overcome this difficulty, we will approximate σ by symbols with compact support. To this end, let us fix a compactly supported infinitely differentiable η defined on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\eta = 1$ near the origin. Then we define

$$\sigma_j(s,\xi) = \sigma(s,\xi)\eta(2^{-j}s, 2^{-j}\xi) \quad \text{with } j \in \mathbb{N}. \tag{9.3}$$

Note that σ_j converges pointwise to σ and $\sigma_j \in S^n_{\rho,\delta}$ uniformly in j. Thus, for any $f \in \mathcal{S}(\mathbb{R}^d; X)$, $T_{\sigma_j} f$ converges to $T_\sigma f$ in $\mathcal{S}(\mathbb{R}^d; X)$ as $j \to \infty$. Since the σ_j's have compact supports, the formula (9.2) works for $T_{\sigma_j} f(s)$. Then we can define the integral (9.2) as follows:

$$T_\sigma f(s) = \lim_{j\to\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma_j(s,\xi) f(t) e^{2\pi i (s-t)\cdot\xi} dt d\xi. \tag{9.4}$$

Proposition 9.3. *Let $0 \leq \delta < 1, 0 \leq \rho \leq 1$ and $n \in \mathbb{R}$. For any $\sigma \in S^n_{\rho,\delta}$, the adjoint of T_σ is continuous on $\mathcal{S}(\mathbb{R}^d; X^*)$.*

Proof. For any $f \in \mathcal{S}(\mathbb{R}^d; X)$ and $g \in \mathcal{S}(\mathbb{R}^d; X^*)$, by the duality relation

$$\langle T_\sigma f, g \rangle = \langle f, (T_\sigma)^* g \rangle,$$

we check that

$$(T_\sigma)^* g(s) = \lim_{j\to\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma_j^*(t,\xi) g(t) e^{2\pi i (s-t)\cdot\xi} dt d\xi. \tag{9.5}$$

By integration by parts, it is clear that $(T_\sigma)^*$ is continuous on $\mathcal{S}(\mathbb{R}^d; X^*)$. □

Since $\mathcal{S}'(\mathbb{R}^d; X^{**}) = (\mathcal{S}(\mathbb{R}^d; X^*))^*$ (see [66, Section 51] for more details of this duality), in the usual way, we extend T_σ to an operator on $\mathcal{S}'(\mathbb{R}^d; X^{**})$.

Definition 9.4. Let $f \in \mathcal{S}'(\mathbb{R}^d; X^{**})$. We define $T_\sigma f$ by the formula

$$\langle T_\sigma f, g \rangle = \langle f, (T_\sigma)^* g \rangle, \quad \forall g \in \mathcal{S}(\mathbb{R}^d; X^*).$$

By Proposition 9.3, $(T_\sigma)^* g \in \mathcal{S}(\mathbb{R}^d; X^*)$ whenever $g \in \mathcal{S}(\mathbb{R}^d; X^*)$. So the bracket on the right hand side of the above definition is well defined. Therefore, $T_\sigma f$ is well defined, and takes value in $\mathcal{S}'(\mathbb{R}^d; X^{**})$ as well.

Proposition 9.5. *Let $0 \leq \delta < 1, 0 \leq \rho \leq 1$ and $n \in \mathbb{R}$. For any $\sigma \in S^n_{\rho,\delta}$, T_σ is continuous on $\mathcal{S}'(\mathbb{R}^d; X^{**})$.*

Proof. For any $f \in \mathcal{S}'(\mathbb{R}^d; X^{**})$, we take a sequence (f_j) such that $f_j \to f$ in $\mathcal{S}'(\mathbb{R}^d; X^{**})$. Then we have

$$\langle T_\sigma f_j, g \rangle = \langle f_j, (T_\sigma)^* g \rangle \longrightarrow \langle f, (T_\sigma)^* g \rangle = \langle T_\sigma f, g \rangle \quad \forall g \in \mathcal{S}(\mathbb{R}^d; X^*).$$

Thus, $T_\sigma f_j$ converges to $T_\sigma f$ in $\mathcal{S}'(\mathbb{R}^d; X^{**})$. So T_σ is continuous on $\mathcal{S}'(\mathbb{R}^d; X^{**})$. □

The pseudo-differential operator defined above has a parallel description in terms of a distribution kernel:

$$T_\sigma f(s) = \int_{\mathbb{R}^d} K(s, s-t) f(t) dt,$$

where K is the inverse Fourier transform of σ with respect to the variable ξ, i.e.

$$K(s,t) = \int_{\mathbb{R}^d} \sigma(s,\xi) e^{2\pi \mathrm{i} t\cdot\xi} d\xi. \tag{9.6}$$

In the sequel, we will focus on the symbols in the class $S^n_{1,\delta}$ with $0 \leq \delta \leq 1$ and $n \in \mathbb{R}$. Similarly to the classical case (see [7], [26], [61] and [64]), we prove that for any operator-valued symbol $\sigma \in S^n_{1,\delta}$, the corresponding kernel K satisfies the following estimates:

Lemma 9.6. *Let $\sigma \in S^n_{1,\delta}$ and $0 \leq \delta \leq 1$. Then the kernel $K(s,t)$ in (9.6) satisfies*

$$\|D_s^\gamma D_t^\beta K(s,t)\|_{B(X)} \leq C_{\gamma,\beta} |t|^{-|\gamma|_1 - |\beta|_1 - d - n}, \quad \forall t \in \mathbb{R}^d \setminus \{0\}, \tag{9.7}$$

$$\|D_s^\gamma D_t^\beta K(s,t)\|_{B(X)} \leq C_{\gamma,\beta,N} |t|^{-N}, \quad \forall N > 0 \ \ \textit{if } |t| > 1. \tag{9.8}$$

Proof. This lemma can be deduced easily from the corresponding scalar-valued results, which can be found in many classical works on pseudo-differential operators, for instance, [65, Lemma 5.1.6]. Given $x \in X$ and $x^* \in X^*$ with norm one, it is clear that $\langle x^*, \sigma(s,t)x\rangle$ is a scalar-valued symbol in $S^n_{1,\delta}$, with distribution kernel $\langle x^*, K(s,t)x\rangle$. Thus, we have

$$\langle x^*, D_s^\gamma D_t^\beta K(s,t)x\rangle = D_s^\gamma D_t^\beta [\langle x^*, K(s,t)x\rangle] \leq C_{\gamma,\beta}|t|^{-|\gamma|_1 - |\beta|_1 - d - n}, \quad \forall t \in \mathbb{R}^d \setminus \{0\}$$

and

$$\langle x^*, D_s^\gamma D_t^\beta K(s,t)x\rangle = D_s^\gamma D_t^\beta [\langle x^*, K(s,t)x\rangle] \leq C_{\gamma,\beta,N}|t|^{-N}, \quad \forall N > 0 \text{ if } |t| > 1.$$

Then, taking the supremum over x and x^* in the above two inequalities, we get the desired assertion. □

In the classical case, the proof the above lemma makes use of the decomposition of the symbol σ into dyadic pieces. Let $(\widehat{\varphi}_k)_{k\geq 0}$ be the resolution of the unit satisfying (1.3). Set

$$\sigma_k(s,\xi) = \sigma(s,\xi)\widehat{\varphi}_k(\xi), \quad \forall (s,\xi) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{9.9}$$

By a similar argument as in the above proof, we also have the following estimates of the corresponding kernels of these pieces σ_k's.

Lemma 9.7. *Let $\sigma \in S^n_{1,\delta}$ and σ_k be as in (9.9) and $K_k(s,t) = \int_{\mathbb{R}^d} \sigma_k(s,\xi) e^{2\pi \mathrm{i} t\cdot\xi} d\xi$. Then*

$$\|D_s^\gamma D_t^\beta K_k(s,t)\|_{B(X)} \lesssim |t|^{-2M} 2^{k(|\beta|_1 + |\gamma|_1 + d - 2M + n)}, \quad \forall M \in \mathbb{N}_0.$$

Now we study the composition of pseudo-differential operators. The following proposition gives a rule of the composition of two pseudo-differential operators. In particular, it shows that the symbol class $S^0_{1,\delta}$ is closed under product.

Proposition 9.8. *Let $0 \leq \delta < 1$ and σ_1, σ_2 be two symbols in $S^{n_1}_{1,\delta}$ and $S^{n_2}_{1,\delta}$ respectively. There exists a symbol σ_3 in $S^{n_1+n_2}_{1,\delta}$ such that*

$$T_{\sigma_3} = T_{\sigma_1} T_{\sigma_2}.$$

Moreover,

$$\sigma_3 - \sum_{|\gamma|_1 < N_0} \frac{(2\pi \mathrm{i})^{-|\gamma|_1}}{\gamma!} D_\xi^\gamma \sigma_1 D_s^\gamma \sigma_2 \in S^{n_1+n_2-(1-\delta)N_0}_{1,\delta}, \quad \forall N_0 \geq 0. \tag{9.10}$$

Proof. Firstly, we assume that σ_1 and σ_2 have compact supports, so we can use (9.2) as an alternate definition of T_{σ_1} and T_{σ_2}. In this way, $T_{\sigma_1}T_{\sigma_2}$ can be written as follows:

$$T_{\sigma_1}(T_{\sigma_2}f)(s) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_3(s,\xi)f(r)e^{2\pi \mathrm{i}(s-r)\cdot\xi}drd\xi,$$

where

$$\begin{aligned}\sigma_3(s,\xi) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_1(s,\eta)\sigma_2(t,\xi)e^{2\pi \mathrm{i}(s-t)\cdot(\eta-\xi)}dtd\eta \\ &= \int_{\mathbb{R}^d} \sigma_1(s,\xi+\eta)\widehat{\sigma}_2(\eta,\xi)e^{-2\pi \mathrm{i}s\cdot\eta}d\eta\end{aligned} \tag{9.11}$$

with $\widehat{\sigma}_2$ the Fourier transform of σ_2 with respect to the first variable. We expand $\sigma_1(s,\xi+\eta)$ by the Taylor formula:

$$\sigma_1(s,\xi+\eta) = \sum_{|\gamma|_1<N_0} \frac{1}{\gamma!}D_\xi^\gamma\sigma_1(s,\xi)\eta^\gamma + \sum_{N_0\le|\gamma|_1<N} \frac{1}{\gamma!}D_\xi^\gamma\sigma_1(s,\xi)\eta^\gamma + R_N(s,\xi,\eta),$$

with the remainder

$$R_N(s,\xi,\eta) = \sum_{|\gamma|_1=N} \frac{1}{\gamma!}\int_0^1 D_\xi^\gamma\sigma_1(s,\xi+\theta\eta)(1-\theta)^N\eta^\gamma d\theta.$$

Now we replace $\sigma_1(s,\xi+\eta)$ in (9.11) by the above Taylor polynomial and remainder. Notice that

$$\frac{1}{\gamma!}\int_{\mathbb{R}^d} D_\xi^\gamma\sigma_1(s,\xi)\eta^\gamma\widehat{\sigma}_2(\eta,\xi)e^{-2\pi \mathrm{i}s\cdot\eta}d\eta = \frac{(2\pi\mathrm{i})^{-|\gamma|_1}}{\gamma!}D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi).$$

Thus,

$$\begin{aligned}\sigma_3(s,\xi) = (&\sum_{|\gamma|_1<N_0} + \sum_{N_0\le|\gamma|_1<N})\frac{(2\pi\mathrm{i})^{-|\gamma|_1}}{\gamma!}D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi) \\ &+ \int_{\mathbb{R}^d} R_N(s,\xi,\eta)\widehat{\sigma}_2(\eta,\xi)e^{-2\pi \mathrm{i}s\cdot\eta}d\eta.\end{aligned} \tag{9.12}$$

For every γ, the term $D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)$ is a symbol in $S_{1,\delta}^{n_1+n_2-(1-\delta)|\gamma|_1}$. Indeed, it is clear that

$$\|D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)\|_{B(X)} \lesssim (1+|\xi|)^{n_1-|\gamma|_1}(1+|\xi|)^{n_2+\delta|\gamma|_1} = (1+|\xi|)^{n_1+n_2-(1-\delta)|\gamma|_1}.$$

Moreover, for any $\beta_1,\beta_2\in\mathbb{N}_0^d$, we have $D_s^{\beta_1}\sigma_1\in S_{1,\delta}^{n_1+\delta|\beta_1|_1}, D_s^{\beta_2}\sigma_2\in S_{1,\delta}^{n_2+\delta|\beta_1|_1}$ and $D_\xi^{\beta_1}\sigma_1\in S_{1,\delta}^{n_1-|\beta_2|_1}, D_\xi^{\beta_2}\sigma_2\in S_{1,\delta}^{n_2-|\beta_2|_1}$. Thus, we get

$$\begin{aligned}\big\|D_s^\beta[D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)]\big\|_{B(X)} &\lesssim \sum_{\beta_1+\beta_2=\beta}\big\|D_s^{\beta_1}D_\xi^\gamma\sigma_1(s,\xi)D_s^{\beta_2}D_s^\gamma\sigma_2(s,\xi)\big\|_{B(X)} \\ &\lesssim (1+|\xi|)^{n_1+n_2-(1-\delta)|\gamma|_1+\delta|\beta|_1},\end{aligned}$$

and

$$\begin{aligned}\big\|D_\xi^\beta[D_\xi^\gamma\sigma_1(s,\xi)D_s^\gamma\sigma_2(s,\xi)]\big\|_{B(X)} &\lesssim \sum_{\beta_1+\beta_2=\beta}\big\|D_\xi^{\gamma+\beta_1}\sigma_1(s,\xi)D_s^\gamma D_\xi^{\beta_2}\sigma_2(s,\xi)\big\|_{B(X)} \\ &\lesssim (1+|\xi|)^{n_1+n_2-(1-\delta)|\gamma|_1-|\beta|_1}.\end{aligned}$$

By the above estimates, we see that when $N_0 \leq |\gamma|_1 < N$, $D_\xi^\gamma \sigma_1(s,\xi) D_s^\gamma \sigma_2(s,\xi) \in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}$.

Now we have to treat the last term in (9.12). For the remainder $R_N(s,\xi,\eta)$, we easily check that for any $|\gamma|_1 = N$ and $0 \leq \theta \leq 1$,

$$\|D_\xi^\gamma \sigma_1(s,\xi+\theta\eta)\|_{B(X)} \leq C_N(1+|\xi|)^{n_1-N}, \quad \text{if } |\xi| \geq 2|\eta|, \tag{9.13}$$

and

$$\|D_\xi^\gamma \sigma_1(s,\xi+\theta\eta)\|_{B(X)} \leq C'_N, \quad \forall\, \eta,\xi \in \mathbb{R}^d. \tag{9.14}$$

For $\widehat{\sigma}_2$, by integration by parts, we see that for any $\beta \in \mathbb{N}_0^d$ such that $|\beta|_1 = \widetilde{N}$,

$$\begin{aligned}(-2\pi\mathrm{i}\eta)^\beta \widehat{\sigma}_2(\eta,\xi) &= \int_{\mathbb{R}^d} (-2\pi\mathrm{i}\eta)^\beta e^{-2\pi\mathrm{i}t\cdot\eta} \sigma_2(t,\xi)dt \\ &= \int_{\mathbb{R}^d} D_t^\beta (e^{-2\pi\mathrm{i}t\cdot\eta}) \sigma_2(t,\xi)dt \\ &= (-1)^\beta \int_{\mathbb{R}^d} e^{-2\pi\mathrm{i}t\cdot\eta} D_t^\beta \sigma_2(t,\xi)dt.\end{aligned}$$

Denote the compact t-support of $\sigma_2(t,\xi)$ by Ω. Then the above calculation immediately implies that

$$\|\widehat{\sigma}_2(\eta,\xi)\|_{B(X)} \lesssim |\Omega|(1+|\eta|)^{-\widetilde{N}}(1+|\xi|)^{n_2+\delta\widetilde{N}}. \tag{9.15}$$

We keep the constant $|\Omega|$ in this inequality for the moment, and will see in the next step that our final result does not depend on this support. Take $\widetilde{N}$ large enough so that

$$\widetilde{N} > \max\Big\{\frac{d}{1-\widetilde{\delta}}, \frac{(1-\delta)N_0}{\widetilde{\delta}-\delta}, \frac{d-n_1+(1-\delta)N_0}{1-2\delta}\Big\},$$

and take $N = \widetilde{\delta}\widetilde{N}$ with $0 \leq \delta < \widetilde{\delta} < 1$. Continuing the estimate of the last term in (9.12), inequalities (9.13) and (9.15) give

$$\begin{aligned}&\Big\|\int_{|\eta|\leq\frac{|\xi|}{2}} \int_0^1 D_\xi^\gamma \sigma_1(s,\xi+\theta\eta)(1-\theta)^N \eta^\gamma \widehat{\sigma}_2(\eta,\xi) e^{-2\pi\mathrm{i}s\cdot\eta} d\theta d\eta\Big\|_{B(X)} \\ &\lesssim \int_{\mathbb{R}^d} |\eta|^N (1+|\eta|)^{-\widetilde{N}} d\eta \cdot (1+|\xi|)^{n_1+n_2-N+\delta\widetilde{N}} \\ &\leq \int_{\mathbb{R}^d} (1+|\eta|)^{(\widetilde{\delta}-1)\widetilde{N}} d\eta \cdot (1+|\xi|)^{n_1+n_2+(\delta-\widetilde{\delta})\widetilde{N}} \\ &\lesssim (1+|\xi|)^{n_1+n_2+(\delta-\widetilde{\delta})\widetilde{N}}.\end{aligned}$$

Moreover, since $\widetilde{N} \geq \frac{(1-\delta)N_0}{\widetilde{\delta}-\delta}$, we have

$$(1+|\xi|)^{n_1+n_2+(\delta-\widetilde{\delta})\widetilde{N}} \leq (1+|\xi|)^{n_1+n_2-(1-\delta)N_0}.$$

By (9.14) and (9.15), we get

$$\begin{aligned}&\Big\|\int_{|\eta|>\frac{|\xi|}{2}} \int_0^1 D_\xi^\gamma \sigma_1(s,\xi+\theta\eta)(1-\theta)^N \eta^\gamma \widehat{\sigma}_2(\eta,\xi) e^{-2\pi\mathrm{i}s\cdot\eta} d\theta d\eta\Big\|_{B(X)} \\ &\lesssim \int_{|\eta|>\frac{|\xi|}{2}} |\eta|^N (1+|\eta|)^{-\widetilde{N}} d\eta \cdot (1+|\xi|)^{n_2+\delta\widetilde{N}} \\ &\lesssim (1+|\xi|)^{n_2+N+d-(1-\delta)\widetilde{N}} \leq (1+|\xi|)^{n_1+n_2-(1-\delta)N_0}.\end{aligned}$$

Therefore, $R_N(s,\xi,\eta) \in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}$. Combining the estimates above, we see that, if we set $R_{N_0}(s,\xi,\eta) = \sum_{N_0\le|\gamma|_1<N} \frac{1}{\gamma!} D_\xi^\gamma \sigma_1(s,\xi)\eta^\gamma + R_N(s,\xi,\eta)$, then $R_{N_0}(s,\xi,\eta) \in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}$. This proves the assertion (9.10) when σ_2 has compact support with respect to the first variable.

Noticing that the above proof depends on the constant $|\Omega|$ in (9.15), we now make use of the resolution of the unit in (7.13) to deal with general symbol σ_2 with arbitrary s-support. For each $k \in \mathbb{Z}^d$, denote $\sigma_{2,k}(s,\xi) = \mathcal{X}_k(s)\sigma_2(s,\xi)$ and

$$\sigma_{3,k}(s,\xi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_1(s,\eta)\sigma_{2,k}(t,\xi)e^{2\pi \mathrm{i}(s-t)\cdot(\eta-\xi)}dtd\eta\,.$$

It has already been established that

$$\sigma_{3,k} - \sum_{|\gamma|_1<N_0} \frac{(2\pi\mathrm{i})^{-|\gamma|_1}}{\gamma!} D_\xi^\gamma\sigma_1 D_s^\gamma\sigma_{2,k} \in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}, \quad \forall\, N_0>0, k\in\mathbb{Z}^d, \tag{9.16}$$

with relevant constants uniform in k. Observe that if two symbols b_1, b_2 in some $S_{1,\delta}^n$ have disjoint s-supports, with

$$\|D_s^\gamma D_\xi^\beta b_i(s,\xi)\|_{B(X)} \le C_{i,\gamma,\beta}(1+|\xi|)^{n+\delta|\gamma|_1-|\beta|_1}, \quad i=1,2,$$

then $b_1+b_2 \in S_{1,\delta}^n$ with

$$\|D_s^\gamma D_\xi^\beta\big(b_1(s,\xi)+b_2(s,\xi)\big)\|_{B(X)} \le \max\{C_{1,\gamma,\beta}, C_{2,\gamma,\beta}\}(1+|\xi|)^{n+\delta|\gamma|_1-|\beta|_1}.$$

For our use, we construct a partition of $\mathbb{Z}^d$ with subsets $U_1, U_2, \cdots, U_{2^d}$ such that for any k_1, k_2 in each U_j, the supports $\operatorname{supp}\mathcal{X}_{k_1}$ and $\operatorname{supp}\mathcal{X}_{k_2}$ are disjoint. More precisely, let π: $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ be the canonical projection sending even integer to 0 and odd integer to 1. Let π^d: $\mathbb{Z}^d \longrightarrow (\mathbb{Z}/2\mathbb{Z})^d$ be the d-fold product of π. Then $(U_j)_{j\in(\mathbb{Z}/2\mathbb{Z})^d} = \big((\pi^d)^{-1}(j)\big)_{j\in(\mathbb{Z}/2\mathbb{Z})^d}$ gives the desired partition of $\mathbb{Z}^d$. Summing over (9.16) in each U_j, we get a symbol still in $S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}$, that is,

$$\sum_{k\in U_j}\sigma_{3,k} - \sum_{k\in U_j}\sum_{|\gamma|_1<N_0} \frac{(2\pi\mathrm{i})^{-|\gamma|_1}}{\gamma!} D_\xi^\gamma\sigma_1 D_s^\gamma\sigma_{2,k} \in S_{1,\delta}^{n_1+n_2-(1-\delta)N_0}.$$

Taking the finite sum over $\{U_j\}_{1\le j\le 2^d}$, we get the asymptotic formula (9.10) in this case.

Finally, let us get rid of the additional assumption that σ_1 and σ_2 have compact supports. We define σ_3^j as follows:

$$T_{\sigma_3^j} = T_{\sigma_1^j}T_{\sigma_2^j}.$$

where $\sigma_1^j(s,\xi) = \sigma_1(s,\xi)\eta(2^{-j}s, 2^{-j}\xi)$ and $\sigma_2^j(s,\xi) = \sigma_2(s,\xi)\eta(2^{-j}s,2^{-j}\xi)$ with η given in (9.3). Notice that the σ_1^j's and the σ_2^j's are in the class $S_{1,\delta}^{n_1}$ and $S_{1,\delta}^{n_2}$ respectively with symbolic constants uniform in j. Therefore, the above arguments ensure that σ_3^j belongs to $S_{1,\delta}^{n_1+n_2}$ and satisfies (9.10) uniformly in j. Passing to the limit, we get that $\sigma_3 \in S_{1,\delta}^{n_1+n_2}$ and satisfies (9.10). Furthermore, by (9.4), we get

$$T_{\sigma_3} = T_{\sigma_1}T_{\sigma_2}.$$

The proof is complete. $\square$

We end this section with the asymptotic formula for the adjoint of a pseudo-differential operator with symbol in the class $S^n_{1,\delta}$.

Proposition 9.9. *Let $0 \leq \delta < 1$, $n \in \mathbb{R}$ and σ be a symbol in $S^n_{1,\delta}$. There exists a symbol $\widetilde{\sigma} \in S^n_{1,\delta}$ such that $T_{\widetilde{\sigma}} = (T_\sigma)^*$. Moreover,*

$$\widetilde{\sigma} - \sum_{|\gamma|_1 < N_0} \frac{(2\pi \mathrm{i})^{-|\gamma|_1}}{\gamma!} D^\gamma_\xi D^\gamma_s \sigma^* \in S^{n-(1-\delta)N_0}_{1,\delta}, \quad \forall N_0 \geq 0.$$

Proof. The proof is similar to that of Proposition 9.8. By (9.5), we get the formal expression of $\widetilde{\sigma}$ that

$$\begin{aligned}\widetilde{\sigma}(s,\xi) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma^*(t,\eta) e^{2\pi \mathrm{i}(s-t)\cdot(\eta-\xi)} dt d\eta \\ &= \int_{\mathbb{R}^d} \widehat{\sigma}^*(\eta, \xi+\eta) e^{2\pi \mathrm{i} s\cdot\eta} d\eta,\end{aligned}$$

where $\widehat{\sigma}^*$ is the Fourier transform of σ^* with respect to the first variable. By the same argument used in the proof of the previous proposition, we may focus on the symbol with compact t-support. Taking the Taylor expression of $\widehat{\sigma}^*(\eta, \xi+\eta)$, we get

$$\widehat{\sigma}^*(\eta,\xi+\eta) = \sum_{|\gamma|_1 < N_0} \frac{1}{\gamma!} D^\gamma_\xi \widehat{\sigma}^*(\eta,\xi)\eta^\gamma + \sum_{N_0 \leq |\gamma|_1 < N} \frac{1}{\gamma!} D^\gamma_\xi \widehat{\sigma}^*(\eta,\xi)\eta^\gamma + R_N(\xi,\eta).$$

As before, we can show that

$$\frac{1}{\gamma!} \int_{\mathbb{R}^d} D^\gamma_\xi \widehat{\sigma}^*(\eta,\xi)\eta^\gamma e^{2\pi \mathrm{i} s\cdot\eta} d\eta = \frac{(2\pi \mathrm{i})^{-|\gamma|_1}}{\gamma!} D^\gamma_\xi D^\gamma_s \widehat{\sigma}^*(s,\xi) \in S^{n-(1-\delta)|\gamma|_1}_{1,\delta}.$$

On the other hand, we can also show that

$$\| \int_{\mathbb{R}^d} R_N(\xi,\eta) e^{2\pi \mathrm{i} s\cdot\eta} d\eta \|_{B(X)} \lesssim (1+|\xi|)^{n-(1-\delta)N_0}$$

by splitting the integral over η into two parts. Moreover, repeating the above procedure to its derivatives, we have $\int_{\mathbb{R}^d} R_N(\xi,\eta) e^{2\pi \mathrm{i} s\cdot\eta} d\eta \in S^{n-(1-\delta)N_0}_{1,\delta}$. Thus, the proposition is proved. $\square$

9.2 Some lemmas

In order to study the boundedness of pseudo-differential operators on the Triebel-Lizorkin spaces, we will use the atomic decomposition stated in the last chapter. In other words, we will focus on the images of the atoms under the action of pseudo-differential operators instead of the images of general functions in the Triebel-Lizorkin spaces.

In the sequel, we will only consider pseudo-differential operators whose symbols take values in $\mathcal{M}$. If we take $X = L_1(\mathcal{M}) + \mathcal{M}$, then $\mathcal{M}$ admits an isometric embedding into $B(X)$ by left or right multiplication. In this way, these $\mathcal{M}$-valued symbols can be seen as a special case of the $B(X)$-valued symbols defined in the previous section. On the other hand, if we embed $\mathcal{M}$ into $B(X)$ by right multiplication, we get another kind of $\mathcal{M}$-valued symbol actions. Accordingly, we define

$$T^c_\sigma f(s) = \int_{\mathbb{R}^d} \sigma(s,\xi) \widehat{f}(\xi) e^{2\pi \mathrm{i} s\cdot\xi} d\xi$$

and

$$T^r_\sigma f(s) = \int_{\mathbb{R}^d} \widehat{f}(\xi)\sigma(s,\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi.$$

All the conclusions proved in the previous section still hold for both T^c_σ and T^r_σ parallel. In the following sections, we mainly focus on the operators T^c_σ.

The first lemma in this section concerns the image of an $(\alpha, Q_{\mu,l})$-subatom under the action of pseudo-differential operators.

Lemma 9.10. *Let $\alpha \in \mathbb{R}$, $\sigma \in S^0_{1,\delta}$ and T^c_σ be the corresponding pseudo-differential operator. In addition, we assume that $K > \frac{d}{2}$. Then for any $(\alpha, Q_{\mu,l})$-subatom $a_{\mu,l}$, we have*

$$\tau\Big(\int_{\mathbb{R}^d}(1+2^\mu|s-2^{-\mu}l|)^{d+M}|D^\gamma T^c_\sigma a_{\mu,l}(s)|^2ds\Big)^{\frac12} \lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}-\frac{|\gamma|_1}{d}}, \quad |\gamma|_1 < K-\frac{d}{2}, \tag{9.17}$$

where $M \in \mathbb{R}$ such that $M < 2L+2$ and the relevant constant depends on M, K, L, γ and d.

Proof. We split the integral on the left hand side of (9.17) into $\int_{4Q_{\mu,l}}$ and $\int_{(4Q_{\mu,l})^c}$. To estimate the term with $\int_{4Q_{\mu,l}}$, we begin with a technical modification of $a_{\mu,l}$. For every $a_{\mu,l}$, we define

$$a = |Q_{\mu,l}|^{-\frac{\alpha}{d}+\frac12}a_{\mu,l}(2^{-\mu}(\cdot + l)).$$

It is easy to see that a is an $(\alpha, Q_{0,0})$-subatom. By translation, we may assume that $l=0$. Then by the Cauchy-Schwarz inequality, for any $s\in\mathbb{R}^d$, we have

$$\begin{aligned}
|T^c_\sigma a_{\mu,l}(s)|^2 &= 2^{-2\mu d}|Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\Big|\int_{\mathbb{R}^d}\sigma(s,\xi)\widehat{a}(2^{-\mu}\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi\Big|^2\\
&= |Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\Big|\int_{\mathbb{R}^d}\sigma(s,2^\mu\xi)\widehat{a}(\xi)e^{2\pi \mathrm{i} s\cdot 2^\mu\xi}d\xi\Big|^2\\
&\le |Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\int_{\mathbb{R}^d}\|\sigma(s,2^\mu\xi)\|^2_{\mathcal{M}}(1+|\xi|^2)^{-K}d\xi\\
&\quad \int_{\mathbb{R}^d}(1+|\xi|^2)^{K}\|\sigma(s,2^\mu\xi)\|^{-2}_{\mathcal{M}}\,\widehat{a}^*(\xi)|\sigma(s,2^\mu\xi)|^2\widehat{a}(\xi)d\xi\\
&\lesssim |Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\int_{\mathbb{R}^d}(1+|\xi|^2)^{-K}d\xi\cdot\int_{\mathbb{R}^d}(1+|\xi|^2)^{K}|\widehat{a}(\xi)|^2d\xi\\
&\lesssim |Q_{\mu,l}|^{2(\frac{\alpha}{d}-\frac12)}\int_{\mathbb{R}^d}|J^K a(t)|^2dt,
\end{aligned}$$

where J^K is the Bessel potential of order K. Combining the second assumption on $a_{\mu,l}$ in Definition 8.18 and the above estimates, we obtain

$$\begin{aligned}
\tau\Big(\int_{4Q_{\mu,l}}|T^c_\sigma a_{\mu,l}(s)|^2(1+2^\mu|s|)^{d+M}ds\Big)^{\frac12} &\lesssim \tau\Big(\int_{4Q_{\mu,l}}|T^c_\sigma a_{\mu,l}(s)|^2ds\Big)^{\frac12}\\
&\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}\tau\Big(\int_{\mathbb{R}^d}|J^K a(t)|^2dt\Big)^{\frac12}\\
&\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}\sum_{|\gamma|_1\le K}\tau\Big(\int_{\mathbb{R}^d}|D^\gamma a(t)|^2dt\Big)^{\frac12} \lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}.
\end{aligned}$$

If $s\in(4Q_{\mu,l})^c$, since $a_{\mu,l}$ has the moment cancellations of order less than or equal to L, we can subtract a Taylor polynomial of degree L from the kernel associated to T^c_σ. Then, applying the estimate (9.7) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|T^c_\sigma a_{\mu,l}(s)|^2 \\
&= \Big|\int_{\mathbb{R}^d} K(s,s-t)a_{\mu,l}(t)dt\Big|^2 \\
&= \Big|\int_{\mathbb{R}^d}[K(s,s-t)-K(s,s)]a_{\mu,l}(t)dt\Big|^2 \\
&= \Big|\int_{\mathbb{R}^d}[\sum_{|\beta|_1=L+1}\frac{L+1}{\beta!}t^\beta\int_0^1(1-\theta)^\beta D^\beta K(s,s-\theta t)d\theta]a_{\mu,l}(t)dt\Big|^2 \\
&\lesssim \sum_{|\beta|_1=L+1}\int_{2Q_{\mu,l}}\Big\|\int_0^1(1-\theta)^\beta D^\beta K(s,s-\theta t)d\theta\Big\|^2_{\mathcal{M}}|t|^{2L+2}dt \\
&\quad\cdot\int_{\mathbb{R}^d}\Big\|\int_0^1(1-\theta)^\beta D^\beta K(s,s-\theta t)d\theta\Big\|^{-2}_{\mathcal{M}}\Big|\int_0^1(1-\theta)^\beta D^\beta K(s,s-\theta t)d\theta\, a_{\mu,l}(t)\Big|^2dt \\
&\le \sum_{|\beta|_1=L+1}\int_{2Q_{\mu,l}}\sup_{0\le\theta\le1}\|D^\beta K(s,s-\theta t)\|^2_{\mathcal{M}}\,|t|^{2L+2}dt\cdot\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt \\
&\lesssim |s|^{-2d-2L-2}\int_{2Q_{\mu,l}}|t|^{2L+2}dt\cdot\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt \\
&\lesssim 2^{-\mu(2L+2+d)}|s|^{-2d-2L-2}\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt.
\end{aligned}\tag{9.18}$$

This estimate implies

$$\begin{aligned}
&\tau\Big(\int_{(4Q_{\mu,l})^c}|T^c_\sigma a_{\mu,l}(s)|^2(1+2^\mu|s|)^{d+M}ds\Big)^{\frac12} \\
&\lesssim 2^{-\mu(L+1-\frac{M}{2})}\Big(\int_{(4Q_{\mu,l})^c}|s|^{-d-2L-2+M}ds\Big)^{\frac12}\cdot\tau\Big(\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt\Big)^{\frac12} \\
&\lesssim 2^{-\mu(L+1-\frac{M}{2})}2^{\mu(L+1-\frac{M}{2})}|Q_{\mu,l}|^{\frac{\alpha}{d}} = |Q_{\mu,l}|^{\frac{\alpha}{d}}.
\end{aligned}$$

If we take $M=-d$ in the above inequality, we have $T^c_\sigma a_{\mu,l}\in L_1(\mathcal{M};L^c_2(\mathbb{R}^d))$. By approximation, we can assume that $\sigma(s,\xi)$ has compact ξ-support, so

$$T^c_\sigma a_{\mu,l}(s)=\int_{\mathbb{R}^d}\sigma(s,\xi)\widehat{a_{\mu,l}}(\xi)e^{2\pi i s\cdot\xi}d\xi$$

is uniformly convergent. Moreover, one can differentiate the integrand and obtain always uniformly convergent integrals. Then, for any $|\gamma|_1<K-\frac{d}{2}$, we have

$$\begin{aligned}
&\tau\Big(\int_{4Q_{\mu,l}}|D^\gamma T^c_\sigma a_{\mu,l}(s)|^2(1+2^\mu|s|)^{d+M}ds\Big)^{\frac12} \\
&\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}\tau\Big(\int_{\mathbb{R}^d}|J^K a(t)|^2dt\Big)^{\frac12}\int_{\mathbb{R}^d}(1+|\xi|)^{2|\gamma|_1-2K}d\xi \\
&\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}\sum_{|\gamma|_1\le K}\tau\Big(\int_{\mathbb{R}^d}|D^\gamma a(t)|^2dt\Big)^{\frac12}\lesssim |Q_{\mu,l}|^{\frac{\alpha}{d}}.
\end{aligned}\tag{9.19}$$

By a similar argument to that of (9.18), we have, for any $\gamma\in\mathbb{N}_0^d$ and $s\in(4Q_{\mu,l})^c$,

$$|D^\gamma T^c_\sigma a_{\mu,l}(s)|^2\lesssim 2^{-\mu(2L+2+d)}|s|^{-2d-2L-2-2|\gamma|_1}\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt.\tag{9.20}$$

Therefore, we deduce that

$$\begin{aligned}
&\tau\Big(\int_{(4Q_{\mu,l})^c} |D^\gamma T_\sigma^c a_{\mu,l}(s)|^2(1+2^\mu|s|)^{d+M}ds\Big)^{\frac{1}{2}} \\
&\lesssim 2^{-\mu(L+1-\frac{M}{2})}\Big(\int_{(4Q_{\mu,l})^c} |s|^{-d-2L-2+M-2|\gamma|_1}ds\Big)^{\frac{1}{2}} \cdot \tau\Big(\int_{\mathbb{R}^d} |a_{\mu,l}(t)|^2dt\Big)^{\frac{1}{2}} \\
&\lesssim 2^{-\mu(L+1-\frac{M}{2})}2^{\mu(L+1-\frac{M}{2}+|\gamma|_1)}|Q_{\mu,l}|^{\frac{\alpha}{d}} = |Q_{\mu,l}|^{\frac{\alpha}{d}-\frac{|\gamma|_1}{d}}.
\end{aligned}$$

Combining the estimates above, we get (9.17). □

On the other hand, we have the following lemma concerning the image of $(\alpha,1)$-atoms under the action of pseudo-differential operators.

Lemma 9.11. *Let $\alpha \in \mathbb{R}$, σ be a symbol in the class $S^0_{1,\delta}$ and T_σ^c be the corresponding pseudo-differential operator. Let $K > \frac{d}{2}$ and b be an $(\alpha,1)$-atom based on the cube $Q_{0,m}$. Then for any $M \in \mathbb{R}$, we have*

$$\tau\Big(\int_{\mathbb{R}^d}(1+|s-m|)^{d+M}|D^\gamma T_\sigma^c b(s)|^2ds\Big)^{\frac{1}{2}} \lesssim 1, \quad |\gamma|_1 < K - \frac{d}{2}, \tag{9.21}$$

where the relevant constant depends on M, K, γ and d.

Proof. The proof of this lemma is similar to that of the previous one. The only difference is that for an $(\alpha,1)$-atom, we do not necessarily have the moment cancellation; in this case, we need to use the extra decay of the kernel when $|t| > 1$.

If $s \in 4Q_{0,m}$, we follow the estimate for subatoms in the previous lemma. Applying the size estimate of b, we get

$$\tau\Big(\int_{4Q_{0,m}}(1+|s-m|)^{d+M}|T_\sigma^c b(s)|^2ds\Big)^{\frac{1}{2}} \lesssim \tau\Big(\int_{\mathbb{R}^d}|J^K b(t)|^2dt\Big)^{\frac{1}{2}} \lesssim 1.$$

If $s \in (4Q_{0,m})^c$ and $t \in 2Q_{0,m}$, we have $|s-t| \geq 1$. Then (9.8) gives

$$\begin{aligned}
|T_\sigma^c b(s)|^2 &= \Big|\int_{\mathbb{R}^d} K(s,s-t)b(t)dt\Big|^2 \\
&\leq \int_{2Q_{0,m}} \|K(s,s-t)\|^2_{\mathcal{M}}dt \int_{2Q_{0,m}} |b(t)|^2dt \\
&\lesssim |s-m|^{-2N}\int_{2Q_{0,m}} |b(t)|^2dt,
\end{aligned}$$

where the positive integer N can be arbitrarily large. Thus

$$\begin{aligned}
&\tau\Big(\int_{(4Q_{0,m})^c}(1+|s-m|)^{d+M}|T_\sigma^c b(s)|^2ds\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_{(4Q_{0,m})^c}|s-m|^{d+M-2N}ds\Big)^{\frac{1}{2}}\tau\Big(\int_{2Q_{0,m}}|b(t)|^2dt\Big)^{\frac{1}{2}} \lesssim 1.
\end{aligned}$$

Then, the estimates obtained above imply that

$$\tau\Big(\int_{\mathbb{R}^d}(1+|s-m|)^{d+M}|T_\sigma^c b(s)|^2ds\Big)^{\frac{1}{2}} \lesssim 1.$$

Similarly, we treat $D^\gamma T_\sigma^c b(s)$ as

$$\tau\Big(\int_{\mathbb{R}^d}(1+|s-m|)^{d+M}|D^\gamma T_\sigma^c b(s)|^2ds\Big)^{\frac{1}{2}} \lesssim 1, \quad |\gamma|_1 < K - \frac{d}{2}.$$

Therefore, (9.21) is proved. □

The following lemma shows that, if the symbol σ satisfies some support condition, we can even establish the $F_1^{\alpha,c}$-norm of the image of $(\alpha, Q_{\mu,l})$-subatom under T_σ^c.

Lemma 9.12. *Let $\sigma \in S^0_{1,\delta}$ and T^c_σ be the corresponding pseudo-differential operator. Assume that $\alpha \in \mathbb{R}$, $K \in \mathbb{N}$ satisfy $K > \frac{d}{2}$ and $K > \alpha + d$. If the s-support of σ is in $(2^{-\mu}l + 4Q_{0,0})^c$, then for any $(\alpha, Q_{\mu,l})$-subatom $a_{\mu,l}$, we have*

$$\|T^c_\sigma a_{\mu,l}\|_{F_1^{\alpha,c}} \lesssim 2^{-\mu(\frac{d}{2}+\iota)},$$

where ι is a positive real number.

Proof. Without loss of generality, we still assume $l = 0$. We keep the assumption of the test functions $\widehat{\Phi} = |\cdot|^N \widehat{\kappa}, \Phi^{(0)} \in \mathcal{S}$ in the proof of Theorem 8.21. In addition, we assume $N \in \mathbb{N}_0$ such that $\alpha + \frac{d}{2} - 1 < N < K - \frac{d}{2}$ and $\operatorname{supp} \Phi_0 \subset Q_{0,0}$. To simplify the notation, we denote $T^c_\sigma a_{\mu,l}$ by $\eta_{\mu,l}$. Then

$$\|\eta_{\mu,l}\|_{F_1^{\alpha,c}} \approx \|\Phi_0 * \eta_{\mu,l}\|_1 + \Big\|\Big(\int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * \eta_{\mu,l}|^2 \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}\Big\|_1.$$

We notice that Φ satisfies the moment cancellation up to order N. It follows that

$$\begin{aligned}\Phi_\varepsilon * \eta_{\mu,l}(s) &= \int_{\mathbb{R}^d} \Phi_\varepsilon(t)[\eta_{\mu,l}(s-t) - \eta_{\mu,l}(s)]dt \\ &= \int_{\mathbb{R}^d} \Phi_\varepsilon(t) \sum_{|\gamma|_1 = N+1} \frac{N+1}{\gamma!}(-t)^\gamma \int_0^1 (1-\theta)^N D^\gamma \eta_{\mu,l}(s-\theta t)d\theta\, dt.\end{aligned} \tag{9.22}$$

Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\Big|\int_{|t|>\frac{|s|}{2}} \Phi_\varepsilon(t) \sum_{|\gamma|_1=N+1} \frac{N+1}{\gamma!} t^\gamma \int_0^1 (1-\theta)^N D^\gamma \eta_{\mu,l}(s-\theta t)d\theta\, dt\Big|^2 \\ &\lesssim \sum_{|\gamma|_1=N+1} \int_{\mathbb{R}^d}\int_0^1 (1-\theta)^{2N}|D^\gamma \eta_{\mu,l}(s-\theta t)|^2 d\theta (1+|t|)^{-d-1}dt \\ &\quad \cdot \int_{\mathbb{R}^d} |\Phi_\varepsilon(t)|^2 |t|^{2N+2}(1+|t|)^{d+1}dt.\end{aligned} \tag{9.23}$$

By (9.20), if $s - \theta t \in (4Q_{0,0})^c$, we have

$$|D^\gamma \eta_{\mu,l}(s-\theta t)|^2 \lesssim 2^{-\mu(2L+2+d)}|s-\theta t|^{-2d-2L-2-2|\gamma|_1} \int_{\mathbb{R}^d} |a_{\mu,l}(r)|^2 dr.$$

Therefore, using the Cauchy-Schwarz inequality again, we have

$$\begin{aligned}&\Big\|\Big(\int_0^1 \varepsilon^{-2\alpha}|\Phi_\varepsilon * \eta_{\mu,l}|^2 \frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}}\Big\|_1 \\ &\lesssim 2^{-\mu(L+1+\frac{d}{2})}\Big(\int_0^1 \varepsilon^{2N-d+2-2\alpha}\frac{d\varepsilon}{\varepsilon}\Big)^{\frac{1}{2}} \int_{(2Q_{0,0})^c} |s'|^{-d-L-N-2}ds' \int_{\mathbb{R}^d}(1+|t|)^{-d-1}dt \\ &\quad \cdot \int_{\mathbb{R}^d} |\Phi(t')|^2|t'|^{2N+2}(1+|t'|)^{d+1}dt' \cdot \tau\Big(\int_{\mathbb{R}^d}|a_{\mu,l}(r)|^2 dr\Big)^{\frac{1}{2}} \\ &\lesssim 2^{-\mu(L+1+\frac{d}{2})}\tau\Big(\int_{\mathbb{R}^d}|a_{\mu,l}(t)|^2dt\Big)^{\frac{1}{2}} \\ &\lesssim 2^{-\mu(L+1+\frac{d}{2}+\alpha)}.\end{aligned} \tag{9.24}$$

It remains to estimate the L_1-norm of $\Phi_0 * \eta_{\mu,l}$, where Φ_0 does not have the moment cancellation. Since $\operatorname{supp} \eta_{\mu,l} \subset (4Q_{0,0})^c$ and by the support assumption of Φ_0, we have $\operatorname{supp} \Phi_0 * \eta_{\mu,l} \subset \{s \in \mathbb{R}^d : |s| \geq \frac{1}{2}\}$. By Lemma 9.11 and the fact that $|\Phi_0(s)| \lesssim (1+|s|)^{-d-R}$ for any $R \in \mathbb{N}$, we have

$$
\begin{aligned}
&|\Phi_0 * \eta_{\mu,l}(s)|^2 = \Big| \int_{\mathbb{R}^d} \Phi_0(s-t)\eta_{\mu,l}(t)dt \Big|^2 \\
&\leq \Big| \int_{|t|\geq \max\{\frac{|s|}{2},1\}} \Phi_0(s-t)\eta_{\mu,l}(t)dt \Big|^2 + \Big| \int_{1\leq |t| < \frac{|s|}{2}} \Phi_0(s-t)\eta_{\mu,l}(t)dt \Big|^2 \\
&\leq \int_{|t|\geq \max\{\frac{|s|}{2},1\}} (1+2^\mu|t|)^{-2d-2R} |\Phi_0(s-t)|^2 dt \cdot \int_{\mathbb{R}^d} (1+2^\mu|t|)^{2d+2R} |\eta_{\mu,l}(t)|^2 dt \\
&\quad + \int_{1\leq |t| < \frac{|s|}{2}} (1+2^\mu|t|)^{-2d-2R} |\Phi_0(s-t)|^2 dt \cdot \int_{\mathbb{R}^d} (1+2^\mu|t|)^{2d+2R} |\eta_{\mu,l}(t)|^2 dt \\
&\lesssim \int_{\mathbb{R}^d} |\Phi_0(t)|^2 dt \, (1+2^\mu|s|)^{-2d-2R} \int_{\mathbb{R}^d} (1+2^\mu|t|)^{2d+2R} |\eta_{\mu,l}(t)|^2 dt \\
&\quad + \int_{|t|\geq 1} (1+2^\mu|t|)^{-2d-2R} dt \, (1+|s|)^{-2d-2R} \int_{\mathbb{R}^d} (1+2^\mu|t|)^{2d+2R} |\eta_{\mu,l}(t)|^2 dt \\
&\lesssim \big[(1+2^\mu|s|)^{-2d-2R} + 2^{-2\mu(d+R)}(1+|s|)^{-2d-2R}\big] \int_{\mathbb{R}^d} (1+2^\mu|t|)^{2d+2R} |\eta_{\mu,l}(t)|^2 dt.
\end{aligned}
$$

Then we can use (9.17) to get, for any $R \in \mathbb{N}$,

$$
\begin{aligned}
\|\Phi_0 * \eta_{\mu,l}\|_1 &\lesssim \Big(\int_{|s|\geq \frac{1}{2}} (1+2^\mu|s|)^{-d-R} ds + 2^{-\mu(d+R)} \int_{|s|\geq \frac{1}{2}} (1+|s|)^{-d-R} ds \Big) \\
&\quad \cdot \tau\Big(\int_{\mathbb{R}^d} (1+2^\mu|t|)^{2d+2R} |\eta_{\mu,l}(t)|^2 dt\Big)^{\frac{1}{2}} \\
&\lesssim 2^{-\mu(d+R+\alpha)}.
\end{aligned}
$$

Combining the estimates above, we get that, there exists $\iota > 0$ such that

$$
\|T_\sigma^c a_{\mu,l}\|_{F_1^{\alpha,c}} = \|\eta_{\mu,l}\|_{F_1^{\alpha,c}} \lesssim 2^{-\mu(\frac{d}{2}+\iota)},
$$

which completes the proof. □

Since every $(\alpha, Q_{k,m})$-atom is a linear combination of subatoms, the above lemma helps us to estimate the image of $(\alpha, Q_{k,m})$-atom under T_σ^c.

Corollary 9.13. *Let $\sigma \in S_{1,\delta}^0$ and T_σ^c be the corresponding pseudo-differential operator. Assume that $\alpha \in \mathbb{R}$, $K \in \mathbb{N}$ satisfy $K > \frac{d}{2}$ and $K > \alpha + d$. If the s-support of σ is in $(2^{-k}m + 6Q_{0,0})^c$, then for any $(\alpha, Q_{k,m})$-atom g,we have*

$$
\|T_\sigma^c g\|_{F_1^{\alpha,c}} \lesssim 1.
$$

Proof. For any $(\alpha, Q_{k,m})$-atom g, it admits the form

$$
g = \sum_{(\mu,l)\leq(k,m)} d_{\mu,l} a_{\mu,l} \quad \text{with} \quad \sum_{(\mu,l)\leq(k,m)} |d_{\mu,l}|^2 \leq |Q_{k,m}|^{-1} = 2^{kd}.
$$

By the support assumption of σ, $\sigma(s,\xi) = 0$ if $s \in 2^{-\mu}l + 4Q_{0,0} \subset 2^{-k}m + 6Q_{0,0}$. Then, we can apply the previous lemma to every $a_{\mu,l}$ with $(\mu, l) \leq (k,m)$. The result is

$$
\|T_\sigma^c a_{\mu,l}\|_{F_1^{\alpha,c}} \lesssim 2^{-\mu(\frac{d}{2}+\iota)} \quad \text{with } \iota > 0.
$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|T^c_\sigma g\|_{F^{\alpha,c}_1} &\leq \sum_{(\mu,l)\leq(k,m)} |d_{\mu,l}|\cdot\|T^c_\sigma a_{\mu,l}\|_{F^{\alpha,c}_1}\\ &\leq \sum_{(\mu,l)\leq(k,m)} |d_{\mu,l}|\cdot 2^{-\mu(\frac{d}{2}+\iota)}\\ &\lesssim \Big(\sum_{(\mu,l)\leq(k,m)} |d_{\mu,l}|^2\Big)^{\frac12}\Big(\sum_{(\mu,l)\leq(k,m)} 2^{-\mu(d+2\iota)}\Big)^{\frac12} \qquad (9.25)\\ &\leq \Big(\sum_{(\mu,l)} |d_{\mu,l}|^2\Big)^{\frac12}\Big(\sum_{\mu\geq k}\frac{|2Q_{k,m}|}{|Q_{\mu,l}|}\cdot 2^{-\mu(d+2\iota)}\Big)^{\frac12}\\ &\lesssim |Q_{k,m}|^{-\frac12}\cdot 2^{-\frac{kd}{2}}=1.\end{aligned}$$

Thus, the assertion is proved. □

Likewise, we can estimate the image of $(\alpha,1)$-atom under the pseudo-differential operator T^c_σ.

Lemma 9.14. *Let $\sigma\in S^0_{1,\delta}$ and T^c_σ be the corresponding pseudo-differential operator. Assume that $\alpha\in\mathbb{R}$, $K\in\mathbb{N}$ satisfy $K>\frac{d}{2}$ and $K>\alpha+d$. If the s-support of σ is in $(k+4Q_{0,0})^c$ for some $k\in\mathbb{Z}^d$, then for any $(\alpha,1)$-atom b such that* $\operatorname{supp} b\subset 2Q_{0,k}$*, we have*

$$\|T^c_\sigma b\|_{F^{\alpha,c}_1}\lesssim 1.$$

Proof. The proof of this lemma is very similar to that of Lemma 9.12; it suffices to apply (the proof of) Lemma 9.11 instead of Lemma 9.10. □

Corollary 9.15. *Let $\sigma\in S^0_{1,\delta}$ and T^c_σ be the corresponding pseudo-differential operator. Given $\alpha\in\mathbb{R}$, $K\in\mathbb{N}$ such that $K>\frac{d}{2}$ and $K>\alpha+d$, then for any $(\alpha,1)$-atom b, we have*

$$\|T^c_\sigma b\|_{F^{\alpha,c}_1}\lesssim 1.$$

Proof. Let $(\mathcal{X}_j)_{j\in\mathbb{Z}^d}$ be the smooth resolution of the unit in (7.13). We decompose $T^c_\sigma b$ as

$$T^c_\sigma b=\sum_{j\in\mathbb{Z}^d}\mathcal{X}_j T^c_\sigma b=\sum_{j\in\mathbb{Z}^d}T^c_{\sigma_j}b,$$

where $\sigma_j=\mathcal{X}_j(s)\sigma(s,\xi)\in S^0_{1,\delta}$ uniformly. Suppose that b is supported in $2Q_{0,k}$ with $k\in\mathbb{Z}^d$. We split the above summation into two parts:

$$T^c_\sigma b=\sum_{j\in k+6Q_{0,0}}\mathcal{X}_j T^c_\sigma b+\sum_{j\notin k+6Q_{0,0}}\mathcal{X}_j T^c_\sigma b. \qquad (9.26)$$

Applying Lemma 9.11 with $M=-d$ to the symbol $\mathcal{X}_j(s)\sigma(s,\xi)$, we get, for any $j\in\mathbb{Z}^d$,

$$\tau\Big(\int_{j+2Q_{0,0}}|D^\gamma(\mathcal{X}_j\,T^c_\sigma b(s))|^2ds\Big)^{\frac12}\lesssim 1,\quad \forall|\gamma|_1\leq[\alpha]+1.$$

Thus, $\mathcal{X}_j\,T^c_\sigma b$ is a bounded multiple of an $(\alpha,1)$-atom. So the first term on the right hand side of (9.26) is a finite sum of $(\alpha,1)$-atoms, and thus has bounded $F^{\alpha,c}_1$-norm. Now we deal with the second term. Note that the s-support of the symbol $\sum_{j\notin k+6Q_{0,0}}\mathcal{X}_j(s)\sigma(s,\xi)$ is in $(k+4Q_{0,0})^c$. Then, it suffices to apply Lemma 9.14 to this symbol, so that

$$\Big\|\sum_{j\notin k+6Q_{0,0}}\mathcal{X}_j T^c_\sigma b\Big\|_{F^{\alpha,c}_1}\lesssim 1.$$

The proof is complete. □

9.3 Regular symbols

In this section, we study the continuity of the pseudo-differential operators with regular symbols in $S^0_{1,\delta}$ $(0 \le \delta < 1)$ on Triebel-Lizorkin spaces. We start by presenting an L_2-theorem. It is a noncommutative analogue of the corresponding classical theorem, which can be found in many works, for instance, [61, 63, 59]. Then, we will use the atomic decomposition obtained in the last chapter to deduce the $F^{\alpha,c}_p$-boundedness. Different from the pseudo-differential operators with the forbidden symbols in $S^0_{1,1}$, which will be treated in the next section, our proof stays at the level of atoms; in other words, we do not need the subtler decomposition that every (α, Q)-atom can be written as a linear combination of subatoms.

Theorem 9.16. *Let $0 \le \delta < 1$, $\sigma \in S^0_{1,\delta}$ and $\alpha \in \mathbb{R}$. Then T^c_σ is bounded on $F^{\alpha,c}_p(\mathbb{R}^d, \mathcal{M})$ for any $1 \le p \le \infty$.*

In order to fully understand the image of an (α, Q)-atom under the action of a pseudo-differential operator, we need to study its $L_1(\mathcal{M}; L^c_2(\mathbb{R}^d))$-boundedness. We will work on the exotic class $S^0_{\delta,\delta}$ with $0 \le \delta < 1$, since we have the inclusion $S^0_{1,\delta} \subset S^0_{\delta,\delta}$. The Cotlar-Stein almost orthogonality lemma plays a crucial role in our proof. Namely, given a family of operators $(T_j)_j \subset B(H)$ with H a Hilbert space, and a positive sequence $\{c(j)\}_j$ such that $\sum_j c(j) = C < \infty$, if the T_j's satisfy:

$$\|T^*_k T_j\|_{B(H)} \le |c(k-j)|^2,$$

and

$$\|T_k T^*_j\|_{B(H)} \le |c(k-j)|^2,$$

then we have

$$\|\sum_j T_j\|_{B(H)} \le C.$$

We begin with a simpler case where $\delta = \rho = 0$. The following lemma is modelled after [61, Proposition VII.2.4]; we include a proof for the sake of completeness.

Lemma 9.17. *Assume $\sigma \in S^0_{0,0}$. Then T^c_σ is bounded on $L_2(\mathcal{N})$.*

Proof. By the Plancherel formula, it is enough to prove the $L_2(\mathcal{N})$-boundedness of the following operator:

$$S^c_\sigma(f)(s) = \int_{\mathbb{R}^d} \sigma(s,\xi) f(\xi) e^{2\pi \mathrm{i} s\cdot\xi} d\xi.$$

Let us make use of the resolution of the unit $(\mathcal{X}_k)_{k\in\mathbb{Z}^d}$ introduced in (7.13) to decompose S^c_σ into almost orthogonal pieces. Denote $\mathrm{k} = (k, k') \in \mathbb{Z}^d \times \mathbb{Z}^d$, and set

$$\sigma_{\mathrm{k}}(s,\xi) = \mathcal{X}_k(s)\sigma(s,\xi)\mathcal{X}_{k'}(\xi),$$

Then, the series $\sum_{\mathrm{k}\in\mathbb{Z}^d\times\mathbb{Z}^d} S^c_{\sigma_{\mathrm{k}}}$ converges in the strong operator topology and

$$S^c_\sigma = \sum_{\mathrm{k}\in\mathbb{Z}^d\times\mathbb{Z}^d} S^c_{\sigma_{\mathrm{k}}}.$$

We claim that $(S^c_{\sigma_{\mathrm{k}}})_{\mathrm{k}}$ satisfies the almost-orthogonality estimates, i.e., for any $N \in \mathbb{N}$,

$$\|(S^c_{\sigma_{\mathrm{k}}})^* S^c_{\sigma_{\mathrm{j}}}\|_{B(L_2(\mathcal{N}))} \le C_N(1 + |\mathrm{k} - \mathrm{j}|)^{-2N},$$

and

$$\|S^c_{\sigma_{\mathrm{k}}}(S^c_{\sigma_{\mathrm{j}}})^*\|_{B(L_2(\mathcal{N}))} \leq C_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N},$$

where the constant C_N is independent of $\mathrm{k}=(k,k')$ and $\mathrm{j}=(j,j')$. Armed with this claim, we can then apply the Cotlar-Stein almost orthogonality lemma stated previously to the operators $(S^c_{\sigma_{\mathrm{k}}})_{\mathrm{k}}$ with $c(\mathrm{j})=(1+|\mathrm{j}|)^{-N}$, $N>2d$. Then, we will have

$$\|S^c_\sigma\|_{B(L_2(\mathcal{N}))} = \|\sum_{\mathrm{k}\in\mathbb{Z}^d\times\mathbb{Z}^d} S^c_{\sigma_{\mathrm{k}}}\|_{B(L_2(\mathcal{N}))} \leq C.$$

Now we prove the claim. Note that for any $f\in L_2(\mathcal{N})$,

$$(S^c_{\sigma_{\mathrm{k}}})^* S^c_{\sigma_{\mathrm{j}}}(f)(\xi) = \int_{\mathbb{R}^d} \sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta) f(\eta) d\eta,$$

where

$$\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta) = \int_{\mathbb{R}^d} \sigma^*_{\mathrm{k}}(s,\xi)\sigma_{\mathrm{j}}(s,\eta) e^{2\pi \mathrm{i} s\cdot(\eta-\xi)} ds. \tag{9.27}$$

By the definition of σ_{k}, we see that if $k-j\notin 2Q_{0,0}$ (recalling that $Q_{0,0}$ is the unit cube centered at the origin), σ_{k} and σ_{j} have disjoint s-support, so

$$\sigma^*_{\mathrm{k}}\sigma_{\mathrm{j}} = 0.$$

When $k-j\in 2Q_{0,0}$, using the identity

$$(1-\Delta_s)^N e^{2\pi \mathrm{i} s\cdot(\eta-\xi)} = (1+4\pi^2|\eta-\xi|^2)^N e^{2\pi \mathrm{i} s\cdot(\eta-\xi)},$$

we integrate (9.27) by parts, which gives

$$\|\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta)\|_{\mathcal{M}} \leq C_N \mathcal{X}_{k'}(\xi)\mathcal{X}_{j'}(\eta)(1+|\xi-\eta|)^{-2N}.$$

Whence,

$$\max\Big\{\int_{\mathbb{R}^d}\|\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta)\|_{\mathcal{M}} d\xi, \int_{\mathbb{R}^d}\|\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta)\|_{\mathcal{M}} d\eta\Big\} \leq C'_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N}. \tag{9.28}$$

For any $f\in L_2(\mathcal{N})$, there exists $g\in L_2(\mathcal{N})$ with norm one such that

$$\|(S^c_{\sigma_{\mathrm{k}}})^* S^c_{\sigma_{\mathrm{j}}} f\|_{L_2(\mathcal{N})} = \Big|\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta) f(\eta)\, d\eta\, g^*(\xi)\, d\xi\Big|.$$

Applying the Hölder inequality and (9.28), we get

$$\begin{aligned}
&\Big|\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta) f(\eta)\, d\eta\, g^*(\xi)\, d\xi\Big| \\
&\leq (\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta)\|_{\mathcal{M}}|f(\eta)|^2 d\eta d\xi)^{\frac{1}{2}} (\tau\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\|\sigma_{\mathrm{k},\mathrm{j}}(\xi,\eta)\|_{\mathcal{M}}|g(\xi)|^2 d\xi d\eta)^{\frac{1}{2}} \\
&\leq C'_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N}\|f\|_{L_2(\mathcal{N})}.
\end{aligned}$$

Thus, $\|(S^c_{\sigma_{\mathrm{k}}})^* S^c_{\sigma_{\mathrm{j}}}\|_{B(L_2(\mathcal{N}))} \leq C'_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N}$. On the other hand, a similar argument also shows that

$$\|S^c_{\sigma_{\mathrm{k}}}(S^c_{\sigma_{\mathrm{j}}})^*\|_{B(L_2(\mathcal{N}))} \leq C'_N(1+|\mathrm{k}-\mathrm{j}|)^{-2N},$$

which proves the claim. □

A weak form of Cotlar-Stein almost orthogonality lemma also plays a crucial role. As before, we suppose that $\sum_j c(j) = C < \infty$. This time we assume that the T_j's satisfy:

$$\sup_j \|T_j\|_{B(H)} \leq C \tag{9.29}$$

and the following conditions hold for $j \neq k$:

$$\|T_j T_k^*\|_{B(H)} = 0 \quad \text{and} \quad \|T_j^* T_k\|_{B(H)} \leq c(j)c(k). \tag{9.30}$$

Then we have

$$\|\sum_j T_j\|_{B(H)} \leq \sqrt{2}C.$$

Lemma 9.18. *Let $\sigma \in S^0_{\delta,\delta}$ with $0 \leq \delta < 1$. Then T^c_σ is bounded on $L_2(\mathcal{N})$.*

Proof. To prove this lemma, we apply Cotlar's lemma as stated above. Let $(\widehat{\varphi}_j)_{j\geq 0}$ be the resolution of the unit defined in (1.3). We can decompose T^c_σ as follows:

$$T^c_\sigma = \sum_{j=0}^{\infty} T^c_{\sigma_j} = \sum_{j \text{ even}} T^c_{\sigma_j} + \sum_{j \text{ odd}} T^c_{\sigma_j},$$

where $\sigma_j(s,\xi) = \widehat{\varphi}_j(\xi)\sigma(s,\xi)$. Note that the symbols in either odd or even summand have disjoint ξ-supports. We will only treat the odd part, since the other part can be dealt with in a similar way. It is clear that $T^c_{\sigma_j}(T^c_{\sigma_k})^* = 0$ if $j \neq k$, since $T^c_{\sigma_j}(T^c_{\sigma_k})^* = T^c_\sigma M_{\widehat{\varphi}_j} M_{\overline{\widehat{\varphi}_k}}(T^c_\sigma)^*$ and $\widehat{\varphi}_j$, $\widehat{\varphi}_j$ have disjoint supports. Now let us estimate the second inequality in (9.30), i.e. the norm of $(T^c_{\sigma_k})^* T^c_{\sigma_j}$. Since

$$(T^c_{\sigma_k})^*(f)(s) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_k^*(t,\xi) f(t) e^{2\pi \mathrm{i}\xi\cdot(s-t)} dt d\xi,$$

and

$$T^c_{\sigma_j}(f)(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_j(t,\eta) f(r) e^{2\pi \mathrm{i}\eta\cdot(t-r)} dr d\eta.$$

Then we have

$$(T^c_{\sigma_k})^* T^c_{\sigma_j}(f)(s) = \int_{\mathbb{R}^d} K(s,r) f(r) dr,$$

with

$$K(s,r) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma_k^*(t,\xi)\sigma_j(t,\eta) e^{2\pi \mathrm{i}[\eta\cdot(t-r)+\xi\cdot(s-t)]} dt d\eta d\xi.$$

Writing

$$e^{2\pi\mathrm{i}(\eta-\xi)\cdot t} = \frac{(1-\Delta_t)^N}{(1+4\pi^2|\xi-\eta|^2)^N} e^{2\pi\mathrm{i}(\eta-\xi)\cdot t},$$

$$e^{2\pi\mathrm{i}(t-r)\cdot \eta} = \frac{(1-\Delta_\eta)^N}{(1+4\pi^2|t-r|^2)^N} e^{2\pi\mathrm{i}(t-r)\cdot \eta},$$

and

$$e^{2\pi\mathrm{i}(s-t)\cdot \xi} = \frac{(1-\Delta_\xi)^N}{(1+4\pi^2|s-t|^2)^N} e^{2\pi\mathrm{i}(s-t)\cdot \xi},$$

we use the integration by parts with respect to the variables t, ξ and η. By standard calculation (see [61, Theorem 2, p. 286] for more details), we get

$$\|K(s,r)\|_{\mathcal{M}} \lesssim 4^{\max(k,j)((\delta-1)N+d)} \int Q(s-t)Q(t-r)dt,$$

where $Q(t) = (1+|t|)^{-2N}$, if $k \neq j$. Denote $K_0(s,r) = \int Q(s-t)Q(t-r)dt$, then

$$\int_{\mathbb{R}^d} K_0(s,r)ds = \int_{\mathbb{R}^d} K_0(s,r)dr = \big(\int_{\mathbb{R}^d}(1+|t|)^{-2N}dt\big)^2 < \infty. \tag{9.31}$$

For any $f \in L_2(\mathcal{N})$, there exists $g \in L_2(\mathcal{N})$ with norm one such that

$$\|(T^c_{\sigma_k})^* T^c_{\sigma_j} f\|_{L_2(\mathcal{N})} = \Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(s,r)f(r)g(s)drds\Big|.$$

Applying the Hölder inequality and (9.31), we get

$$\begin{aligned}
&\Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(s,r)f(r)g(s)drds\Big| \\
&\leq \big(\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|K(s,r)\|_{\mathcal{M}}|f(r)|^2 dsdr\big)^{\frac12}\big(\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|K(s,r)\|_{\mathcal{M}}|g(s)|^2 dsdr\big)^{\frac12} \\
&\lesssim 4^{\max(k,j)((\delta-1)N+d)}\|f\|_{L_2(\mathcal{N})},
\end{aligned}$$

which implies that

$$\|(T^c_{\sigma_k})^* T^c_{\sigma_j}\|_{B(L_2(\mathcal{N}))} \lesssim c(j)c(k), \quad j \neq k,$$

with $c(j) = 2^{j((\delta-1)N+d)}$. If we take $N > \frac{d}{1-\delta}$, the sequence $\big(c(j)\big)_j$ is summable.

In order to apply Cotlar-Stein's lemma, it remains to show that $T^c_{\sigma_j}$'s satisfy (9.29). To this end, we do some technical modifications. Set

$$\widetilde{\sigma}_j = \sigma_j(2^{-j\delta}\cdot, 2^{j\delta}\cdot).$$

We can easily check that the $\widetilde{\sigma}_j$'s belong to $S^0_{0,0}$, uniformly in j. Then, by Lemma 9.17, the $T^c_{\widetilde{\sigma}_j}$'s are bounded on $L_2(\mathcal{N})$ uniformly in j. If Λ_j denotes the dilation operator given by

$$\Lambda_j(f) = f(2^{j\delta}\cdot),$$

then, we can easily verify that

$$T^c_{\sigma_j} = \Lambda_j T^c_{\widetilde{\sigma}_j}\Lambda_j^{-1}.$$

Thus,

$$\|T^c_{\sigma_j}\|_{B(L_2(\mathcal{N}))} \leq \|T^c_{\widetilde{\sigma}_j}\|_{B(L_2(\mathcal{N}))} < \infty.$$

Therefore, $(T^c_{\sigma_j})_{j\geq 0}$ satisfy the assumptions of Cotlar's lemma. So we get

$$\|T^c_{\sigma}\|_{B(L_2(\mathcal{N}))} = \|\sum_{j=0}^{\infty} T^c_{\sigma_j}\|_{B(L_2(\mathcal{N}))} < \infty.$$

Thus, T^c_σ is bounded on $L_2(\mathcal{N})$. □

Let $0 \leq \delta < 1$. Since we have the inclusion $S^0_{1,\delta} \subset S^0_{\delta,\delta}$, then we clearly have the following corollary:

Corollary 9.19. *Let $\sigma \in S^0_{1,\delta}$ with $0 \leq \delta < 1$. Then T^c_σ is bounded on $L_2(\mathcal{N})$.*

Furthermore, we can deduce the boundedness of T^c_σ on $L_1\big(\mathcal{M}; L^c_2(\mathbb{R}^d)\big)$ from the above corollary.

Lemma 9.20. *Let $\sigma \in S^0_{1,\delta}$ with $0 \leq \delta < 1$. Then T^c_σ is bounded on $L_1\big(\mathcal{M}; L^c_2(\mathbb{R}^d)\big)$.*

Proof. Since $0 \leq \delta < 1$, Proposition 9.9 tells us that the adjoint $(T^c_\sigma)^*$ of T^c_σ is still in the class $S^0_{1,\delta}$. Thus, by duality, it is enough to prove the boundedness of $(T^c_\sigma)^*$ on $L_\infty(\mathcal{M}; L^c_2(\mathbb{R}^d))$. Indeed, there exists $u \in L_2(\mathcal{M})$ with norm one such that

$$\Big\|\Big(\int_{\mathbb{R}^d} |(T^c_\sigma)^*(f)(s)|^2 ds\Big)^{\frac12}\Big\|_{\mathcal{M}} = \Big(\int_{\mathbb{R}^d} \langle |(T^c_\sigma)^*(f)(s)|^2 u, u\rangle_{L_2(\mathcal{M})} ds\Big)^{\frac12}$$
$$= \Big(\int_{\mathbb{R}^d} \|(T^c_\sigma)^*(fu)(s)\|^2_{L_2(\mathcal{M})} ds\Big)^{\frac12}.$$

Then, applying Corollary 9.19 to $(T^c_\sigma)^*$, we get

$$\Big(\int_{\mathbb{R}^d} \|(T^c_\sigma)^*(fu)(s)\|^2_{L_2(\mathcal{M})} ds\Big)^{\frac12} \lesssim \Big(\int_{\mathbb{R}^d} \|f(s)u\|^2_{L_2(\mathcal{M})} ds\Big)^{\frac12} \leq \Big\|\Big(\int_{\mathbb{R}^d} |f(s)|^2 ds\Big)^{\frac12}\Big\|_{\mathcal{M}}.$$

Thus, we conclude that T^c_σ is bounded on $L_1(\mathcal{M}; L^c_2(\mathbb{R}^d))$. □

Now we are ready to prove the main theorem in this section.

Proof of Theorem 9.16. Step 1. We begin with the special case $p = 1$ and $\alpha = 0$. Since $F^{0,c}_1(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}^c_1(\mathbb{R}^d, \mathcal{M})$ with equivalent norms, the assertion is equivalent to saying that when $\sigma \in S^0_{1,\delta}$ with $0 \leq \delta < 1$, T^c_σ is bounded on $\mathrm{h}^c_1(\mathbb{R}^d, \mathcal{M})$. By the atomic decomposition introduced in Theorem 7.6, it suffices to prove that, for any atom b based on a cube with side length 1 and any atom g based on a cube with side length less than 1, we have

$$\|T^c_\sigma b\|_{\mathrm{h}^c_1} \lesssim 1 \quad \text{and} \quad \|T^c_\sigma g\|_{\mathrm{h}^c_1} \lesssim 1.$$

Corollary 9.15 tells us that

$$\|T^c_\sigma b\|_{\mathrm{h}^c_1} \lesssim 1.$$

Thus, it remains to consider the atom g based on cube Q with $|Q| < 1$. Without loss of generality, we may assume that Q is centered at the origin. Let $(\mathcal{X}_j)_{j\in\mathbb{Z}^d}$ be the resolution of the unit defined in (7.13) and $\mathcal{X}^Q_j = \mathcal{X}_j(l(Q)^{-1}\cdot)$ for $j \in \mathbb{Z}^d$. Then, we have $\operatorname{supp} \mathcal{X}^Q_j \subset l(Q)j + 2Q$. Now, set $h_1 = \sum_{j\in 4Q_{0,0}} \mathcal{X}^Q_j$ and $h_2 = \sum_{j\notin 4Q_{0,0}} \mathcal{X}^Q_j$. By the support assumption of $\mathcal{X}^Q_j$, it is obvious that $\operatorname{supp} h_1 \subset 6Q$, $\operatorname{supp} h_2 \subset (4Q)^c$. Moreover,

$$h_1(s) + h_2(s) = 1 \quad , \forall\, s \in \mathbb{R}^d.$$

Now we decompose σ into two parts:

$$\sigma(s,\xi) = h_1(s)\sigma(s,\xi) + h_2(s)\sigma(s,\xi) \overset{\text{def}}{=} \sigma^1(s,\xi) + \sigma^2(s,\xi).$$

Note that σ^1 and σ^2 are still in the class $S^0_{1,\delta}$ and

$$T^c_\sigma g = T^c_{\sigma^1} g + T^c_{\sigma^2} g.$$

Firstly, we deal with the symbol σ^1 which has compact s-support. We consider the adjoint operator $(T^c_\sigma)^*$ of T^c_σ. Since $\delta < 1$, by Proposition 9.9, there exists $\widetilde{\sigma} \in S^0_{1,\delta}$ such that

$$(T^c_\sigma)^* = T^c_{\widetilde{\sigma}}.$$

If we take $\zeta_j(s) = \widetilde{\sigma^1_j}(s,0)^* = \overline{\mathcal{X}^Q_j(s)\widetilde{\sigma}(s,0)^*}$ for $j \in 4Q_{0,0}$, then ζ_j is an $\mathcal{M}$-valued infinitely differentiable function with all derivatives belonging to $L_\infty(\mathcal{N})$. Therefore, we have

$$\operatorname{supp} m^c_{\zeta_j} g \subset l(Q)j + 2Q,$$

and

$$\tau\Big(\int_{\mathbb{R}^d} |m^c_{\zeta_j} g(s)|^2 ds\Big)^{\frac{1}{2}} \lesssim |Q|^{-\frac{1}{2}}. \tag{9.32}$$

This indicates that, except for the vanishing mean property, $m^c_{\zeta_j} g$ coincides with a bounded multiple of an h^c_1-atom defined in Definition 7.1. Now let us set $\sigma^1_j(s,\xi) = \mathcal{X}^Q_j(s)\sigma(s,\xi)$ for $j \in 4Q_{0,0}$ and set $T^c_j = T^c_{\sigma^1_j} - m^c_{\zeta_j}$. It is clear that $\operatorname{supp} T^c_j g \subset l(Q)j + 2Q$. Since $(m^c_{\zeta_j})^* = m^c_{\zeta_j{}^*}$ and $(T^c_{\sigma^1_j})^* x = \widetilde{\sigma^1_j}(s,0)x = \zeta_j{}^* x$ for every $x \in \mathcal{M}$, then we have

$$\tau\Big(\int_{l(Q)j+2Q} T^c_j g(s) ds \cdot x\Big) = \langle T^c_j g, x\rangle = \langle g, (T^c_j)^* x\rangle = \langle g, (T^c_{\sigma^1_j} - m^c_{\zeta_j})^* x\rangle = 0.$$

Hence, $T^c_j g$ has vanishing mean. Moreover, applying Lemma 9.20 and (9.32), we get

$$\begin{aligned}\tau\Big(\int_{l(Q)j+2Q} |T^c_j g(s)|^2 ds\Big)^{\frac{1}{2}} &\leq \tau\Big(\int_{l(Q)j+2Q} |T^c_{\sigma^1_j} g(s)|^2 ds\Big)^{\frac{1}{2}} + \tau\Big(\int_{l(Q)j+2Q} |m^c_{\zeta_j} g(s)|^2 ds\Big)^{\frac{1}{2}} \\ &\lesssim \tau\Big(\int_{2Q} |g(s)|^2 ds\Big)^{\frac{1}{2}} + |Q|^{-\frac{1}{2}} \lesssim |Q|^{-\frac{1}{2}}.\end{aligned}$$

Combining the above estimates with Remark 7.3, we see that T^c_j maps h^c_1-atoms to h^c_1-atoms. Thus, T^c_j is bounded on $\mathrm{h}^c_1(\mathbb{R}^d,\mathcal{M})$ and so is $T^c_{\sigma^1}$.

Step 2. Now let us consider $T^c_{\sigma^2}$. By Theorem 8.21, we may assume that g has moment cancellations of order $L > \frac{d}{2} - 1$. Note that $\operatorname{supp} T^c_{\sigma^2} g \subset (4Q)^c$. And if $s \in (4Q)^c$, following the argument in (9.18) with g in place of $a_{\mu,l}$, we get

$$|T^c_{\sigma^2} g(s)|^2 \lesssim l(Q)^{2L+2+d} |s|^{-2d-2L-2} \int_{2Q} |g(t)|^2 dt.$$

Then for $M < 2L+2$,

$$\begin{aligned}&\tau\Big(\int_{(4Q)^c} |T^c_{\sigma^2} g(s)|^2 (1 + l(Q)^{-1}|s|)^{d+M} ds\Big)^{\frac{1}{2}} \\ &\lesssim l(Q)^{L+1-\frac{M}{2}} \Big(\int_{(4Q)^c} |s|^{-d-2L-2+M} ds\Big)^{\frac{1}{2}} \cdot \tau\Big(\int_{2Q} |g(t)|^2 dt\Big)^{\frac{1}{2}} \\ &\lesssim l(Q)^{L+1-\frac{M}{2}} l(Q)^{-L-1+\frac{M}{2}} |Q|^{-\frac{1}{2}} = |Q|^{-\frac{1}{2}}.\end{aligned} \tag{9.33}$$

Moreover, we claim that $T^c_{\sigma^2} g$ can be decomposed as follows:

$$T^c_{\sigma^2} g = \sum_{m\in\mathbb{Z}^d} \nu_m H_m,$$

where $\sum_m |\nu_m| \lesssim 1$ and the H_m's are h^c_1-atoms. Then, by Theorem 7.2, we will get $\|T^c_{\sigma^2} g\|_{\mathrm{h}^c_1} \lesssim 1$. Now let us prove the claim. Since $L > \frac{d}{2} - 1$, we can choose M such that $M > d$ and $M < 2L+2$. Take $\nu_m = |Q|^{-\frac{1}{2}}(1 + l(Q)^{-1}|m|)^{-\frac{d+M}{2}}$ and $H_m = \nu_m^{-1} \mathcal{X}_m T^c_{\sigma^2} g$, where $(\mathcal{X}_m)_{m\in\mathbb{Z}^d}$ denotes again the smooth resolution of the unit (7.13), i.e.

$$1 = \sum_{m\in\mathbb{Z}^d} \mathcal{X}_m(s), \quad \forall\, s \in \mathbb{R}^d.$$

Applying (9.33), we have

$$\begin{aligned}&\tau\Big(\int_{2Q_{0,m}} |H_m(s)|^2 ds\Big)^{\frac{1}{2}} \\ &\lesssim \nu_m^{-1} (1 + l(Q)^{-1}|m|)^{-\frac{d+M}{2}} \tau\Big(\int_{(4Q)^c} |T^c_{\sigma^2} g(s)|^2 (1 + l(Q)^{-1}|s|)^{d+M} ds\Big)^{\frac{1}{2}} \lesssim 1.\end{aligned}$$

And the normalizing constants ν_m satisfy

$$\begin{aligned}\sum_m |\nu_m| &= |Q|^{-\frac{1}{2}} \sum_m (1+l(Q)^{-1}|m|)^{-\frac{d+M}{2}} \\ &\leq |Q|^{-\frac{1}{2}} \int_{\mathbb{R}^d} (1+l(Q)^{-1}|s|)^{-\frac{d+M}{2}} ds \lesssim 1.\end{aligned}$$

Combining the estimates of $T^c_{\sigma_1} g$ and $T^c_{\sigma_2} g$, we conclude that $\|T^c_\sigma g\|_{\mathrm{h}_1^c} \lesssim 1$. Thus, T^c_σ is bounded on $\mathrm{h}_1^c(\mathbb{R}^d, \mathcal{M})$.

Step 3. For the case where $p = 1$ and $\alpha \neq 0$, we use the lifting property of Triebel-Lizorkin spaces stated in Proposition 8.5. By the property of the composition of pseudo-differential operators stated in Proposition 9.8, we see that

$$T^c_{\sigma^\alpha} = J^\alpha T^c_\sigma J^{-\alpha}$$

is still a pseudo-differential operator with symbol σ^α in $S^0_{1,\delta}$. Then for $f \in F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$, we have

$$\|T^c_\sigma f\|_{F_1^{\alpha,c}} = \|J^{-\alpha} \circ T^c_{\sigma^\alpha} \circ J^\alpha f\|_{F_1^{\alpha,c}} \approx \|T^c_{\sigma^\alpha} \circ J^\alpha f\|_{\mathrm{h}_1^c} \lesssim \|J^\alpha f\|_{\mathrm{h}_1^c} \approx \|f\|_{F_1^{\alpha,c}}.$$

Hence, T^c_σ is bounded on $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.

Step 4. Finally, we deal with the case $1 < p \leq \infty$. By the previous steps, $(T^c_\sigma)^* = T^c_{\widetilde{\sigma}}$ is bounded on $F_1^{-\alpha,c}(\mathbb{R}^d, \mathcal{M})$ with $\alpha \in \mathbb{R}$, then it is clear that T^c_σ is bounded on $F_\infty^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. Given $1 < p < \infty$ and $\alpha \in \mathbb{R}$, by interpolation

$$\big(F_\infty^{\alpha,c}(\mathbb{R}^d, \mathcal{M}), F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})\big)_{\frac{1}{p}} = F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M}),$$

we get the boundedness of T^c_σ on $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. □

Remark 9.21. A special case of Theorem 9.16 is that if the symbol is scalar-valued, then $\int_{\mathbb{R}^d} \sigma(s,\xi)\widehat{f}(\xi) e^{2\pi \mathrm{i} s\cdot\xi} d\xi = \int_{\mathbb{R}^d} \widehat{f}(\xi)\sigma(s,\xi) e^{2\pi \mathrm{i} s\cdot\xi} d\xi$. In this case, T^c_σ is also bounded on $\mathrm{h}_p^r(\mathbb{R}^d, \mathcal{M})$ for any $1 \leq p \leq \infty$. By Proposition 6.15, we deduce that T^c_σ is bounded on $L_p(\mathcal{N})$.

Corollary 9.22. *Let $n, \alpha \in \mathbb{R}$, $0 \leq \delta < 1$ and $\sigma \in S^n_{1,\delta}$. Then T^c_σ is bounded from $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ to $F_p^{\alpha-n,c}(\mathbb{R}^d, \mathcal{M})$ for any $1 \leq p \leq \infty$.*

Proof. Recall that the Bessel potential of order n maps $F_p^{\alpha,c}$ isomorphically onto $F_p^{\alpha-n,c}$. If $\sigma \in S^0_{1,\delta}$, by Proposition 9.8, we see that

$$\sigma(s,\xi)(1+|\xi|^2)^{\frac{n}{2}} \in S^n_{1,\delta},$$

and its corresponding pseudo-differential operator is given by $T^c_\sigma \circ J^n$. Then the assertion follows obviously from Theorem 9.16. □

9.4 Forbidden symbols

The purpose of this section is to extend the boundedness results obtained in the previous one to the pseudo-differential operators with forbidden symbols, i.e. the symbols in the class $S^n_{1,1}$. There are two main differences between these operators and those with symbols in $S^n_{1,\delta}$ with $0 \leq \delta < 1$. The first one is that when $\sigma \in S^0_{1,1}$, T^c_σ is not necessarily bounded on $L_2(\mathcal{N})$. The second one is that $S^0_{1,1}$ is not closed under the products and adjoints. Fortunately, if the function spaces have a positive degree of smoothness, the operators with symbols in $S^0_{1,1}$ will be bounded on them. In the classical theory, the regularity of operators with forbidden symbols on Sobolev spaces $H^\alpha_p(\mathbb{R}^d)$, Besov spaces $B^\alpha_{pq}(\mathbb{R}^d)$ and Triebel Lizorkin spaces $F^\alpha_{pq}(\mathbb{R}^d)$ with $\alpha > 0$ has been widely investigated, see [44, 45, 4, 58, 65].

Our first result in this section concerns the regularity of pseudo-differential operators with forbidden symbols on the operator-valued Sobolev spaces. Let us give some background on these function spaces.

For $\alpha \in \mathbb{R}, 1 \le p \le \infty$ and a Banach space X, the potential Sobolev space $H^\alpha_p(\mathbb{R}^d; X)$ is the space of all distributions in $S'(\mathbb{R}^d; X)$ which have finite Sobolev norm $\|f\|_{H^\alpha_p} = \|J^\alpha f\|_{L_p(\mathbb{R}^d;X)}$. It is well known that the potential Sobolev spaces are closely related to Besov spaces. In our case, we still use the resolution of the unit $(\varphi_k)_{k\ge 0}$ introduced in (1.3) to define Besov spaces. Given $\alpha \in \mathbb{R}^d$ and $1 \le p, q \le \infty$, the Besov space $B^\alpha_{p,q}(\mathbb{R}^d; X)$ is defined to be the subspace of $S'(\mathbb{R}^d; X)$ consisting of all f such that

$$\|f\|_{B^\alpha_{p,q}} = \Big(\sum_{k\ge 0} 2^{qk\alpha} \|\varphi_k * f\|^q_{L_p(\mathbb{R}^d;X)} \Big)^{\frac{1}{q}} < \infty.$$

The above vector-valued Besov spaces $B^\alpha_{p,q}(\mathbb{R}^d; X)$ have been studied by many authors, see for instance [1]. Instead of the above defined Banach-valued spaces, we prefer to study the operator-valued spaces $H^\alpha_p(\mathbb{R}^d; L_p(\mathcal{M}))$ and $B^\alpha_{p,q}(\mathbb{R}^d; L_p(\mathcal{M}))$. Obviously, the main difference is that the Banach space X varies for different p. The following inclusions are easy to check for every $1 \le p \le \infty$,

$$B^\alpha_{p,1}(\mathbb{R}^d; L_p(\mathcal{M})) \subset H^\alpha_p(\mathbb{R}^d; L_p(\mathcal{M})) \subset B^\alpha_{p,\infty}(\mathbb{R}^d; L_p(\mathcal{M})).$$

Besov spaces are stable under real interpolation. More precisely, if $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \ne \alpha_1$ and $0 < \theta < 1$, then

$$\big(B^{\alpha_0}_{p,q_0}(\mathbb{R}^d; L_p(\mathcal{M})), B^{\alpha_1}_{p,q_1}(\mathbb{R}^d; L_p(\mathcal{M}))\big)_{\theta,q} = B^\alpha_{p,q}(\mathbb{R}^d; L_p(\mathcal{M})), \tag{9.34}$$

for $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1,\ p, q, q_0, q_1 \in [1, \infty]$. This result can be deduced from its Banach-valued counterpart in [1]; similar results in operator-valued setting can be found in [72].

The following lemma states the regularity of pseudo-differential operators with forbidden symbols on $H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$ for $\alpha > 0$.

Lemma 9.23. *Let $\sigma \in S^0_{1,1}$. Then T^c_σ is bounded on $H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$ for any $\alpha > 0$.*

Proof. Let $(\varphi_j)_{j\ge 0}$ be the resolution of the unit satisfying (1.3). It is straightforward to show that $H^\alpha_2(\mathbb{R}^d; L_2(\mathcal{M}))$ admits an equivalent norm:

$$\|f\|_{H^\alpha_2(\mathbb{R}^d;L_2(\mathcal{M}))} \approx \big(\sum_{j\ge 0} 2^{2j\alpha} \|\varphi_j * f\|^2_{L_2(\mathcal{N})} \big)^{\frac{1}{2}} = \|f\|_{B^\alpha_{2,2}(\mathbb{R}^d;L_2(\mathcal{M}))}. \tag{9.35}$$

Let σ_k with $k \in \mathbb{N}_0$ be the dyadic decomposition of σ given in (9.9). By the support assumptions of $\widehat{\varphi}$ and $\widehat{\varphi}_0$, we have

$$T^c_{\sigma_k}(f) = T^c_{\sigma_k}(f_k),$$

where $f_k = (\varphi_{k-1} + \varphi_k + \varphi_{k+1}) * f$ for $k \ge 1$, and $f_0 = (\varphi_0 + \varphi_1) * f$. Applying Lemma 9.7 to K_k with $M = 0$, we get

$$\int_{|s-t|\le 2^{-k}} \|D^\gamma_s K_k(s, s-t)\|_{\mathcal{M}} dt \lesssim \int_{|s-t|\le 2^{-k}} 2^{k(|\gamma|_1 + d)} dt \approx 2^{k|\gamma|_1}.$$

If $d+1$ is even, applying Lemma 9.7 again to K_k with $2M = d+1$, we get

$$\int_{|s-t|>2^{-k}} \|D_s^\gamma K_k(s,s-t)\|_{\mathcal{M}} dt \lesssim \int_{|s-t|>2^{-k}} 2^{k(|\gamma|_1-1)}|s-t|^{-d-1}dt \approx 2^{k|\gamma|_1};$$

if $d+2$ is even, letting $2M = d+2$ in Lemma 9.7, we get the same estimate. Therefore, summing up the above estimates of $\int_{|s-t|\le 2^{-k}}$ and $\int_{|s-t|>2^{-k}}$, we obtain

$$\int_{\mathbb{R}^d} \|D_s^\gamma K_k(s,s-t)\|_{\mathcal{M}} dt \lesssim 2^{k|\gamma|_1}.$$

Since the estimate of $\|D_s^\gamma K_k(s,s-t)\|_{\mathcal{M}}$ is symmetric in s and t, the same proof also shows that

$$\int_{\mathbb{R}^d} \|D_s^\gamma K_k(s,s-t)\|_{\mathcal{M}} ds \lesssim 2^{k|\gamma|_1}.$$

For any $f \in H_2^\alpha(\mathbb{R}^d; L_2(\mathcal{M}))$ and $k \in \mathbb{N}_0$, there exists $g_k \in L_2(\mathcal{N})$ with norm one such that $\|D_s^\gamma T_{\sigma_k}^c(f)\|_{L_2(\mathcal{N})} = \tau \int_{\mathbb{R}^d} D_s^\gamma T_{\sigma_k}^c(f)(s) g_k^*(s) ds$. Then,

$$\begin{aligned}
&\|D_s^\gamma T_{\sigma_k}^c(f)\|^2_{L_2(\mathcal{N})} \\
&= \Big|\tau \int_{\mathbb{R}^d} D_s^\gamma T_{\sigma_k}^c(f)(s) g_k^*(s) ds\Big|^2 \\
&= \Big|\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D_s^\gamma K_k(s,s-t) f_k(t) dt\, g_k^*(s)\, ds\Big|^2 \qquad (9.36)\\
&\le \tau \int_{\mathbb{R}^d} \|D_s^\gamma K_k(s,s-t)\|_{\mathcal{M}} |g_k(s)|^2 dt ds \cdot \tau \int_{\mathbb{R}^d} \|D_s^\gamma K_k(s,s-t)\|_{\mathcal{M}} |f_k(t)|^2 ds dt \\
&\lesssim 2^{2k|\gamma|_1} \cdot \|f_k\|^2_{L_2(\mathcal{N})}.
\end{aligned}$$

Taking $\gamma = 0$, the above calculation implies that

$$\|T_\sigma^c(f)\|_{L_2(\mathcal{N})} \le \sum_{k\ge 0} \|T_{\sigma_k}^c(f)\|_{L_2(\mathcal{N})} \lesssim \sum_{k\ge 0} \|f_k\|_{L_2(\mathcal{N})} \lesssim \|f\|_{B^0_{2,1}}, \qquad (9.37)$$

which says that T_σ^c is bounded from $B^0_{2,1}(\mathbb{R}^d; L_2(\mathcal{M}))$ to $L_2(\mathcal{N})$.

On the other hand, if we take

$$a_0 = \varphi_0, \quad a_j(\xi) = (1-\varphi_0(\xi))\frac{\xi_j}{|\xi|^2},$$

then we get

$$1 = a_0(\xi) + \sum_{j=1}^d a_j(\xi)\xi_j, \quad \forall\, \xi \in \mathbb{R}^d.$$

This identity implies

$$1 = (a_0(\xi) + \sum_{j=1}^d a_j(\xi)\xi_j)^l = \sum_{|\gamma|_1\le l} \sigma_\gamma(\xi)\xi^\gamma, \quad \forall\, l \in \mathbb{N}_0, \forall\, \xi \in \mathbb{R}^d,$$

where the $\sigma_\gamma(\xi)$'s are symbols in $S_{1,0}^{-|\gamma|_1} \subset S_{1,1}^{-|\gamma|_1}$. The above identity allows us to decompose the term $\varphi_j * T_{\sigma_k}^c(f)$ in the following way:

$$\varphi_j * T_{\sigma_k}^c(f) = \sum_{|\gamma|_1\le l} T_{\sigma_\gamma}^c(\varphi_j * D_s^\gamma T_{\sigma_k}^c(f)) = \sum_{|\gamma|_1\le l} T_{\sigma_\gamma^j}^c(D_s^\gamma T_{\sigma_k}^c(f)), \qquad (9.38)$$

where $\sigma_\gamma^j = \sigma_\gamma \widehat{\varphi}_j$. Note that the symbol $\sigma_\gamma^j \in S_{1,0}^{-|\gamma|_1}$ for any j, and if $|\gamma|_1 < l$, $\sigma_\gamma^j \neq 0$ if and only if $j = 0$ and $j = 1$. If $j \leq k+1$, by the Plancherel formula and (9.36), we have

$$2^{j\alpha}\|\varphi_j * T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})} \lesssim 2^{j\alpha}\|T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})} \lesssim 2^{j\alpha}\|f_k\|_{L_2(\mathcal{N})} \lesssim 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})}.$$

If $j \geq k+2$, adapting the proof of (9.36) with σ_γ^j in place of σ_k, we deduce that

$$\|T^c_{\sigma_\gamma^j}(D_s^\gamma T^c_{\sigma_k}(f))\|_{L_2(\mathcal{N})} \leq C_\gamma 2^{-j|\gamma|_1}\|D_s^\gamma T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}. \tag{9.39}$$

For any $|\gamma|_1 < l$, by the previous observation, $\sigma_\gamma^j = 0$. Therefore, estimates (9.36), (9.38) and (9.39) imply that

$$\begin{aligned}\|\varphi_j * T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})} &= \|\sum_{|\gamma|_1=l} T^c_{\sigma_\gamma^j}(D_s^\gamma T^c_{\sigma_k}(f))\|_{L_2(\mathcal{N})}\\ &\lesssim \sum_{|\gamma|_1=l} 2^{-jl}\|D_s^\gamma T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})}\\ &\lesssim \sum_{|\gamma|_1=l} 2^{(k-j)l}\|f_k\|_{L_2(\mathcal{N})}.\end{aligned}$$

Thus, if we take l to be the smallest integer larger than α, we have

$$2^{j\alpha}\|\varphi_j * T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})} \lesssim 2^{(j-k)(\alpha-l)}2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})} \leq 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})}.$$

Combining the above estimate for $j \geq k+2$ and that for $j \leq k+1$, we get

$$\sup_{j\in\mathbb{N}_0} 2^{j\alpha}\|\varphi_j * T^c_{\sigma_k}(f)\|_{L_2(\mathcal{N})} \lesssim 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})},$$

whence,

$$\|T^c_{\sigma_k}(f)\|_{B^\alpha_{2,\infty}} \lesssim 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})}.$$

Then by the triangle inequality, we have

$$\|T^c_\sigma(f)\|_{B^\alpha_{2,\infty}} \leq \sum_{k\geq 0}\|T^c_{\sigma_k}(f)\|_{B^\alpha_{2,\infty}} \lesssim \sum_{k\geq 0} 2^{k\alpha}\|f_k\|_{L_2(\mathcal{N})} \lesssim \|f\|_{B^\alpha_{2,1}}, \tag{9.40}$$

which says that T^c_σ is bounded from $B^\alpha_{2,1}(\mathbb{R}^d; L_2(\mathcal{M}))$ to $B^\alpha_{2,\infty}(\mathbb{R}^d; L_2(\mathcal{M}))$.

Applying (9.37), (9.40) and the real interpolation (9.34) with $p = 2$, $q = 2$ and $\alpha_0 = 0$, $\alpha_1 = \alpha$, we obtain the following boundedness:

$$\|T^c_\sigma(f)\|_{B^\beta_{2,2}} \lesssim \|f\|_{B^\beta_{2,2}}, \quad \forall\beta > 0.$$

Finally, (9.35) together with the above inequality yields the desired assertion. □

Remark 9.24. Even though it is not the main subject of this paper, the regularity of pseudo-differential operators on operator-valued Besov spaces is already obtained in the above proof. Let us record it specifically in the below. Let $1 \leq p, q \leq \infty$.

(i) If $\sigma \in S^0_{1,\delta}$ for some $0 \leq \delta \leq 1$, then T^c_σ is bounded from $B^0_{p,1}(\mathbb{R}^d; L_p(\mathcal{M}))$ to $L_p(\mathcal{N})$, and bounded on $B^\alpha_{p,q}(\mathbb{R}^d; L_p(\mathcal{M}))$ for any $\alpha > 0$.

(ii) If $\sigma \in S^0_{1,\delta}$ with $0 \leq \delta < 1$, then T^c_σ is bounded on $B^\alpha_{p,q}(\mathbb{R}^d; L_p(\mathcal{M}))$ for any $\alpha \in \mathbb{R}$.

Indeed, the argument in (9.36) still works for all $1 \leq p \leq \infty$. Then we get the boundedness of T_σ^c from $B_{p,1}^0(\mathbb{R}^d; L_p(\mathcal{M}))$ to $L_p(\mathcal{N})$ as in (9.37). Likewise, we can deduce the L_p version of (9.40), i.e. the boundedness from $B_{p,1}^\alpha$ to $B_{p,\infty}^\alpha$ for $\alpha > 0$. Thus, for $\alpha > 0$, the boundedness of T_σ^c on $B_{p,q}^\alpha(\mathbb{R}^d; L_p(\mathcal{M}))$ is ensured by interpolation. If $\delta < 1$, by Proposition 9.8 and the lifting property of Besov spaces, we get the boundedness for general $\alpha \in \mathbb{R}$. Finally, we note that, different from the Triebel-Lizorkin spaces, the above assertions hold for T_σ^r as well.

Since for $\sigma \in S_{1,1}^0$, T_σ^c is not necessarily bounded on $L_2(\mathcal{N})$, we cannot expect its boundedness on $L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$. However, by Lemma 9.23, we are able to prove its boundedness on $L_1(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c)$ when $\alpha > 0$. Note that the classical Sobolev space $H_2^\alpha(\mathbb{R}^d)$ is a Hilbert space with the inner product $\langle f, g\rangle = \int_{\mathbb{R}^d} J^\alpha f(s)\overline{J^\alpha g(s)}ds$. By the definition of Hilbert-valued L_p-spaces, we see that $f \in L_1(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c)$ if and only if $J^\alpha f \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$.

Lemma 9.25. *Let $\sigma \in S_{1,1}^0$. Then T_σ^c is bounded on $L_1(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c)$ for any $\alpha > 0$.*

Proof. Following the argument for lemma 9.20 by replacing $(T_\sigma^c)^*$ with $J^\alpha T_\sigma^c$, we see that T_σ^c is bounded on $L_\infty(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c)$. Let $f \in L_1(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c)$ and $A = (\int_{\mathbb{R}^d} |J^\alpha f(s)|^2 ds)^{\frac{1}{2}}$. By approximation, we may assume that A is invertible. Thus, f admits the decomposition

$$f = fA^{-1}A,$$

where $\|A\|_{L_1(\mathcal{M})} = \|f\|_{L_1\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)}$ and $\|fA^{-1}\|_{L_\infty\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)} = 1$. From this decomposition, we establish the $L_1(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c)$-norm of $T_\sigma^c(f)$ as follows:

$$\begin{aligned}
\|T_\sigma^c(f)\|_{L_1\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)} &= \|T_\sigma^c(fA^{-1})A\|_{L_1\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)} \\
&\leq \|T_\sigma^c(fA^{-1})\|_{L_\infty\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)} \|A\|_{L_1(\mathcal{M})} \\
&\lesssim \|fA^{-1}\|_{L_\infty\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)} \|A\|_{L_1(\mathcal{M})} \\
&= \|f\|_{L_1\left(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c\right)},
\end{aligned}$$

which implies that T_σ^c is bounded on $L_1(\mathcal{M}; H_2^\alpha(\mathbb{R}^d)^c)$. □

Based on the previous lemma and the atomic decomposition obtained in Theorem 8.21, we are able to study the boundedness of pseudo-differential operators with forbidden symbols on the operator-valued Triebel-Lizorkin spaces $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.

Theorem 9.26. *Let $\sigma \in S_{1,1}^0$ and $\alpha > 0$. Then T_σ^c is bounded on $F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$.*

Proof. Let $f \in F_1^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$. We fix K, L to be two integers such that $K > \alpha + d$ and $L > d$. By the atomic decomposition in Theorem 8.21, f can be written as

$$f = \sum_{j=1}^{\infty} (\mu_j b_j + \lambda_j g_j),$$

where the b_j's are $(\alpha, 1)$-atoms and the g_j's are (α, Q)-atoms, μ_j and λ_j are complex numbers such that

$$\sum_{j=1}^{\infty} (|\mu_j| + |\lambda_j|) \approx \|f\|_{F_1^{\alpha,c}}.$$

In order to prove the assertion, by the above atomic decomposition, it suffices to prove that

$$\|T^c_\sigma b\|_{F_1^{\alpha,c}} \lesssim 1 \quad \text{and} \quad \|T^c_\sigma g\|_{F_1^{\alpha,c}} \lesssim 1,$$

for any $(\alpha,1)$-atom b and (α,Q)-atom g. We have shown in Corollary 9.15 that

$$\|T^c_\sigma b\|_{F_1^{\alpha,c}} \lesssim 1. \tag{9.41}$$

Thus it remains to consider $T^c_\sigma g$. This is the main part of the proof which will be divided into several steps for clarity.

Step 1. By translation, we may assume that the supporting cube Q of the atom g is centered at the origin. We begin with a split of the symbol σ: Let h_1, h_2 be two nonnegative infinitely differentiable functions on $\mathbb{R}^d$ such that $\operatorname{supp} h_1 \subset (Q)^c$, $\operatorname{supp} h_2 \subset 2Q$ and

$$1 = h_1(\xi) + h_2(\xi), \quad \forall\, \xi \in \mathbb{R}^d.$$

For any $(s,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, we write

$$\sigma(s,\xi) = h_1(\xi)\sigma(s,\xi) + h_2(\xi)\sigma(s,\xi) \stackrel{\text{def}}{=} \sigma_1(s,\xi) + \sigma_2(s,\xi).$$

It is clear that σ_1 and σ_2 are still two symbols in $S^0_{1,1}$, and

$$\|T^c_\sigma g\|_{F_1^{\alpha,c}} \leq \|T^c_{\sigma_1} g\|_{F_1^{\alpha,c}} + \|T^c_{\sigma_2} g\|_{F_1^{\alpha,c}}.$$

First, we consider the case where the cube Q is of side length one, i.e. $Q = Q_{0,0}$, and deal with the term $\|T^c_{\sigma_1} g\|_{F_1^{\alpha,c}}$ in the above split. Let $(\mathcal{X}_j)_{j\in\mathbb{Z}^d}$ be the resolution of the unit defined in (7.13) and $\widetilde{\mathcal{X}}_j = \mathcal{X}_j(2\cdot)$ for $j \in \mathbb{Z}^d$. We write

$$\begin{aligned} T^c_{\sigma_1} g &= \sum_{j\in 8Q_{0,0}} T^c_{\sigma_1^j} g + \sum_{j\notin 8Q_{0,0}} T^c_{\sigma_1^j} g \\ &\stackrel{\text{def}}{=} G_1 + H_1, \end{aligned} \tag{9.42}$$

where $\sigma_1^j(s,\xi) = \sigma_1(s,\xi)\widetilde{\mathcal{X}}_j(s)$.

We claim that for every $j \in \mathbb{Z}^d$, $T^c_{\sigma_1^j} g$ is the bounded multiple of an $(\alpha, Q_{0,\frac{j}{2}})$-atom (with the convention $Q_{0,\frac{j}{2}} = \frac{j}{2} + Q_{0,0}$). No loss of generality, we prove the claim just for $j=0$. Applying Lemma 9.25 to the symbol σ_1^0, we get

$$\tau\Big(\int_{\mathbb{R}^d} |J^\alpha T^c_{\sigma_1^0} g(s)|^2 ds\Big)^{\frac12} \lesssim \tau\Big(\int_{\mathbb{R}^d} |J^\alpha g(s)|^2 ds\Big)^{\frac12} \lesssim |Q_{0,0}|^{-\frac12}.$$

Thus, in order to prove the claim, it remains to show that $T^c_{\sigma_1^0} g$ can be written as the linear combination of subatoms and the coefficients satisfy a certain condition. By Definition 8.18, g admits the following representation:

$$g = \sum_{(\mu,l)\leq(0,0)} d_{\mu,l} a_{\mu,l}, \tag{9.43}$$

where the $a_{\mu,l}$'s are $(\alpha, Q_{\mu,l})$-subatoms and the coefficients $d_{\mu,l}$'s are complex numbers with $\sum_{(\mu,l)\leq(0,0)} |d_{\mu,l}|^2 \leq 1$. Then we have

$$T^c_{\sigma_1^0} g = \sum_{(\mu,l)\leq(0,0)} d_{\mu,l} T^c_{\sigma_1^0} a_{\mu,l}.$$

Given $\mu \in \mathbb{N}_0$, let $(\mathcal{X}_{\mu,m})_{m\in\mathbb{Z}^d}$ be a sequence of infinitely differentiable functions on $\mathbb{R}^d$ such that

$$1 = \sum_{m\in\mathbb{Z}^d} \mathcal{X}_{\mu,m}(s), \quad \forall\, s \in \mathbb{R}^d, \tag{9.44}$$

and each $\mathcal{X}_{\mu,0}$ is nonnegative, supported in $2Q_{\mu,0}$ and $\mathcal{X}_{\mu,m}(s) = \mathcal{X}_{\mu,0}(s - 2^{-\mu}m)$. It is the $2^{-\mu}$-dilated version of the resolution of the unit in (7.13). We decompose $T^c_{\sigma_1^0} g$ in the following way:

$$T^c_{\sigma_1^0} g = \sum_{\mu=0}^{\infty} \sum_m \mathcal{X}_{\mu,m} \sum_l d_{\mu,l} T^c_{\sigma_1^0} a_{\mu,l}. \tag{9.45}$$

Observe that the only m's that contribute to the above sum $\sum_m$ are those $m \in \mathbb{Z}^d$ such that $2Q_{\mu,m} \cap Q_{0,0} \neq \emptyset$, so $Q_{\mu,m} \subset 2Q_{0,0}$. Thus, we obtain the decomposition

$$T^c_{\sigma_1^0} g = \sum_{(\mu,m)\leq(0,0)} D_{\mu,m} G_{\mu,m}, \tag{9.46}$$

where

$$\begin{aligned} D_{\mu,m} &= \big(\sum_l |d_{\mu,l}|^2 (1 + |m - l|)^{-(d+1)}\big)^{\frac{1}{2}}, \\ G_{\mu,m} &= \frac{1}{D_{\mu,m}} \mathcal{X}_{\mu,m} \sum_l d_{\mu,l} T^c_{\sigma_1^0} a_{\mu,l}. \end{aligned}$$

It is evident that

$$\big(\sum_{(\mu,m)\leq(0,0)} |D_{\mu,m}|^2\big)^{\frac{1}{2}} \lesssim \big(\sum_{(\mu,l)\leq(0,0)} |d_{\mu,l}|^2\big)^{\frac{1}{2}} \leq 1.$$

Now we show that the $G_{\mu,m}$'s are bounded multiple of $(\alpha, Q_{\mu,m})$-subatoms. Firstly, we have $\operatorname{supp} G_{\mu,m} \subset \operatorname{supp} \mathcal{X}_{\mu,m} \subset 2Q_{\mu,m}$. Secondly, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\tau\big(\int_{2Q_{\mu,m}} |\sum_l d_{\mu,l} T^c_{\sigma_1^0} a_{\mu,l}(s)|^2 ds\big)^{\frac{1}{2}} \\ &\lesssim \big(\sum_l |d_{\mu,l}|^2 (1 + |m - l|)^{-(d+1)}\big)^{\frac{1}{2}} \\ &\quad \cdot \sum_l (1 + |m - l|)^{\frac{1-M}{2}} \tau\big(\int_{2Q_{\mu,m}} (1 + 2^\mu (s - 2^{-\mu} l))^{d+M} |T^c_{\sigma_1^0} a_{\mu,l}(s)|^2 ds\big)^{\frac{1}{2}}. \end{aligned} \tag{9.47}$$

If we take $M = 2L + 1$, since $L > d$, we have $\frac{1-M}{2} < -d$. Applying Lemma 9.10, we get

$$\tau\big(\int_{\mathbb{R}^d} |G_{\mu,m}(s)|^2 ds\big)^{\frac{1}{2}} \lesssim \sum_l (1 + |m - l|)^{\frac{1-M}{2}} |Q_{\mu,l}|^{\frac{\alpha}{d}} \lesssim |Q_{\mu,m}|^{\frac{\alpha}{d}}.$$

Similarly, the derivative estimates in Lemma 9.10 ensure that

$$\tau\big(\int |D^\gamma G_{\mu,m}(s)|^2 ds\big)^{\frac{1}{2}} \lesssim |Q_{\mu,m}|^{\frac{\alpha}{d} - \frac{|\gamma|_1}{d}}, \quad \forall |\gamma|_1 \leq [\alpha] + 1.$$

Since $\alpha > 0$, no moment cancellation for subatoms is required. Thus, we have proved that the $G_{\mu,m}$'s are bounded multiple of $(\alpha, Q_{\mu,m})$-subatoms, whence the claim. Therefore, G_1 in (9.42) is the finite sum of $(\alpha, Q_{0,j})$-atoms, which yields $\|G_1\|_{F_1^{\alpha,c}} \lesssim 1$ by Theorem 8.21.

The term H_1 in (9.42) is much easier to handle. Observe that H_1 corresponds to the symbol $\sigma(s,\xi)\sum_{j\notin 8Q_{0,0}}\widetilde{\mathcal{X}}(s)$, whose s-support is in $(6Q_{0,0})^c$. Thus, we apply Corollary 9.13 directly to get that

$$\|H_1\|_{F_1^{\alpha,c}} \lesssim 1.$$

Step 2. Let us consider now the case where the supporting cube Q of g has side length less than one. As above, we may still assume that Q is centered at the origin. Let g be an $(\alpha, Q_{k,0})$-atom with $k\in\mathbb{N}$. Then g is given by

$$g = \sum_{(\mu,l)\le(k,0)} d_{\mu,l}a_{\mu,l} \quad \text{with } \sum_{(\mu,l)} |d_{\mu,l}|^2 \le |Q_{k,0}|^{-1} = 2^{kd}.$$

We normalize g as

$$\begin{aligned} h &= 2^{k(\alpha-d)}g(2^{-k}\cdot) \\ &= \sum_{(\mu,l)\le(k,0)} 2^{-\frac{kd}{2}}d_{\mu,l}2^{k(\alpha-\frac{d}{2})}a_{\mu,l}(2^{-k}\cdot) \\ &= \sum_{(\mu,l)\le(k,0)} \widetilde{d}_{\mu,l}\widetilde{a}_{\mu,l}, \end{aligned}$$

where $\widetilde{a}_{\mu,l} = 2^{k(\alpha-\frac{d}{2})}a_{\mu,l}(2^{-k}\cdot)$ and $\widetilde{d}_{\mu,l} = 2^{-\frac{kd}{2}}d_{\mu,l}$. Then it is easy to see that each $\widetilde{a}_{\mu,l}$ is an $(\alpha, Q_{\mu-k,l})$-subatom and h is an $(\alpha, Q_{0,0})$-atom. Define $\sigma_{1,k}(s,\xi) = \sigma_1(2^{-k}s, 2^k\xi)$, then we have

$$\begin{aligned} T^c_{\sigma_1}g(s) &= \int_{\mathbb{R}^d} \sigma_1(s,\xi)\widehat{g}(\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi \\ &= 2^{-k\alpha}\int_{\mathbb{R}^d} \sigma_1(s,\xi)\widehat{h}(2^{-k}\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi \\ &= 2^{k(d-\alpha)}\int_{\mathbb{R}^d} \sigma_{1,k}(2^k s,\xi)\widehat{h}(\xi)e^{2\pi \mathrm{i}\, 2^k s\cdot\xi}d\xi \\ &= 2^{k(d-\alpha)}T^c_{\sigma_{1,k}}h(2^k s). \end{aligned}$$

Since the ξ-support of σ_1 is away from the origin, we have

$$\|D_s^\gamma D_\xi^\beta \sigma_{1,k}(s,\xi)\|_{\mathcal{M}} \le C_{\gamma,\beta}|\xi|^{|\gamma|_1-|\beta|_1} \approx C_{\gamma,\beta}(1+|\xi|)^{|\gamma|_1-|\beta|_1}, \quad \forall k\in\mathbb{N}.$$

Thus, $\sigma_{1,k}$ is still a symbol in the class $S^0_{1,1}$. Then, applying the result for $(\alpha, Q_{0,0})$-atoms obtained in Step 1 to the symbol $\sigma_{1,k}$, we get $\|T^c_{\sigma_{1,k}}h\|_{F_1^{\alpha,c}} \lesssim 1$. Moreover, since $\alpha>0$, we can apply the homogeneity argument stated after Corollary 8.11 to get

$$\|T^c_{\sigma_1}g\|_{F_1^{\alpha,c}} \lesssim \|T^c_{\sigma_{1,k}}h\|_{F_1^{\alpha,c}} \lesssim 1.$$

Step 3. It remains to deal with the symbol σ_2. Note that $\sigma_2 = h_2(\xi)\sigma(s,\xi)$ with $\sigma\in S^0_{1,1}$ and $\operatorname{supp} h_2 \in 2Q$. Then for $\delta<1$, say $\delta=\frac{9}{10}$, we have $\sigma_2\in S^0_{1,\delta}$. Indeed, by definition, we have, for every $s\in\mathbb{R}$,

$$\begin{aligned} \|D_s^\gamma D_\xi^\beta \sigma_2(s,\xi)\|_{\mathcal{M}} &\lesssim \sum_{\beta_1+\beta_2=\beta} \|D_s^\gamma D_\xi^{\beta_1}\sigma(s,\xi)\cdot D^{\beta_2}h_2(\xi)\|_{\mathcal{M}} \\ &\le \sum_{\beta_1+\beta_2=\beta} C_{\gamma,\beta_1}(1+|\xi|)^{|\gamma|_1-|\beta_1|_1}\cdot|D^{\beta_2}h_2(\xi)|. \end{aligned}$$

But since h_2 is an infinitely differentiable function with support $2Q$, it is clear that for $\xi \in 2Q$,

$$(1+|\xi|)^{|\gamma|_1-|\beta_1|_1} \leq C_\gamma (1+|\xi|)^{\frac{9}{10}|\gamma|_1-|\beta_1|_1}, \quad \text{and} \quad |D^{\beta_2}h_2(\xi)| \leq C_{\beta_2}(1+|\xi|)^{-|\beta_2|_1}.$$

Putting these two inequalities into the estimate of $\|D_s^\gamma D_\xi^\beta \sigma_2(s,\xi)\|_{\mathcal{M}}$, we obtain

$$\|D_s^\gamma D_\xi^\beta \sigma_2(s,\xi)\|_{\mathcal{M}} \leq C_{\gamma,\beta}(1+|\xi|)^{\frac{9}{10}|\gamma|_1-|\beta|_1},$$

which yields $\sigma_2 \in S^0_{1,\frac{9}{10}}$. Therefore, it follows from Theorem 9.16 that $\|T^c_{\sigma_2} g\|_{F_1^{\alpha,c}} \lesssim \|g\|_{F_1^{\alpha,c}}$ for $g \in F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$. Combining this with the estimates in the first two steps, we complete the proof of the theorem. $\square$

If $\sigma \in S^0_{1,1}$, it is not true in general that $(T^c_\sigma)^*$ corresponds to a symbol in the class $S^0_{1,1}$. However, if we assume additionally this last condition, duality and interpolation arguments will give the following boundedness of T^c_σ:

Theorem 9.27. *Let $1 < p < \infty$ and let $\sigma \in S^0_{1,1}$, $\alpha \in \mathbb{R}$. If $(T^c_\sigma)^*$ admits a symbol in the class $S^0_{1,1}$, then T^c_σ is bounded on $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$.*

A similar argument as in the proof of Corollary 9.22 gives the following results concerning the symbols in $S^n_{1,1}$ with $n \in \mathbb{R}$.

Corollary 9.28. *Let $n \in \mathbb{R}$, $\sigma \in S^n_{1,1}$ and $\alpha > 0$. If $\alpha > n$, then T^c_σ is bounded from $F_1^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ to $F_1^{\alpha-n,c}(\mathbb{R}^d,\mathcal{M})$.*

Corollary 9.29. *Let $n, \alpha \in \mathbb{R}$, $\sigma \in S^n_{1,1}$ and $1 < p < \infty$. If $(T^c_\sigma)^*$ admits a symbol in the class $S^n_{1,1}$, then T^c_σ is bounded from $F_p^{\alpha,c}(\mathbb{R}^d,\mathcal{M})$ to $F_p^{\alpha-n,c}(\mathbb{R}^d,\mathcal{M})$.*

Chapter 10

Applications

10.1 Applications to tori

We will first recall the definitions and relevant results of the operator-valued Triebel-Lizorkin spaces on tori stated in [72]. Then we extend the results of the pseudo-differential operators in the previous chapter to the torus case. The main idea is to reduce the torus case to the Euclidean one that we discussed previously by a periodization argument.

In this section, $\mathcal{M}$ still denotes a von Neumann algebra with a normal semifinite faithful trace τ, but $\mathcal{N} = L_\infty(\mathbb{T}^d)\overline{\otimes}\mathcal{M}$.

We identify $\mathbb{T}^d$ with the unit cube $\mathbb{I}^d = [0,1)^d$ via $(e^{2\pi \mathrm{i} s_1}, \cdots, e^{2\pi \mathrm{i} s_d}) \leftrightarrow (s_1, \cdots, s_d)$. Under this identification, the addition in $\mathbb{I}^d$ is the usual addition modulo 1 coordinatewise; an interval of $\mathbb{I}^d$ is either a subinterval of $\mathbb{I}$ or a union $[b,1] \cup [0,a]$ with $0 < a < b < 1$, the latter union being the interval $[b-1, a]$ of $\mathbb{I}$ (modulo 1). So the cubes of $\mathbb{I}^d$ are exactly those of $\mathbb{T}^d$. Accordingly, functions on $\mathbb{T}^d$ and $\mathbb{I}^d$ are identified too.

Recall that φ is a Schwartz function satisfying (1.1). Then for every $m \in \mathbb{Z}^d \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}m) = \sum_{j\geq 0} \varphi(2^{-j}m) = 1.$$

This tells us that in the torus case $\{\varphi(2^{-j}\cdot)\}_{j\geq 0}$ gives a resolvent of the unit. According to this, we make a slight change of the notation that we used in the previous chapters :

$$\varphi^{(j)} = \varphi(2^{-j}\cdot), \ \forall j \geq 0.$$

Let $\varphi_j = \mathcal{F}^{-1}(\varphi^{(j)})$ for any $j \geq 0$. Now we periodize φ_j as

$$\widetilde{\varphi}_j(z) = \sum_{m\in\mathbb{Z}^d} \varphi_j(s+m) \quad \text{with} \quad z = (e^{2\pi \mathrm{i} s_1}, \ldots, e^{2\pi \mathrm{i} s_d}).$$

Then, we can easily see that $\widetilde{\varphi}_j$ admits the following Fourier series:

$$\widetilde{\varphi}_j(z) = \sum_{m\in\mathbb{Z}^d} \varphi(2^{-j}m) z^m. \tag{10.1}$$

Thus, for any $f \in \mathcal{S}'(\mathbb{R}^d; L_1(\mathcal{M}) + \mathcal{M})$, whenever it exists,

$$\widetilde{\varphi}_j * f(z) = \int_{\mathbb{T}^d} \widetilde{\varphi}_j(zw^{-1}) f(w) dw = \sum_{m\in\mathbb{Z}^d} \varphi(2^{-j}m)\widehat{f}(m) z^m \quad z \in \mathbb{T}^d.$$

The following definition was given in [72, Section 4.5].

Definition 10.1. Let $1 \le p < \infty$ and $\alpha \in \mathbb{R}^d$. The column operator-valued Triebel-Lizorkin space $F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$ is defined to be

$$F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M}) = \{f \in \mathcal{S}'(\mathbb{T}^d; L_1(\mathcal{M})) : \|f\|_{F_p^{\alpha,c}} < \infty\},$$

where

$$\|f\|_{F_p^{\alpha,c}} = \|\widehat{f}(0)\|_{L_p(\mathcal{M})} + \|(\sum_{j\ge 0} 2^{2j\alpha} |\widetilde{\varphi}_j * f|^2)^{\frac{1}{2}}\|_{L_p(\mathcal{N})}.$$

The row and mixture spaces $F_p^{\alpha,r}(\mathbb{T}^d, \mathcal{M})$ and $F_p^{\alpha}(\mathbb{T}^d, \mathcal{M})$ are defined similarly to the Euclidean case.

By the discussion before (10.1), if we identify a function f on $\mathbb{T}^d$ as a 1-periodic function f_{pe} on $\mathbb{R}^d$, then the convolution $\widetilde{\varphi}_j * f$ on $\mathbb{T}^d$ coincides with the convolution $\varphi_j * f_{\mathrm{pe}}$ on $\mathbb{R}^d$. More precisely:

$$\widetilde{\varphi}_j * f(z) = \varphi_j * f_{\mathrm{pe}}(s) \quad \text{with} \quad z = (e^{2\pi \mathrm{i} s_1}, \cdots, e^{2\pi \mathrm{i} s_d}).$$

By the almost orthogonality of the Littlewood-Paley decomposition given in (1.3), we get the following easy equivalent norm of $F_p^{\alpha,c}(\mathbb{I}^d, \mathcal{M})$:

$$\|f_{\mathrm{pe}}\|_{F_p^{\alpha,c}(\mathbb{I}^d,\mathcal{M})} \approx \|\phi_0 * f_{\mathrm{pe}}\|_p + \|(\sum_{j\ge 0} 2^{2j\alpha} |\varphi_j * f(z)|^2)^{\frac{1}{2}}\|_p,$$

where $\widehat{\phi}_0(\xi) = 1 - \sum_{j\ge 0} \varphi(2^{-j}\xi)$. Since $\widehat{\phi}_0$ is supported in $\{\xi : |\xi| \le 1\}$ and $\widehat{\phi}_0(\xi) = 1$ if $|\xi| \le \frac{1}{2}$, it then follows that

$$\|\check{\phi}_0 * f_{\mathrm{pe}}\|_p = \|\widehat{f}(0)\|_p \,.$$

Hence, combining the estimates above, we have

$$\|f\|_{F_p^{\alpha,c}(\mathbb{T}^d,\mathcal{M})} \approx \|f_{\mathrm{pe}}\|_{F_p^{\alpha,c}(\mathbb{I}^d,\mathcal{M})} \tag{10.2}$$

Thus $F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$ embeds into $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$ isomorphically. The equivalence (10.2) allows us to reduce the treatment of $\mathbb{T}^d$ to that of $\mathbb{R}^d$; and by periodicity, all the functions considered now are restricted on $\mathbb{I}^d$.

We are not going to state the properties of $F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$ specifically, and refer the reader to [72, Section 4.5]. But we note that the characterizations for $F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$ are better than the ones obtained in the previous chapter for $F_p^{\alpha,c}(\mathbb{R}^d, \mathcal{M})$, since in the torus case, we do not need to worry about the properties of the test functions near the origin.

Let us turn to the study of toroidal symbols. In the discrete case, the derivatives degenerate into discrete difference operators. Let $\sigma : \mathbb{Z}^d \to \mathcal{M}$. For $1 \le j \le d$, let e_j be the j-th canonical basis of $\mathbb{R}^d$. We define the forward and backward partial difference operators Δ_{m_j} and $\overline{\Delta}_{m_j}$:

$$\Delta_{m_j}\sigma(m) := \sigma(m + e_j) - \sigma(m), \quad \overline{\Delta}_{m_j}\sigma(m) := \sigma(m) - \sigma(m - e_j),$$

and for any $\beta \in \mathbb{N}_0^d$,

$$\Delta_m^{\beta} := \Delta_{m_1}^{\beta_1} \cdots \Delta_{m_d}^{\beta_d}, \quad \overline{\Delta}_m^{\beta} := \overline{\Delta}_{m_1}^{\beta_1} \cdots \overline{\Delta}_{m_d}^{\beta_d}.$$

Definition 10.2. Let $0 \le \delta, \rho \le 1$ and $\gamma, \beta \in \mathbb{N}_0^d$. Then the toroidal symbol class $S_{\rho,\delta}^n(\mathbb{T}^d \times \mathbb{Z}^d)$ consists of those $\mathcal{M}$-valued functions $\sigma(s, m)$ which are smooth in s for all $m \in \mathbb{Z}^d$, and satisfy

$$\|D_s^{\gamma} \Delta_m^{\beta} \sigma(s, m)\|_{\mathcal{M}} \le C_{\alpha,\beta,m} (1 + |m|)^{n - \rho|\beta|_1 + \delta|\gamma|_1}.$$

Definition 10.3. Let $\sigma \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$. For any $f \in \mathcal{S}'(\mathbb{T}^d; L_1(\mathcal{M}))$, we define the corresponding toroidal pseudo-differential operator as follows:

$$T^c_\sigma f(s) = \sum_{m \in \mathbb{Z}^d} \sigma(s,m)\widehat{f}(m)e^{2\pi \mathrm{i} s\cdot m}.$$

When studying the toroidal pseudo-differential operators T^c_σ on $\mathbb{T}^d$, especially its action on operator-valued Triebel-Lizorkin spaces on $\mathbb{T}^d$, a very useful tool is to extend the toroidal symbol to a symbol defined on $\mathbb{T}^d \times \mathbb{R}^d$, which reduces the torus case to the Euclidean one. This allows us to use the arguments in the last section. The extension of scalar-valued toroidal symbol has been well studied in [59]. With some minor modifications, the arguments used in [59] can be adjusted to our operator-valued setting.

The following lemma is taken from [59]. Denote by $\delta_0(\xi)$ the Kronecker delta function at 0, i.e., $\delta_0(0) = 1$ and $\delta_0(\xi) = 0$ if $\xi \neq 0$.

Lemma 10.4. *For each $\beta \in \mathbb{N}_0^d$, there exists a function $\phi_\beta \in \mathcal{S}(\mathbb{R}^d)$ and a function $\zeta \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\sum_{k\in\mathbb{Z}^d} \zeta(s+k) \equiv 1,$$
$$\widehat{\zeta}\,|_{\mathbb{Z}^d}(\xi) = \delta_0(\xi) \quad \text{and} \quad D^\beta_\xi(\widehat{\zeta})(\xi) = \overline{\Delta}^\beta_\xi \phi_\beta(\xi),$$

for any $\xi \in \mathbb{R}^d$.

The following lemma is the operator-valued analogue of Theorem 4.5.3 in [59].

Lemma 10.5. *Let $0 \le \rho, \delta \le 1$ and $n \in \mathbb{R}$. A symbol $\sigma \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$ is a toroidal symbol if and only if there exists an Euclidean symbol $\widetilde{\sigma} \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$ such that $\sigma = \widetilde{\sigma}\,|_{\mathbb{T}^d\times\mathbb{Z}^d}$.*

Proof. We first prove the "if" part. Let $\widetilde{\sigma} \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$. If $|\beta|_1 = 1$, then by the mean value theorem for vector-valued functions, we have

$$\|\Delta^\beta_m D^\gamma_s \sigma(s,m)\|_{\mathcal{M}} \le \sup_{0\le\theta\le1} \left\|\partial^\beta_\xi D^\gamma_s \widetilde{\sigma}(s, m+\theta\beta)\right\|_{\mathcal{M}}.$$

For a general multi-index $\beta \in \mathbb{N}_0^d$, we use induction. Writing $\beta = \beta' + \delta_j$ and using the induction hypothesis, we get

$$\begin{aligned}
\|\Delta^\beta_m D^\gamma_s \sigma(s,m)\|_{\mathcal{M}} &= \|\Delta^{\delta_j}_m(\Delta^{\beta'}_m D^\gamma_s \widetilde{\sigma}(s,m))\| \\
&\le \sup_{0\le\theta\le1} \|\partial_j(\Delta^{\beta'}_m D^\gamma_s \widetilde{\sigma}(s, m+\theta\delta_j))\|_{\mathcal{M}} \\
&= \sup_{0\le\theta\le1} \|\Delta^{\beta'}_m(\partial_j D^\gamma_s \widetilde{\sigma}(s, m+\theta\delta_j))\|_{\mathcal{M}} \\
&\le \sup_{0\le\theta'\le1} \|D^{\beta'}_\xi \partial_j D^\gamma_s \widetilde{\sigma}(s, m+\theta'\beta)\|_{\mathcal{M}} \\
&= \sup_{0\le\theta'\le1} \|D^\beta_\xi D^\gamma_s \widetilde{\sigma}(s, m+\theta'\beta)\|_{\mathcal{M}}.
\end{aligned}$$

Thus we deduce that

$$\begin{aligned}
\|\Delta^\beta_m D^\gamma_s \sigma(s,m)\|_{\mathcal{M}} &\le \sup_{0\le\theta'\le1} \|D^\beta_\xi D^\gamma_s \widetilde{\sigma}(s, m+\theta'\beta)\|_{\mathcal{M}} \\
&\le C'_{\alpha,\beta,m}(1+|m|)^{n-\rho|\beta|_1+\delta|\gamma|_1}.
\end{aligned}$$

Now let us show the "only if" part. In the proof of Theorem 4.5.3 in [59], the desired Euclidean symbol is constructed with the help of the functions in Lemma 10.4. We can transfer directly the arguments in [59] to our setting. But we still include a proof for completeness. Let $\zeta \in \mathcal{S}(\mathbb{R}^d)$ be as in Lemma 10.4. Define a function $\widetilde{\sigma} : \mathbb{T}^d \times \mathbb{R}^d \to \mathcal{M}$ by

$$\widetilde{\sigma}(s,\xi) = \sum_{m\in\mathbb{Z}^d} \widehat{\zeta}(\xi - m)\sigma(s,m).$$

Thus, $\sigma = \widetilde{\sigma}\,|_{\mathbb{T}^d\times\mathbb{Z}^d}$. Moreover, using summation by parts, we have

$$\begin{aligned}
\|D_s^\gamma D_\xi^\beta \widetilde{\sigma}(s,\xi)\|_{\mathcal{M}} &= \Big\| \sum_{m\in\mathbb{Z}^d} D_\xi^\beta \widehat{\zeta}(\xi - m) D_s^\beta \sigma(s,m) \Big\|_{\mathcal{M}} \\
&= \Big\| \sum_{m\in\mathbb{Z}^d} \overline{\Delta}_\xi^\beta \phi_\beta(\xi - m) D_s^\gamma \sigma(s,m) \Big\|_{\mathcal{M}} \\
&= \|(-1)^{|\beta|_1} \sum_{m\in\mathbb{Z}^d} \phi_\beta(\xi - m) \Delta_m^\beta D_s^\gamma \sigma(s,m)\|_{\mathcal{M}} \\
&\lesssim \sum_{m\in\mathbb{Z}^d} |\phi_\beta(\xi - m)|(1+|m|)^{n-\rho|\beta|_1+\delta|\beta|_1} \\
&\lesssim \sum_{m\in\mathbb{Z}^d} |\phi_\beta(\xi - m)|(1+|\xi - m|)^{n-\rho|\beta|_1+\delta|\gamma|_1}(1+|\xi|)^{n-\rho|\beta|_1+\delta|\gamma|_1} \\
&\lesssim (1+|\xi|)^{n-\rho|\beta|_1+\delta|\gamma|_1},
\end{aligned}$$

whence, $\widetilde{\sigma} \in S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$. □

Theorem 10.6. *Let $\sigma \in S^0_{1,\delta}(\mathbb{T}^d \times \mathbb{Z}^d)$ and $\alpha \in \mathbb{R}$. Then*

- *If $0 \le \delta < 1$, then T^c_σ is a bounded operator on $F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$ for every $1 \le p \le \infty$.*
- *If $\delta = 1$ and $\alpha > 0$, then T^c_σ is a bounded operator on $F_1^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$.*
- *If $\delta = 1$ and $(T^c_\sigma)^*$ admits a symbol in the class $S^0_{1,1}(\mathbb{T}^d \times \mathbb{Z}^d)$, then T^c_σ is bounded on $F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$ for any $1 < p < \infty$.*

Proof. By Lemma 10.5, there exists $\widetilde{\sigma}$ in $S^n_{\rho,\delta}(\mathbb{T}^d \times \mathbb{R}^d)$ such that $\sigma = \widetilde{\sigma}\,|_{\mathbb{T}^d\times\mathbb{Z}^d}$. Let $f \in F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$. By the identification $\mathbb{T}^d \approx \mathbb{I}^d$, for any $z \in \mathbb{T}^d$, there exists $s \in \mathbb{I}^d$ such that

$$\begin{aligned}
T^c_\sigma f(z) &= \sum_{m\in\mathbb{Z}^d} \sigma(s,m)\widehat{f}(m)e^{2\pi \mathrm{i} s\cdot m} \\
&= \int_{\mathbb{R}^d} \widetilde{\sigma}(s,\xi)\widehat{f}_{\mathrm{pe}}(\xi)e^{2\pi \mathrm{i} s\cdot\xi}d\xi = T^c_{\widetilde{\sigma}} f_{\mathrm{pe}}(s).
\end{aligned}$$

Now we apply Theorems 9.16, 9.26 and 9.27 to the symbol $\widetilde{\sigma}$ and f_{pe}. Then by the equivalence (10.2), we get the boundedness of T^c_σ on $F_p^{\alpha,c}(\mathbb{T}^d, \mathcal{M})$. □

10.2 Applications to Quantum tori

We now apply the results of the previous section to the quantum case. To this end, we first recall the relevant definitions. Let $d \ge 2$ and $\theta = (\theta_{kj})$ be a real skew symmetric

$d \times d$-matrix. The associated d-dimensional noncommutative torus $\mathcal{A}_\theta$ is the universal C*-algebra generated by d unitary operators $U_1, \ldots, U_d$ satisfying the following commutation relation

$$U_k U_j = e^{2\pi \mathrm{i} \theta_{kj}} U_j U_k, \quad j, k = 1, \ldots, d.$$

We will use standard notation from multiple Fourier series. Let $U = (U_1, \cdots, U_d)$. For $m = (m_1, \cdots, m_d) \in \mathbb{Z}^d$, define

$$U^m = U_1^{m_1} \cdots U_d^{m_d}.$$

A polynomial in U is a finite sum

$$x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m \quad \text{with} \quad \alpha_m \in \mathbb{C}.$$

The involution algebra $\mathcal{P}_\theta$ of all such polynomials is dense in $\mathcal{A}_\theta$. For any polynomial x as above, we define

$$\tau(x) = \alpha_0.$$

Then τ extends to a faithful tracial state τ on $\mathcal{A}_\theta$. Let $\mathbb{T}_\theta^d$ be the w^*-closure of $\mathcal{A}_\theta$ in the GNS representation of τ. This is our d-dimensional quantum torus. The state τ extends to a normal faithful tracial state on $\mathbb{T}_\theta^d$ that will be denoted again by τ. Note that if $\theta = 0$, then $\mathbb{T}_\theta^d = L_\infty(\mathbb{T}^d)$ and τ coincides with the integral on $\mathbb{T}^d$ against normalized Haar measure dz.

Any $x \in L_1(\mathbb{T}_\theta^d)$ admits a formal Fourier series:

$$x \sim \sum_{m \in \mathbb{Z}^d} \widehat{x}(m) U^m \;\text{ with }\; \widehat{x}(m) = \tau((U^m)^* x).$$

In [48], a transference method has been introduced to overcome the full noncommutativity of quantum tori and to use methods of operator-valued harmonic analysis. Let $\mathcal{N}_\theta = L_\infty(\mathbb{T}^d) \overline{\otimes} \mathbb{T}_\theta^d$, equipped with the tensor trace $\nu = \int dz \otimes \tau$. For each $z \in \mathbb{T}^d$, define π_z to be the isomorphism of $\mathbb{T}_\theta^d$ determined by

$$\pi_z(U^m) = z^m U^m = z_1^{m_1} \cdots z_d^{m_d} U_1^{m_1} \cdots U_d^{m_d}. \tag{10.3}$$

This isomorphism preserves the trace τ. Thus for every $1 \le p < \infty$,

$$\|\pi_z(x)\|_p = \|x\|_p, \; \forall x \in L_p(\mathbb{T}_\theta^d).$$

The main points of the transference method are contained in the following lemma from [6].

Lemma 10.7. (1) *Let $1 \le p \le \infty$. For any $x \in L_p(\mathbb{T}_\theta^d)$, the function $\widetilde{x} : z \mapsto \pi_z(x)$ is continuous from $\mathbb{T}^d$ to $L_p(\mathbb{T}_\theta^d)$ (with respect to the w^*-topology for $p = \infty$).*

(2) *If $x \in L_p(\mathbb{T}_\theta^d)$, then $\widetilde{x} \in L_p(\mathcal{N}_\theta)$ and $\|\widetilde{x}\|_p = \|x\|_p$, that is, $x \mapsto \widetilde{x}$ is an isometric embedding from $L_p(\mathbb{T}_\theta^d)$ into $L_p(\mathcal{N}_\theta)$.*

(3) *Let $\widetilde{\mathbb{T}_\theta^d} = \{\widetilde{x} : x \in \mathbb{T}_\theta^d\}$. Then $\widetilde{\mathbb{T}_\theta^d}$ is a von Neumann subalgebra of $\mathcal{N}_\theta$ and the associated conditional expectation is given by*

$$\mathbb{E}(f)(z) = \pi_z\Big(\int_{\mathbb{T}^d} \pi_{\overline{w}}[f(w)] dw \Big), \quad z \in \mathbb{T}^d, \; f \in \mathcal{N}_\theta.$$

Moreover, $\mathbb{E}$ extends to a contractive projection from $L_p(\mathcal{N}_\theta)$ onto $L_p(\widetilde{\mathbb{T}_\theta^d})$ for $1 \le p \le \infty$.

To avoid complicated notation, we will use the same notation for the derivation for the quantum tori $\mathbb{T}_\theta^d$ as for functions on $\mathbb{T}^d$. For every $1 \le j \le d$, define the derivation to be the operator ∂_j satisfying:

$$\partial_j(U_j) = 2\pi \mathrm{i} U_j \quad \text{and} \quad \partial_j(U_k) = 0 \text{ for } k \neq j.$$

Given $m \in \mathbb{N}_0^d$, the associated partial derivation D^m is $\partial_1^{m_1} \cdots \partial_d^{m_d}$. We keep using the resolvent of unit given by functions in (10.1). The Fourier multiplier on $\mathbb{T}_\theta^d$ with symbol $\varphi(2^{-j}\cdot)$ is then

$$\widetilde{\varphi}_j * x = \sum_{m \in \mathbb{Z}^d} \varphi(2^{-j}m)\, \widehat{x}(m) U^m.$$

The analogue of Schwartz class on the quantum torus is given by

$$\mathcal{S}(\mathbb{T}_\theta^d) = \{ \sum_{m \in \mathbb{Z}^d} a_m U^m : \{a_m\}_{m \in \mathbb{Z}^d} \text{ rapidly decreasing}\}.$$

This is a w^*-dense $*$-subalgebra of $\mathbb{T}_\theta^d$ and contains all polynomials. It is equipped with a structure of Fréchet $*$-algebra, and has a locally convex topology induced by a family of semi-norms. We denote the tempered distribution on $\mathbb{T}_\theta^d$ by $\mathcal{S}'(\mathbb{T}_\theta^d)$ which is the space of all continuous linear functional on $\mathcal{S}(\mathbb{T}_\theta^d)$. Then by duality, both partial derivations and the Fourier transform extend to $\mathcal{S}'(\mathbb{T}_\theta^d)$. Triebel-Lizorkin spaces on the quantum torus are defined and well studied in [72]. Let us recall the definition.

Definition 10.8. Let $1 \le p < \infty$ and $\alpha \in \mathbb{R}^d$. The column Triebel-Lizorkin space $F_p^{\alpha,c}(\mathbb{T}_\theta^d)$ is defined by

$$F_p^{\alpha,c}(\mathbb{T}_\theta^d) = \{x \in \mathcal{S}'(\mathbb{T}_\theta^d) : \|x\|_{F_p^{\alpha,c}} < \infty\},$$

where

$$\|x\|_{F_p^{\alpha,c}} = |\widehat{x}(0)| + \big\|(\sum_{j \ge 0} 2^{2j\alpha} |\widetilde{\varphi}_j * x|^2)^{\frac{1}{2}}\big\|_p.$$

The row space $F_p^{\alpha,r}(\mathbb{T}_\theta^d)$ and mixture space $F_p^{\alpha}(\mathbb{T}_\theta^d)$ are then defined similarly.

The transference method in Lemma 10.7 allows us to connect $F_p^{\alpha}(\mathbb{T}_\theta^d)$ with operator-valued spaces $F_p^{\alpha,c}(\mathbb{T}^d, \mathbb{T}_\theta^d)$. The result is

Lemma 10.9. *For any $1 \le p < \infty$, the map $x \mapsto \widetilde{x}$ extends to an isometric embedding from $F_p^{\alpha,c}(\mathbb{T}_\theta^d)$ to $F_p^{\alpha,c}(\mathbb{T}^d, \mathbb{T}_\theta^d)$ with complemented image.*

Let us turn to the definition of pseudo-differential operators on $\mathbb{T}_\theta^d$.

Definition 10.10. Let $0 \le \delta, \rho \le 1$, $n \in \mathbb{R}$ and $\gamma, \beta \in \mathbb{N}_0^d$ be multi-indices. Then the toroidal symbol class $S^n_{\mathbb{T}_\theta^d, \rho, \delta}(\mathbb{Z}^d)$ consists of those functions $\sigma : \mathbb{Z}^d \to \mathbb{T}_\theta^d$ which satisfy

$$\|D^\beta(\Delta_m^\gamma \sigma(m))\| \le C_{\beta,\gamma}(1 + |m|)^{n - \rho|\gamma|_1 + \delta|\beta|_1}, \quad \forall m \in \mathbb{Z}^d.$$

Definition 10.11. Let $\sigma \in S^n_{\mathbb{T}_\theta^d, \rho, \delta}(\mathbb{Z}^d)$. For any $x \in \mathbb{T}_\theta^d$, we define the corresponding toroidal pseudo-differential operator on $\mathbb{T}_\theta^d$ as follows:

$$T_\sigma^c x = \sum_{m \in \mathbb{Z}^d} \sigma(m) \widehat{x}(m) U^m.$$

Now we are ready to prove the mapping property of pseudo-differential operators on quantum torus.

Theorem 10.12. *Let $\sigma \in S^0_{\mathbb{T}^d_\theta,1,\delta}(\mathbb{Z}^d)$ and $\alpha \in \mathbb{R}$. Then*

- *If $0 \le \delta < 1$, then T^c_σ is a bounded operator on $F^{\alpha,c}_p(\mathbb{T}^d_\theta)$ for every $1 \le p \le \infty$.*
- *If $\delta = 1$ and $\alpha > 0$, then T^c_σ is a bounded operator on $F^{\alpha,c}_1(\mathbb{T}^d_\theta)$.*
- *If $\delta = 1$ and $(T^c_\sigma)^*$ admits a symbol in the class $S^0_{\mathbb{T}^d_\theta,1,\delta}(\mathbb{Z}^d)$, then T^c_σ is bounded on $F^{\alpha,c}_p(\mathbb{T}^d_\theta)$ for any $1 < p < \infty$.*

Proof. Recall that π_z denotes the isomorphism of $\mathbb{T}^d_\theta$ determined by (10.3). We claim that, given $m \in \mathbb{Z}^d$, the function $z \mapsto \pi_z(\sigma(m))$ from $\mathbb{T}^d$ to $\mathbb{T}^d_\theta$ satisfies

$$\|D^\gamma_z \Delta^\beta_m \pi_z(\sigma(m))\| \le C_{\gamma,\beta}(1+|m|)^{n+\delta|\gamma|_1-\rho|\beta|_1}. \tag{10.4}$$

Since π_z commutes with the derivations on $\mathbb{T}^d_\theta$, we have $D^\gamma\Delta^\beta\pi_z\sigma(m) = \pi_z(D^\gamma\Delta^\beta\sigma(m))$. Therefore,

$$\|D^\gamma\Delta^\beta\pi_z\sigma(m)\| = \|\pi_z(D^\gamma\Delta^\beta\sigma(m))\| \le \|D^\gamma\Delta^\beta\sigma(m)\| \le C_{\gamma,\beta}(1+|m|)^{n+\delta|\gamma|_1-\rho|\beta|_1}.$$

Denote $\widetilde{\sigma}(z,m) = \pi_z(\sigma(m))$ for $(z,m) \in \mathbb{T}^d \times \mathbb{Z}^d$ and consider the pseudo-differential operator $T^c_{\widetilde{\sigma}}$. Combining (10.4) and Theorem 10.6, we obtain the boundedness of $T^c_{\widetilde{\sigma}}$ on $F^{\alpha,c}_p(\mathbb{T}^d,\mathbb{T}^d_\theta)$. Moreover, for any polynomial x on $\mathbb{T}^d_\theta$ and $f(z) = \pi_z(x)$, we have

$$\begin{aligned} T^c_{\widetilde{\sigma}} f(z) &= \sum_{m\in\mathbb{Z}^d} \widetilde{\sigma}(z,m)\widehat{f}(m) z^m \\ &= \sum_{m\in\mathbb{Z}^d} \pi_z(\sigma(m))\widehat{x}(m)U^m z^m \\ &= \sum_{m\in\mathbb{Z}^d} \pi_z(\sigma(m)\widehat{x}(m)U^m) = \pi_z(T^c_\sigma(x)). \end{aligned}$$

Finally, by Lemma 10.9 and Theorem 10.6, we have

$$\begin{aligned} \|T^c_\sigma(x)\|_{F^{\alpha,c}_p(\mathbb{T}^d_\theta)} &= \|\pi_\cdot(T^c_\sigma(x))\|_{F^{\alpha,c}_p(\mathbb{T}^d,\mathbb{T}^d_\theta)} = \|T^c_{\widetilde{\sigma}} f\|_{F^{\alpha,c}_p(\mathbb{T}^d,\mathbb{T}^d_\theta)} \\ &\lesssim \|f\|_{F^{\alpha,c}_p(\mathbb{T}^d,\mathbb{T}^d_\theta)} = \|x\|_{F^{\alpha,c}_p(\mathbb{T}^d_\theta)}. \end{aligned}$$

The three assertions are proved. □

Finally, let $S^n_{\rho,\delta}(\mathbb{Z}^d)$ be the scalar-valued toroidal symbol class, consisting of those functions $\sigma : \mathbb{Z}^d \to \mathbb{C}$ which satisfy

$$|D^\beta(\Delta^\gamma_m\sigma(m))| \le C_{\beta,\gamma}(1+|m|)^{n-\rho|\gamma|_1+\delta|\beta|_1}, \quad \forall m \in \mathbb{Z}^d.$$

In this setting, it is evident that T^c_σ and T^r_σ give the same pseudo-differential operator on $\mathbb{T}^d_\theta$, denoted by T_σ simply. Then, we have the following

Corollary 10.13. *Let $\sigma \in S^0_{1,\delta}(\mathbb{Z}^d)$ and $\alpha \in \mathbb{R}$. Then*

- *If $0 \le \delta < 1$, then T_σ is a bounded operator on $F^{\alpha,c}_p(\mathbb{T}^d_\theta)$, $F^{\alpha,r}_p(\mathbb{T}^d_\theta)$ and $F^\alpha_p(\mathbb{T}^d_\theta)$ for every $1 \le p \le \infty$.*
- *If $\delta = 1$ and $\alpha > 0$, then T_σ is a bounded operator on $F^{\alpha,c}_1(\mathbb{T}^d_\theta)$, $F^{\alpha,r}_1(\mathbb{T}^d_\theta)$ and $F^\alpha_1(\mathbb{T}^d_\theta)$.*

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