Analyse des EDP non linéaires de type hyperbolique et parabolique dégénéré, approximation numérique par volumes finis, et applications

Mémoire de synthèse
en vue de l'obtention de l'habilitation à diriger les recherches

Analysis of nonlinear PDEs of hyperbolic and degenerate parabolic type, numerical approximation by finite volume methods, and applications

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Besançon and Moscow, May-June 2011

À ma famille et à mes amis.

À la mémoire de Stanislav Nikolaevich Kruzhkov et de Philippe Bénilan ;
à tous ceux qui m'ont guidé et me guident dans l'apprentissage des mathématiques.

À mes collaborateurs sans qui rien n'aurait été fait,
avec un sincère "merci" pour les semaines, mois et années de travail et d'amitié
que nous avons partagés.

Enfin, aux marseillais du LATP et aux bisontins du Labo de Maths,
pour tous les moments vécus ensemble.

Présenté le 24 Nov. 2011 à l'UFR des Sciences et Techniques, Besançon

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The research works that I conducted since the beginning of my PhD were concerned with several tightly related topics, unified mainly by the common analysis tools used to approach the problems. All of them were devoted to "solving" partial differential equations. Most of these equations are nonlinear evolution equations governed by differential operators that are accretive in L^1. This includes various reaction-convection-diffusion problems such as scalar conservation laws, porous medium or fast diffusion problems, Leray-Lions kind problems, fractional (nonlocal) diffusions, and mixed problems including a sum of different operators. Many of the problems I considered should be seen as singular limits of more regular parabolic problems. I also analyzed some systems of reaction-diffusion equations and some hyperbolic systems of conservation laws. My main activity is the study of relevancy of different solution concepts; it usually leads to results on existence, uniqueness and structural stability of the appropriately defined solutions to these problems. While the methods of analysis "inside the domain" were often already well established, in a number of works I treated the questions of taking into account boundary conditions, interface coupling, or the behaviour of solutions at infinity. Most of the problems under study are of rather academic character, though strongly motivated by applications from fluid mechanics, hydrogeology and petroleum engineering, traffic modelling, population dynamics, electrocardiology, etc. For some of these problems, I participated to the development of finite volume discretization techniques and the related "discrete functional analysis" tools, with a focus on approximation of nonlinear or anisotropic diffusion operators and on interface coupling of finite volume schemes for conservation laws. These techniques permitted to prove convergence of finite volume schemes designed for several academic and applied problems.

The HDR manuscript, the publications and conferences that constitute the basis of the HDR thesis can be found at `http://lmb.univ-fcomte.fr/Boris-Andreianov`

Analyse des EDP non linéaires de type hyperbolique et parabolique dégénéré, approximation numérique par volumes finis, et applications

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Les travaux de recherche que j'ai menés depuis le début de ma thèse ont été dédiés à une série de questions proches les unes des autres, essentiellement reliées par des outils d'analyse mathématique communs utilisés dans l'approche des problèmes, et visant toutes la "résolution" d'équations aux dérivées partielles. La plupart de celles-ci sont des équations d'évolution non linéaires gouvernées par des opérateurs différentiels accrétifs dans L^1. Ceci concerne en particulier des équations de réaction-convection-diffusion tels que les lois de conservation, les équations de milieux poreux et de diffusion rapide, les problèmes du type Leray-Lions, les problèmes de diffusions fractionnaires (c'est-à-dire non locales), ainsi que des problèmes mixtes faisant intervenir une somme de différents opérateurs. Plusieurs de ces problèmes doivent être vus comme les limites singulières de problèmes paraboliques plus réguliers. J'ai également analysé certains systèmes de réaction-diffusion et de lois de conservation hyperboliques. Mon activité principale est d'étudier la pertinence de différentes notions de solution ; les résultats obtenus peuvent alors conduire à l'établissement de l'existence, de l'unicité et de la stabilite structurelle des solutions définies d'une façon bien adaptée au problème. Alors que les méthodes d'analyse "à l'intérieur du domaine" étaient la plupart du temps déjà bien établies, je me suis intéressé dans une série de travaux à la prise en compte des conditions aux limites, du couplage à travers une interface, ou encore du comportement des solutions à l'infini. Les problèmes que j'ai étudiés, bien que souvent de caractère académique, ont toutefois été, à l'origine, fortement motivés par des applications provenant des domaines de la mécanique des fluides, de l'hydrogéologie et de l'ingénierie pétrolière, de la modélisation du trafic routier, de la dynamique des populations, de l'électrocardiologie, etc. Pour certains de ces problèmes, j'ai participé au développement de techniques de discrétisation par les volumes finis et d'outils d'"analyse fonctionnelle discrète" associés, en mettant l'accent sur l'approximation d'opérateurs de diffusion non linéaires et anisotropes, et sur le couplage par une interface de schémas de volumes finis pour les lois de conservation. Ces techniques ont permis de démontrer la convergence des schémas de volumes finis pour divers problèmes académiques et appliqués.

Le manuscrit présentant l'HDR ainsi que les publications et les conférences qui constituent la base de cette HDR peuvent être consultés à l'adresse suivante :
`http://lmb.univ-fcomte.fr/Boris-Andreianov`

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Translation (from Russian)

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Chapter 1

Introduction

The research works that I conducted since the beginning of my PhD were concerned with several tightly related topics, unified mainly by the common analysis tools used to approach the problems. All of them were devoted to "solving" partial differential equations. Most of these equations were evolution equations governed by differential operators in space that are accretive in L^1: this includes various reaction-convection-diffusion problems such as scalar conservation laws, porous medium or fast diffusion problems, Leray-Lions kind problems, fractional Laplace diffusions, and mixed problems including a sum of different operators. Many of the problems I considered should be seen as singular limits of more regular parabolic problems. While the methods of analysis "inside the domain" often were already well established, in a number of works I treated questions of taking into account boundary conditions, or interface coupling, or the behaviour of solutions at infinity. I also analyzed some systems of reaction-diffusion equations and some hyperbolic systems of conservation laws. All of the problems under study are of rather academic character, though strongly motivated by applications. For some of these problems, I participated to development of finite volume discretization techniques and in analysis of convergence of finite volume methods, with a focus on approximation of nonlinear or anisotropic diffusion operators and on interface coupling of finite volume schemes for conservation laws.

My interest went, specifically, to questions of definition of solution and of establishing their fundamental properties: existence, uniqueness, stability with respect to the data, structural stability or singular limits. Theory of PDEs of the aforementioned type uses notions of weak or variational solutions, entropy solutions (in various contexts), kinetic solutions, renormalized solutions, and also the abstract notions of mild or integral solutions for evolution equations. Establishing appropriate definitions, analyzing existence, uniqueness, comparison and continuous dependence properties (including structural stability, which means dependence of solutions on perturbation of coefficients and nonlinearities present in the problem), proving convergence of approximations, and applying such analysis techniques to some more concrete problems was my principal activity. The key tools I used were the classical functional analysis and PDE methods: as keywords, let me mention Lebesgue, Sobolev and BV spaces, variable exponent spaces, a priori estimates, compactness theorems and passage to the limit in nonlinearities, Young measures and related weak compactness arguments, fixed-point or topological degree arguments, weak and strong boundary traces, variational methods, convex analysis tools, approximation by regularization or by discretization, nonlinear semigroups of contraction, comparison arguments and maximum principles, self-similar or reference solutions, reduction to ODEs or differential inequalities, entropies, truncation test functions, regularity and maximal regularity results, chain rules and duality, structure-preserving numerical schemes, discrete duality properties and other discrete functional analysis tools.

Let me briefly present a state-of-the-art that was the background of the work that I carried out in collaboration with a number of co-authors.

A state of the art

In this paragraph, I simply call different PDEs by their names, and freely employ a number of notions and concepts well known to the specialists of the subject of conservation laws, nonlinear degenerate parabolic equations, applications of nonlinear semigroups, and finite volume approximation. For less specialized readers, I tried to provide in footnotes the precise form of equations under study, and a very brief and still heuristic interpretation of the key-words appearing in the text. More detailed information can be inferred from the subsequent Chapters.

Considered separately, purely hyperbolic[1] and purely parabolic (possibly singular and degenerate) [2] quasilinear scalar problems, as well as nonlinear Leray-Lions kind problems[3], were well understood since years (see e.g. the monographs Serre [147], Ladyzhenskaya et al. citeLadSolUral, Vázquez [155], Lions [116]). The fundamental theory of entropy solutions[4] of conservation laws was established by Kruzhkov [108][5], and re-interpreted by Crandall [60] and Bénilan [21] using nonlinear semigroup methods[6]. The degenerate elliptic-parabolic case[7] was deeply investigated by Alt and Luckhaus [6], the hyperbolic-parabolic theory[8] reposed for a long time upon the work of Vol'pert an Hudjaev [157]. Yet in the late 1990ies, important progress was observed at the interface of the hyperbolic and the parabolic theories, due to

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[1] Hyperbolic conservation law: $u_t + \operatorname{div} \mathfrak{f}(t,x,u) = s$

[2] Parabolic quasilinear equation: $u_t + \operatorname{div}\left(\mathfrak{f}(t,x,u) + a(t,x,u)\nabla\varphi(u)\right) = s$, with a a map into the space of uniformly bounded and coercive symmetric matrices and φ a strictly increasing function. When φ is non-Lipschitz we speak of a singular parabolic problem; and when φ' may have zeros, we say that the parabolic problem is degenerate.

[3] Nonlinear Leray-Lions problems: $u_t + \operatorname{div} \mathfrak{a}(t,x,u;\nabla u) = s$, with $\mathfrak{a}$ satisfying a series of pseudomonotonicity, coercivity and growth conditions that permit to set up the problem in the duality framework of $W^{1,p} - W^{-1,p'}$ spaces. The prototype example is the p-laplacian, defined below. Such nonlinear parabolic equations are singular if $1 < p < 2$, and degenerate if $p > 2$.

[4] By an entropy solution of the scalar conservation law $u_t + \operatorname{div} \mathfrak{f}(u) = 0$ we mean an L^∞ function u satisfying, in the weak sense, the equation and the family of "Kruzhkov entropy inequalities"

$$\forall k \in \mathbb{R} \;\; \partial_t |u-k| + \operatorname{div}\left[\operatorname{sign}(u-k)(f(u)-f(k))\right] \leq 0.$$

It should be stressed that the functions $\eta_k : r \mapsto |r-k|$ are selected like a "basis" in the set of all convex functions (called "entropies"), and the associated "entropy flux" $q_k : r \mapsto \operatorname{sign}(r-k)(f(r)-f(k)$ should be seen as a primitive of $f'(r)\eta'(k)$. For smooth solutions, the entropy inequalities actually hold with the equality sign; this is a particular case of the renormalization (see below). The choice of the "basis" entropies $r \mapsto |r-k|$ permits to deduce uniqueness using the "doubling of variables" method of Kruzhkov.
The definition of entropy solutions has a deep physical motivation: the conservation law can be understood as the singular limit of non-degenerate parabolic "viscous" problems $u^\varepsilon_t + \operatorname{div} \mathfrak{f}(u^\varepsilon) = \varepsilon \operatorname{div} a(u^\varepsilon)\nabla u^\varepsilon$. The "entropy dissipation", encoded by the "$\leq$" sign of the entropy inequalities, is induced by dissipative processes that take place at sharp transitions of the viscous solution u^ε.

[5] it was a pleasure for me to contribute to the publication, for western readers, of S.N. Kruzhkov's lectures [Tr] compiled and complemented by G.A. Chechkin and A.Yu. Goritsky

[6] The nonlinear semigroup theory is based on time-implicit discretization of abstract evolution equations $u_t + Au = s$ for nonlinear, possibly multi-valued operators A for which the resolvents $(A+\lambda I)^{-1}$ are non-expansive operators with dense domain.

[7] Degenerate elliptic-parabolic case: $b(v)_t + \operatorname{div} \mathfrak{a}(t,x,v;\nabla v) = s$ with continuous non-decreasing b

[8] Degenerate parabolic-hyperbolic case: $u_t + \operatorname{div} \mathfrak{a}(t,x,u;\nabla\varphi(u)) = s$ with continuous non-decreasing φ

the founding work of J. Carrillo [45]. It allowed to extend the notion and the technical tools of Kruzhkov entropy solutions [108] to the general elliptic-parabolic-hyperbolic setting[9]. The theory carries on even to nonlinear Leray-Lions diffusion operators (see Carrillo and Wittbold [46]) of which the p-laplacian[10] is the main example, although the issue of dependence of the diffusion operator in space variable remains a diffuculty for the doubling-of-variables techniques. Extensions to anisotropic problems[11] were provided by Bendahmane and Karlsen [17]. An argument of doubling of the time variable[12] was put forward by Otto [127], which made complete the theory of weak energy (or variational) solutions[13] of elliptic-parabolic problems without making appeal to the semigroup theory (cf. Bénilan and Wittbold [29]). The existence analysis was later complemented by bi-monotonone approximation[14] arguments developed by Ammar and Wittbold [8].

From another perspective, the same family of problems was treated using general methods of nonlinear semigroups governed by accretive operators[15] in the space L^1 (see, e.g., Bénilan, Crandall and Pazy [25]). Mild/integral solutions[16] given by the nonlinear semigroup the-

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[9] Elliptic-parabolic-hyperbolic case: $b(v)_t + \operatorname{div} \mathfrak{a}(t,x,v;\nabla\varphi(v)) = s$ with continuous non-decreasing b and φ; $\mathfrak{a}$ can be quasilinear, $a(t,x,v)\nabla v$, or Leray-Lions.

[10] The p-laplacian operator: $\Delta_p : u \mapsto -\Delta_p u := \operatorname{div} |\nabla u|^{p-2}\nabla u$; skipping the issue of boundary conditions, let us say that it acts from $W^{1,p}_{loc}(\Omega)$ to its dual space in $W^{-1,p'}_{loc}$. Here $1 < p < \infty$ and $p' = p/(p-1)$ is the conjugate exponent of p. This is the prototype of nonlinear Leray-Lions operators. Operators of 1-laplacian and ∞-laplacian can be defined, using deeper tools of convex analysis, Radon measures and BV spaces.

[11] We always consider anisotropic convection terms $\operatorname{div} \mathfrak{f}(t,x;u)$; the anisotropy is stressed when elliptic operators of the form $\sum_{i,j=1}^N \partial_i a_i(t,x,u;\partial_j\varphi_j(u))$ are considered.

[12] Doubling of variables: this term refers to the procedure of obtention of a term like $\int |u(t)-\hat{u}(t)|\xi(t)\,dt$ as a limit of $\int\int |u(t)-\hat{u}(s)|\xi(t)\delta_n(t,s)\,dtds$ as the sequence $(\delta_n)_n$ of test functions concentrates to the Dirac measure supported on $\{s=t\}$. The method is used in the context of entropy solutions since the introduction of both ideas by Kruzhkov [108]. A different, though related meaning is given to this term in the context of viscosity solutions of Hamilton-Jacobi equations.

[13] By finite energy or variational solutions to elliptic and/or parabolic problems we mean the solutions that can be taken themselves as test functions in the corresponding weak formulations. Typical examples are given by solutions of Euler-Lagrange equations for minimization of convex coercive continuous or lower semi-continuous functionals such as $J : u \mapsto \int \big(\frac{1}{p}|\nabla u|^p - su\big)$ over Sobolev spaces like $W^{1,p}_0(\Omega)$.

[14] By a bi-monotone approximation of, say, $s \in L^1$ we mean a sequence $s^{n,m}$ of, say, $L^1 \cap L^\infty$ functions with the property $s^{n,m} \uparrow_{n\to\infty}\downarrow_{m\to\infty} s$.

[15] A possibly nonlinear and multi-valued operator A on a Banach space X is given by its graph; it is said accretive if for all $\lambda > 0$ the resolvent $(A+\lambda I)^{-1}$ (also given by its graph) is a non-expansive operator on X. In the case the domain of the resolvent is the whole of X, accretive operator is said m-accretive.

[16] Mild solutions of the evolution problem $u_t + Au = s$, $u(0) = u(0)$, given by the Crandall-Liggett Theorem, are $C([0,T];X)$ limits of solutions $u_\varepsilon(t) := \sum_{i=1}^{N_\varepsilon} u^i_\varepsilon 1\!\!1_{(t^{i-1}_\varepsilon, t^i_\varepsilon]}(t)$ to time-implicit semi-discretized problems

$$\forall i = 1..N_\varepsilon \quad \frac{u^i_\varepsilon - u^{i-1}_\varepsilon}{t^i_\varepsilon - t^{i-1}_\varepsilon} + Au^i_\varepsilon = s^i_\varepsilon$$

with a consistent approximation of $s(t)$ by $\sum_{i=1}^{N_\varepsilon} s^i_\varepsilon 1\!\!1_{(t^{i-1}_\varepsilon, t^i_\varepsilon]}(t)$.
Integral solutions of $u_t + Au = s$ defined by Bénilan [21] are abstract functions satisfying the family of differential "infinitesimal contraction" inequalities

$$\forall (\hat{u},\hat{s}) \in A \quad \partial_t \|u(t)-\hat{u}\|_X \le [u(t)-\hat{u}, s(t)-\hat{s}]_X,$$

where the bracket $[F,G]_X$ is the map from $X \times X$ to $\mathbb{R}$ which describes the derivative of $\|\cdot\|_X$ at the point F in the direction G. For instance, if $X = L^1$ then $[F,G]_{L^1} = \int (\operatorname{sign}_0 F)G + \int_{[F=0]} |G|$.
According to the general theory of nonlinear semigroups, mild and integral solutions exist, are unique and coincide in the case of an m-accretive operator.

ory were considered as abstract objects for many years (these solutions can be obtained by closure or passage to the limit from sequences of more conventional solutions, such as the variational ones). Yet in the 90ies, the ideas of truncation[17] and renormalization[18] (Boccardo and Gallouet [32], Lions and Murat [124], Bénilan et al. [22, 23]) allowed to characterize these solutions intrinsically, in a way acceptable for the PDE community[19], and even to extend the well-posedness theory to some measure data (see Dal Maso, Murat, Orsina and Prignet [65]). The corresponding notions of entropy and renormalized solutions[20] became classical in a few years.

Another approach, that had a different motivation but that achieves the same goal of giving sense to mild solutions, was the kinetic approach[21] developed by several authors, firstly for scalar conservation laws (see Lions, Perthame and Tadmor [117], Perthame [138]), then for anisotropic parabolic problems (Chen and Perthame [51]). Presently, the kinetic interpretation remains limited to quasilinear convection-diffusion problems[22], but it allows for a deeper insight into the local structure of solutions. At the same time, the idea of H-measures[23]

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[17] Truncation: the idea of using nonlinear composition of solutions by functions $T_k : r \mapsto \max\{\min\{r,k\},-k\}$.

[18] Renormalization: the idea of deriving, from a PDE with unknown scalar function u, a family of "accompanying" PDEs satisfied by different functions $T(u)$ (e.g., T can be an entropy $|\cdot - k|$, a truncation T_k, a general bounded smooth function, etc.). In the elliptic and parabolic context, the original PDE may be eventually *replaced* by the "accompanying" PDEs, which yields a weaker notion of solution (a renormalized solution may even not be a distributional solution).

[19] Given a function u, the definition of a mild solution just cannot be checked. Although the situation with integral solutions is much better (at least potentially, one could check whether the inequalities defining integral solutions hold for u), this notion of solution does not have a direct relation to the PDE in hand. Entropy and renormalized solutions' notions are both verifiable and PDE-based.

[20] Entropy solution: the idea is to consider *unbounded* "solutions" by writing a formulation with "test functions" $T_k(u-\xi)$, ξ being bounded and T_k being a truncation; the procedure leads to a series of variational inequalities. It should be stressed that entropy solutions of conservation laws (in the sense of Kruzhkov) are quite different in their spirit from entropy solutions of degenerate elliptic or parabolic problems (in the sense of Bénilan et al.).

Renormalized solutions: the idea is to consider unbounded (or "infinite energy") "solutions" u as functions that verify a wide family of renormalized formulations for $T(u)$ with bounded nonlinearities T. While the formal equation with $T = Id$ is meaningless, the renormalized formulation contains some information relevant also to the limits $T \to 1$ and $T \to Id$.

A key feature is that energy (variational) solutions are both entropy and renormalized; and an entropy/renormalized solution which is of finite energy turns out to be a weak one.

The identification between entropy and renormalized solutions is often indirect; in fact, both can be seen as the unique limit of a common approximation procedure.

[21] Kinetic approach: a solution u of a "macroscopic model" can be obtained by averaging in an additional "microscopic" variable ξ: for the case we are interested in, $u = \int \xi \chi_u(\xi)\, d\xi$ where χ (the "kinetic function") writes as $\chi_u(\xi) = 1\!\!1_{[0<u<\xi]} - 1\!\!1_{[\xi<u<0]}$. This implies, in particular, that $\mathfrak{f}(u) = \int \mathfrak{f}(\xi)\chi_u(\xi)\, d\xi$.

Then the kinetic formulation of, e.g., a hyperbolic scalar conservation law is obtained by substituting the *nonlinear* PDE $u_t + \operatorname{div} \mathfrak{f}(u) = 0$ by the one-parameter family of the *linear* PDEs $\chi(\xi;\cdot)_t + \mathfrak{f}'(\xi)\cdot\nabla\chi(\cdot;\xi) = -\mu_\xi$. Then $u(\cdot)$ is reconstituted from $(\chi(\cdot;\xi))_\xi$ by averaging. The right-hand side here is a "kinetic measure" μ_ξ; it possesses properties that permit to link this notion of solution to the one of entropy solution.

Let us stress that the kinetic approach was inspired by the gas dynamics context, where ξ is the velocity variable and χ is related to the maxwellian.

[22] which means, Leray-Lions nonlinear operators cannot be treated in this way

[23] The idea of Young measure, as a description of weakly-* convergent sequences in L^∞, was used in the classical theory of conservation laws in order to construct solutions. H-measures, produced from sequences of vectors, give a better description of weak convergence properties because they capture the oscillation frequencies

was adapted by Panov [132] so that to be used on entropy solutions of conservation laws. Entropy-process formulations[24] (see Gallouët and Hubert [85], Panov [132]), kinetic interpretation of entropy solutions, and the H-measures techniques led to a better understanding of the pointwise behaviour of solutions or sequences of solutions; in particular, compactification results from the nonlinearity (such as were first obtained with Young measure and compensated compactness techniques by Tartar and by DiPerna) were extended, or their use was simplified considerably[25].

Use of weak normal boundary traces[26] of the flux for conservation laws was initiated by Otto in [128], and allowed for a complete solution of the Dirichlet boundary problem; it gave an occasion to revisit the theory of divergence-measure fields[27] and to deepen the understanding of weak boundary traces and integration-by-parts arguments (see Chen and Frid [47]). Then, as another outcome of the aforementioned kinetic interpretation of solutions and the compactification techniques of H-measures, existence of strong traces[28] was shown for merely L^∞ entropy solutions of conservation laws (Vasseur [154], Panov [133]).

While classical Kruzhkov theory of scalar conservation laws seemed essentially complete in 1970ies, new trends were constantly appearing in 1990ies. In particular, a series of works concerned with the effect of infinite speed of propagation[29] for a non-Lipschitz flux function was conducted by Panov, Kruzhkov, Bénilan (see, e.g., [26]). Further, in some applications, necessity of considering non-Kruzhkov shocks[30] was progressively made clear, due in particular to the contributions of LeFloch et al. [113] and to the investigation of problems presenting

in the sequence. Paramertized families of H-measures, that can be seen as H-measures of a vector of size $\mathbb{R}$, allow to study simultaneous oscillations in the quantities $(|u-k|)_{k\in\mathbb{R}}$.

[24] Entropy-process formulation: the Young measure description is substituted by description in terms of the measure distribution function: $f(u)(\cdot)=\int f(\lambda)\,d\nu_\cdot(\lambda)\equiv\int_0^1\mu(\cdot,\alpha)\,d\alpha$.

[25] "Compensated compactness" arguments combine bounds in functional spaces (that only imply weak convergence) and a family of differential constraints (equalities, inequalities...) in order to "convert" weak convergence into the strong one, and thus to "pass to the limit in nonlinearities". Young measures, entropy-process solutions and H-measures are technical tools that can be used along with the differential constraints coming from entropy inequalities.

[26] E.g. if u solves $\operatorname{div}\mathfrak{f}(u)=0$ in Ω, one can give sense to the weak trace $\gamma_w\mathfrak{f}(u)\cdot\nu$ on the boundary Ω with ν the normal to the boundary; this is achieved by applying the Green-Gauss integration-by-parts formula in subdomains Ω_h of Ω, with a test function ξ, and passing to the limit with $\Omega_h\to\Omega$.

[27] These are L^p fields, $1\le p\le\infty$, of which the divergence is a measure.

[28] By strong trace of u, we mean that $u_h:=u|_{\partial\Omega_h}$ converge to $u|_{\partial\Omega}$ in L^1_{loc} and a.e.; as for the case of weak normal boundary traces, a sense is given to this convergence by "lifting" $\partial\Omega$ to a family $\partial\Omega_h$ with a common coordinate x' such that $x\leftrightarrow(x',h)$ is a regular enough map on a neigbourhood of $\partial\Omega$.

[29] For a conservation law $u_t+\operatorname{div}\mathfrak{f}(u)=0$, modifying the initial data at the distance $T\|\mathfrak{f}'\|_\infty$ from a point x, we cannot affect the value $u(t,x)$ as long as $t<T$. This is the finite speed of propagation effect; in a weaker form, it can be observed also for quasilinear and nonlinear parabolic equations (porous medium equation, p-laplacian heat equation). But when $\mathfrak{f}'$ is not (locally) Lipschitz, "information can escape to infinity or arrive from infinity" instanteneously.

[30] Typical entropy solutions of scalar conservation law are piecewise smooth, and the curves of discontinuity, called shocks, should obey a conservation property (called Rankine-Hugoniot condition) and an entropy dissipation condition that can be derived from the entropy inequalities. A Rankine-Hugoniot shock is said "Kruzhkov" if it satisfies the entropy dissipation condition, and "non-Kruzhkov" otherwise.

a space-dependent discontinuous in space flux function[31] conducted by several authors (see in particular Gimse and Risebro [88], Towers [150], Karlsen, Risebro, Towers [104], Audusse and Perthame [13], Adimurthi, Mishra and Gowda [2], Diehl [67] and the references given in these works and in [112]).

Non-local diffusion operators[32] in convection-diffusion problems, of which fractional powers of the laplacian[33] are the prototype, gained attention thanks to their use in modelling of phenomena such as gas detonation. A theory of these operators (called Lévy operators) was developed since a long time as a part of the theory of stochastic processes. PDE approaches to such problems are more recent; in particular, the definition by Alibaud [4] of entropy solutions of the fractional Burgers equation[34] and analysis of shock creation by Alibaud, Droniou and Vovelle [5] made it clear that entropy methods should be used on such problems, at least in the convection-dominated case[35].

Theory of Leray-Lions operators[36] as exposed by Lions [116] is a very classical one. Applications in modelling of electrorheological fluids[37] (Růžička [144]), then applications to other non-Newtonian fluids and to image restoration problems[38] came along with an intense revival of interest to the so-called variable exponent diffusion problems. The setting uses a generalization of p-laplacian operators: namely, the exponent p is allowed to vary (see Zhikov [160] for a pioneering work on such problems). Functional-analytic framework[39], variational aspects of the problem, regularity of energy minimizers were analyzed in 1990ies and 2000nd; a number of references in these and other directions can be found in the survey paper [68]. Some of the aspects of the PDE theory, such as entropy and renormalized solutions, were explored starting from the mid 2000nd.

Finite volume numerical approximation[40] of convection and diffusion operators is a long-

[31] The model case of conservation law with discontinuous flux is:

$$u_t + \operatorname{div} \mathfrak{f}(x;u) = 0 \;\text{ with }\; \mathfrak{f}(x;\cdot) = f^l(\cdot)1\!\!1_{[x<0]} + f^r(\cdot)1\!\!1_{[x>0]}.$$

[32] Nonlocal (Lévy) diffusion operators arise from stochastic modelling, using Lévy jump processes in the place of the Brownian motion. These operators can be expressed under the form of integral operators with singular kernel, $(\mathcal{L}_\pi[u])(x) = -v.p. \int_{\mathbb{R}^N} (u(x+z) - u(x))\, d\pi(z)$ where $d\pi$ is an ad hoc measure. The case $d\pi(z) = C_{N,\lambda} \frac{dz}{|z|^{N+\lambda}}$ corresponds to the fractional laplacian $(-\Delta u)^{\lambda/2}$, $0 < \lambda < 2$.

[33] Fractional powers of the laplacian can be defined either in terms of Fourier transform, or (in more generality) by the Lévy-Khintchine integral formula, given in the previous footnote.

[34] Fractional Burgers equation: $\partial_t u + \partial_x (u^2/2) + (-\Delta)^{\lambda/2}[u] = 0$. In arbitrary dimension and for general convection flux $\mathfrak{f}$, the analogous equation is called fractional (or fractal) conservation law.

[35] In the context of the fractional conservation laws, one distinguishes the diffusion-dominated ($1 < \lambda < 2$), the critical ($\lambda = 1$) and the convection-dominated ($0 < \lambda < 1$) cases.

[36] See previous footnotes, including the example of p-laplacian

[37] In electrorheological fluids, physical properties of the flow change according to the strength of the surrounding electromagnetic field. The $p(x)$-laplacian $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a basic ingredient of the electrorheological models.

[38] In image restoration, one important direction is to find a compromise between edge preservation (that can be achieved using 1-laplacian kind operators: total variation flow, mean curvature flow...) and image denoising (achieved e.g. with the classical laplacian diffusion). Variable exponent $p(x)$-laplacian offers such a compromise.

[39] Lebesgue and Sobolev variable exponent spaces $L^{p(x)}$, $W^{1,p(x)}$.

[40] Finite volume approximation of a diffusion operator $-\operatorname{div} \mathcal{F}$ consists in "integrating" the divergence over each part K of a partition of the space domain; the Green-Gauss formula then yields $-\int_K \operatorname{div} \mathcal{F} = -\int_{\partial K} \mathcal{F} \cdot n$, where the normal flux of $\mathcal{F}$ through the boundary of K is then reconstructed from the degrees of freedom

standing subject; a systematic approach to mathematical analysis of these schemes culminated in the monograph [76] of Eymard, Gallouët and Herbin, written in late 1990ies. Since then, discretization of anisotropic diffusion operators on general 2D, and then 3D meshes[41] has been a center of interest of a wide community of numerical analysts; some references can be found in the benchmark papers summarized by Herbin and Hubert in [92]. The difficulties of approximation of these problems are the same as the difficulties that appear while nonlinear diffusion problems of the p-laplacian kind are approximated. One solution was proposed independently by Hermeline [94] and by Domelevo and Omnès [70]; it later assumed the somewhat pretentious name of Discrete Duality Finite Volumes[42]. For simpler handling of sequences of approximate solutions produced by numerical schemes, different versions of compactness, embedding, passage-to-the-limit arguments were developed by Gallouët et al. (see, e.g., [76, 77, 86]).

Finite volume methodology was successfully used on several practical problems, of which the porous medium problems, related to petroleum engineering and to hydrogeology, were of prime importance (see, e.g., Eymard, Gallouët, Herbin, Michel [79]). Applications to doubly degenerate convection-diffusion problems of the kind appearing in sedimentation, to cardiac electric activity simulation, and to population dynamics systems were natural, in view of the divergence structure of these problems.

In conclusion of this state-of-the-art section, let me mention the outstanding development of the theory of hyperbolic systems of conservation laws which took place in the 1990ies in the school of Trieste around A. Bressan (see [37]). Although I have never published any work in this direction, I followed this “revolution” with much fascination; and the beauty of these mathematics provided a strong inspiration for many years of my own work.

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of the method. Finite volume methods are related both to the finite differences and (non-conformal) finite elements, but possess a mathematical machinery of their own. They are quite popular in approximation of hyperbolic conservation laws, of industrial convection-diffusion problems (petroleum engineering, sedimentation, hydrogeology), and of population dynamics problems.

[41] Discretization of isotropic diffusion operators on so-called orthogonal meshes is a much easier task: roughly speaking, the flux reconstruction is straightforward in this case, and the stencil of the resulting numerical scheme is much reduced.

[42] DDFV (Discrete Duality Finite Volumes) possess a kind of exact integration-by-parts formula, which allows for preserving the structure of diffusion operators in the discretization step. This feature greatly simplifies the mathematical analysis of the resulting scheme; but, as a matter of fact, different forms of exact or approximate Discrete Duality property are fulfilled for many of known finite volume schemes. One recent example is the quite general concept of Gradient Schemes, of which the DDFV schemes are an example.

Outline

For the readers' convenience, different chapters of the manuscript correspond to a classification "per research topic", though several papers just cannot be assigned to only one category. Such classification is somewhat artificial, and lacks to stress interactions between the ideas and techniques used in these different contexts. For instance, a large part of my work was devoted to taking into account boundary or interface conditions in nonlinear convection-diffusion problems, including pure conservation laws. The works on non-local diffusion-convection equations can be situated somewhere in between conservation laws and parabolic equations. To give another example, finite volume methods were used in many papers, not only for the sake of numerical analysis itself but also for the sake of proving existence of solutions; and the methods for proving convergence of numerical approximations parallel the methods used to prove structural stability of the approximated problem. Despite the difficulties of classification, I have to provide one; hope that cross-comments could help the reader to perceive some unity in the works I carried out.

The first and the longest chapter is devoted to different questions relevant to conservation laws. I first describe briefly the works [97, 99^1, 99^2, 03] of my PhD thesis devoted to the Riemann problem for the scalar conservation law and for the p-system of gas dynamics and nonlinear elasticity, both in Langrangian and Eulerian coordinates. In these works, I considered explicit passage to the limit in Dafermos (self-similar) viscosity approximation of the problem, by using elementary but robust methods. In particular, both the elliptic-hyperbolic case in Lagrangian coordinates (related to description of phase transitions) and the case with vacuum in Eulerian coordinates were treated successfully. Further, I describe the results of another work from my PhD, obtained with Philippe Bénilan and Stanislav Nikolaevich Kruzhkov in [00]. It contained several refinements of results of Ph. Bénilan, S.N. Kruzhkov and Evgueni Panov on scalar conservation laws with infinite speed of propagation, including a study of uniqueness of solutions decreasing to zero at infinity. Another part of this work exploits monotonicity assumptions on the flux; this part is based upon an extensive use of the nonlinear semigroup theory. Next, the works [08^1, 07^4, ^{6}] with Karima Sbihi on general dissipative boundary conditions for conservation laws are presented. In these works, the boundary conditions are encoded by a maximal monotone graph, that allows to consider Dirichlet, Neumann, Robin, obstacle boundary conditions and their mixtures. This is a singular limit problem, in the sense that the formal boundary condition, meaningful for numerical or viscosity approximations, should be relaxed at the hyperbolic limit. We interpret the celebrated Bardos-LeRoux-Nédélec results on the Dirichlet problem in terms of a projection operator for maximal monotone graphs, and generalize the projection procedure so that to include every dissipative boundary condition. Indeed, to a formal boundary condition (given by a maximal monotone graph on $\mathbb{R}$) there corresponds an effective boundary condition (given by the projected graph). Then we establish well-posedness and stability by perturbation for the definition of entropy solution that includes the effective boundary condition. Next, the works [11^2, 10^8] with Kenneth Hvistendahl Karlsen and Nils Henrik Risebro on scalar conservation laws with discontinuous flux are presented. These works contain a kind of unifying theory for model conservation law with discontinuous in space flux function. We put forward a notion of L^1-dissipative germ, responsible for the coupling at the flux discontinuities, and provide general techniques to exploit the germ structure so that to get existence and uniqueness of the entropy solutions associated with the germ. We manage to encompass in one presentation a number of examples and applications previously treated in the literature; in particular, the case of the standard vanishing viscosity is described in [10^8], in multiple space dimensions. The extension to multi-D of the technique of [11^2] is technical but it follows essentially the same guidelines as for the model 1D case; this is an on-going work. Three new applications of the theory are presented next. In the work [10^5] with Paola Goatin and Nicolas Seguin, we consider a road traffic model with point constraint, corresponding e.g. to a road light. While a notion of entropy solution was previously formulated, we un-cover the germ that underlies the coupling at the location of the road light, and produce a strikingly simple finite volume scheme to approximate solutions. In the works [12^2, 10^7, ^5] with N. Seguin, Frédéric Lagoutière and Takeo Takahashi, we analyze a particle-in-Burgers model which was proposed in a previous work of my co-authors. A first step in the analysis consists in treating the Burgers equation with singular source term $-u\,\delta_0(x)$ located at $x = 0$ (see [12^2]). Using again the concept of L^1-dissipative germ, we describe the specific non-conservative coupling of conservation laws across the interface $\{x = 0\}$, and then across the free boundary located at the particle path. We construct in particular efficient finite volume schemes in order to approximate solutions, first for an auxiliary un-coupled problem obtained from the fixed-point formulation, and then for the fully coupled problem. Using in addition wave-front tracking algorithm to establish uniform BV bounds even in presence of a moving particle, we establish well-posedness of the coupled problem. Finally, in the note [^2] with Clément Cancès we apply the theory to classify admissibility notions for the one-dimensional conservation law known as the Buckley-Leverett equation arising in the porous medium context. According to the configuration of flux and capillary pressure curves, we point out the adequate notion of solution (cf. [101] where particular cases were treated at the price of much heavier calculations; cf. [44]). Let us stress that in the discontinuous flux Buckley-Leverett equation, the form of the capillary pressure curves does affect the coupling of solutions at the interface between different rocks. Finally, I present the paper [10^6] with Nathaël Alibaud dedicated to fractional (or fractal) conservation laws in the convection-dominated case; this means that the non-local diffusion operator, although present in the equation, is of order below 1 so that it is unable to induce the parabolic regularity properties of solutions. We demonstrate that the Alibaud's notion of entropy solution to fractal conservation laws is fully adequate, because we are able to construct non-unique weak solutions.

Chapter 3 is concerned with degenerate parabolic-hyperbolic problems. The paper [04^2] with Fouzia Bouhsiss, as well as a part of the survey work [11^3] with Noureddine Igbida, were concerned with the Neumann problem for convection-diffusion equations (without hyperbolic degeneracy) in a bounded domain, yet they use the entropy methods proper to the hyperbolic framework. Two other works [06^2, 07^3] with Igbida, also commented in the survey [11^3], are devoted to the case of Dirichlet boundary conditions (homogeneous and non-homogeneous); also here, we avoid true hyperbolic degeneracy, but the methods in use allow for such an extension. The on-going work with Mohamed Gazibo and Guy Vallet [^9] is devoted to the Dirichlet boundary condition in the general parabolic-hyperbolic case. Next, paper [10^1] with Mohamed Maliki is devoted to taking into account non-Lipschitz fluxes in unbounded domains for quiasilinear parabolic-hyperbolic equations; they are in relation with the work [00] on conservation laws, and are continued in [^7]. This work studies classes of well-posedness of weak and very weak solutions to elliptic fast diffusion equations; in particular, we establish that bounded solutions exist and are unique, using three different approaches. Namely, we use weighted L^1 spaces with power-like or with exponential weights; and we exploit the L^1_{loc} framework following the founding work of Brézis on quasilinear diffusion equations in $\mathbb{R}^N$.

Finally, paper [10^2] with Mostafa Bendahmane and K.H. Karlsen is concerned with existence and convergence of numerical approximations for doubly nonlinear parabolic-hyperbolic problems in a bounded domain with zero Dirichlet boundary condition: we prove well-posedness and perform numerical analysis by means of a DDFV scheme (see Chapter 5) in a quite general situation, including non-Lipschitz convection flux. The works of Chapter 3 are closely related to those of the next chapter.

Chapter 4 is concerned with parabolic or parabolic-elliptic problems, in various contexts. The chapter opens with a brief description of the unpublished work from my PhD concerned with structural stability for elliptic-parabolic systems involving Leray-Lions operators, with an explicit time dependence in the evolution term. Leray-Lions operators appeared in the works of the preceding chapter ([^{11}, 10^2, 06^2]); and a large part of Chapter 5 is devoted to their numerical approximation. The work [09] with M. Bendahmane, K.H. Karlsen and Stanislas Ouaro is a theoretical counterpart of the work [10^2] which, moreover, explores the triply nonlinear elliptic-parabolic-hyperbolic framework and addresses the question of structural stability. In the sequel, I placed a work with K. Sbihi and Petra Wittbold that describes the action of an inhomogeneous absorption term in a Leray-Lions kind parabolic equation, giving a notion of solution and proving well-posedness. The key contribution here is a functional framework: being unable to exploit parabolic capacity theories, we manage to use, pointwise in time, the elliptic capacity related to the Sobolev space $W_0^{1,p}$. Then I describe the works [10^3, 10^4] with M. Bendahmane and S. Ouaro devoted to structural stability of $p(x)$-laplacian kind problems and to well-posedness of the even more exotic $p(u)$-laplacian equation. Some coupled systems involving $p(v)$-laplacian of u are also discussed, under strong regularity assumptions on the component v of the solution (u, v). In these works, we exploit the notion of renormalized solution and the Young measures techniques in order to overcome the difficulties of "moving functional spaces": indeed, different solutions u_n are naturally estimated in different Sobolev spaces $W^{1,p_n(\cdot)}$. Next, the note [10^9] with N. Alibaud and M. Bendahmane is presented, that casts a basis of a theory of renormalized solutions for non-local Lévy diffusion operators. The main difficulty resided in finding a pertinent generalization of the objects and hints used in the theory of renormalized solutions of local elliptic problems; we benefited from the experience of many previous works on this subject in order to provide a rather short existence and uniqueness proof. The ideas carry on to the parabolic framework, see [^{8}]. Finally, the work [12^4] with P. Wittbold on convergence of approximations for elliptic-parabolic problems "without structure condition" is presented; here, the difficulty is the lack of time compactness estimates. We somehow enforce time compactness by exhibiting a relation of general approximations to approximations by monotone sequences of solutions that were already well understood by K. Ammar and P. Wittbold and by A. Zimmermann.

Chapter 5 accounts on the works which main objective was the development and mathematical analysis of finite volume methods for nonlinear (Leray-Lions kind) and then for linear anisotropic diffusion operators. I've got interested in finite volume approximation of the p-laplacian starting from the collaboration with Michaël Gutnic and P. Wittbold that was a part of my PhD. In [04^1] (see also [01]), we have developed a new approach to convergence of finite volume schemes that uses a "continuous" formulation of the discrete equations obtained by means of lifting operators. We applied this approach to proving convergence of approximations for the elliptic-parabolic p-laplacian equation, and proposed a variant of co-volume scheme for practical use. Later, in a series of joint works with Franck Boyer and Florence Hubert, we explored finite volume approximation of the p-laplacian on cartesian meshes. In [04^3], we described a family of structure-preserving schemes with partial gradient reconstruction; we proved convergence and established the basic error estimates. The three subsequent works explored, in two complementary directions, the optimality of error estimation techniques in the case of uniform cartesian meshes. In [05[1]] (see also [07[1]] for the non-uniform mesh case), we used a Besov space approach (which corresponds to the minimal regularity of solutions) and in [06[1]] we used higher-order consistency properties generated by symmetries and by the elevated regularity of the solutions considered. Then, in [07[2]] (see also [05[2]]) we demonstrated that the full gradient reconstruction in the "Discrete Duality" finite volume (DDFV) schemes offers great facilities for approximation of Leray-Lions kind problems; we obtained the basic error estimates for DDFV schemes on quite general 2D meshes, established some key consistency and compactness lemmas, and treated in detail non-homogeneous Dirichlet boundary conditions. In the note [08[3]], together with M. Bendahmane and K.H. Karlsen we presented a 2D "magical formula" for gradient reconstruction which implies discrete duality for the so-called co-volume schemes. As a consequence, we get the 3D generalization of 2D DDFV schemes that we used for the work [10[2]] with the same co-authors (the consistency of this scheme was reduced to consistency of a 2D co-volume scheme). Pursuing the analysis of this 3D scheme in collaborations with M. Bendahmane, F. Hubert, K.H. Karlsen, Stella Krell, and Charles Pierre, we produced a detailed description and numerical tests [12[1]] of the 3D DDFV scheme (see also [11[7]], for a brief account); an application [11[4]] to cardiac bidomain simulation (see Chapter 6); and the paper [[4]] that summarizes the discrete functional analysis tools and their use for the elliptic-parabolic p-laplacian problem of the early works [04[1], 01]. Using finite volume discretizations on the aforementioned elliptic-parabolic problems and on some of the problems discussed in Chapter 6, I was led to summarize in the note [11[6]] the different time compactness techniques for their finite volume discretization of parabolic and degenerate parabolic PDEs. The note presents a lemma due to Kruzhkov used in [[11], 04[1]], a refinement of the variational translation techniques of Alt and Luckhaus, and the monotonicity hints of [12[4]].

Chapter 6 is devoted to miscellaneous works inspired by more concrete applications (notice that the works [10[5]], [[2]] and [12[2], 10[7], [5]] described in Chapter 2 also fall within this class); most of the works contain a large part devoted to numerical approximation. In the works [[1]] and [12[3]] with Robert Eymard, Mustapha Ghilani and Nouzha Marhraoui, we study the singular limit of the two-phase flow in porous media, in absence of gravity and under special assumptions on the source term, for the case where one of the two phases (air) becomes infinitely mobile with respect to the other phase (water). Using a carefully designed finite volume scheme which benefits from a kind of global flux formulation, we prove existence of solutions that possess estimates robust with respect to the air mobility parameter μ. At the limit $\mu \to \infty$, we un-cover a one-phase equation that coincides with the classical Richards model at least in the cas source terms are absent (see [12[3]]). In [[1]], we provide numerical illustrations for different μ and a comparison to the Richards equation. Next, in the work [11[4]] with M. Bendahmane, K.H. Karlsen and Ch. Pierre, already mentioned in Chapter 5, we analyse a simplified bidomain model of cardiac electric activity. We first provide a variational formulation that appears to be somewhat new, and then describe a 3D DDFV space discretization strategy, prove the convergence of the associated scheme, and provide numerical examples. The three last works are devoted to reaction-diffusion systems. In the work [11[1]] with M. Bendahmane and Ricardo Ruiz Baier, we singled out a class of 2×2 cross-diffusion systems that can be treated using the traditional estimates-and-compactness approach. Yet the problem does not fall within the standard variational framework (in the sense that the solutions are not admissible as test functions in the formulation), and the compactness arguments should be carefully chosen. We establish existence and then reproduce the arguments at the discrete level, proving convergence of finite volume approximations. In passing, we were led to establish some lemmas of general interest for two-point finite volume approximations (namely, optimal Sobolev embeddings for the case of Neumann boundary conditions and a discrete result mimicking the Kruzhkov time compactness lemma). We also discuss and illustrate the instabilities of the system and make a comparison of solutions of cross-diffusion and of self-diffusion systems. In the work [11[5]] with M. Bendahmane and Mazen Saad, we study finite volume discretizations for a version of the Keller-Segel model for chemoattraction, in the degenerate parabolic case where the overcrowding is prevented. We prove convergence of the scheme which uses an upwind (or directional splitting) discretization of the convection term, and provide numerical simulations showing the behaviour expected from the model.

Finally, in the work [11[9]] with Halima Labani, we obtain attractor type L^∞ estimates for a class of reaction-diffusion systems motivated by a concrete example (3×3 and 5×5 systems describing blood oxigenation). This work uses the classical tools of linear semigroup theory, more recent maximal regularity and L^p techniques, and a hint of preconditioning that allows for treating different boundary conditions on different components of the solution.

The concluding chapter contains some research perspectives for my future work.

Presentation of the Chapters

Each Chapter is constructed along the same guidelines. Sections or subsections account for each of the different works.

For each work or series of works, I give a short presentation of the questions under study and describe the results we were aimed at, pointing out the interest of the questions and the difficulties related to them. When it is possible to give some background without entering into too technical details, I do so. Then I attempt to describe the main ideas that underlie the publications. I describe or state informally the most important of the obtained results, skipping technical assumptions or going to prototype cases. In conclusion, further open questions related to each work may be presented.

The full statements and proofs can be inferred from the original publications, accessible through the HAL preprint server of CNRS *http://hal.archives-ouvertes.fr/* or through my webpage on *http://lmb.univ-fcomte.fr/*.

Chapter 2

Conservation laws

2.1 Dafermos viscosity and the Riemann problem

According to the modern theory of hyperbolic conservation laws in one space dimension, construction of solutions (both practical: numerical methods, and theoretical: Glimm scheme, wave-front tracking) requires the building blocks that are solutions of the so-called Riemann problem (the Cauchy problem with a simple-jump initial function). Solutions are self-similar, i.e., they depend on the ratio x/t, and it is expected that they be obtained as limits of "viscosity regularized" parabolic problems. For systems, physical viscosity is not necessarily present in all the equations of the system; this makes the regularized problem degenerate parabolic. Viscosity with specific scaling, used by C. Dafermos in the 1970ies, allow to approximate self-similar solutions by self-similar ones, and thus reduce the PDE problems to ODEs.

As usual, concrete systems modelling some real phenomena arise more interest; and they permit to use very specific methods relying on some particular structure of the equations.

On the other hand, the topic of conservation laws with non-Lipschitz (merely continuous) flux function was quite active at the beginning of my PhD; and we were interested, with my advisor S.N. Kruzhkov, in finding analysis methods robust enough to be used on as general flux functions as possible.

Therefore I have studied the viscous approximation of the Riemann problem with merely continuous flux functions: firstly for the scalar conservation law in one space dimension; then for the p-system (isentropic gas dynamics, nonlinear elasticity) eventually allowing for elliptic zones (phase transitions); and finally, for the isentropic gas dynamics in Eulerian coordinates (allowing for vacuum creation from non-vacuum data).

The scalar conservation law

In the case of a scalar conservation law $\partial_t u + \partial_x f(u) = 0$ with Riemann initial datum $u_0(x) = u^l 1\!\!1_{[x<0]} + u^r 1\!\!1_{[x>0]}$, it is well known that the solution should be constructed by inverting (it the graph sense) the derivative of the convex envelope F of f on the interval $[u^l, u^r]$ (one takes the concave envelope if $u^l > u^r$). Indeed, e.g. in the case f is convex, function $u(t,x) = U(x/t)$ with $-\xi U'(\xi) + f'(U(\xi))U'(\xi) = 0$ provides a solution; thus finding $u(\cdot,\cdot)$ amounts to solving the equation $f'(U(\xi)) = \xi$ for $U(\cdot)$.

The goal of the work [97, 99[1]] was to obtain the formula by using the approximation by "self-similar" or "Dafermos" viscosity:

$$\partial_t u + \partial_x f(u) = \varepsilon t \partial_{xx} u.$$

The equation allows to look for viscosity profiles U^ε that only depend on $\xi = x/t$. Then the idea of the work is to "enforce" the structure of the limit $U = [F']^{-1}$, where F is a convex or concave envelope of f. Therefore, we use this form as a pattern for U_ε, fulfill the inverse transformation, and study F_ε such that $U_\varepsilon = [F'_\varepsilon]^{-1}$. We deduce a second-order equation on F_ε that satisfies a maximum principle; then, using elementary methods, we show that concave or convex functions F_ε converge to the limit F, along with the derivatives, as $\varepsilon \to 0$. Consequently,

Viscosity regularized Riemann problem $\partial_t u + \partial_x f(u) = \varepsilon \partial_{xx} u$, $u_0(x) = u^l 1\!\!1_{[x<0]} + u^r 1\!\!1_{[x>0]}$ admits a unique self-similar solution $u(t,x) = U(x/t)$ that converges to the unique admissible solution of the Riemann problem for the conservation law as $\varepsilon \downarrow 0$.

The method works for a general, merely continuous, flux function f.

The p-system of isentropic gas dynamics

For general systems of conservation laws, works of A. Tzavaras and al. on the approximation by Dafermos viscosity led to quite general results on the Riemann problem. The key point of these works is to establish existence of heteroclinic orbits for the ODE system describing the viscosity profiles; general tools of dynamical systems are in use.

But, for some particular systems, more information can be inferred from using more explicit methods for finding heteroclinic orbits. Specifically, for the Riemann problem for the so-called p-system with "physical" viscosity

$$(VP_{syst}) \qquad \left\{ \begin{array}{l} u_t - v_x = 0 \\ v_t - f(u)_x = \varepsilon t v_{xx}, \end{array} \right. \qquad (u,v)|_{t=0} = \left\{ \begin{array}{ll} (u_-, v_-), & x<0 \\ (u_+, v_+), & x>0, \end{array} \right.$$

for $\varepsilon = 0$ a formula for solution can be given, somewhat similar to the formula $U = [F']^{-1}$ of the scalar case (Leibovich, [114]). Mimicking the approach of my previous works [97, 99^1], in [99^2] I derived for the p-system the result completely analogous to the one of the scalar case:

Under the mere assumption of continuity and strict monotonicity of the nonlinearity $f(\cdot)$, the Riemann problem for the viscosity regularized p-system (VP_{syst}) admits a unique self-similar solution $(u(t,x), v(t,x)) = (U(x/t), V(x/t))$ that converges to the unique admissible solution of the Riemann problem for the p-system as $\varepsilon \downarrow 0$.

Somewhat later I discovered that the problem can contain an additional challenge: f in the p-system need not be increasing, it may contain a zone with reversed monotonicity (the corresponding models include phase transitions, e.g. for the Van der Waals gazes and for nonlinear elasticity models). Adaptation of the method of [99^2] to the case with phase transitions was straightforward (see [Th]):

The previous result holds without the monotonicity assumption on f

(we need some assumption to ensure existence for each data, e.g., $f(\pm\infty) = \pm\infty$ is enough).

Isentropic gas dynamics with vacuum in Eulerian coordinates

The Lagrangian formulation of isentropic gas dynamics, that leads to the p-system above, is not suitable for description of vacuum zones in the solutions. Yet for the Riemann problem, is is particularly interesting to observe the formation of vacuum states from non-vacuum initial states. Therefore the Euler formulation of the same system should be considered:

$$(VEul) \left\{ \begin{array}{l} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = \varepsilon t u_{xx} \end{array} \right. \quad (\rho, u)|_{t=0} = \left\{ \begin{array}{lll} (\rho_-, u_-), & x < 0 & \rho_\pm > 0, \\ (\rho_+, u_+), & x > 0, & u_\pm \in \mathbb{R}. \end{array} \right.$$

Treating the isentropic gas dynamics in Eulerian coordinates proved possible with the same arsenal of methods. The key point is, once more, to obtain a solution pattern (derived from the known "lagrangian case" formula) and to exploit this pattern as a fit for the solutions of the viscosity regularized system. As previously, the method proved robust enough to treat the very degenerate situation where intermediate vacuum states appear in the solutions. I have obtained that

> *Under the mere assumption of continuity and strict monotonicity of the nonlinearity $f(\cdot)$, the Riemann problem for the viscosity regularized Eulerian system $(VEul)$ admits a unique self-similar solution $(\rho(t,x), u(t,x)) = (R(x/t), U(x/t))$ that converges to the unique admissible solution of the Riemann problem as $\varepsilon \downarrow 0$. The limit solution may contain vacuum states and it is expressed by an explicit formula involving $f(\cdot)$, $\rho_\pm, u_\pm$.*

2.2 Multi-dimensional conservation laws with merely continuous flux

In general, from the definition of Kruzhkov entropy solutions of conservation laws

$$\partial_t u + \operatorname{div} f(u) = 0 \ \text{ in } (0, +\infty) \times \mathbb{R}^N, \ \ u|_{t=0} = u_0$$

using the doubling of variables device one deduces the so-called Kato inequality

$$\partial_t |u - \hat{u}| + \operatorname{div} q(u, \hat{u}) \leq 0, \ \ |u - \hat{u}||_{t=0} = |u_0 - \hat{u}_0| \ \text{ in } \mathcal{D}'([0, \infty) \times \mathbb{R}^N).$$

When the flux function $\mathfrak{f}$ governing the conservation law is locally Lipschitz continuous, the proof of uniqueness of entropy solutions from the Kato inequality is straightforward, using the idea of finite speed of propagation (see Kruzhkov [108]).

Now, what happens if $\mathfrak{f}'$ is unbounded (or even not defined)? Sufficient uniqueness conditions were given by Bénilan and Kruzhkov, Hil'debrand in early 1970ies; in particular, in the one-dimensional case uniqueness is always true. Then in the beginning of 1990ies, Panov constructed an example of non-uniqueness of L^∞ solutions in two space dimensions. Heuristically, information "comes from infinity" in this example, due to the infinite speed of propagation. Kruzhkov and Panov first formulated an "anisotropic" condition on the moduli of continuity of the directional components f_i of the vector flux function $\mathfrak{f}$, which yields uniqueness:

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon^{N-1}} \prod_{i=1}^{N} \omega_i(\varepsilon) < +\infty, \ \ \text{where } |f_i(z) - f_i(\hat{z})| \leq \omega_i(|z - \hat{z}|).$$

Combined with the Panov's counterexample [109, 110], the condition appears to be a sharp one.

In [26], Bénilan and Kruzhkov established modern techniques for the problem, and showed that the monotonicity of one of the flux components f_i may also be sufficient for uniqueness in the $2D$ case, provided the solutions are in $L^1 \cap L^\infty$. Then the following question was asked: what could be the optimal use of flux monotonicity in this framework ? What can be said on uniqueness of solution "decaying to zero" at infinity ? I participated to this work started by Bénilan and Kruzhkov, which resulted in the paper [00]. The answers we give are the following:

– *decaying to zero entropy solutions are unique if their moduli of continuity at the origin (and not the global ones, on the interval of values of solutions) satisfy the anisotropic conditions of Panov-Kruzhkov-Bénilan type;*
– *for the case of $L^1 \cap L^\infty$ data, uniqueness is ensured whenever $(N-1)$ flux components (with respect to some orthogonal basis in $\mathbb{R}^N$) are monotone strictly increasing.*

The first point was not inexpected and it is mainly technical. An ODE for quantities that look like $\phi(R) = \int_{|x|<R} |u - \hat{u}|$ is derived from Kato inequality with the help of the test functions introduced by Bénilan and Kruzhkov; then a kind of inverse Hölder inequality is used to show that if $\phi(R)$ is non zero then it grows "too quickly" as R goes to infinity (this gives a kind of Liouville principle).

This question was revisited in the more general parabolic-hyperbolic and even purely parabolic setting (see Chapter 3); the new idea is the new choice of test functions, that brings a shorter proof of the basic result (if $\mathfrak{f}$ is $1-\frac{1}{N}$ Hölder continuous, uniqueness holds). Currently, with N. Alibaud we are looking at this question for fractional (non-local) conservation laws.

As to the second point of the above statement, the argument is quite beautiful: one argues by induction in the space dimension N, going back-and-forth between the (evolution) conservation law in k or $k-1$ dimensions and the stationary problem of the kind $u + \operatorname{div} \tilde{\mathfrak{f}}(u) = s$ in k dimensions. The link is provided by the fine machinery of the nonlinear semigroup theory: we prove that non-uniqueness for the evolution problem in $\mathbb{R}^N$ is equivalent to non-uniqueness of the stationary problem in $\mathbb{R}^N$, then on the stationary problem we make the change of variables $v_i = f_i(u)$ with monotone f_i so that to find an evolution problem in $N-1$ dimensions, and so forth. This study requires a deep analysis of the abstract operator on L^1 formally corresponding to the expression $u \mapsto \operatorname{div} \mathfrak{f}(u)$, and of its resolvent.

We show in particular that, although uniqueness is not always achieved, it is generic in the sense that the set of data for which uniqueness fails is a very small subset of all possible data; moreover,

there always exist the maximal and the minimal solutions to the problem,
the "maximal and minimal solution" operators generate nonlinear semigroups of contraction in L^1 and thus yield maximal and minimal solutions of the evolution problem.

The results on maximal and minimal solutions actually concern the more general framework of decaying to zero at infinity solutions (e.g., solutions in $L^\infty \cap L^p$). The idea here is simply to approximate the initial datum u_0 from above and from below by $u_0 \pm \delta$; because the data u_0 and $u_0 + \delta$ are "separated enough", we manage to make work the comparison principle that would not work in general. Then maximal and minimal solutions are created as monotone limits of u^δ as $\delta \to 0^\pm$.

2.3 Dissipative boundary conditions for conservation laws

While scalar conservation laws and systems with the Dirichlet boundary condition (BC, in the sequel) received much attention, almost no theoretical results are available for other BC. Even in the very important for applications zero-flux (Neumann) BC, we are only aware of the result of Bürger, Frid and Karlsen [40] where the special assumption $\mathfrak{f}(0) = 0 = \mathfrak{f}(1)$ ensures that the zero-flux condition is meaningful 'as it stands" for $\partial_t u + \operatorname{div} \mathfrak{f}(u) = 0$. This contrasts with the essential feature known in the Dirichlet case: namely, the "formal" BC (that can be imposed, e.g., for the viscous approximation or for a numerical approximation scheme) must be relaxed to an "effective" BC; see Bardos, LeRoux and Nédélec [15]. In this case approximate solutions develop a boundary layer responsible for the transition from the formal to the effective BC. It was pointed out by Dubois and LeFloch [74] that the Bardos-LeRoux-Nédélec condition can be stated in terms of a graph; in this section, we claim that this is "the good point of view" on the general BC problem.

To treat the Dirichlet BC rigorously, different authors used either the BV framework (thus somewhat regular flux f and initial and boundary data u_0, u^D) of [15] or the weak traces and boundary entropies' framework of Otto [128]. Yet the recent advances of the theory (kinetic solutions of Lions, Perthame and Tadmor [117] and parametrized family of H-measures of Panov [132]) brought a considerable technical simplification for study of boundary-value problems. Namely, Vasseur [154] and then Panov [133] showed that entropy solutions in a domain Q of space-time admit *strong* traces on ∂Q. Thus the BV technique could be used without the restrictive BV regularity ([154]). To be precise, while strong traces of u should not exist in general, "the traces one may need" (those of the normal flux $\mathfrak{f}(u) \cdot \nu$ and of normal entropy fluxes $\mathfrak{q}(u) \cdot \nu$) do exist: this fact is somewhat hidden in the statement the main theorem in [133]; cf. [08[1], [6]].

Further, Karima Sbihi, in a first part of her PhD, adressed the question of general non-linear boundary conditions for elliptic and parabolic problems $\partial_t b(v) + \operatorname{div} \mathfrak{a}(u, \nabla u) = 0$; such conditions are "dissipative" if they take the form $(\beta(u) - \mathfrak{a}(u, \nabla u) \cdot \nu)|_{\partial\Omega} \ni 0$ with some maximal monotone graph β on $\mathbb{R}$. The Dirichlet BC case corresponds to $\beta = \{0\} \times \mathbb{R}$; and the Neumann (zero-flux) case is $\beta = \mathbb{R} \times \{0\}$.

Thus for the second part of Karima Sbihi's PhD, we considered the problem of scalar conservation law with general dissipative boundary condition. The project included a conjecture ("what effective BC should correspond to a formal BC given by a graph β ?"); a proof of uniqueness of the associated entropy solutions using the Panov's strong trace techniques; and an extensive justification of the effective *interpretation* of the formal BC, using different stability arguments (such as the passage to the limit from the viscosity regularized problem or stability by perturbation of β).

In [08[1]] we've announced our conjecture an the first results, later published in the Sbihi's PhD [145]. The key informations of the note [08[1]] are the following:

- *the effective BC graph* $\tilde{\beta}$ *is a* projection *of the formal BC graph* β *(projection that we visualized geometrically and expressed pointwise in a rather awkward way);*
- *the corresponding definition of entropy solution says that the strong trace* γu *of solution* u *satisfies* $\big(\gamma u, \mathfrak{f}(\gamma u) \cdot \nu\big) \in \tilde{\beta}$ *pointwise on* $(0, T) \times \partial\Omega$*, and this readily yields uniqueness;*
- *in the case of a constant in* (t, x) *graph* β *and of a flat boundary, entropy solutions in the above sense are limits of a "natural" approximation procedure.*

The approximation procedure worked under a "quick growth" assumption on β at $\pm\infty$, needed to ensure uniform L^∞ estimates. The procedure included a viscous regularization of the stationary problem $u+\mathrm{div}\,\mathfrak{f}(u)=s$, $\beta(u)-\mathfrak{f}(u)\cdot\nu\ni 0$ in Ω, and a subsequent approximation of the evolution problem by the nonlinear semigroup technique. In the subsequent note [07[4]] with Sbihi we've added one more level of approximation (monotone approximation of β) and, roughly speaking, dropped the growth assumption on β; as an example, we gave a general interpretation of the zero-flux (Neumann) BC thus going beyond the restrictive framework of the Bürger, Frid and Karlsen [40] result. We also added a more interesting definition of solution that does not involve pointwise trace constraints on the boundary, but that uses global (up-to-the-boundary) entropy inequalities with incorporated remainder terms supported on the boundary. This allowed to formulate a corresponding notion of measure-valued (entropy-process) solution.

The main restriction of the existence techniques of [08[1], 07[4]] is that we needed strong compactness of the sequence of traces $(\gamma u^\varepsilon)_\varepsilon$ on the boundary; clearly, compactness argument 'inside the domain" cannot provide this. Therefore we used approximation methods *enforcing strong compactness on the boundary* (whence the assumption of a flat domain and autonomous β, for translation invariance and thus compactness in space; the semigroup techniques, for compactness in time; and monotone approximation arguments, for using monotone convergent sequences).

In the work [[6]] with Sbihi we managed to make several important steps towards understanding the general problem. First, we found two additional points of view on the projection operation $\tilde{}:\ \beta\mapsto\tilde\beta$ (the operation transforming the formal BC graph into the effective BC graph). We can now state it heuristically as follows:

$$Proj(\beta)\qquad \begin{array}{l}\tilde\beta \textit{ is the "closest" to } \beta \textit{ maximal monotone subgraph}\\ \textit{of the graph } \big\{(r,\mathfrak{f}(r)\cdot\nu)\,|\,r\in\mathbb{R}\big\} \textit{ of the function } \mathfrak{f}_\nu=\mathfrak{f}\cdot\nu\\ \textit{that contains all the points of crossing of } \beta \textit{ with the graph of } \mathfrak{f}_\nu.\end{array}$$

The graph $\tilde\beta$ can be seen as a combination of upper envelopes of $\mathfrak{f}_\nu$ (on the subintervals of $\mathbb{R}$ where $\beta\geq\mathfrak{f}_\nu$) and of lower envelopes (on the intervals where $\beta\leq\mathfrak{f}_\nu$).

Second, we found a new definition of entropy solutions (equivalent to the previous ones) in terms of global entropy inequalities, which writes as follows (with $\beta=\beta_{(t,x)}$ a general Carathéodory non-autonomous graph):

$$(EI)\qquad \begin{array}{l}\forall k\in\mathbb{R}\ \ \forall\xi\in\mathcal{D}([0,T)\times\overline{\Omega})^+\\ \displaystyle\int_0^T\!\!\int_\Omega\big(-(u-k)^\pm\xi_t-q^\pm(u,k)\cdot\nabla\xi\big)-\int_\Omega(u_0-k)^\pm\xi(0,\cdot)\\ \displaystyle\qquad\leq\int_0^T\!\!\int_\Omega \mathrm{sign}\,(u-k)^\pm f\,\xi+\iint_\Sigma \min\big\{\,C_k\,,\,\big(\beta_{(t,x)}(k)-\mathfrak{f}_{\nu(x)}(k)\big)^\mp\big\}\,\xi(t,x).\end{array}$$

Here, C_k is a constant that depends on $\|u\|_\infty$ and on k[1]. Let us stress that this definition *directly involves the formal BC graph* β and not the projected graph $\tilde\beta$; it does not involve explicitly the boundary traces of u; and as a matter of fact, it is well adapted to justification of the passage-to-the-limit in the vanishing viscosity method. Thus the different technical restrictions (flat boundary, (t,x)-independent β, growth assumptions) of our previous results with Sbihi are eventually dropped.

[1]truncation by C_k makes the right-hand side be finite, since we extend $\beta_{(t,x)}$ to an $\overline{\mathbb{R}}$-valued graph.

Eventually, we arrived in [[6]] to a well-posedness result for conservation laws with general dissipative boundary condition:

– *There exists a unique entropy solution of the boundary-value problem*
$\partial_t u + \operatorname{div} \mathfrak{f}(u) = s,\ \ u|_{t=0} = u_0,\ \ \mathfrak{f}(u)\cdot\nu|_{\partial\Omega} \in \beta_{(t,x)}(u)$
given by a maximal monotone graph $\beta_{(t,x)}(\cdot)$, in the sense of global entropy inequalities (EI);
– *the formal boundary condition should be expressed - in terms of strong boundary traces - as "$\big(\gamma u, \mathfrak{f}(\gamma u)\cdot\nu\big) \in \tilde\beta_{(t,x)}$ pointwise on $(0,T)\times\partial\Omega$" (this is implicitly contained in (EI)), where the graph $\tilde\beta_{(t,x)}$ is the projection $(Proj(\beta)$ of $\beta_{(t,x)}$ on the graph of $\mathfrak{f}\cdot\nu_x$;*
– *the entropy solution depends continuously on the data and on the graph β, and it can be obtained in several approximation steps from the viscosity regularized problem.*

The result can be used on mixed Dirichlet, Robin, Neumann, obstacle boundary conditions.

The main unsatisfactory point is that we were unable to bypass the flux genuine nonlinearity assumption (except for the 1D case where we adapted to the "viscosity approximation setting" the recent BV_{loc} estimation technique of Bürger, García, Karlsen and Towers [41, 43] conceived for numerical schemes). Indeed, we cannot obtain existence of solutions from a *weakly compact* sequence of approximations. As a matter of fact, in our techniques the notion of measure-valued (entropy-process) solution does appear but it can be exploited only if the existence of an entropy solution is already known (due to some approximation with *strong compactness* properties)[2]. This drawback is also the one of the next section, where interface coupling of conservation laws is studied using similar concepts.

2.4 L^1 theory of conservation laws with discontinuous flux

Think of an incompressible immiscible two-phase flow in a porous medium which is "fractured" (i.e., the medium is a juxtaposition of two rocks with different permeability and capillarity properties). In the hyperbolic regime, in one space dimension, the phenomenon should be described by a discontinuous-flux Buckley-Leverett equation; here, "discontinuous flux" means that $\mathfrak{f}(x;u)$ is a piecewise constant or piecewise regular in x, continuous in u function. Another source of analogous problems were sedimentation models, see, e.g., Diehl [67] and references therein.

These were the first motivations to look at conservation laws with discontinuous flux:

$$\partial_t u + \operatorname{div} \mathfrak{f}(t,x;u) = 0$$

with, in general, a Caratheodory flux function $\mathfrak{f}$. For the time being, the Caratheorory case was only accessible in the case where the problem can be reduced to the standard Kruzhkov case

$$\partial_t b(x;u) + \operatorname{div} \mathfrak{g}(u) = 0$$

by a change of variables; this is the framework of Audusse and Perthame [13] as it was re-interpreted by Panov [135]. In general, one hopes for well-posedness results for BV in (t,x) fluxes $\mathfrak{f}$; in what concerns uniqueness, only the piecewise smooth case was truly investigated.

The multi-dimensional case is interesting because it was poorly studied; together with K.H. Karlsen and N.H. Risebro we'll pursue in [[10]] the adaptation of the below results to the

[2] In other words, we can "compare" an entropy-process solution to an entropy solution, but we were not able to compare two entropy-process solutions.

multi-D setting. We have already illustrated some hints and difficulties of the multi-D case in the work [10[8]]. But let us focus here on the "playground" which is the one-dimensional problem

$$(MP) \quad \partial_t u + \partial_x \mathfrak{f}(x;u) = 0 \ \text{ with } \ \mathfrak{f}(x;\cdot) = f^l(\cdot)1\!\!1_{[x<0]} + f^r(\cdot)1\!\!1_{[x>0]}.$$

This was indeed the playground of an important number of preceding works (let me cite [88, 67, 14, 150, 2]. We claim that in [11[2]], we have uncovered a general structure beyond families of entropy solutions to such equations; subsequent adaptation to the general piecewise regular in (t,x), multidimensional discontinuous flux $\mathfrak{f}(t,x;\cdot)$ is a matter of technique. Several applications that we developed (vanishing viscosity approximation in [10[8]], together with Karslen and Risebro; porous medium with vanishing capillarity effects in [[2]] together with C. Cancès; road traffic model with point constraint in [10[5]], with P. Goatin and N. Seguin; a non-conservative 1D fluid-particle interaction model in [10[7], 12[2], [5]] with N. Seguin and F. Lagoutière, T. Takahashi) show that the ideas of [11[2]] make it easy to investigate and classify different conservation flux models with discontinuous flux.

Here it should be stressed that different consistent notions of solution may co-exist for discontinuous flux problems, see Adimurthi, Mishra and Veerappa Gowda [2] and Bürger et al. [42]. Different semigroups of (entropy) solutions may correspond to different physical dissipation processes taking place at the flux discontinuity locations (we call them "interfaces").

What we show is that the admissibility of (entropy) solutions is fully reduced to the question of coupling of *piecewise constant* solutions across the interface[3].

We put forward the notion of admissibility germ *that is the set $\mathcal{G}$ of couples $(c^l,c^r)\in\mathbb{R}^2$ such that $c(x) := c^l 1\!\!1_{[x<0]} + c^r 1\!\!1_{[x<0]}$ is considered as an admissible solution to the discontinuous flux equation. This includes the Rankine-Hugoniot condition $f^l(c^l) = f^r(c^r)$ and the* interface entropy dissipation *condition:* $\forall (c^l,c^r),(b^l,b^r)\in\mathcal{G} \;\; q^l(c^l,b^l) \geq q^r(c^r,b^r)$,

with $q^{l,r}(u,k) := \operatorname{sign}(u-k)(f^{l,r}(u)-f^{l,r}(k))$ the Kruzhkov entropy fluxes on each side from the interface $\Sigma=\{x=0\}$. Whenever no new couple could be added to $\mathcal{G}$ in such a way that the above interface entropy dissipation inequality hold, we say that the germ is *maximal*; a maximal germ will be denoted by $\mathcal{G}^*$. Whenever any Riemann problem (Cauchy problem with $u_0(x) = u_- 1\!\!1_{[x<0]} + u_+ 1\!\!1_{[x>0]}$) can be solved using Kruzhkov wave fans in $[x<0]$, states (u^l,u^r) in $\mathcal{G}^*$, and Kruzhkov wave fans in $[x>0]$, we say that the maximal germ $\mathcal{G}^*$ is *complete*.

Different properties of germs and relations between them are studied in length in [11[2]] (see also the [10[8], Appendix]); these relations are extensively used to classify the known applications to concrete problems and to develop new ones.

As one application, we construct a counterexample to uniqueness of weak solutions satisfying the well-known entropy condition of Towers et al. [150, 104] in the case where the 'crossing condition" of Karlsen, Risebro and Towers [104] fails.

As another application, we fully describe the "vanishing viscosity germ" (see also the next paragraph) and prove convergence of the viscosity approximation for our model one-dimensional problem; note that its equivalent description was obtained recently by Diehl [67].

[3]In the sequel, I do not attempt to describe in detail the techniques used in [11[2]]; but I concentrate on explaining the key ideas. Readers interested in a more detailed introduction to the theory of [11[2]] may take a look at the short survey note [11[8]] and at the appendix of [10[8]].

A maximal germ $\mathcal{G}^*$ being given, the notion of solution is the following:

A function $u \in L^\infty$ is a $\mathcal{G}^*$-entropy solution of the problem if it is a Kruzhkov solution away from the interface, and if the strong left and right traces $\gamma^l u$, $\gamma^r u$ verify $\big((\gamma^l u)(t),(\gamma^r u)(t)\big) \in \mathcal{G}^*$ pointwise at the interface.

Equivalently, following the idea of [14, 13, 43] the notion of solution can be stated under the form of *adapted entropy inequalities* in which $k \in \mathbb{R}$ is replaced with *a priori* admissible stationary solutions:

Setting $c(x) = c^l 1\!\!1_{[x<0]} + c^r 1\!\!1_{[x<0]}$ with $(c^l,c^r) \in \mathbb{R}^2$,
a function $u \in L^\infty$ is a $\mathcal{G}^*$-entropy solution if it verifies

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^N} |u(t,x)-c(x)|\,\xi_t + \mathfrak{q}(x;u(t,x),c(x))\cdot\nabla\xi\,dxdt \\ -\int_{\mathbb{R}^N}|u_0(x)-c(x)|\,\xi(0,x)\,dx + \int_\Sigma R_{VV}\left(\left(c^l,c^r\right)\right)\xi(\sigma)\,d\sigma \geq 0,$$

with some "remainder function" $R_{VV} : \Sigma\times\mathbb{R}^2 \longrightarrow \mathbb{R}^+$ which, roughly speaking, is "large enough" among functions small enough to satisfy $\forall\ \left(c^l,c^r\right) \in \mathcal{G}^*,\ R((c^l,c^r)) = 0$.

This formulation with incorporated remainder terms is suitable for multi-dimensional setting of [10[8]] and for time-dependent germs used in [10[5]]; a simpler formulation for the model problem does not include general couples $(c^l,c^r) \in \mathbb{R}^2$ but only those in $\mathcal{G}^*$, and thus the remainder term is not needed (cf. the formulation of Carrillo [45] designed to avoid remainder terms).

The key statement of the theory is the following:

For every complete maximal L^1D germ $\mathcal{G}^*$, problem (MP) is well-posed in the framework of $\mathcal{G}^*$-entropy solutions with L^∞ initial data.

With this notion of solution, uniqueness follows directly from the interface entropy dissipation assumption on the germ. To prove existence for complete germs, one uses a monotone finite volume with the Godunov solver at the discontinuity. Technical hypotheses include either the Lipschitz assumption on $f^{l,r}$ in order to exploit the BV_{loc} technique of [41, 43], or the genuine nonlinearity assumption on the fluxes $f^{l,r}$.

Perhaps, the most practically useful part of the work [11[2]] is the investigation of "definite" parts of maximal complete germs; it turns out that in many important situations, a smaller[4] part $\mathcal{G}$ of $\mathcal{G}^*$ encodes the whole germ $\mathcal{G}^*$. We say that $\mathcal{G}$ is a *definite* germ if it admits only one maximal extension (which is $\mathcal{G}^*$).

For instance, in the work [43] of Bürger, Karlsen and Towers and in the works [10[5], [2], [3]] of the author with P. Goatin and N. Seguin and with C. Cancès only one global entropy inequality is needed in the definition of $\mathcal{G}^*$-entropy solution, because $\mathcal{G}$ is a singleton. The benefit for proving convergence of approximation procedures and existence is remarkable. In practice, assume we want to obtain a semigroup $\mathcal{S}$ as limit of semigroups $\mathcal{S}_\varepsilon$ of approximate solutions.

[4]sometimes, a much smaller part: $\mathcal{G}$ can be just a singleton, as this is the case for the Buckley-Leverett equation with discontinuous flux... it can even be the empty set, in one very degenerate situation !

We have the following general principle:

whenever the approximation procedure satisfies the two properties:

- *the semigroups $\mathcal{S}_\varepsilon$ are well defined and they are L^1-contractive;*
- (Id) *each semigroup $\mathcal{S}_\varepsilon$ contains the functions $c(x) = c^l 1\!\!1_{[x<0]} + c^r 1\!\!1_{[x<0]}$ with $(c^l, c^r) \in \mathcal{G}$, where $\mathcal{G}$ is a* definite *germ,*

compactness of the family $(\mathcal{S}_\varepsilon u_0)_\varepsilon$ implies convergence to the $\mathcal{G}^*$-entropy solution $\mathcal{S}u_0$.

E.g., in the work [12[2]] the whole existence proof is based upon convergence of a numerical scheme well-balanced in respect of stationary solutions $c(x) = c^l 1\!\!1_{[x<0]} + c^r 1\!\!1_{[x<0]}$ with $(c^l, c^r) \in \mathcal{G}$ where $\mathcal{G}$ is considerably smaller than $\mathcal{G}^*$.

The ideas of the long work [11[2]] are outlined in some detail in the survey paper [11[8]]. Now we discuss the applications of the method, that somewhat confirmed its usefulness.

2.5 Applications of the theory for discontinuous flux

Let me describe the results obtained as applications of the general theory of [11[2]]. In all cases, we un-cover the germ governing the interface coupling, and arrive to a well-posedness result. The last application (the particle-in-Burgers model) goes beyond the framework of the previous chapter: firstly, the interface coupling there is non-conservative, and secondly, we have to deal with a PDE-ODE system using many additional tools.

2.5.1 Vanishing viscosity limit of multi-dimensional conservation laws with discontinuous flux

In the work [10[8]] with K.H. Karlsen and N.H. Risebro, we have studied the multi-dimensional case with the notion of solution given by the standard (not necessarily physical, this depends on the applicative context) viscous regularization:

an admissible solution u is the a.e. limit, as $\varepsilon \downarrow 0$, of solutions u^ε of
$\partial_t u^\varepsilon + \operatorname{div}(\mathfrak{f}(x, u^\varepsilon)) = \varepsilon \Delta u^\varepsilon, \quad u^\varepsilon|_{t=0} = u_0.$

We consider genuinely nonlinear (non-affine on any interval) fluxes of the form

$$\mathfrak{f} : (x,z) \in \mathbb{R}^N \times \mathbb{R} \mapsto \begin{cases} \mathfrak{f}^l(z) & x \in \Omega^l \\ \mathfrak{f}^r(z) & x \in \Omega^r, \end{cases} \quad \mathfrak{f}^{l,r} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}), \ \left(\mathfrak{f}^{l,r}\right)' \neq 0 \text{ a.e.} \tag{2.1}$$

Here for $\Phi : \mathbb{R}^{N-1} \to \mathbb{R}$ a C^2 function[5], we denote

$$\Omega^l := \mathbb{R}^+ \times \left\{(x_1, x') \in \mathbb{R}^N \ \mid \ x_1 < \Phi(x')\right\},$$
$$\Omega^r := \mathbb{R}^+ \times \left\{(x_1, x') \in \mathbb{R}^N \ \mid \ x_1 > \Phi(x')\right\},$$

and $\Sigma := \overline{\Omega^l} \cap \overline{\Omega^r}$. For $\sigma \in \Sigma$, denote by $\nu(\sigma)$ the unit vector normal to Σ pointing from Ω^l to Ω^r. For $\sigma \in \Sigma$, $f^{l,r}(\sigma; \cdot)$ denotes the normal component $\mathfrak{f}^{l,r}(\cdot) \cdot \nu(\sigma)$ on Σ of $\mathfrak{f}^{l,r}(\cdot)$.

[5] This is a simplifying assumption, more general case will be considered in [[10]]

Our results are:

The above notion of solution (as singular limit of vanishing viscosity approximation) is equivalent to saying that u is a Kruzhkov solution away from Σ, and a.e. on Σ, the strong traces couple $(\gamma^l u, \gamma^r u)(\sigma)$ belongs to the "vanishing viscosity" germ $\mathcal{G}_{VV}(\sigma)$ which is a set of couples defined from the normal flux components $f^{l,r}(\sigma, \cdot)$.

The germ $\mathcal{G}_{VV}$ is described explicitly e.g. by the "Oleinik-like chord condition" ([67]):

$$(u^l, u^r) \in \mathcal{G}_{VV} \quad \text{if } f^l(u^l) = f^r(u^r) \quad \text{and there exists } u^o \in \mathrm{ch}(u^l, u^r) \text{ such that}$$
$$(u^r - u^o)(f^r(z) - f^r(u^r)) \geq 0 \ \forall z \in \mathrm{ch}(u^r, u^o),$$
$$(u^o - u^l)(f^l(z) - f^l(u^l)) \geq 0 \ \forall z \in \mathrm{ch}(u^l, u^o),$$
where for $a, b \in \mathbb{R}$, $ch(a, b)$ denotes the interval $[\min\{a, b\}, \max\{a, b\}]$.

Moreover, $\mathcal{G}_{VV}(\sigma)$ is completely determined by the smaller set $\mathcal{G}^0_{VV}(\sigma)$ of states (u^l, u^r) that can be connected by a viscosity profile across the tangent to Σ hyperplane of $\mathbb{R}^{N+1}$.

The latter notion of solution can therefore be called $\mathcal{G}_{VV}$-entropy solution. Assuming in addition any hypothesis that brings uniform L^∞ estimate on $(u^\varepsilon)_\varepsilon$ (such as the assumption $\mathfrak{f}(0) = 0 = \mathfrak{f}(1)$, for $[0, 1]$-valued data) we get the following claim:

For all L^∞ initial datum there exists a unique $\mathcal{G}_{VV}$-entropy solution.
The solution map is an order-preserving semigroup, L^1 contractive on $L^1 \cap L^\infty$.

Uniqueness, comparison, contraction proof is straightforward from the Kato inequality, plus strong traces. plus the dissipativity of the germ. Existence and convergence of vanishing viscosity approximations is much subtler: we combine a construction of *one-dimensional* viscous profiles in the normal direction to Σ, and the definition of entropy solution with remainder terms. Unless Σ is itself a hyperplane, this argument involves lengthy technicalities; but the idea behind it is just (Id) of the previous section.

2.5.2 Buckley-Leverett equation with discontyinuous flux as a vanishing capillarity limit

In the works Kaasschieter [101] and Cancès [44], the authors particular cases of vanishing capillarity limit for one-dimensional model of flow in porous medium composed of two different rocks. The notion of solution should be as follows:

an admissible solution u is the a.e. limit, as $\varepsilon \downarrow 0$, of solutions u^ε of
$$\partial_t u^\varepsilon + \mathrm{div}\,(\mathfrak{f}(x, u^\varepsilon)) = \varepsilon \partial_x(\lambda(x, u^\varepsilon)\partial_x \pi(x, u^\varepsilon)), \quad u^\varepsilon|_{t=0} = u_0,$$

where $\mathfrak{f}(x, \cdot) = f^l(\cdot)1\!\!1_{[x<0]} + f^r(\cdot)1\!\!1_{[x>0]}$ and $\lambda(x, \cdot)$, $\pi(x, \cdot)$ take an analogous form; the left and right "capillarities" π^l and π^r are strictly increasing functions, and $\lambda^{l,r}$ are strictly positive. The functions $f^{l,r}$ corresponding to Buckley-Leverett model with gravity are "bell-shaped", i.e., $f^l(0) = f^r(0)$, $f^l(1) = f^r(1)$ and each one has a unique local maximum within $(0, 1)$.

When the discontinuity is absent, it is known that the notion on solution is the same as Kruzhkov solution and thus it does not depend on the *form* of the capillarity π.

In the discontinuous flux setting, not only the presence of π "before the limit" counts, but also the form of $\pi^{l,r}$ intervenes at the level of interface coupling.

In the work [2] with C. Cancès we completely classify the notions of solution possibly obtained as vanishing capillarity limits with respect to possible configurations of fluxes $f^{l,r}$ and of capillarities $\pi^{l,r}$. According to the theory of Bürger, Karlsen and Towers [43] and the one of [112],

each notion of solution is determined by a choice of a definite germ $\mathcal{G}_{(A,B)}$
that is a singleton consisting of one "connection" (A,B).

Here, a *connection* is a couple such that $f^l(A) = f^r(B)$ and the Lax "in-going characteristics condition" is violated from both sides from the interface[6]. Then the associated maximal germ $\mathcal{G}^*_{(A,B)}$ consists of (A,B) and of all couples satisfying $f^l(u^l) = f^r(u^r) \leq F$ with $F = f^l(A) = f^r(B)$ and such that the shock has in-going characteristics at least from one side from the interface.

While in the works of Cancès [44] the two extreme cases (the "optimal connection" which maximizes F and the "barrier connection" that minimizes F) were investigated, in [2] we show that

every connection is a vanishing capillarity limit for some $\pi^{l,r}$*;*
given $\pi^{l,r}$*, the associated connection (denoted* (u^l_π, u^r_π)*) is obtained*
by intersecting two monotone curves in $[0,1] \times [0,1]$ *(one decreasing, the other increasing)*;
these two curves can be interpreted as two interface coupling constraints.

Roughly speaking, we need both the Rankine-Hugoniot flux continuity constraint $f^l(u^l_\pi) = f^r(u^r_\pi)$ and the constraint $\pi^l(u^l) = \pi^r(u^r)$ which makes the capillarities connected. In fact, the latter constraint should be carefully relaxed, which corresponds to an interface layer phenomenon.

Then the consequence of the general theory is:

for general datum u_0*, the vanishing capillarity limit* u *of* u^ε *is the* $\mathcal{G}_{(u^l_\pi, u^r_\pi)}$*-entropy solution.*

While this result could have been expected, our point here is that

as soon as we have found the one admissible connection (u^l_π, u^r_π)*,*
no calculation is needed to justify that convergence of u^ε *to the*
$\mathcal{G}_{(u^l_\pi, u^r_\pi)}$*-entropy solution* u *of Buckley-Leverett equation holds* for all initial datum.

Our instrument is merely (Id) of the previous theoretical section. This approach contrasts with the heavy calculations of [101] where, in fact, only a particular case was achieved by a painstaking study of all possible viscous profiles.

Furthermore, using the notion of flux limitation introduced in [55] (see the forthcoming section), in the note [3] with C. Cancès we give a simple practical formula for the Godunov scheme at the interface, which is applied in the context of [2]. Let us point out that the numerical comparison of the hyperbolic Buckley-Leverett problem with interface coupling and of the full parabolic problem with small capillarity shows very close coincidence of numerical solutions (except for a possible boundary layer at the interface developed by the parabolic approximation); at the same time, the Buckley-Leverett discontinuous flux model exhibits a speed-up of factor close to 800 (!) compared to the parabolic one (see [2]).

[6] We say that the Lax condition is violated even if the characteristics is parallel to the interface, making it a contact discontinuity; this allows to treat simultaneously all the possible cases

2.5.3 A road traffic model with point constraint

In the road traffic model introduced by Colombo and Goatin in [55], a standard conservation law $\partial_t u + \partial_x f(u) = 0$ with flux f of the kind $f(u) = u(1-u)$ is supplemented by a formal point constraint "$\gamma f(u) \leq F$" where $\gamma f(u)$ is the trace of the flux at the location $\{x = 0\}$. This may model road lights (in which case we need $F = F(t)$) or different road obstacles (pay tolls, customs, etc.).

The work carried out with P. Goatin and N. Seguin in [10[5]] demonstrates that this problem is very close to the one of the previous paragraph. It is enough to assume that $f^l \equiv f^r$; then the interface coupling is entirely determined by the choice of the level F (namely, we have here a "connection" (u^l_F, u^r_F) with $f(u^{l,r}_F) = F$). We need to incorporate more technical tools because the germ $\mathcal{G}_{(u^l_F, u^r_F)}$ actually depends on time variable t.

Actually, a notion of entropy solution, uniqueness and existence proof (for BV data) were given by Colombo and Goatin in [55]. Our contribution is

– *a "better" notion of entropy solution where the constraint is incorporated by means of an adapted entropy inequality (as in [43]; see also [11[2]]);*
– *L^∞ well-posedness results with simple uniqueness proof; stability wrt $F(\cdot)$;*
– *existence, by proving convergence of a very simple monotone finite volume scheme.*

To achieve the first two points, we have just analyzed the entropy formulation of [55] for constant in time F, and inferred the "connection" (u^l_F, u^r_F) allowed by this formulation. Then the theory of [11[2]] was applied, in its version that allows for time-dependent family of germs (we work with the family of definite germs $\big(\mathcal{G}_{(u^l_{F(t)}, u^r_{F(t)})}\big)_{t>0}$). To this end, we must use adapted entropy inequalities with remainder term that can take the simple form $R(t, (c^l, c^r)) = Const\, \mathrm{dist}((c^l, c^r), (u^l_{F(t)}, u^r_{F(t)}))$.

For the third point,

we start with a monotone consistent two-point finite volume scheme for unconstrained equation $\partial_t u + \partial_x f(u) = 0$ with a numerical flux $g(\cdot, \cdot)$ (Godunov, Rusanov, Lax-Friedrichs,...) and limit this numerical fluxat the interface $\{x = 0\}$ by setting $g_F(u_{-1/2}, u_{1/2}) := \min\{F(t), g(u_{-1/2}, u_{1/2})\}$.

Notice that the scheme is well-balanced in the sense that it preserves stationary solutions $c(x) = c^l 1\!\!1_{[x<0]} + c^r 1\!\!1_{[x>0]}$ with (c^l, c^r) that belong to a large enough part of the maximal germ $\mathcal{G}^*_{(u^l_F, u^r_F)}$. We then deduce discrete adapted entropy inequalities from the discrete contraction, and bring to use the notion of entropy-process solution. Notice that this approach, which is interesting by itself, can be simplified by exploiting the BV_{loc} bounds following the method developed by Bürger, García, Karlsen and Towers [41, 43].

Simulations with this scheme allow to reproduce, e.g., the interesting phenomenon of "green waves" : for a sequence of road lights with a well-chosen time lag, each light is modelled by a time-periodic point constraint with values $F(t) \in \{0, \max f\}$.

2.5.4 A one-dimensional fluid-particle interaction model

The D'Alembert paradox says that, if one neglects completely the viscous effects in a fluid model, the fluid cannot exert a force on a body that moves within. Yet it is sometimes interesting to model these viscous effects only at the interaction location, neglecting them in the remaining fluid. This leads to *drag force* models.

Such a model, in the simple one-dimensional situation, was studied by F. Lagoutière, N. Seguin and T. Takahashi in [112]; it takes the form of a Burgers equation with singular source term:

$$\partial_t u + \partial_x \frac{u^2}{2} = -\lambda(u - h'(t))\delta_0(x - h(t))$$

for the fluid, coupled with the ODE

$$mh''(t) = \lambda(u(t, h(t)) - h'(t))$$

for the point particle of mass m moving along the (unknown) trajectory $x = h(t)$. The linear drag force term can be replaced by the quadratic drag force term $\lambda(u - h'(t))|u - h'(t)|$, which yields a simpler problem.

Clearly, the above setting is a formal one, because the Burgers equation involves a product of distributions as a source term, and the ODE involves a value of u on the trajectory $\{x = h(t)\}$ that is not defined neither. Both difficulties were addressed in [112]. The first one was solved by giving a sense to the non-conservative product using a LeRoux approximation of the "straight particle model" $\partial_t u + \partial_x \frac{u^2}{2} = -\lambda(u - V)\delta_0(x - Vt)$ (V being constant). As a consequence, all possible trace couples (u^l, u^r) on the interface were described; what we show is that

these couples form a maximal L^1-dissipative (but non-conservative) germ $\mathcal{G}_\lambda(V)$,

in a sense completely analogous to the one of [11[2]]. Moreover, the dependence on V is a trivial one: $\mathcal{G}_\lambda(V) = (V, V) + \mathcal{G}_\lambda$ where $\mathcal{G}_\lambda$ corresponds to the "building block" problem

$$(BBP) \quad \partial_t u + \partial_x \frac{u^2}{2} = -\lambda u \delta_0(x)$$

The difficulty of interpreting the ODE governing the particle movement is resolved by writing

$$mh''(t) = \gamma^l(u, \frac{u^2}{2}) \cdot \nu(t) - \gamma^r(u, \frac{u^2}{2}) \cdot \nu(t),$$

where $\gamma^{l,r}$ denote the left- and right-sided traces of the normal to the curve $\{x = h(t)\}$ component of the flux of the Burgers equation. This actually means that the particle is "moved by the defect of flux conservativity" at the location of the particle path.

The contribution of the paper [12[2]] with N. Seguin is solving the problem (BBP) and building a simple finite volume scheme for its approximation. Namely, we show that

– *for all $u_0 \in L^\infty$ there exists a unique $\mathcal{G}_\lambda$-entropy solution of (BBP);*
– *for approximating the solution, one can use a monotone consistent two-point flux $g(\cdot,\cdot)$ with the following modification at the interface:*
$$g_0^-(u_{-1/2}, u_{1/2}) = g(u_{-1/2}, \phi^-(u_{1/2})), \quad g_0^+(u_{1/2}, u_{-1/2}) = g(u_{1/2}, \phi^+(u_{-1/2})),$$
where $\phi^\pm(\cdot)$ are specific mappings such that for all $u^{l,r} \in \mathbb{R}$, $(\phi^-(u^r), u^r)$ and $(u^l, \phi^+(u^l))$ belong to the part $\mathcal{G}_\lambda^0 := \{(c, c - \lambda) \mid c \in \mathbb{R}\}$ of the germ $\mathcal{G}_\lambda$.

The uniqueness part is an application of the theory of [11[2]], based on a verification of the fact that $\mathcal{G}_\lambda$ is indeed a maximal L^1-dissipative germ (NB: the Rankine-Hugoniot condition should now be discarded from the germ definition, because the interface coupling is non-conservative).

Notice that, in the place of a tedious case study we use again the idea (Id) (the germ $\mathcal{G}_\lambda$ being defined from the LeRoux approximation that leads to a contractive semigroup of solutions).

The existence part could have been greatly simplified if we simply used the LeRoux approximation or a finite volume scheme with the Godunov solver on the interface (notice that the Riemann solver described by Lagoutière, Seguin and Takahashi is quite intricate, and we tend to avoid its use within the scheme). Our point was to construct a scheme which could be as simple and flexible as possible[7]. Then we decomposed the germ $\mathcal{G}_\lambda$ into a well-chosen union $\mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda \cup \mathcal{G}^3_\lambda$ and enforced the well-balance property of the scheme wrt to stationary solutions $c(x) = c1\!\!1_{[x<0]} + (c-\lambda)1\!\!1_{[x>0]}$ (i.e., solutions with $(u^l, u^r) \in \mathcal{G}^1_\lambda$). This would have been enough if $\mathcal{G}^1_\lambda$ was a definite germ, unfortunately, this is not the case (but this works for the case of quadratic drag force). Then we manage to show that $\mathcal{G}^2_\lambda$ is preserved not exactly, but asymptotically as the discretization step goes to zero. The part $\mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda$ being definite, from the discrete contraction property we infer enough of adapted entropy inequalities to show that an accumulation point of the sequence of discrete solutions is a $\mathcal{G}_\lambda$-entropy solution. Existence of such accumulation point is ensured by the BV_{loc} technique of [41, 43].

In the survey paper [10⁷] with F. Lqgoutière, N. Seguin and T. Takahashi we have announced the results of the aforementioned paper [12²] and embedded them into the theoretical and numerical study of the coupled particle-in-Burgers problem. Based on the above results,

- *we define entropy solutions for equation* $\partial_t u + \partial_x \frac{u^2}{2} = -\lambda(u - h'(t))\delta_0(x - h(t))$ *with fixed* $W^{1,\infty}$ *particle path* $h(\cdot)$*, and then also for the equation coupled with the ODE* $mh''(t) = \gamma^l(u, \frac{u^2}{2}) \cdot \nu(t) - \gamma^r(u, \frac{u^2}{2}) \cdot \nu(t)$*;*
- *we prove well-posedness for the fixed-path problem decoupled from the ODE*;
- *we deduce existence, for* L^∞ *data, for the coupled problem;*
- *based on the simple well-balanced solver of the auxiliary problem* (BBP)*, we construct a Glimm-type random-choice scheme for the coupled problem and compare it numerically to the analogous scheme that uses the intricate interface Riemann solver of [112].*

Adaptation of the definitions is straightforward, using in particular the adapted entropy inequalities with remainder terms. Existence is obtained by fixed-point argument: we separate $\mathcal{B}$ the "frozen-particle solution" operator for the Burgers equation with prescribed singular source, and $\mathcal{C}$ the "frozen-fluid solution" operator for the ODE with prescribed drag force. Proving the continuity of the composition of the two operators is delicate; again, we use a variant of the idea (Id). Finally, the construction of the new (well-balanced) scheme for the coupled problem uses an accurate combination of particle advancement and random-choice for the conservation law; the singular source is taken into account only by the adaptation of numerical fluxes at the interface, using the maps $\phi^\pm$ from the model problem (BBP).

The work [10⁷] is continued in [⁵] with the same co-authors; the goal is a well-posedness theory for the coupled problem. In [⁵]

we prove the additional BV *estimate*[8] *on problem* (BBP),

using the technique of wave-front tracking approximations. This allows the most careful control of wave interactions at the interface. Such a control is necessary in order to prove

[7] This constraint is due to the fact that problem (BBP) is an intermediate step in theoretical and numerical study of the coupled particle-in-Burgers problem.

[8] this is an up-to-the-interface estimate, contrarily to the BV_{loc} estimate of [41, 43])

that these interactions decrease the total variation (for the constant h' case) or that the decrease can be controlled by the variations of $h'(\cdot)$ when the particle path is not straight.

Then we use either splitting in time, of fixed-point arguments on the coupled problem. We make the change of variable $\bar{x} = x - h(t)$ which makes appear the h-dependent flux $\bar{f} : u \mapsto \frac{u^2}{2} - h'(t)u$, and the BV control permits to use the results of [34, 103] on continuous dependence on the flux for scalar conservation laws. This eventually leads to a first well-posedness for the coupled problem:

there exists a unique solution to the particle-in-Burgers model with BV data.

Note that uniqueness of solutions corresponding to general L^∞ data remains an open question.

2.6 Non-uniqueness of weak solutions for convection-dominated fractal conservation laws

Fractional (or fractal) conservation law is given by

$$\partial_t u + \operatorname{div} \mathfrak{f}(u) + (-\Delta)^{\lambda/2}[u] = 0 \quad \text{in } (0,T) \times \mathbb{R}^N;$$

here $(-\Delta)^{\lambda/2}$ is the fractional Laplace operator of order $\lambda \in (0,2)$, defined e.g. through the Fourier transform. A more general definition uses the Lévy-Khintchine singular integral formula (see, e.g., [4]).

For $\lambda > 1$, it was shown by Droniou, Gallouët and Vovelle [72] that smooth solutions can be constructed using the Duhamel formula (the convection term is treated as source term for the fractional heat equation, $\partial_t u + (-\Delta)^{\lambda/2}[u] = -\operatorname{div} \mathfrak{f}(u)$). Early attempts to define entropy solutions (supposedly needed for the case $\lambda < 1$) tried to use the nonlocal entropy dissipation inequality $\eta(u)(-\Delta)^{\lambda/2}[u] \leq (-\Delta)^{\lambda/2}[\eta(u)]$ for convex functions (entropies) η; as it appears now, in these inequalities too much information is lost[9]. Then Alibaud in [4] invented a notion of entropy solution which looks as follows:

an L^∞ function u is an entropy solution of the fractional conservation law if for all regular entropy-entropy flux couple $(\eta, \mathfrak{q})$, and for all $r > 0$

$$\partial_t \eta(u) + \operatorname{div} \mathfrak{q}(u) - C_{\lambda,N} \int_{|z|>r} \eta'(u(t,x)) \, \frac{u(t,x+z) - u(t,x)}{|z|^{1+\lambda}} \, dz - C_{\lambda,N} \int_{|z|\leq r} \frac{\eta(u)(t,x+z) - \eta(u)(t,x)}{|z|^{1+\lambda}} \, dz \;\leq\; 0.$$

The key point here is the splitting of the singular Lévy-Khintchine integral into a (vanishing, as $r \to 0$) singular part on which entropy dissipation is used, and on the non-singular term for which one retains the information that would be lost if the dissipation inequality was used. Notice that the same idea was already used by Jakubowski and Wittbold [100] in the context of *time-nonlocal* conservation laws with memory terms.

[9] The situation is entirely similar to the one with degenerate parabolic-hyperbolic problems. The breakthrough happened when Carrillo [45] developed a technique that *keeps some information* from the degenerate diffusion operator $-\operatorname{div} \nabla\varphi(u)$ under the form of the measure term $\limsup_{\varepsilon\to 0} \int_{[|\varphi(u)-\varphi(k)|<\varepsilon]} |\nabla\varphi(u)|^2$, information to be exploited in the entropy inequalities. See Chapter 3.

Looking at the Kruzhkov doubling of variables, one easily sees that the above entropy formulation is well adapted to the technique; thus it leads to a Kato inequality and uniqueness (at least, for regular flux $\mathfrak{f}$). Therefore the notion of entropy solution yields a nice well-posedness theory; still, in the case $\lambda > 1$ we know from the results of [72] that entropy and weak solutions must coincide.

The question is then: to which extent is the theory of entropy solutions necessary?

Heuristically, in the convection-dominated fractional equation one "expects" both shock creation from regular data and non-uniqueness of discontinuous solutions (simply because this was the case for the pure convection conservation law). Unlike in the pure convection case, constructing explicitly any non-constant solution seems to be an extremely difficult task; this is due to the presence of two very different terms in the fractal conservation laws (local, propagating along characteristics; and non-local, propagating by convolution with the fractional heat kernel). Qualitative methods should be used instead.

The phenomenon of shock creation was demonstrated by Alibaud, Droniou and Vovelle [5] using a careful analysis of the non-local term along the characteristic curves of the local part of the equation.

Then in our joint work [10^6] with N. Alibaud, we have shown that

> *There is non-uniqueness of weak solutions to the fractional Burgers equation in the convection-dominated case $\lambda < 1$.*

Indeed, in [10^6] we constructed (implicitly) a non-entropy stationary solution of the fractional Burgers equation by the vanishing viscosity approximation with a vanishing singular term:

$$\varepsilon(v_\varepsilon - \partial^2_{xx} v_\varepsilon) + \partial_x \left(\frac{v_\varepsilon^2}{2} \right) + \mathcal{L}_\lambda[v_\varepsilon] = -2\varepsilon\, \partial_x(\delta_0) \quad \text{in } \mathcal{D}'(\mathbb{R}),$$

δ_0 being the Dirac delta. Odd, discontinuous at zero solutions with traces $v_\varepsilon(0^\pm) = \pm 1$ of this problem exist (the proof is an accurate application of Shauder fixed-point argument), they converge to a limit v. We show that the jump discontinuity of the origin persists. By construction, this discontinuity fails to satisfy the Oleinik condition that we have extended to the fractional Burgers framework; and the limit is a weak solution, because the very singular right-hand side $-2\varepsilon\, \partial_x(\delta_0)$ vanishes as $\varepsilon \to 0$. This yields a weak, non-entropy solution v.

The work contains a number of useful technical lemmas related to nonlocal Lévy operators of order $\lambda < 1$ on the space of odd, discontinuous at zero functions. We also developed an alternative proof, based on suitably constructed barrier functions and on the comparison arguments that use "adapted entropy inequalities", in the spirit of Audusse and Perthame [13]. We believe that these arguments may prove useful for a further study of special solutions to fractional conservation laws.

Chapter 3

Degenerate parabolic-hyperbolic problems and boundary conditions

Numerous works exist already on the subject of nonlinear convection-diffusion equations of parabolic-hyperbolic type with Dirichlet boundary conditions; yet both in the Dirichlet and in the Neumann case, optimal results and techniques are not available yet. In the seminal contribution [45] of J. Carrillo (see also Carrillo and Wittbold [46]) several techniques were established for treating scalar nonlinear degenerate parabolic equations. One technique allowed to obtain entropy inequalities inside the domain"; the other one, to use these inequalities together with the Kruzhkov doubling of variables method in order to deduce the "local Kato inequalities"; and there was an ingenious series of arguments that permitted to treat the problem "up to the boundary", in the case of the *homogeneous Dirichlet boundary condition*. General boundary conditions now appear as one of the bottlenecks of the method[1].

This chapter summarizes several related contributions to the subject of well-posedness and approximation of such problems with different boundary conditions. Notice that there is a strong intersection with the next chapter: actually, both chapters treat of different cases of degenerate elliptic-parabolic-hyperbolic problems. Both chapters are a fruit of collaboration with a number of mathematicians from Philippe Bénilan's school: Fouzia Bouhsiss, Noureddine Igbida, Mohamed Maliki, Stanislas Ouaro, Karima Sbihi, Petra Wittbold, as well as with Nathael Alibaud, Mostafa Bendahmane, Kenneth H. Karlsen, and Guy Vallet.

In the present chapter I have collected the problems where the focus and the main technical difficulties are related to proving uniqueness of solutions and to taking into account different boundary conditions, or conditions at infinity.

The main ideas of the first three sections below were summarized in the recent survey paper [11[3]] written with N. Igbida[2]. While we do not always treat the case of hyperbolic degeneracy, we put ourselves in the situation where the notion of entropy solution must be used; and as a matter of fact, one of the main goals was to prepare the ground for studying problems with true hyperbolic degeneracy (this work is in progress, in collaboration with G. Vallet [[9]] and with my Ph.D. student M. Gazibo).

The last chapter is different because it also investigates issues of theoretical numerical

[1]The other bottleneck is, using doubling of variables on diffusion operators with explicit dependence on the space variable, such as the $p(x)$-laplacian discussed in Chapter 4.

[2]The contributions of these sections are particularly technical, and despite some effort of explaination, I suggest that the reader take a look at the longer, but much more self-contained survey [11[3]]

analysis of the problem. We treat a degenerate parabolic problem with hyperbolicity regions and nonlinear Leray-Lions diffusion, but we take the simplest boundary condition. While uniqueness was already well understood, the multi-fold nonlinearity of the equation made it delicate to construct a suitable numerical scheme and prove convergence of numerical approximations. We developed a finite volume scheme and several analysis tools that are described and generalized in Chapter 5 below.

3.1 Scalar nonlinear convection-diffusion equations with Neumann boundary condition

The work [04^2] was a last chapter of the PhD thesis of Fouzia Bouhsiss. I only assisted Fouzia in finishing her work when our PhD advisor Philippe Bénilan passed away.

The motivation was to adapt the uniqueness method of Carrillo [45] to the case of Neumann boundary conditions; in order to avoid the difficult question of interpretation of the boundary condition for hyperbolic conservation laws (see Section 2.3) we looked at the problem governed by the non-degenerate convection-diffusion operator $\operatorname{div} F(v) - \Delta v$; we had to obtain rather delicate *a priori* estimation of solutions related to the degenerate time-evolution term $b(v)_t$. The problem is to study *weak* solutions in a bounded domain for

$$(PbN) \qquad \begin{cases} \partial_t b(v) + \operatorname{div} F(v) - \Delta v = f \\ (F(v) - \nabla v) \cdot \nu|_{\partial\Omega} = 0, \quad b(u)|_{t=0} = b_0 \end{cases}$$

with "finite energy data" b_0, f in the sense used by Alt and Luckhaus [6] and Otto [127]. A growth condition limits the growth of $|F(z)|^2$ at infinity in terms of the functions $zb(z)$ and z^2, moreover, the surjectivity assumption on b is needed: $b(\pm\infty) = \pm\infty$. Further, the assumption that makes pertinent our study of (PbN) is:

F is not Lipschitz nor $1/2$-Hölder continuous.

Indeed, in the case F is Lipschitz the proof of uniqueness of a weak solution of (PbN) is straightforward, taking $H_\alpha(v - \hat{v})$ for the test function ($v, \hat{v}$ being weak solutions, and $H_\alpha(\cdot)$ being the approximation of sign function as used by Carrillo [45]). The 1/2-Hölder case is slightly more technical, and was fully treated by Otto [127] using in particular the original idea of doubling of the time variable. In general only $H_\alpha(v-k)$, $k = const$, can be used; therefore one seeks to exploit the doubling of variables tools of Kruzhkov-Carrillo, with the necessary adaptation to Neumann boundary conditions. Yet, because of this boundary condition a straightforward adaptation runs into a major technical difficulty: namely,

using the doubling of variables method we can easily derive
the up-to-the-boundary Kato inequality and uniqueness in the case the boundary condition $(F(v) - \nabla v) \cdot \nu|_{\partial\Omega} = 0$ *is satisfied in the sense of a* strong *boundary trace.*

In other words, uniqueness and comparison principle clearly hold for "regular enough" (say,

C^1 up to the boundary) weak solutions of (PbN). In [04²] we have found the following remedy:

- *de-symmetrize the test functions* $\xi_n : \Omega \times \Omega \mapsto \mathbb{R}$ *of the Kruzhkov method so that it be zero on* $\Omega \times \partial\Omega$ *(and non-zero on* $\partial\Omega \times \Omega$*, so that* $\xi_n \to \delta_{x=y}$ *as* $n \to \infty$*); then we get the Kato inequality by requiring that* only one solution *be "regular enough"*;
- *deduce the* L^1 *contraction inequality* $\int_\Omega |b(v) - b(\hat{v})|(t) \le \int_\Omega |b^0 - \hat{b}^0| + \int_0^T\int_\Omega |f - \hat{f}|$ *for a general weak solution* v *and for a "regular enough" solution* $\hat{v}$;
- *impose assumptions on the problem that allow to get "regular enough" solutions for an* L^1*-dense set of data*;
- *by density, extend the contraction inequality to the case of two general weak solutions* v, $\hat{v}$.

We were able to carry out this programm by proving L^∞ bounds on solutions with L^∞ data (Moser's technique) and byusing the regularity result of Lieberman [115] on *elliptic* problems:

under the assumption that F *is locally Hölder continuous of some order* $\theta > 0$*,*
the stationary *problem* $b(v) + \operatorname{div} F(v) - \Delta v = f$, $(F(v) - \nabla v) \cdot \nu|_{\partial\Omega} = 0$
with source term $f \in L^\infty$ *admits a "regular enough" solution (namely,* $v \in C^1(\overline{\Omega})$*).*

The conclusion follows using the device of integral solution of the nonlinear semigroup theory (see Bénilan [21] and the book [25])[3]. Indeed, the tool of integral solutions is based precisely on the comparison of a solution to the evolution equation with a solution of the associated stationary problem. Thus, the final argument is the following:

- *by the formal expression* $v \mapsto \operatorname{div} F \circ b^{-1}(v) - \Delta b^{-1}(v)$ *we define a multi-valued operator on a subset of* $C^1(\overline{\Omega})$ *(i.e., on "regular enough" functions) and prove that it is accretive, densely defined, with* m*-accretive closure*;
- *we prove that if* v *is a weak solution of the degenerate parabolic problem, then* $u = b(v)$ *is an integral solution of the abstract evolution equation* $\partial_t u + Au = f$, $u|_{t=0} = b^0$*;*
- *by nonlinear semigroup theory we readily get uniqueness of* $b(v)$*, and* L^1 *contractivity.*

Eventually, we get well-posedness for the problem under study:

if $F \in C^\theta_{loc}(\mathbb{R})$*, for all finite energy data there exists a unique weak solution to problem* (PbN).

Together with N. Igbida and S. Soma, we are currently adapting the method of [04²] to mixed Dirichlet-Neumann boundary conditions for more general elliptic-parabolic problem. Further, with the PhD student M. Gazibo, we exploit analogous ideas on the parabolic-hyperbolic problem, using also the ideas of Bürger, Frid, Karlsen [40]. The method does work in one space dimension. In general, existence of a large set of "regular enough" solutions is a rather strong limitation for the method of [04²]. Yet the idea found further nice applications; see, e.g., Section 4.1.6 below.

3.2 Scalar nonlinear convection-diffusion equations with Dirichlet boundary conditions

Two techniques were developed for scalar degenerate parabolic-hyperbolic equations with Dirichlet boundary condition. The one of Carrillo [45] works on the homogeneous boundary

[3] It has been noticed by Ph. Bénilan and P. Wittbold in [29] that the doubling of the time variable method of Otto [127] can be replaced by the standard tools of the semigroup theory; in [04²] we have used precisely the same idea

condition[4]. A second one, due to Otto [128], is to use weak trace techniques; it was extended to parabolic-hyperbolic problems by Mascia, Porretta, Terracina [122], Michel and Vovelle [123] and Vallet [153].

In papers [06[2]], [07[3]] and in the survey [11[3]], all of them written with N. Igbida, we revisited several aspects of this problem. For the sake of clarity, let us look at the equation

$$\partial_t v + \operatorname{div}\big(F(\varphi(v)) - \mathfrak{a}(\nabla\varphi(v))\big) = s$$

with non-strictly increasing φ and with Dirichlet BC; this is a Stefan-like (not hyperbolic) problem. What is expected is that weak solutions are unique. But, as in the previous section, in order to achieve general results we need to prove that weak solutions satisfy entropy inequalities, and then use the doubling of variables method and arrive to the Kato inequality for a couple of solutions $u, \hat{u}$ (say, with coinciding initial and source data):

$$(KI) \quad \int_0^T\!\!\int_\Omega \big(-|v-\hat{v}|\partial_t\xi \; - \; \operatorname{sign}(w-\hat{w})(F(w)-F(\hat{w}))\cdot\nabla\xi \\ + \operatorname{sign}(w-\hat{w})(\mathfrak{a}(\nabla w)-\mathfrak{a}(\nabla\hat{w}))\cdot\nabla\xi\big) \le 0$$

for a non-negative test function ξ (we'll precise later whether ξ is zero on $\partial\Omega$ or not).

Actually, it is not trivial to get "$|v-\hat{v}|$" as the first term of (KI): the term that appears naturally is "$\operatorname{sign}(w-\hat{w})(v-\hat{v})$" where $w=\varphi(v)$. When $\varphi(\cdot)$ is not strictly increasing, getting to (KI) is a delicate issue (see Carrillo [45] and the work [99] by Igbida and J.M. Urbano). In [06[2]], for the case of the homogeneous Dirichlet problem, our contribution is:

the technique for getting Carrillo's entropy inequalities is simplified and generalized.

The proof is simple but tricky. It permits to bypass all restriction on F besides its continuity. We also show that the regularity of the domain Ω in [45] can be relaxed.

Now, let us stress that for the stationary problem $v + \operatorname{div}(F(\varphi(v)) - \mathfrak{a}(\nabla\varphi(v))) = f$, a simple and elegant technique for getting entropy and Kato inequalities (at least, inside the domain) is to use the test functions of Blanchard and Porretta [31]; these are the functions $H_\alpha(w-\hat{w}+\alpha\pi)$ with smooth π that would approximate $\operatorname{sign}(v-\hat{v})$. The survey [11[3]] also contains several new techniques and results, and among them,

we adapt the idea of Blanchard and Porretta [31] in order to give a simpler and more general proof of entropy and Kato inequalities "inside the domain" for the evolution problem[5].

With this tool in hand, we readily get (KI) for $\xi \in \mathcal{D}((0,T)\times\Omega)$ (i.e., for $\xi = 0$ on the boundary of Ω). Let us stress that the doubling of variables "inside the domain" is relatively simple, but working near the boundary, by the method of [45] or by those of [122, 123, 153], is a very delicate issue. Then our contribution in [07[3]] (see also [11[3]]) is:

– *we manage to* deduce *the up-to-the-boundary Kato inequality* (KI) *from the Kato inequality inside the domain, by using a specially designed sequence* $(\xi_h)_h$ *of functions truncating a neighbourhood of the boundary*;
– *we do so under very mild regularity assumptions on the boundary.*

[4]A partial adaptation to the general case is presented by Ammar, Carrillo and Wittbold [7] and in a series of subsequent works of K. Ammar.

[5]The original technique of Blanchard and Porretta [31] was developed for evolution equations with space-time dependent diffusion operators; in [12], a first step was made towards the adaptation of the idea to the doubling-of-variables method. Our variant can be seen as a simplification of the techniques of [31], it has the weakness of being restricted to the time independent diffusion operators.

The idea is to take $\xi \in D([0,T)\times\Omega)$ in (KI) and to "send ξ to 1"; it is not difficult to make vanish the second term (coming from convection) in (KI), but the last one (coming from diffusion) does present a difficulty. Actually, our technique only works when $\mathfrak{a}$ is linear, indeed, it is based on the idea to *put all the derivatives* on the test function. At the limit "as $\xi \to 1$" in (KI), one expects to drop the last term by writing $\operatorname{sign}(w-\hat{w})(\nabla w - \nabla\hat{w}) = \nabla|w-\hat{w}|$ and by claiming that

"since $W = |w-\hat{w}| \geq 0$ is zero on the boundary, then $\partial_n W$ is non-positive."

To do so rigourously, some amount of extra regularity of W is needed, see [142] of Gagneux and Rouvre. In [07[3]], we circumvent this *formal* argument by constructing the sequence $(\xi_h)_h$ in such a way that $\int_\Omega \nabla W \cdot \nabla \xi_h$ stays non-negative *already for a fixed h*. Functions ξ_h are constructed simply by solving auxiliary Laplace problems in an h-neighbourhood of the boundary[6]. A careful examination of the properties of $(\xi_h)_h$ permits to enlarge the class of domains for which the method works; for instance, we are able to include domains with cracks. More importantly,

by the technique of [07[3]] we treat a general[7] Dirichlet boundary condition w^D on $w = \varphi(v)$.

Finally, let us mention that the above uniqueness techniques can be transferred to the framework of renormalized solutions with mere L^1 data. The original idea, eventually appeared in print in the paper [98] of Igbida, Sbihi and Wittbold, was to *reduce* the issue of uniqueness of renormalized solutions to the L^1 contraction property for weak solutions. Indeed, in the definition of a renormalized solution, one writes down a family of PDEs satisfied by some functions $S_n(\cdot)$ of the solution v; typically, in the case of a Laplacian diffusion we arrive to the weak formulation of some PDE of the form

$$\partial_t S_n(v) + \operatorname{div} F_n(v) - \Delta\varphi_n(v) = s\, S_n'(v) - S_n''(v)|\nabla v|^2,$$

and in addition, the source term of this PDE tends to s as S_n converges in a suitable way to the identity function. Then comparing *weak solutions* $S_n(v)$ and $S_n(\hat{v})$ by a contraction inequality, as $n\to\infty$ we recover the inequality of the kind $\|v-\hat{v}\|_{L^1} \leq \|s-\hat{s}\|_{L^1}$. In [07[3]], using this idea

we extended the above results to the setting of renormalized solutions.

Unfortunately, the method only works if the diffusion is a homogeneous operator (thus $S_n'(v)\mathfrak{a}(\nabla\varphi(v))$ is converted into $\mathfrak{a}(\nabla\varphi_n(v))$ for some new nonlinearity φ_n). To cope with the general case,

in [11[3]] we also proposed a variant of the doubling of variables techniques that can be used on diffusion operators of the form $\operatorname{div}[\,k(v)\mathfrak{a}(\nabla\varphi(v))\,]$*; consequently, uniqueness of renormalized solutions becomes a byproduct of weak solutions' uniqueness results.*

As it has been said in the introduction to the chapter, we've treated the "not truly degenerate" case where weak solutions are unique, but entropy methods are necessary to prove it. Combining the ideas of [07[3]] (for the diffusion terms) and of Section 2.3 (for the convection terms), in the forthcoming joint work [[9]] with M. Gazibo and G. Vallet we'll treat general Dirichlet BC for the "truly degenerate" parabolic-hyperbolic problem.

[6]A related construction was the essential ingredient of the work Mascia, Porretta, Terracina [122]

[7]in comparison with the works [122, 153], we do not need that Δw^D be a measure

3.3 Scalar nonlinear convection-diffusion and fast diffusion problems in the whole space

The published work [10[1]] and the forthcoming preprint [[7]] written with M. Maliki originate from the question of "optimal" conditions for uniqueness of entropy solutions of the problem

$$(Pb_{\mathbb{R}^N})\ \ \partial_t v + \operatorname{div} F(v) - \Delta\varphi(v) = s \quad \text{in the whole space,} \quad v|_{t=0} = v_0$$

recall that entropy solutions, in general, lie in L^∞. In [118], Maliki and Touré extended the Bénilan-Kruzhkov technique discussed in Section 2.2 to this parabolic-hyperbolic case, proving that

uniqueness of entropy solutions of problem $(Pb_{\mathbb{R}^N})$ *holds true under the condition*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \prod_{i=1}^{N} \left[\omega_{F_i}(\varepsilon) + \sqrt{\varepsilon\omega_\varphi(\varepsilon)}\right] = 0$$

where $|F_i(z) - F_i(\hat z)| \le \omega_{F_i}(|z-\hat z|), \quad |\varphi(z)-\varphi(\hat z)| \le \omega_\varphi(|z-\hat z|)$.

The above condition is anisotropic; a sufficient isotropic condition is then, $F \in C^{1-\frac{1}{N}}$, $\varphi \in C^{1-\frac{2}{N}}$. The cases $N = 1$ and $N = 2$ are special, e.g., for $N = 1$ mere continuity of F, φ is enough for uniqueness of entropy solutions.

It is known (see Section 2.2) that the above anisotropic condition is, in a sense, optimal in the case $\varphi \equiv 0$. With M. Maliki, we've looked at the case $F \equiv 0$ in order to understand how optimal the condition is wrt regularity of φ. Although the "porous medium/fast diffusion" stationary problem

$$(Pb^{diff}_{\mathbb{R}^N})\ \ v - \Delta\varphi(v) = f \quad \text{in the whole space,} \quad v|_{t=0} = v_0$$

is extremely well studied, we have not found a ready-to-use result for weak L^∞ solutions, and started the investigation of the problem in several directions. The answer was that any regularity condition on φ is superfluous:

L^∞ distributional *solutions of* $(Pb^{diff}_{\mathbb{R}^N})$ *are unique*
for a merely continuous non-decreasing nonlinearity φ,

and we actually proved this in three different ways. Indeed, we have investigated $(Pb^{diff}_{\mathbb{R}^N})$ in the following three settings:

- L^1_{loc} solutions with L^1_{loc} data, for a uniformly continuous φ satisfying the so-called Keller-Osserman condition;
- solutions in weighted Lebesgue spaces $L^1(\rho)$ with exponentially decaying weights ρ and uniformly continuous φ;
- solutions in the weighted Lebesgue spaces with the specific weights $\rho_R(x) = \frac{1}{\max\{R,|x|\}^{N-2}}$, $N \ge 3$ and uniformly continuous φ.

Our point is:

every of the above three settings yields a well-posedness class for problem $(Pb^{diff}_{\mathbb{R}^N})$.

In any of the three cases, considering L^∞ solutions we may assume that φ is uniformly continuous on $\mathbb{R}$, which answers the question we were asking. More generally, for L^1_{loc} or $L^1(\rho)$ solutions we must have φ *uniformly continuous* on $\mathbb{R}$ and therefore, φ is at least sublinear. This means that we work in the "fast diffusion" framework ($\varphi(z) = |z|^{m-1}z$ with $0 < m \leq 1$ is the prototype of fast diffusions).

Actually, only the technique using the weights ρ_R (truncated *fundamental solutions* of the Laplacian !) proved to be robust enough to incorporate the convection term $\operatorname{div} F(v)$; and not surprisingly, we have a restriction on the regularity of F. The result of [10¹] reads,

entropy (thus L^∞) solutions of the evolution convection-diffusion problem $(Pb_{\mathbb{R}^N})$ are unique for a merely continuous non-decreasing φ and $1 - \frac{1}{N}$-Hölder continuous F, for $N \geq 3$

(the cases $N = 1$ and $N = 2$ are already known from the work [118]).

> *this gives a new (rather simple) proof of uniqueness for scalar conservation laws under the well-known* isotropic *Hölder continuity condition on the flux F.*

Let us briefly present the ideas of the work [7]. Our main tools are: moduli of continuity of φ; the Kato inequality for $(Pb^{diff}_{\mathbb{R}^N})$:

$$\int (v - \hat{v})^+ \xi \leq \int \operatorname{sign}^+(v - \hat{v})(f - \hat{f})\xi + \int |\varphi(v) - \varphi(\hat{v})| \Delta\xi \quad \text{with } \xi \geq 0,\ \xi \in \mathcal{D}(\mathbb{R}^N);$$

and a careful choice of test functions to make ξ converge to 1 in the Kato inequality.

We start by establishing the Kato inequality in much generality (non-autonomous φ, possibly mere distributional solutions, etc.) using different hints such as the Blanchard-Porretta technique (see the previous section). We denote by ω the modulus of continuity of φ on $\mathbb{R}$. It remains to choose a sequence $(\xi_n)_n$ of test functions.

In the first setting (L^1_{loc} solutions) we are inspired by the pioneering work of Brézis [38] and its generalization by Gallouët and Morel [87]. The idea is: under the so-called Keller-Osserman condition[8] the diffusion is so fast that the mere fact that a solution is globally defined in $\mathbb{R}^N$ becomes a severe restriction. For instance, the unique globally defined solution of $u = \Delta |u|^{m-1}u$, $0 < m < 1$, is identically zero. Applied to the Kato inequality, this kind of argument brings uniqueness.

In [7], we revisit the techniques of [87], dropping some unnecessary assumptions on the shape of the graph φ and introduce a generalized Keller-Osserman condition in terms of the modulus of continuity of φ. We complement the existence part by using extensively the order-preservation feature of the problem that comes along with the uniqueness (cf. Section 4.1.2). Let me stress here the beautiful fact, of which I was not aware before the work [7]:

continuous dependence for order-preserving PDEs is an immediate consequence of uniqueness.

To prove this, an $\liminf/\limsup$ approach should be used; it yields a purely *qualitative* continuous dependence result.

[8] for a concave on $\mathbb{R}^+$, even and increasing on $\mathbb{R}$ function φ, the Keller-Osserman condition can be stated as $\int_1^\infty \frac{dz}{z\varphi^{-1}(z)} < \infty$.

In contrast, in the settings of weighted $L^1(\rho)$ spaces also quantitative continuous dependence information will be obtained. Let us describe a second well-posedness class:

in the setting of $L^1(\rho)$ solutions for exponentially decaying weights ρ,
we actually use the "generalized Kato inequalities"

$$\int (v-\hat{v})S'(w-\hat{w})\xi \le \int S'(w-\hat{w})(f-\hat{f})\xi + \int S(|w-\hat{w})|)\Delta\xi, \;\; w=\varphi(v), \; \hat{w}=\varphi(\hat{v}).$$

The nonlinearity $S(\cdot)$ is selected using the interplay between the modulus of continuity ω_φ of φ and the *a priori* weighted-L^1 assumption on $\varphi(v), \varphi(\hat{v})$. Indeed, we "convert" $v-\hat{v}$ into $w-\hat{w}$ by writing $(v-\hat{v})S'(w-\hat{w}) \ge \omega_\varphi^{-1}(w-\hat{w})S'(w-\hat{w})$. Then

the idea of the construction is, roughly speaking, to "linearize" the resulting Kato inequality

by making $\omega_\varphi^{-1}(z)S'(z)$ and $S(z)$ look similar. While this is impossible for technical reasons, we introduce an additional truncation parameter k and work with a family of truncations $S_k(\cdot)$ that "almost linearize" the Kato inequality wrt the quantity $|w-\hat{w}|$. From this construction, we eventually deduce that $w-\hat{w}$ must be identically zero if source terms coincide. If $s-\hat{s}$ is not zero, we establish a kind of "weighted contraction inequality" (with solution-dependent weights) which provides some *quantitative* continuous dependence result.

For a third well-posedness class,

in the $L^1(\rho_R)$ setting with "truncated fundamental solution" weights ρ_R,
we put the test function[9] ρ_R in the classical Kato inequality, using that $\Delta\rho_R \le 0$.

In this way the last term of the Kato inequality can be dropped, and

we get weighted L^1 contraction[10] inequalities for distributional solutions of $(Pb^{diff}_{\mathbb{R}^N})$

which are actually valid for rather general sub-harmonic weights. Based on this contraction result,

we define mild solutions of the evolution problem $\partial_t v - \Delta\varphi(v) = s$, $v(0) = v_0$
and establish partial uniqueness results for the weak solutions
of the associated fast diffusion evolution PDE in weighted L^1 spaces,

using the arguments of nonlinear semigroup theory.

The adaptation of the results of the latter setting to problem $(Pb_{\mathbb{R}^N})$ consists in sending R to infinity, while controlling the new term coming from $\operatorname{div} F(v)$ with a Hölder modulus of continuity of $F(\cdot)$. We are currently looking with N. Alibaud at the case of fractional conservation law (see Section 2.5.4) using the method of [10[1]].

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[8] following Brézis [38], the generalized Kato inequality comes from test functions $S'(\varphi(v)-\varphi(\hat{v}))$ with $S(\cdot)$ monotone; the classical Kato inequality is the particular case corresponding to $S(z)=z^+$ with the additional hint from [31].

[9] actually, one needs careful approximation arguments in order to obtain ρ_R as limit of $\mathcal{D}'$ test functions ξ of the Kato inequality while controlling the remainder terms

[10] this remarkable fact was pointed out already in the paper [24] of Bénilan and Crandall, in a slightly different setting; we were not aware of this result brought to our attention by J.L. Vázquez.

3.4 Entropy solutions of doubly nonlinear parabolic-hyperbolic problems and their numerical approximation

The doubly nonlinear parabolic-hyperbolic problem

$$(PbPH)\ \ \partial_t v + \operatorname{div}\big(F(v) - \mathfrak{a}(\nabla w)\big) = s, \ \ w = \varphi(v)$$

in a bounded domain with homogeneous Dirichlet boundary condition is similar, in many aspects, to the elliptic-parabolic problem studied in the Carrillo and Wittbold paper [46]. For both problems, uniqueness stems from the method of [45] adapted to Leray-Lions diffusions in [46]. But existence for $(PbPH)$ is considerably more difficult, because strong compactness of approximated solutions $(v_n)_n$ is not straightforward at all.

In the work [10[2]] with M. Bendahmane and K.H. Karlsen, not only we prove well-posedness, but

we construct a finite volume numerical method, and prove its convergence to the unique entropy solution of problem $(PbPH)$ with homogeneous Dirichlet BC.

The (many) specific issues related to the construction and analysis of the numerical scheme are explained in Chapter 5. Roughly speaking, we follow the same steps in the discrete setting as in the continuous setting. Here I mainly explain the PDE aspects of the problem.

In the definition of an entropy solution, we take into account the BC by means of the semi-Kruzhkov entropies $\operatorname{sign}^{\pm}(v-k)$ as in [45]. Following Bendahmane and Karlsen [17, 18] we use regularized semi-Kruzhkov entropies and exploit systematically the "chain rules in space". For existence, we regularize the problem by a viscosity term, and prove standard bounds on v_n (in L^∞) and on $w_n = \varphi(v_n)$ (in $L^2(0,T;W_0^{1,p}(\Omega))$; this is an "energy" bound). Further,

consider a sequence of approximate solutions $(v_n)_n$ with natural L^∞ and "energy" bounds; for merely continuous F and φ, carefully exploiting the modulus of continuity of φ we derive uniform time translation estimates in L^1 on the sequence $(w_n)_n$; eventually, we have strong L^1 compactness of $(w_n)_n$ and weak L^p compactness of ∇w_n.

The next step is to use the Minty-Browder argument in order to pass to the limit in the nonlinear term $\mathfrak{a}(\nabla w_n)$. The key observation here is that the formal term "$\int_\Omega(\operatorname{div} F(v_n))\varphi(v_n)$" vanishes due to a chain rule, integration by parts and to the homogeneous Dirichlet boundary condition. Unfortunately, for the limit formulation we can only write the convection term as a nonlinear weak-* limit, using the device of Young measures (or, more precisely, of entropy-process solutions in the sense of Gallouët et al. [85, 76, 79]): $\lim \int_\Omega F(v_n)\xi = \int_0^1\!\int_\Omega F(\mu(\cdot;\alpha))\,d\alpha$. The above chain rule argument does support this, namely, the term "$\int_\Omega(\int_0^1 \operatorname{div} F(\mu(\cdot;\alpha))\,d\alpha) w(\cdot)$" with $w(\cdot) = \int_\Omega \varphi(\mu(\cdot;\alpha))\,d\alpha$ also vanishes. We deduce strong convergence of $(\nabla w_n)_n$ and achieve an entropy-process formulation similar to the one of Eymard, Gallouët, Herbin and Michel [79]. Then we embark on the doubling-of-variables procedure:

we have rewritten[11] in [10[2], Appendix A] the Carrillo doubling of variables arguments for entropy-process solutions of the homogeneous Dirichlet problem which eventually permit us to "reduce" the entropy-process solution: we get $\mu(\cdot,\alpha)\equiv v(\cdot)$, where v is the unique entropy solution to the problem.

The convergence proof for finite volume method contains an additional complication, namely, because we use DDFV "double" finite volume scheme, the discrete solution has to components both of which converge weakly-*. Then we need to introduce the notion of entropy-"double-process" solution and cope with this additional technicality within the doubling-of-variables method.

The theoretical part of the work [10^2] is continued in the paper [09] with S. Ouaro and the same co-authors; and the finite volume part of [10^2] eventually led to the new "CeVe-DDFV" scheme in 3D presented in [08^3] and in [12^1, ^{4}, 11^4].

[11] The Carrillo paper [45] is reputed to be difficult to read, and we hope that a different presentation of the same ideas can be useful at least for newcomers to the theory. Some of the original arguments of [45] are re-arranged, some arguments are borrowed from other sources (E.Yu. Panov's private communication, etc.).

Chapter 4

Variational, entropy and renormalized solutions of nonlinear or nonlocal parabolic and elliptic-parabolic problems: well-posedness, structural stability

This chapter is closely related to the previous one; scalar nonlinear degenerate parabolic equations are treated, but the hyperbolic degeneracy is replaced with (or complemented by) the elliptic one. Another focus of this chapter is on "renormalized" (in the sense of Lions and Murat [124]) and "entropy" (in the sense of Bénilan et al. [22]) solutions that are more general than the standard "variational" ones. Not only these solutions are of interest of their own; in Section 4.1.4 we show that delicate convergence results for weak solutions are easier obtained if we embed weak solutions into the more general renormalized/entropy setting.

The first and the longest section of this chapter is devoted to studying convergence of approximate solutions; we call this the "structural stability" issue. The difficulty of the second section lies in treating a very irregular absorption term; here, a careful choice of the functional-analytic framework was the essential ingredient of the well-posedness result. The third section introduces the notion of renormalized solution for fractional diffusion equation; we give a condensed existence and uniqueness proof that benefits from the experience of many previous papers on the subject of local diffusion equations.

4.1 Structural stability and numerical approximation

This section is on "structural stability" issue, i.e., on continuous dependence of solutions on the nonlinearities that appear in the equation. It should be stressed that techniques for structural stability can be re-used for proving convergence of numerical approximations, see, e.g., Section 4.1.2.

4.1.1 Structural stability for time-dependent elliptic-parabolic problems

I started to work on Leray-Lions (or, more precisely, Alt-Luckhaus) kind problems in the second part of my PhD; Philippe Bénilan suggested me to re-visit the celebrated well-posedness result of [6] by bringing into analysis the L^1 time-compactness argument:

*(S.N. Kruzhkov [107]) if $(u_n)_n$ and $(F_n^\alpha)_n^\alpha$ are bounded in $L^1_{loc}(Q)$, $Q=(0,T)\times\Omega$,
if the evolution PDEs $\partial_t u_n = \sum_{|\alpha|\leq m} D^\alpha F_n^\alpha$ are fulfilled in $\mathcal{D}'(Q)$,
and if $(u_n)_n$ are "compact in space" in the sense of a uniform L^1_{loc} space translation estimate,
then $(u_n)_n$ is also "compact in time" (consequently, it is $L^1_{loc}(Q)$ compact).*

To illustrate the flexibility of this argument as compared to the original time translation estimate used by Alt and Luckhaus [6], we treated the case with explicit time dependence in the evolution term:

$$(SysAL)\ \ \partial_t b(t,x,V) - \operatorname{div} \mathfrak{a}(t,x,V,\nabla V) = s, \quad b(t,x,V)|_{t=0} = b^0$$

in a bounded domain with homogeneous Dirichlet or Neumann boundary conditions.

We actually aimed at structural stability result under the heuristic form

(SS) *the set of solutions of perturbed problems is compact,
and every accumulation point is solution of the limit problem.*

Let us stress that $(SysAL)$ is an $m\times m$ system of equations, where $b(t,x,\cdot)$ is a cyclically monotone vector field on $\mathbb{R}^m$ (this means, b is a gradient of some convex scalar potential), and $\mathfrak{a}$ is a Leray-Lions operator from $(W_0^{1,p})^m$ to $(W^{-1,p'})^m$. While uniqueness is not guaranteed unless additional restrictions are imposed, the method of Alt and Luckhaus [6] brings a general existence result through convergence of Galerkin approximations, in the case $b=b(V)$.

Thus my goal in [11] was to extend and simplify[1] the existence technique for $(SysAL)$, using in particular the Kruzhkov compactness lemma. The work [11] remained unpublished, and recently I've understood that it contained a serious error: namely, the Kruzhkov argument is perfectly fit for structural stability analysis, but it is not adapted to proving compactness of Galerkin approximations[2]. Thus existence proof was incomplete. This difficulty is bypassed, e.g., by using a finite volume discretization for constructing approximate solutions, in the place of the Galerkin method. With the tools presented in Chapter 5, finite volume semi-discretization in space and the discrete ("finite volume") Kruzhkov lemma lead to an existence result along the same lines as in the structural stability proof that we now discuss.

The work [11] as presented in [Th] contains the following steps:

- we identify the restrictions on the nonlinearities and data under which t-dependent nonlinearity $b(\cdot)$ can be considered;
- we prove a version of the "Mignot-Bamberger/Alt-Luckhaus" chain rule (see [6, 127, 46] for different versions of this argument) adapted to this explicit time dependence;

[1] here I mean a simplification "at the level of ideas" used in the proof; clearly, introducing additional time dependence in b makes some calculations longer and harder than in the model autonomous case

[2] indeed, it is delicate to write down the PDE verified by Galerkin approximations: one only has access to a projection of this PDE on some finite-dimensional subspace; yet the Kruzhkov argument requires a weak PDE formulation with arbitrary test function

- we give a slight modification of the original L^1 compactness argument of Kruzhkov;
- we combine this compactness argument, fine properties of Nemytskii operators, and the Minty trick in order to deduce the result (SS).

Although quite technical, this work was an excellent initiation to new ideas and techniques; I investigated a number of related problems in the subsequent years.

Let me also point out that one of the conditions necessary for the method to work is the "structure condition" stating that, roughly speaking, $\mathfrak{a}(t,x,z,\xi) = \tilde{\mathfrak{a}}(t,x,b(t,x,z),\xi)$; indeed, we can get time compactness on $(b(t,x,V_n))_n$ and not on $(V_n)_n$. The question asked (and partially answered) in 1996 by Bénilan and Wittbold [30] was:

$(QStrCond)$ *do we have existence and structural stability* without *the "structure condition"?*

The next section is devoted to this question.

4.1.2 Approximation of elliptic-parabolic equations "without the structure condition"

Question $(QStrCond)$ of the previous section was, to a large extent, answered by K. Ammar and P. Wittbold in [8]. Following [29], let us concentrate on the model situation:

$$\partial_t b(v) + \operatorname{div} F(v) - \Delta v = s, \;\; b(v)|_{t=0} = b^0, \;\; \text{with homogeneous Dirichlet BC.}$$

The main idea of [8] was:

create a monotone *sequence of approximate solutions.*

Indeed, time compactness of $(v_n)_n$ was the bottleneck in the previous section; it comes for granted if the sequence is monotone.

In practice, the method yields existence in the case of a scalar equation

(monotonicity can be obtained as the L^1 contraction principle), but it is not applicable to general systems of the kind $(SysAL)$. A typical application is, existence of renormalized solutions as limits of *bi-monotone* sequence of solutions $v^{n,m}$ of problems corresponding to $L^1 \cap L^\infty$ data $s^{n,m}$ with $s^{n,m} \downarrow_{m\to\infty}\uparrow_{n\to\infty} s$.

But what about convergence of "natural" approximations ?

such as, for instance, numerical approximations of the problem of approximations using time-implicit semi-discretization[3]?

The answer that we give in the work [12[4]] with P. Wittbold is in fact very simple:

in the case uniqueness of a solution v can be established and
if the approximation procedure is order-preserving, one can use $\liminf - \limsup$ *tricks*
in order to reduce the discussion to the case of a monotone sequence of approximations.

[3] time-implicit semidisretization is the basic tool of the nonlinear semigroup theory

The precise construction includes an additional layer of approximation, namely,

we use penalization of the PDE by a carefully chosen absorption term $\psi^\varepsilon(v)$
which brings local on $(0,T]$ *time translation estimates*[4] *on discrete solutions* $(v_n^\varepsilon)_n$;
then we use the order-preservation feature of the PDE and of the approximation scheme to infer that $v=\lim_{\varepsilon\to 0^-}\lim_{n\to\infty} v_n^\varepsilon \le \liminf_{n\to\infty} v_n \le \limsup_{n\to\infty} v_n \le \lim_{\varepsilon\to 0^+}\lim_{n\to\infty} v_n^\varepsilon = v$.

For a practical application,

we prove that the assumptions of the above method
(uniqueness of a solution + an order-preserving approximation scheme) are fulfilled, e.g.,
if F is Lipschitz continuous and a monotone finite volume scheme is used for approximation.

Similarly, for a Lipschitz continous F in [12⁴] we prove convergence of ε-discretizations to a mild solution of the problem, thus complementing the pioneering result of Bénilan and Wittbold [30] on the problem "without structure condition".

4.1.3 Structural stability for triply nonlinear degenerate parabolic problems

By "triply nonlinear", we mean a problem of the form

$$(TNL)\;\; \partial_t b(v) - \operatorname{div}\mathfrak{a}(v,\nabla\varphi(v)) = s, \;\; b(v)|_{t=0} = b^0$$

that we supplement by the homogeneous Dirichlet boundary condition, for the sake of simplicity[5]. As usual, b and φ are continuous non-strictly increasing nonlinearities and $\mathfrak{a}$ satisfies the pseudomonotonicity assumption of the Leray-Lions kind.

I was initiated to this problem by S. Ouaro, who came for a research stay to Besançon in 2005. In the Ouaro and Ouaro and Touré previous works [129, 130] on the subject, the one-dimensional case was understood (many ideas come back to the work of Bénilan and Touré [28]). Treating the general multi-dimensional case proved difficult. Ouaro and myself finally came out with the joint work [09] with M. Bendahmane and K.H. Karlsen; it exploited extensively the experience from the "doubly nonlinear" framework of the paper [10²].

We had to impose mild structure restrictions, such as the bijectivity of $b+\varphi$, but we end up with a complete well-posedness and structure stability result:

Entropy solutions of problem (TNL) exist and form an L^1-contractive semigroup;
solutions depend continuously on the data and the nonlinearities $b,\varphi,\mathfrak{a}$.

The proof is technical; it is a careful combination of different hints of [45, 46], [8], [129], [10²]. Besides the combination itself, an important original element of the paper [09] is:

we prove a new simple estimate that allows to "cut off"
any set of values of v where $\varphi(v)$ has a small variation.

[4]the translation estimates we use were obtained by A. Zimmermann in [162]

[5]the simplification is not merely technical: as we have seen in Chapter 3, for treating Neumann or non-homogeneous Dirichlet boundary conditions we were led to reduce the generality of our assumptions on the data and nonlinearities of the problem

Due to this estimate, we are able to neglect the set of (t, x) where v belongs to a small neighbourhood of the "flat regions" of φ; then

we use the Minty argument "piecewise", in the complementary of the cut-off regions.

Let me mention that several works in the same direction were carried out by K. Ammar and H. Redwane, with a focus on the non-homogeneous Dirichlet condition.

4.1.4 Structural stability and approximation of $p(x)$ and $p(u)$-laplacian kind problems

The "variable exponent" elliptic and parabolic Leray-Lions problems became the object of intense world-wide research activity in the last decade. While the problem was introduced by V.V. Zhikov in the mid-1980ies, the revival of interest came from modelling applications such as electro-rheological fluids and image restoration.

The prototype example of "variable exponent" problem is the $p(x)$-laplacian, say,

$$(PxPb)\ \ u - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f.$$

The problem has to be set in the *ad hoc* Sobolev space, denoted $W^{1,p(x)}$. While many results on existence, regularity, multiplicity of solutions of this kind of equations were obtained, with S. Ouaro and M. Bendahmane we've got interested in one particular aspect of the problem:

how to treat sequences of solutions (u_n) corresponding to different exponents p_n?

Applications to numerical analysis of the $p(x)$-laplacian are evident: one has to discretize $p(\cdot)$ while discretizing the equation ! An application that is, may be, even more important is to coupled problems where the dependence of $p(x)$ on x is governed, may be indirectly, by the solution u itself. Let us give two (very academic!) examples of such problems:

$$(PuPb)\ \ u - \operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u\right) = f,$$

$$(CouplPb)\ \ \begin{cases} u - \operatorname{div}\left(|\nabla u|^{p(x,v)-2}\nabla u\right) = f, \\ v - \Delta v = g(u, v). \end{cases}$$

In order to prove existence of solutions for such problems, one has to answer the above question; which means, one has to analyze the issue of structural stability (dependence on the nonlinearities) of the variable exponent Leray-Lions kind problems. That's what we have done in the two-parts work [10[3], 10[4]] with M. Bendahmane and S. Ouaro.

The main difficulty is:

get rid of the functional analysis arguments that require a "fixed space" framework like $W_0^{1,p(x)}$.

Indeed, actually solutions u_n belong to different spaces (say, $W_0^{1,p_n(x)}(\Omega)$); for instance, on cannot use the abstract Minty argument on such a sequence of solutions.

The main tool of our work is the description of weakly L^1 convergent sequences in terms of Young measures, and their reduction using the monotonicity of the nonlinearity $\xi \mapsto |\xi|^{p-2}\xi$.

The Minty argument is then replaced by a Young measures' reduction argument previously used by Hüngerbuhler et al. [69, 97][6].

Using this tool, we "pull everything down to the one-and-only space L^1"

and skip the difficulty of having ∇u_n lying in different $L^{p_n(x)}$ spaces[7].

4.1.5 Structural stability for $p(x)$-problems; broad/narrow solutions

We are interested, in principle, in considering

merely measurable exponents $p : \Omega \mapsto [p_-, p_+] \subset (1, \infty)$,

but this leads to an obstruction. Namely, in general

two spaces appear in the analysis:
the "broad" space $\dot{E}^{p(x)} = \{u \in W_0^{1,1}(\Omega) \,|\, \nabla u \in L^{p(x)}(\Omega)\}$ *and the "narrow" space* $W_0^{1,p(x)}(\Omega)$;
it is feasible to show that a limit u of $(u_n)_n$ belongs to the "wide space"
and that the limit PDE is satisfied with test functions in the narrow" space.

Thus u is what we call "incomplete" solution: incomplete solutions are not variational solutions, and their uniqueness is not guaranteed.

Therefore we limit our considerations to variational "narrow" or "broad" solutions
and prove that narrow (respectively, wide) solutions are stable
by approximation of $p(\cdot)$ from above (respectively, from below).

It should be stressed that we study stability of renormalized solutions (or Bénilan et al. entropy solutions: the two notions are equivalent) for both broad and narrow frameworks, which has in particular the following advantage:

Due to the use of renormalized solutions, we were able to prove the structural stability
under the assumption of the mere weak L^1 convergence of the sequence $(f_n)_n$ of source terms

Indeed, setting up a notion of weak convergence of, say, $f_n \in L^{p'_n(x)}$ to $f \in L^{p'(x)}$ is quite a technical matter. Thus, on this example we see that

here, the use of renormalized solutions yields optimal results on the weak ones !

Further, from the structural stability result, it follows that

problems of kind $(PxPb)$ are well posed for weak or renormalized narrow solutions;
it is also well-posed for weak or renormalized broad solutions.

To be specific, using Galerkin approximations we deduce existence of renormalized broad and renormalized narrow solutions; these become weak broad and weak narrow solutions, respectively, under suitable restrictions on the source term.

Let us also stress that because of the generality of the structural stability result
it is very easy to deduce existence for a more general problem
$b(u) - div\mathfrak{a}(x, u, \nabla u) = f$, *provided the exponent p remains dependent on x only.*

---

[6] to my opinion, this Young measures' reduction argument is much more natural that the Minty trick

[7] yet the Minty trick can be used as well to resolve the difficulty: in [161] Zhikov employed a combination of cut-off and Minty arguments to get a proof of structural stability that is both much shorter and a bit more general than the ours. In a way, Zhikov's argument also works by "pulling everything down to L^1.

Uniqueness is a straightforward reproduction of the L^1 contraction argument known for the constant-exponent setting. We cannot generalize it to the u-dependent problem mentioned above unless strong restrictions on this dependence are imposed. Indeed, to my knowledge, no adaptation of the doubling-of variables argument was yet proposed for variable exponent convection-diffusion operators.

A first motivation for the work [10^3] was the theoretical numerical analysis of problems of kind $(PxPb)$. We have announced in [10^3] a joint work with Bendahamane and R. Ruiz Baier on finite volume approximation of the $p(x)$-laplacian; this project has not yet been completed, mainly because of lack of time and of some lack of originality in its numerical analysis part. Indeed, using any discrete duality finite volume scheme (e.g., the DDFV scheme of [07^2]), with suitable discretization of $p(\cdot)$ we easily get convergence of approximates to a narrow solution (cf. Chapter 5 for discretization of p-laplacian with constant exponent p). In contrast, approximation of broad solutions seems to be a difficult question!

In conclusion, let us point out that at least for one rather typical example,

we show in [10^3, Appendix] that narrow and broad solutions coincide "generically".

That is, the set of data for which broad and narrow solutions may not coincide is very small compared to the set of all possible data.

4.1.6 On $p(u)$-laplacian problems and coupled variable exponent problems

In [10^4], we have applied the tools of [10^3] in order to study u-dependent ("auto-rheological") variable exponent problems such as $(PuPb)$ and $(CouplPb)$.

Denote by p_∞ the exponent $p(u)$ of $(PuPb)$ or the exponent $p(x,v)$ of $(CouplPb)$. The main difficulty that we have encountered is the following:

we are unable to establish existence of a solution without restrictions on p_∞
(of which the "log-Hölder continuity" is the most practical one).

More specifically, we easily get existence of "incomplete" solutions, which is not satisfactory; thus we are doomed to the framework where broad and narrow solutions would coincide.

Thus the technique leads us to impose severe restrictions:

in the case of problem $(PuPb)$, we need that the dimension N be greater than $p_+ = \sup p(\cdot)$.

Note that this assumption makes renormalized solutions needless (it is imposed in order to ensure the Hölder continuity, and thus the boundedness, of a solution u). Under this restriction, we prove that

$(PuPb)$ is well-posed in the setting of weak (broad≡narrow) solutions,
moreover, the map $S : f \mapsto u$ is an order-preserving L^1 contraction semigroup.

The latter part (uniqueness, L^1 contraction) was rather unexpected for us. Its proof borrows the idea of F. Bouhsiss used in Section 3.1: we start by comparing a general solution u to a "regular enough" solution $\hat{u}$, then we conclude "by density" of regular enough solutions. Existence of a sufficiently large set of "regular enough" solutions stems from the Hölder regularity results of Acerbi, Mingione [1], Fan [83] for $p(x)$-laplacian kind problems.

For the coupled problem, we also have to ensure the log-Hölder continuity of the exponent $p_\infty = p(x,v)$; here the conditions are less restrictive (due to the fact that v is, generically, Hölder regular thanks to the classical properties of the second equation in $(CouplPb)$). We prove that

there exists a solution to problem $(CouplPb)$, under some restrictions on nonlinearities.

This result is nothing more than an example of application of techniques of [10[3]]; the technique may apply to different problems where the variable nonlinearity depends on u in a non-local way.

4.2 Parabolic equations with general absorption terms

The work [08[2]] was a part of the PhD thesis of Karima Sbihi, and it continued the study that Sbihi and P. Wittbold conducted for elliptic problems with x-dependent irregular absorption term. The background on this topic is provided by the work [159] by Wittbold. The associated parabolic problem is

$$(AbsPb)\ \ \partial_t v - \operatorname{div} \mathfrak{a}(v, \nabla v) + \beta(x,v) \ni s, \quad v|_{t=0} = v_0$$

(in a bounded domain with Dirichlet boundary condition, to be specific). Here $\mathfrak{a}$ is of Leray-Lions type, with some (not variable) exponent p. Further, $\beta(x,\cdot)$ is a maximal monotone graph; typically, β may represent an obstacle condition for a solution u. The family of graphs $(\beta(x,\cdot))_x$ may be too irregular in x. In the elliptic case, one follows the approach of Bouchitté [33] to give sense to the formal term[8] "$\beta(x,v)$" as a measure μ not charging the sets of zero p-capacity. Let us denote by $\mathcal{M}_0$ the set of all such measures.

Thus according to the previous works on the subject, in the elliptic case the absorption term $\beta(x,v)$ should be understood as an element of $\mathcal{M}_0$. In the parabolic setting,

the chief obstacle towards formulating the appropriate notion of solution to $(AbsPb)$ was to "separate" the measure parts in the sum $\partial_t v + \beta(t,v)$.

Indeed, a natural way to study problem $(AbsPb)$ would be to develop a "parabolic capacity" theory in order to reproduce, for this parabolic problem, the approach from the elliptic one (see [159]). For linear diffusion, such a theory was constructed by M. Pierre in [131]. The theory of Droniou, Porretta and Prignet [73] is a nonlinear one, but the fundamental difficulty of separating $\partial_t v + \beta(x,v)$ precluded us from using it.

In [08[2]] we circumvent both the above difficulty and the use of parabolic capacities. Indeed,

we put forward the space $L^1(0,T;\mathcal{M}_0)$ and use elliptic capacities t-almost everywhere.
By careful approximation we manage to show that the measure μ representing $\beta(x,v)$ actually lies in the space $L^1(0,T)$ with values in the elliptic measure space $\mathcal{M}_0$.
As a consequence, we prove a "maximal regularity" result:
both $\partial_t v$ and $\beta(x,v)$ lie in the space $L^1(Q) + L^p(0,T;W^{-1,p})$,

[8]Following [33], the formal equality "$\beta(x,v) = \mu$" is understood in a relaxed sense: namely, μ lies in the subdifferential of an *ad hoc* convex s.c.i. functional $\tilde{j} := j + I_{[\gamma_-,\gamma_+]}$. Here $\beta = \partial j$ and the indicator function $I_{[\gamma_-,\gamma_+]}$ restricts the domain of j so that to comply with the generic p-quasicontinuity regularity of $W^{1,p}$ functions. If $(\beta(x,\cdot))_x$ is regular enough, then $\operatorname{Dom} j = \operatorname{Dom} \tilde{j}$ and the additional term can be dropped.

and starting from this point, the standard techniques (such as the chain rule lemma of Mignot-Bamberger/Alt-Luckhaus) apply. Roughly speaking, without this $L^1(0,T;\mathcal{M}_0)$-regularity property we could not apply the Droniou, Porretta and Prignet parabolic capacity theory; and with this regularity in hand, we do not need to use parabolic capacities.

Besides this essential question of taking into account the absorption term, the work [08[2]] was highly technical. To be brief,

we prove well-posedness in the setting of entropy solutions of Bénilan et al.,

i.e., we have considered general L^1 data and used a truncation procedure to define solutions. Both existence and uniqueness proofs use several approximation steps: Yosida regularization of the absorption term, penalization by an additional absorption $\psi^{n,m}(v)$ (cf. Section 4.1.2), the Crandall-Ligett theorem, etc.. Notice that, as in the pioneering work of Bénilan et al. [22], the notion of entropy solution permits to give a purely PDE interpretation of the solution given by the Crandall-Liggett theorem. In the same vein, in [08[2]]

we achieve an intrinsic characterization of the mild solution of the abstract evolution problem $\partial v + A_\beta v \ni s,\ v|_{t=0} = v_0$*; here, following P. Wittbold [159],* A_β *is the operator associated with the formal expression* $-\mathrm{div}\,\mathfrak{a}(\cdot,\nabla\cdot) + \beta(x,\cdot)$*;* A_β *was defined in the elliptic framework.*

It should be stressed that our method is not general enough to truly settle the question of definition of solutions of $(AbsPb)$: for instance, the additional $L^1(0,T;\mathcal{M}_0)$-regularuty that we prove for the measure part of $\beta(x,v)$ need not hold for time-dependent absorption term $\beta(t,x,v)$.

4.3 Renormalized solutions of non-local diffusion problems

Consider the Laplacian diffusion problem $-\Delta u = f$, say with Dirichlet boundary conditions in a bounded domain. When $f \in H^{-1}$, the notion of variational solution is appropriate. For general L^1 of event measure data, the notions of entropy (in the sense of Bénilan et al. [22]) and renormalized (in the Lions and Murat sense) were developed in order to provide a coherent well-posedness theory.

The goal of the work [10[9]] with N. Alibaud and M. Behdahmane was to provide a generalization of these notions of solution to the non-local framework of Section 2.5.4. We have started with the notion of renormalized solutions (treating entropy solutions is not more difficult, see the forthcoming work [[8]]) in the "pure fractional diffusion" framework

$$(FDPb) \quad b(v) + (-\Delta)^{\lambda/2} v = f \quad \text{in } \mathbb{R}^n.$$

The result is:

we adapt the notion of renormalized solution to $(FDPb)$ *and justify existence as well as uniqueness,* L^1*-contraction and comparison property for renormalized solutions.*

Let us stress that usually, working with renormalized solutions is quite technical; one interesting feature of our proof is:

based on the many previously known hints, we give a combined existence&uniqueness proof that benefits from different hints presented in the previous sections[9]. These ideas are not specific to the non-local context, and can be used to shorten the presentation of a number of preceding works.

What is specific to the non-local nature of $(FDPb)$
is the necessity to circumvent the use of chain rules.

To explain the idea, let us write down the renormalized formulation of problem $-\Delta u = f$ in $\mathbb{R}^N$: it reads[10]

$$\begin{array}{l} \text{for all } k>0 \quad T_k(u)\in H^1_0(\mathbb{R}^N), \quad \lim_{m\to\infty}\int_{[k<|u|<k+1]}|\nabla u|^2=0, \text{ and} \\ \text{for all } S\in W^{2,\infty}(\mathbb{R}) \text{ with compactly supported } S', \\ \displaystyle\int_{\mathbb{R}^N} S''(u)|\nabla u|^2\xi+\nabla S(u)\cdot\nabla\xi=\int_{\mathbb{R}^N} fS'(u)\xi \quad \text{holds for } \xi\in\mathcal{D}(\mathbb{R}^N) \end{array}$$

where $T_k : z \mapsto \mathrm{sign}\,(z)\min\{|z|,k\}$ is the truncation function. The above formulation strongly exploits the chain rule proper to the local framework: in particular, the fact that $\nabla T_k(u)$ is supported within the set $[x : u(x) \in (-k,k)]$ is used (we can also replace T_k and $(-k,k)$ by S and by the support of S', respectively).

In order to state the renormalized formulation for $(-\Delta)^{\lambda/2}u = f$,

the main tool is the representation[11]
$\int v\,(-\Delta)^{\lambda/2} = \frac{1}{2}\iint (v(y)-v(x))(u(y)-u(x))\,d\pi(x,y)$
where π *is the* ad hoc *measure on* $\mathbb{R}^{2N}$.

We borrowed this approach from the work of Cifani, Jakobsen and Karlsen [53] on fractional diffusion equations. Based on this "bilinear form hint", we introduce a symmetrization device in order to formulate a definition of renormalized equation for the fractional laplacian comparable, term per term, to the above local definition[12]:

$$\begin{array}{l} \textit{for all } k>0, \qquad \iint_{\mathbb{R}^{2N}}(u(x)-u(y))\,(T_k(u)(x)-T_k(u)(y))\,d\pi(x,y) \;<\; +\infty; \\ \textit{moreover,} \quad \lim_{k\to+\infty}\iint_{[(u(x),u(y))\in A_k]}|u(x)-u(y)|\,d\pi(x,y)=0, \\ \textit{and for all compactly supported } S'\in W^{1,\infty}(\mathbb{R}), \\ \displaystyle\iint_{\mathbb{R}^{2N}}(u(x)-u(y))\,\big(S'(u)(x)-S'(u)(y)\big)\frac{\xi(x)+\xi(y)}{2}\,d\pi(x,y) \\ \displaystyle+\iint_{\mathbb{R}^{2N}}(u(x)-u(y))\,\big(\xi(x)-\xi(y)\big)\frac{S'(u)(x)+S'(u)(y)}{2}\,d\pi(x,y)=\int_{\mathbb{R}^N} f\,S'(u)\,\xi. \end{array}$$

where $A_k := \{(u,v)\in\mathbb{R}^2 \mid\ k+1\le\max\{|u|,|v|\} \text{ and } (\min\{|u|,|v|\}\le k \text{ or } uv<0)\ \}$.

Use of the set A_k replaces, in the non-local setting, the chain-rule-based integrability constraint $\|1\!\!1_{[k<|u|<k+1]}|\nabla u|^2\|_{L^1} \to 0$ of the local formulation. Accordingly, our proof of well-posedness for renormalized solutions uses a partition of $\mathbb{R}^{2N}$ into suitable subsets.

[9] let me invite the interested reader to look at the note [10[9]] where the steps of the combined existence&uniqueness proof are exhibited in two pages

[10] due to the integrability constraints of the formulation, all the terms in the renormalized equation have a precise meaning; for instance, $S'(u) = S' \circ T_K(u)$ for K large enough, thus $S'(u) \in H^1$ and $S''(u)|\nabla u|^2$ is given the sense $\nabla S'(u)\cdot\nabla T_K(u)$

[11] This representation stems from the Lévy-Khintchine formula, see Section 2.5.4

[12] as in the local case, a careful examination permits to say that all the terms of the renormalized equation are indeed meaningful, under the integrability constraints given in the definition

Because we deduce that the associated elliptic operator $A := (-\Delta)^{\lambda/2}$ has an m-accretive in L^1 closure, it follows that

the associated abstract evolution equation $\partial_t u + Au = f \in L^1(\mathbb{R}^N)$ *has a unique mild solution.*

Study of solutions of this evolution equation from the PDE viewpoint is the object of the forthcoming work [8].

Chapter 5

Some finite volume schemes and their analysis

Approximating convection-diffusion operators with finite volume methodology is by now a classical subject. In the last years, there was an increasing demand for schemes that allow to treat anisotropic and heterogeneous diffusion problems on general meshes. My focus was on finite volume approximation of *nonlinear* diffusion operators, but the difficulties are essentially the same as for the linear anisotropic case.

I was concerned with three classes of "structure-preserving" methods, namely the so-called co-volume schemes, the DDFV schemes, and schemes specifically designed for cartesian meshes. The main issues were : construction of the schemes (always in view of certain applications, or in order to keep particular structural properties the main of which was the "discrete duality"); analysis of consistency, stability, and convergence properties; obtaining *a priori* convergence orders; and proving lemmas known as "discrete functional analysis tools" for different finite volume schemes.

5.1 A "continuous approach" to analysis of finite volume schemes

This work corresponds to the last chapter of my thesis [Th], its ideas was published in the note [01] and the full paper appeared later in [04[1]]. Together with M. Gutnic and P. Wittbold, we have given ourselves the objective to investigate finite volume approximation of elliptic-parabolic equations (or systems); for the sake of being definite, let us stick to

$$(EqFV)\quad \partial_t b(v) - \mathrm{div}\,(|\nabla v|^{p-2}\nabla v) = s,\;\; b(v)|_{t=0} = b_0,$$

(see Diaz and De Thélin [66] for a generalization) with a homogeneous Dirichlet condition in a bounded domain.

The idea was to benefit from the experience of the work [[11]] (from my side) and of the numerical analysis work [78] (from my collaborator's side) and to re-transcript the structural stability proof for $(EqFV)$ as a convergence proof for approximations. Naturally, we've adopted the principle that

a "good" discretization of a PDE should preserve its key structural properties, namely, in the case of $(EqFV)$ the discrete diffusion operator should have the same monotonicity, coercivity, growth properties as the underlying continuous Leray-Lions operator.

Only much later, in [07[2]] and more specifically, in [[4]] we have obtained all these properties as consequence of the one "discrete duality" property, see Section 5.3.1, constructed a scheme having these properties, and proposed a "fully discrete" convergence approach following the classical paradigm (it is systematically used by Eymard, Gallouët and Herbin in the classical book [76]). Indeed, recall that finite volume methods use *exterior* space approximation. Classically, one proves discrete counterparts of the key technical lemmas used in the existence and structural stability analysis for the PDE in hand; then one assembles them in the same way as for analysis of the continuous problem. In [01, 04[1]] we were more than "inspired" by the continuous proof: we proposed to use it essentially as it stands. More precisely,

we have introduced an original "continuous" approach
for studying convergence of finite volume schemes:

in the place of producing discrete counterparts of "continuous" arguments
we lift the discrete objects into the "continuous" space
and then use without adaptation, the arguments of the "continuous" setting.

This we have built a collection of techniques that are rather interesting by themselves, and that eventually led to a proof that

under mild proportionality assumptions on the meshes,
structure-preserving finite volume approximations of $(EqFV)$ converge to the unique solution.

The starting point was to invert the operator of projection of a $W^{1,p}$ function onto the space of piecewise constant functions[1] (the lifting operator should be bounded from the "discrete $W^{1,p}$ space" to the continuous one). We used convolution techniques but also solved local PDE problems in order to find liftings with interesting properties. Then, we worked with the lifted sequence $(\tilde{v}^h)_h$ in the place of the original sequence $(v_h)_h$ of discrete solutions. Details of the construction are given in [04[1]], while an outline of the proof can be inferred from [01].

I believed that the "continuous approach" would prove useful for proving more difficult results, but the experience we had with the problem partially solved in [05[1], 07[1]] (see Section 5.2.3) is disappointing. Indeed, "continuous" approach seemed very natural for extending the results of [05[1], 07[1]] to unstructured meshes (the goal was to obtain discrete Besov estimates, that are stated in terms of translations). It turned out that lifted functions $\tilde{v}_h$ were not helpful for translation arguments. Thus, the specific approach of [04[1]] did not prove particularly useful, and as a matter of fact I used the classical "fully discrete" paradigm in all the subsequent works.

Yet, besides the "continuous approach" and one of the first proofs of convergence of finite volume methods for Leray-Lions operators, paper [04[1]] contained several elements that proved useful later on. Namely,

we proved a Poincaré inequality without *any proportionality condition on the meshes*,
we proposed a finite volume scheme that is a close relative of "co-volume" schemes[2],
and we proved a first version of the "reconstruction formula"
that eventually laid the basis for a 3D generalization of the "DDFV" scheme.

We refer to Section 5.4 for further development of these subjects.

[1]A somewhat similar reconstruction procedure is used since several years for analysis of Discontinuous Galerkin Methods

[2]In particular, for treating the nonlinear diffusion problem $(PbFV)$, we have reconstructed the whole

5.2 Approximation of the p-laplacian on cartesian meshes

Arriving as a Maître de Conférences to Marseilles, I continued to work on finite volume approximation of p-laplacian, due to the encounter with F. Boyer and F. Hubert. This collaboration resulted in a series of works on *a priori* error estimates with schemes on cartesian meshes. These works are complementary: after having established the "basic" error estimates, we looked at estimates that are optimal either in respect of the convergence order, or in respect of the generic regularity of solutions. The works [04[3], 06[1], 05[1], 07[1]] are quite technical, and I only present the essential ideas of these works.

5.2.1 A family of 2D finite volume schemes on cartesian meshes. Basic error estimates.

In [04[3]], we have introduced a family of nine-point finite volume schemes on cartesian meshes for the elliptic p-laplacian problem. Our scheme could be classified as a co-volume scheme (see [91]): the discrete functions $u^{\mathfrak{T}}$ are piecewise constant on a primal mesh, and the discrete gradients[3] $|\nabla^{\mathfrak{T}} u^{\mathfrak{T}}|$ are piecewise constant on the dual mesh. With respect to the work [04[1]], to the framework of co-volume or DDFV schemes we used later on, and with respect to the recent unifying notion of gradient schemes ([82]), our approach in [04[3]] was different:

we have not reconstructed the vector $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ of discrete gradient,
but we have reconstructed separately the length $|\nabla^{\mathfrak{T}} u^{\mathfrak{T}}|$
and the normal components of $\nabla^{\mathfrak{T}} u^{\mathfrak{T}}$ on edges of the primal mesh.

Indeed, a finite volume methods approximates, for all control volume K of the primal mesh $\mathfrak{T}$, the value $\int_K \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \int_{\partial K} |\nabla u|^{p-2}(\nabla u \cdot \nu)$. Thus, reconstructing the absolute value per dual volume plus the normal components on edges of the gradient is enough to formulate a scheme.

To give an example, if we denote by u_1, u_2, u_3, u_4 the vertex values (numbered, e.g., counterclockwise) of $u^{\mathfrak{T}}$ on a square dual volume K^* of side length h, then the simplest of our schemes gives

$$|\nabla_{K^*} u^{\mathfrak{T}}|^2 = \frac{1}{2}\Big(\big(\frac{(u_2-u_1)^2}{h^2} + \frac{(u_3-u_4)^2}{h^2}\big) \;+\; \big(\frac{(u_4-u_1)^2}{h^2} + \frac{(u_3-u_2)^2}{h^2}\big)\Big)$$

and the normal components are (up to a sign)

$$\frac{1}{2}\Big(\frac{u_2-u_1}{h} + \frac{u_3-u_4}{h}\Big), \qquad \frac{1}{2}\Big(\frac{u_4-u_1}{h} + \frac{u_3-u_2}{h}\Big).$$

Actually, this scheme has properties very similar to the "symmetrized co-volume" scheme used by Handlovičová and Mikula [91] in the image processing context.

Actually, we wanted our scheme to possess nice structure properties, therefore the choice of the two reconstructions is not independent.

We imposed a symmetry restriction on the method that led to a relation
between the choice of two reconstructions; as a consequence, we get
a one-parameter family of schemes satisfying a kind of "discrete duality" property.

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vector of discrete gradient (and not only its normal component on edges of control volumes); the question of suitable reconstruction has been studied extensively over the last decade, and the ideology of "full gradient reconstruction" (or Multi-Point Flux Approximations) is well established by now.

[3]as we state it just below, discrete gradients are not fully defined

By that time, "discrete duality" properties were not formalized as much as they are now[4] but actually this structure property implied that

– *there is a kind of summation-by-parts property for the scheme*
– *the scheme is equivalent to minimization of a discrete energy functional built from* $|\nabla^{\mathfrak{T}} u^{\mathfrak{T}}|^p$.

Notice that the latter property readily implies existence and uniqueness (from strict convexity of the energy functional) of a discrete solution. More importantly, it permits to use descent methods (conjugate gradient,. . .) to solve in practice the nonlinear algebraic system that we obtain. Later, we have realized that the coordination-decomposition method of Glowinski and Marrocco [89] gives even better numerical results; in contrast, we have ruled out the Newton method because of the practical impossibility to provide an accurate enough initial guess.

The main result of [04[3]] is:

if the problem $-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = f$ *(with homogeneous Dirichlet BC)*
has a solution u_e *that is known to be* $W^{2,p}$*-regular, then our scheme approximates* u_e
at the rate of $h^{\min\{p-1,1/(p-1)\}}$ *at least, in the "discrete* $W^{1,p}$*" norm.*

The proof is based on comparing of $u^{\mathfrak{T}}$ with the projection $u_e^{\mathfrak{T}}$ on the mesh of the exact solution u_e; we see $u_e^{\mathfrak{T}}$ as an approximate solution of the scheme, subtract and analyze the two discrete equations that we formally write as

$$\text{“}-\mathrm{div}^{\mathfrak{T}}\mathfrak{a}(\nabla^{\mathfrak{T}} u^{\mathfrak{T}}) = f^{\mathfrak{T}}\text{” and “}-\mathrm{div}^{\mathfrak{T}}\mathfrak{a}(\nabla^{\mathfrak{T}} u_e^{\mathfrak{T}}) = f^{\mathfrak{T}} + r^{\mathfrak{T}}\text{”}.$$

The convergence order $\min\{(p-1), 1/(p-1)\}$ comes from the two (local) Hölder continuity properties:

$$\mathfrak{a} : \xi \mapsto |\xi|^{p-2}\xi \quad \begin{array}{l}\text{is (locally) } \alpha := \min\{1,(p-1)\}\text{-Hölder continuous}\\ \text{with a (locally) } \beta := \min\{\frac{1}{p-1},1\}\text{-Hölder continuous inverse.}\end{array}$$

It is the product $\alpha \times \beta$ that appears in the above error estimate. Indeed, the exponent α of $\mathfrak{a}$ comes from the remainder term $r^{\mathfrak{T}}$ of the equation "$-\mathrm{div}^{\mathfrak{T}}\mathfrak{a}(\nabla^{\mathfrak{T}} u_e^{\mathfrak{T}}) = f^{\mathfrak{T}} + r^{\mathfrak{T}}$"; we have $r^{\mathfrak{T}} \sim h^\alpha$ in case u_e is $W^{2,p}$-regular. The exponent β appears in the estimate of $|\nabla^{\mathfrak{T}} u_e^{\mathfrak{T}} - \nabla^{\mathfrak{T}} u^{\mathfrak{T}}|$ via the scalar product $(\mathfrak{a}(\nabla^{\mathfrak{T}} u_e^{\mathfrak{T}}) - \mathfrak{a}(\nabla^{\mathfrak{T}} u^{\mathfrak{T}})) \cdot (\nabla^{\mathfrak{T}} u_e^{\mathfrak{T}} - \nabla^{\mathfrak{T}} u^{\mathfrak{T}})$.

The paper [04[3]] is concluded with numerical examples that show that the predicted orders are pessimistic, especially as p approaches 1 or ∞. Moreover, the above convergence result is a conditional one: for $p \geq 2$, no regularity condition on the right-hand side f is known that ensures the $W^{2,p}$ regularity of u_e. This two facts led us to continue, in two different directions, the work on error estimates for the scheme.

5.2.2 Error estimates for regular solutions on uniform meshes

The first direction was:

since in [04[3]] we have used a non-justified regularity result anyway,
let us assume as much regularity as needed to arrive to superconvergence rates.

[4]for DDFV schemes of Section 5.3.1, gradient schemes of [82] and the mimetic finite difference schemes (see, e.g., [39]), discrete duality property is an essential feature or even a part of the definition of the scheme

Superconvergence (convergence with the h^2 rate) is often observed in practice for finite volume approximation of diffusion operators[5], but it is difficult to justify.

On uniform meshes and for isotropic operators, superconvergence estimates can be proved theoretically, by using cancelations in Taylor expansions; yet on a nonlinear operator, one needs to expand $\mathfrak{a}(\nabla u_e)$ at order four, and the nonlinearity of $\mathfrak{a}$ makes a direct calculation extremely painful. In [06[1]] (with F. Boyer and F. Hubert)

we managed to organize these heavy calculations in a rather readable[6] way.

The hint was, introduce reflection operators T_x, T_y: $u(-x, y) = T_x u(x, y)$, etc., and keep record of the cancelations using parity or imparity properties of T_x, T_y, their compositions and derivatives.

Recall that for $p \geq 2$, the basic $W^{1,p}$ convergence order obtained in [04[3]] is in $h^{1/(p-1)}$. The results of [06[1]] are, roughly speaking, the following:

for a $W^{4,1}$ solution u_e, an $h^{2/(p-1)}$ convergence rate is shown for $p \geq 4$; for $p = 2$, we get a convergence rate of $h^2|\ln h|$; finally, for $3 < p < 4$ rates intermediate between $h^{1/(p-1)}$ and $h^{2/(p-1)}$ are obtained.

Roughly speaking, the convergence order doubles with respect to the result of [04[3]], but it still vanishes as $p \to \infty$. The result is interesting even for $p = 2$:

in particular, we get superconvergence estimates for nine-point finite volume approximations of the Laplacian on uniform cartesian meshes, under a verifiable[7] assumption on the source f.

One cannot use the method for $p < 2$ if ∇u_e happens to be zero, because $\mathfrak{a}$ is singular at the origin. Then, we asked the question of

what orders can be obtained for solutions without singular points ?

Clearly, solutions without critical points cannot exist when homogeneous Dirichlet boundary condition is imposed; therefore we consider non-homogeneous boundary condition.

Then, guided by a very pertinent remark of the referee of the first version of the paper, we obtained the most accurate results by interpolating between the general case and the case of solutions without critical points. Namely, following the idea of Barrett and Liu [16]

we say that a solution u_e of the p-laplacian is non-degenerate if $|\nabla u_e| \geq c > 0$, and u_e is ν-weakly degenerate if we have $|\nabla u_e|^{-\nu} \in L^1$. In [16], a sufficient condition for ν-weak degeneracy is stated in terms of integrability assumptions on $|f|^{-1}$.

[5] for anisotropic operators and unstructured meshes, one often observes convergence orders intermediate between h^1 and h^2; see e.g. the 2D and 3D benchmarks [92, 80] of the FVCA conference series

[6] *relatively* readable...

[7] recall that one weakness of the result of [04[3]] was the fact that the $W^{2,p}$ regularity could not be inferred from any known regularity assumption on f. For $p = 2$, we were able to use some Grisvard's results [90] to justify the $W^{4,1}$ regularity of a solution in the unit square; the difficulty only comes from the corners. Although we were not able to fully justify analogous results for the p-laplacian, by bootstrapping the regularity argument we can justify at least the local $W^{1,4}$ regularity of u_e when u_e is non-degenerate ($\equiv$without critical points); for the case where ∇u_e is periodic, up-to-the boundary $W^{4,1}$ regularity of u_e can be justified under a verifiable assumption on f.

Then we use the family of inequalities[8] of [16, Lemma 2.1]: for $\xi, \eta \in \mathbb{R}^2$,

$$\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\big) \geq C(p,t)|\xi - \eta|^t \big(|\xi| + |\eta|\big)^{p-t} \text{ as } p > 1,\ t \geq 2.$$

and adapt the choice of exponent $(p-t)$ to the given integrability of $|\nabla u_e|^{-\nu}$;

the optimal choice of the parameter t gives much stronger error estimates.

We refer to [06¹] for the long list of detailed statements; here, let us mention that

– *improved orders (including asymptotically stable ones, as $p \to \infty$) are obtained in $W^{1,q}$ norms (e.g. with $q \leq 2$) for non-degenerate or ν-weakly degenerate solutions*;
– *superconvergence*[9] *in $W^{1,\min\{2,p\}}$ and in L^∞ is true for non-degenerate solutions, for all p.*

In conclusion, the numerical experiments confirmed as optimal the h^2 convergence rate on very regular non-degenerate solutions of the p-laplacian.

5.2.3 Besov regularity and optimal error estimates for the p-laplacian on cartesian meshes

In this section, still based upon [04³] we take a direction opposite to the previous section:

Assuming only that $f \in L^{p'}$, we adapt the error estimation techniques to the generic regularity of solutions of the p-laplacian, $p > 2$.

The pre-history of this work was to benefit from the idea of Tyukhtin [151] that we found in the paper of Chow [52]:

Using the minimization properties of both *continuous and discrete solutions, one obtains better error estimates than those of the "traditional" method used in [04³].*

Yet it is clear that the method requires some extra regularity (with respect to the basic $W^{1,p}$ regularity) of a solution u_e; and one needs that the discrete solution possess the same regularity. But then,

the problem is, the a priori *assumption $u_e \in W^{2,p}$ made in [04³] tells us nothing about the "regularity" of the associated discrete solution $u^{\mathfrak{T}}$;*

we just cannot apply the technique of [151, 52] for the (hypothetic) $W^{2,p}$ solutions. We should rely on some *method of proof* of the extra regularity of u_e, method that one must also reproduce at the discrete level.

The appropriate regularity is a Besov regularity[10] : $u_e \in B^{1,1/(p-1)}_{\infty}$, obtained by Simon [148]. The argument to re-transcript into the discrete framework is the translation method as used by Simon.

[8] The Hölder continuity properties of $\mathfrak{a} : \xi \mapsto |\xi|^{p-2}\xi$ and of its inverse were the simplest particular case of these inequalities

[9] To be precise, we have not obtained h^2 rate but the rate of $h^2|\ln h|$

[10] The order $\frac{1}{p-1}$ is explained by the $1/(p-1)$-Hölder regularity of the inverse of the map $\mathfrak{a} : \xi \mapsto |\xi|^{p-2}\xi$, see Section 5.2.1

Thus, in the work [05[1]], F. Boyer, F. Hubert and myself carried out the following program:

- *define a discrete analogue of the Besov space* $B_\infty^{1,1/(p-1)}$;
- *prove the Besov regularity of the discrete solution* $u^{\mathfrak{T}}$ *by the translation method, and also prove the Besov regularity of the projection* $u_e^{\mathfrak{T}}$ *of the exact solution using methods of approximation theory*;
- *implement the estimation technique of [151, 52] (with adaptation to the Besov regularity) and deduce error estimates in* $h^{\frac{2}{p(p-1)}}$ *(for* $p \geq 3$*) for our scheme on uniform cartesian meshes.*

In the subsequent note [07[1]], we discussed the case of non-uniform cartesian meshes. Clearly, there is a difficulty to apply translation arguments on a non-uniform mesh;

in [07[1]] we managed to extend the above results
to non-uniform, but "smoothly refined" cartesian meshes.

The idea was to use variable translation vector fields, as in the paper [148] of Simon. Typically, the proof works when the mesh in use is the image of a uniform cartesian mesh by a sufficiently smooth coordinate transformation $(x, y) \mapsto (M(x), N(y))$. Let us stress that we still do not know how to define and use discrete Besov spaces on general unstructured meshes.

In conclusion, let us stress that the numerical tests in [05[1]], performed on radial solutions having almost precisely the Besov $B_\infty^{1,1/(p-1)}$ regularity, confirmed that the above theoretical convergence order is optimal.

5.3 Discrete duality finite volume schemes

Somewhat surprisingly, finite volume schemes for nonlinear diffusion and convection-diffusion PDEs continued to be my "second speciality" after my departure from Marseilles, along with (and in close interaction with) the PDE questions of the preceding Chapters. One focus of these works was on the so-called DDFV (Discrete Duality Finite Volume) schemes, first in 2D and then in 3D. I also looked at the 2D co-volume schemes, that are much simpler and still possess the same "discrete duality" property.

I do not attempt neither to present technical details[11], giving only the ideas of the construction and its main properties.

5.3.1 DDFV ("double") scheme in 2D and application to Leray-Lions elliptic problems

It seems to be a general rule that a numerical scheme is invented several times. The idea of "cell+vertex"-centered finite volume schemes appeared in the school of R. Nikolaïdes in early 1990ies, but then it was re-invented, in a much more developed form, by F. Hermeline [94] in late 1990ies and by K. Domelevo and P. Omnès [70] in early years 2000. The perspective of Hermeline was a rather practical one. Domelevo and Omnès presented their scheme from a theoretical numerical analysis viewpoint, and stated the remarkable "discrete duality" property of the scheme: the discrete operators $\nabla^{\mathfrak{T}}$ and $-\mathrm{div}^{\mathfrak{T}}$ of this scheme are duals of each

[11]The interested reader may consult the short note [11[7]] which, along with numerical results, contains a succinct presentation of the 3D DDFV scheme and of the associated formalism.

other, wrt well-chosen L^2 scalar products. In other words, a discrete Green-Gauss formula is available within the DDFV framework.

Discrete duality implies the essential structure-preservation properties listed in Section 5.1, therefore, the 2D "DDFV" (Discrete Duality Finite Volume) scheme of [70] is a scheme that perfectly illustrates the work [04[1]]. And with respect to the cartesian schemes of [04[3], 06[1], 05[1]], the DDFV scheme has the clear advantage since it works on very general meshes and it requires a simpler notational formalism. Therefore we continued the collaboration with F. Boyer and F. Hubert, now looking at the 2D DDFV scheme for approximation of general Leray-Lions elliptic problem

$$(LLFV) \qquad -\operatorname{div} \mathfrak{a}(x, \nabla u) = f \quad \text{with general Dirichlet BC.}$$

Being a Leray-Lions operator means that $\mathfrak{a}$ is a generalization of the p-laplacian case $\mathfrak{a}(\xi) = |\xi|^{p-2}\xi$ in respect of coercivity, growth and monotonicity conditions.

Let me explain very briefly the idea of the DDFV scheme in 2D, and provide the notation necessary for stating some sample results. I discuss the properties that I find most useful for understanding convergence[12].

The DDFV scheme is inspired by diamond schemes of Coudière, Vila, Villedieu [59] with the difference that the vertex values are not interpolated from the center values, but kept as independent unknowns.

A mesh (called primal) is selected; it can be quite general: non-orthogonal, non-conforming, etc.. Some mesh called dual is constructed;

centers of the primal mesh are vertices of the dual mesh and vice versa;
an unknown is attached to centres of primal and to centres of dual mesh.

Thus, with respect to the initial mesh the scheme is Cell+Vertex-centered. Two partitions into primal and dual volumes induce a third partition: the diamonds.

a diamond is constructed on a couple of primal centers x_K, x_L and a couple of dual centers x_{K^*}, x_{L^*}

where K, L are neighbours, K^*, L^* are dual neighbours, and $[x_{K^*}, x_{L^*}] = \partial K \cap \partial L$. We denote the (primal+double) mesh by $\mathfrak{T}$, and the associated diamond mesh, by $\mathfrak{D}$.

Then the discrete gradient $\nabla^{\mathfrak{T}}$ is reconstructed diamond-wise:

In a diamond, the vector $\nabla_D u^{\mathfrak{T}}$ is reconstructed from the four entries $u_K, u_L, u_{K^*}, u_{L^*}$ of $u^{\mathfrak{T}}$ as the unique vector having the projections $\frac{u_L - u_K}{|\overrightarrow{x_K x_L}|} \frac{\overrightarrow{x_K x_L}}{|\overrightarrow{x_K x_L}|}$ and $\frac{u_{L^*} - u_{K^*}}{|\overrightarrow{x_{K^*} x_{L^*}}|} \frac{\overrightarrow{x_{K^*} x_{L^*}}}{|\overrightarrow{x_{K^*} x_{L^*}}|}$ in the directions $\overrightarrow{x_K x_L}$ and $\overrightarrow{x_{K^*} x_{L^*}}$, respectively.

In other words, one component of the 2D gradient vector $\nabla_D u^{\mathfrak{T}}$ is reconstructed as the divided difference from the primal mesh values u_K, u_L, and another direction is reconstructed analogously from the dual mesh values u_{K^*}, u_{L^*}.

Denoting by $\mathbb{R}^{\mathfrak{T}}$ the space of discrete functions (consisting per one value u_K per each primal volume K and one value u_{K^*} per each dual volume K^*), denoting by $(\mathbb{R}^2)^{\mathfrak{D}}$ the space of

[12] I tacitly mean that a family $(\mathfrak{T}_h)_h$ of meshes is given, parametrized by the mesh size h, that there exist discrete solutions $u^{\mathfrak{T}_h}$ of some PDE discretized on the mesh $\mathfrak{T}_h$, and we are interested in studying the convergence of $u^{\mathfrak{T}_h}$ to an exact solution u of the PDE

discrete vector fields (consisting of one value per diamond D)[13], we therefore have the discrete gradient operator

$$\nabla^{\mathfrak{T}} : \mathbb{R}^{\mathfrak{T}} \mapsto (\mathbb{R}^2)^{\mathfrak{D}}.$$

Moreover, by the standard finite volume construction, for each primal volume K one defines the value $\mathrm{div}_K \mathcal{F}^{\mathfrak{T}}$ of a discrete field $\mathcal{F}^{\mathfrak{D}}$ by integrating the normal component of the piecewise constant field $\mathcal{F}^{\mathfrak{T}}$ on the piecewise flat boundary ∂K. The same construction is used on dual volumes. This defines the discrete divergence operator

$$\mathrm{div}^{\mathfrak{T}} : (\mathbb{R}^2)^{\mathfrak{D}} \mapsto \mathbb{R}^{\mathfrak{T}}.$$

To be precise, boundary conditions should be accounted for: the Dirichlet ones in the definition of discrete gradient operator, the Neumann ones, in the definition of discrete divergence operator.

Now, the key fact of the theory is:

upon introducing natural scalar products $\left[\!\left[\cdot, \cdot \right]\!\right]_\Omega$ *on* $\mathbb{R}^{\mathfrak{T}}$ *and* $\left\{\!\!\left\{ \cdot, \cdot \right\}\!\!\right\}_\Omega$ *on* $(\mathbb{R}^2)^{\mathfrak{T}}$,
the operators $\nabla^{\mathfrak{T}}$ *and* $-\mathrm{div}^{\mathfrak{T}}$ *are dual to each other, in the sense:*
$\forall u^{\mathfrak{T}} \in \mathbb{R}^{\mathfrak{T}}\ \forall \mathcal{F}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{D}} \quad \left[\!\left[-\mathrm{div}^{\mathfrak{T}} u^{\mathfrak{T}}, \mathcal{F}^{\mathfrak{T}} \right]\!\right]_\Omega = \left\{\!\!\left\{ \nabla^{\mathfrak{T}} u^{\mathfrak{T}}, \mathcal{F}^{\mathfrak{T}} \right\}\!\!\right\}_\Omega$
if one of the two discrete objects is zero on the boundary.

This fact assesses a far-reaching analogy between the discrete and the continuous frameworks. Namely,

the discrete duality implies that the DDFV discretization $u^{\mathfrak{T}} \mapsto -\mathrm{div}^{\mathfrak{T}} \mathfrak{a}(\nabla^{\mathfrak{T}} u^{\mathfrak{T}})$
of a Leray-Lions operator $u \mapsto -\mathrm{div}\, \mathfrak{a}(\nabla u)$ *is "structure-preserving".*

To be specific, the discrete operators fulfill the same coercivity, growth and monotonicity properties as the continuous one[14]. Moreover, if the continuous operator derived from minimization of a potential $u \mapsto \int_\Omega \Phi(\nabla u)$, then the discrete operator derives from minimization of a discrete potential $u^{\mathfrak{T}} \mapsto \int_\Omega \Phi(\nabla^{\mathfrak{T}} u^{\mathfrak{T}})$.

Another cornerstone for the analysis is, consistency properties. For the two scalar products,

$$\text{for } u \in L^p,\ v \in L^{p'},\ \left[\!\left[\mathbb{P}^{\mathfrak{T}} u, \mathbb{P}^{\mathfrak{T}} v \right]\!\right]_\Omega \to \int_\Omega uv,$$
$$\text{for } \mathcal{F} \in L^p,\ \mathcal{G} \in L^{p'},\ \left\{\!\!\left\{ \mathbb{P}^{\mathfrak{D}} \mathcal{F}, \mathbb{P}^{\mathfrak{D}} \mathcal{G} \right\}\!\!\right\}_\Omega \to \int_\Omega \mathcal{F} \cdot \mathcal{G},$$

where $\mathbb{P}^{\mathfrak{T}}$ (respectively, $\mathbb{P}^{\mathfrak{D}}$) is an operator of projection of L^1_{loc} functions (respectively, fields) on the space of discrete functions (respectively, of discrete fields). The projection may use the mean value per mesh element (in case of merely integrable functions or fields) of some point value per element (in the case of regular functions of fields).

[13] Whenever convenient, we tacitly identify $\mathcal{F}^{\mathfrak{T}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ with a piecewise constant function on the diamond mesh $\mathfrak{D}$ (i.e., it is constant per diamond).
For discrete functions, the identification is more delicate: we may identify a discrete function $u^{\mathfrak{T}}$ with the couple $(u^{\mathfrak{T},\circ}, u^{\mathfrak{T},*})$ where $u^{\mathfrak{T},\circ}$ is piecewise constant on the primal mesh, and $u^{\mathfrak{T},*}$ is piecewise constant on the dual mesh. But we also show in [07²] that the natural object to look at is the function $\frac{1}{2}u^{\mathfrak{T},\circ} + \frac{1}{2}u^{\mathfrak{T},*}$ on Ω.

[14] moreover, these properties are quantified with constants independent of the size of the mesh $\mathfrak{T}$

For the discrete gradient, we have consistency "in the strong sense":

$$\text{for } u \in C^1, \quad \nabla^{\mathfrak{T}_h}\mathbb{P}^{\mathfrak{T}_h}u \text{ (lifted to a function on } \Omega) \text{ converges to } \nabla u \text{ as } h \to 0,$$

where we take the projection $(\mathbb{P}^{\mathfrak{T}}u)_K = u(x_K)$. This property is due to the consistency of the reconstruction formula on affine functions. Then it follows from the discrete duality property and the Green-Gauss formula that the discrete divergence operator is consistent "in the weak sense"[15],

$$\text{for } \mathcal{F} \in C^1,\, u \in C^1, \quad \Big[\!\Big[\operatorname{div}^{\mathfrak{T}_h}\mathbb{P}^{D_h}\mathcal{F}, \mathbb{P}^{\mathfrak{T}_h}u\Big]\!\Big]_\Omega \to \int (\operatorname{div}\mathcal{F})\, u \text{ as } h \to 0.$$

One has analogous results for $W^{1,p}$ functions and fields, under stronger proportionality restrictions on the meshes.

We skip many details on the consistency issues: the interested reader may consult [07[2]] and [[4]]. In particular,

in [07[2]] we carefully treat the case of non-homogeneous Dirichlet BC.

The main results of [07[2]] (see also the short note [05[2]]) are:

- *mathematical framework and tools for analyzing the 2D DDFV operators and schemes: consistency, asymptotic compactness*[16]*, Poincaré inequalities, etc. (see Section 5.4)*;
- *convergence proof for DDFV schemes on Leray-Lions operators, for general solutions*
- *"basic" error estimates, of the same kind as in [04[3]], for $W^{2,p}$-regular solutions.*

F. Boyer and F. Hubert then pursued the work on 2D DDFV schemes for the case of piecewise regular heterogeneous Leray-Lions operator $u \mapsto -\operatorname{div}\mathfrak{a}(x, \nabla u)$ and created the so-called m-DDFV scheme [36]. DDFV schemes found many applications in the context of linear anisotropic problems on general meshes, and in some nonlinear problems. Their numerical behaviour, as compared to other new and well-established schemes, was illustrated by the FVCA5 benchmark (see [92] and the subsequent papers of the same volume). For implementation issues, including the proof of convergence of the "fully practical" coordination-decomposition algorithm, I refer to Boyer and Hubert [36].

My interest went rather to application of DDFV schemes for doubly nonlinear convection-diffusion equations ([10[2]]) and to generalizing DDFV schemes to three space dimensions.

5.3.2 A gradient reconstruction formula in 2D and discrete duality co-volume scheme

The origin of the 3D generalization[17] that we found of the 2D DDFV scheme was the "gradient reconstruction identity" of the work [04[1]] with M. Gutnic and P. Wittbold. The formula of

[15] For 3D CeVe-DDFV schemes on uniform cartesian meshes (this case is particularly interesting for 3D image processing applications), we have shown in [[4]] that the discrete divergence is strongly consistent.

[16] by asymptotic compactness we mean properties of the kind:
if $(u^{\mathfrak{T}_h})_h$ (where the size h of $\mathfrak{T}_h$ goes to zero) is a family bounded in some "coherent" discrete norms associated to $\mathfrak{T}_h$, then the family of the associated discrete function on Ω converges to u in the *ad hoc* sense.

[17] Here let me stress that there exist several generalizations established by several authors or groups of authors. The ours, now baptized "3D CeVe-DDFV", was discovered independently by F. Hermeline ([96], but see also [95]) and also by M. Bendahmane, K.H. Karlsen and myself while working on the paper [10[2]]. Actually,

[04[1]] stated that

$$(Rec_\perp)\quad \begin{array}{l} \textit{if } \nu_1,\ \nu_2,\ \nu_3 \textit{ are the unit vectors parallel to three sides of a triangle } T, \\ \textit{the point } O \textit{ is the center of the circumscribed circle of } T, \\ \textit{and } T_1, T_2, T_3 \textit{ are the three subtriangles of } T \textit{ with vertex } O, \\ \textit{then the operator } \frac{2}{|T|}\big(|T_1|\, Proj_{\nu_1} + |T_2|\, Proj_{\nu_2} + |T_3|\, Proj_{\nu_3}\big) \textit{ is the identity map on } \mathbb{R}^2. \end{array}$$

Here, $|T_i|$ is the area of T_i (if O falls outside T, we take the signed area).

Replacing the projection operators Proj_{ν_i} in $(Rec_\perp)$ by $\frac{u_{i+1}-u_i}{|x_{i+1}-x_i|}\,\nu_i$ we get a reconstruction formula that takes divided differences along triangle T's sides and yields the gradient of the affine function interpolating the three values.

Reconstruction formula coming from $(Rec_\perp)$ was used in the 3D scheme of [10[2]] in order to define the discrete gradient on every face of the primal mesh (each face is a triangle, its vertices $x_{K_i^*}$, $i = 1, 2, 3$, are centers of the dual mesh, thus with the three values $u_{K_i^*}$ we reconstruct the projection of the discrete gradient on the face). Gradient reconstructed by this formula leads indeed to a discrete duality scheme completely analogous to the 2D scheme of the previous section. Due to $(Rec_\perp)$, the gradient reconstruction is consistent with affine functions.

The above reconstruction formula works on "orthogonal" meshes in 2D. Starting from the year 2000, I was asking myself the question:

what could be a generalization of $(Rec_\perp)$ *?*

By generalization, I mean higher-dimensional analogues or two-dimensional properties with general choice of the point O and general polygon (not necessarily a triangle). It was clear that the property still holds for an inscriptible polygon; and that replacing 2D triangle T by a 3D tetrahedron, we do not find any straightforward analogue of $(Rec_\perp)$ (see [10[2], Appendix B] where we explain the problem). Some straightforward 3D generalizations are true on very structured meshes (e.g., on uniform cartesian meshes). In 2007, letting O to be an arbitrary point of an arbitrary polygon T

we eventually found "the good point of view" on $(Rec_\perp)$ *that allowed for a generalization*:
choosing O *a point to partition an* m*-sided polygon* T *into* m *triangles* T_i,
we discovered[18] *a remarkable identity of the form*

$$(Rec_{gen})\qquad \forall r \in \mathbb{R}^2 \qquad r = \frac{2}{|T|}\sum_{i=1}^m |T_i|(r\cdot\nu_i)\,\tau_i$$

where the vector τ_i is constructed in some precise way, see [08[3]] (τ_i coincides with ν_i if O is the circumcenter of O). The remarkable identity (Rec_{gen}) is used in the same way as $(Rec_\perp)$, and

the identity (Rec_{gen}) *plays, for vertex-centered schemes, the same role*
that the celebrated "magical formula" (see [71]) plays for cell-centered schemes.

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we first understood the case of tetrahedral mesh with orthogonality condition, but then, thanks to discussions with F. Hubert, we have found the gradient reconstruction formula for the general case, see [08[3]].
Recently, it was understood in collaboration with Ch. Pierre and Y. Coudière that the CeVe-DDFV scheme is essentially the pioneering 3D DDFV scheme from the PhD thesis of Pierre [139] and the work [58]. For a detailed description of the 3D CeVe-DDFV scheme and comparison to other schemes, we refer to [12[1]] and the short note [11[7]].

[18]this property in an elementary geometrical identity which was certainly demonstrated long time ago; but to the best of my knowledge, it was never exploited in finite volumes context, and I have found no reference proving or citing this identity

As previously, replacing $(r \cdot \nu_i)$ in (Rec_{gen}) by divided differences $\frac{u_{i+1}-u_i}{|x_{i+1}-x_i|}$ we find a "reconstruction formula" for a discrete gradient $\nabla_T u^{\mathfrak{T}}$ on a polygon T in terms of the values of $u^{\mathfrak{T}}$ given at the vertices of T.

As an illustration of the usefulness of this "reconstruction formula", with M. Bendahmane and K.H. Karlsen

in [08[3]] we have presented the 2D "discrete duality" co-volume scheme for which the diamond mesh of polygons T with arbitrary centers is used for representing vector fields, and its "median dual mesh" (or "Donald mesh") is used to represent discrete functions.

This scheme is known in many particular cases, see, e.g., Afif and Amaziane [3], Handlovičovà and Mikula [91], and this is a simple alternative to 2D DDFV schemes. It the diamond mesh is made of triangles, the method is closely related to mixed finite elements method.

Thus we have another example of discrete duality scheme in 2D, on very general meshes.

5.3.3 A 3D Cell-Vertex DDFV scheme

From the reconstruction formula (Rec_{gen}) of the previous section we have derived in [08[3]] the generalization of the 3D DDFV scheme of [10[2]], now called 3D CeVe-DDFV scheme. The detailed description (including generalizations that allow the centers of volumes and faces be not contained within the corresponding volume or face, etc.), proof of discrete duality and numerical tests on linear diffusion problems can be found in [12[1], 11[7]] (see [10[2]] for the case of orthogonal tetrahedral mesh). Let us give a brief description:

- *in 3D CeVe-DDFV, we start with an arbitrary (possibly non conformal) primal polygonal mesh with arbitrary cell and face centers, we use* middlepoints *for edge centers, and we construct the dual mesh which vertices are cell, face, or edge centers*;
- *unknowns are attached only to cell centers and vertices (=centers of dual volumes)*;
- *diamonds are constructed on a face* $K|L$ *and on the two primal cell centers* x_K, x_L;
- *discrete gradient is assembled from the projection on the direction* $\overrightarrow{x_K x_L}$ *(treated as in* 2*D) and from the projection on the face* $K|L$*, which is reconstructed using* (Rec_{gen});
- *the discrete duality property holds, with the* ad hoc *re-definition of scalar product* $\left[\!\left[\cdot,\cdot\right]\!\right]_\Omega$.

Numerical examples analyzed in the benchmark [80] show that the CeVe-DDFV scheme supports comparison with respect to many other schemes designed for anisotropic linear diffusion problems on general meshes; it has a clear disadvantage at the level of number of unknowns (per fixed primal mesh!) and of the stencil, but it is rather robust, and provides a good approximation of the gradient.

Convergence analysis for the 3D CeVe-DDFV scheme is quite similar to the 2D analysis of [07[2]], and we discuss some of the tools in more detail in the next section. In the paper [11[4]], with M. Bendahmane, K.H. Karlsen and Ch. Pierre we have applied the scheme for approximation of the so-called bidomain model of cardiac electric activity, proving convergence and giving numerical examples (see Section 6.2 for details). In [[4]], we come back to problem $(EqFV)$ of Section 5.1 and prove convergence of the associated 3D CeVe-DDFV scheme, as an illustration for use of discrete analysis techniques that we now discuss.

5.4 Some analysis tools for finite volume schemes

Convergence proofs for finite volume schemes that we developed in the works already discussed in this Chapter have a common feature: they tend to be presented, as much as possible, along the lines of the corresponding "PDE" proofs. While doing this, one necessarily uses discrete versions of different "continuous" results, such as Poincaré or Sobolev embedding inequalities, compactness criteria such as the Aubin-Lions-Simon Lemma or the Kruzhkov lemma of Section 4.1.1, chain rule properties such as the Mignot-Bamberger/Alt-Luckhaus lemma, etc. Developing this kind of general tools was one of my main activities in analysis of finite volume methods. Below, I briefly describe these contributions.

5.4.1 Discrete Poincaré and Sobolev inequalities

The Poincaré and Sobolev inequalities were proved, in particular, by Eymard, Gallouët and Herbin in [76] (see also [56]), under some uniform proportionality assumptions on the meshes. Later, in the Appendix of the paper [77] of the same authors the Sobolev inequalities for the case of Dirichlet boundary condition were deduced from the original Nirenberg approach, by mimicking the "continuous" case.

I contributed to further development of these two technical results.

In [04[1]], with M. Gutnic and P. Wittbold we have shown that

> *the Poincaré inequality for standard (e.g., two-point) finite volume schemes*
> *with the homogeneous Dirichlet boundary condition*
> *holds* without *any proportionality assumptions on the mesh.*

The idea of the proof is to separate the case where u_K, u_L are of similar order of magnitude and the case where one of the two is considerably larger; in the second case, we have $|u_K|+|u_L| \sim |u_K - u_L|$ and we estimate $|u_K - u_L|$ by using (or rather re-using) the discrete gradient. Analogous result holds for DDFV schemes, see [07[2]].

In [11[1], Appendix B], with M. Bendahmane and R. Ruiz Baier we have extended the approach of Eymard, Gallouët and Herbin [77, Appendix] to cover the important case of Neumann boundary conditions in Sobolev embedding inequalities:

> *we gave a simple proof of the discrete inequality* $\|u^{\mathfrak{T}}\|_{p^*} \leq C(\|\nabla^{\mathfrak{T}} u^{\mathfrak{T}}\|_p + |(\bar{u})^{\mathfrak{T}}|)$
> *where* $(\bar{u})^{\mathfrak{T}}$ *is the mean value of* $u^{\mathfrak{T}}$ *over some subdomain*,

and p is the critical Sobolev exponent. The argument starts by *using* the compact embedding of BV in L^1, which applies directly at the discrete level. Then we exploit the Poincaré inequality of [76] for the Neumann case to get a first estimate of the mean value $(\bar{u})^{\mathfrak{T}}$; finally, we use the bootstrap method known in the continuous case, following closely the ideas of [77, Appendix].

5.4.2 Time compactness tools for evolution PDEs in divergence form

I have already discussed the Kruzhkov time compactness Lemma (see [107]) in Sections 4.1.1 and 5.1. One point of the work [04[1]] was, precisely, to use the original ("continuous") Kruzhkov Lemma for compactness of *discrete* solutions. Later on, I have arrived to the conclusion that it is more efficient to provide a discrete version of the Kruzhkov lemma, and we done so in [11[1]] (with M. Bendahmane and R. Ruiz Baier, for cell-centered finite volume schemes) and in [[4]] (with M. Bendahmane and F. Hubert, for DDFV schemes).

The Kruzhkov lemma is a very convenient tool, namely because it is a fully L^1 (or L^1_{loc}) based result; no particular discrete space is needed to state the result. This makes quite simple the assumptions of the discrete version. Consider space-and-time discrete functions $u^{\mathfrak{T},\Delta t}$ where Δt is a time discretization step, and the discrete evolution equations are satisfied [19]:

$$(DEvEq) \quad \frac{b(u^{\mathfrak{T},n}) - b(u^{\mathfrak{T},n-1})}{\Delta t} = \operatorname{div}^{\mathfrak{T}} \mathcal{F}^{\mathfrak{T},n}$$

with the obvious meaning of notation. We have proved the following:

under three rather non-restrictive assumptions on meshes and discrete operators[20]*,*
an L^1_{loc}-bounded family of discrete functions $(b(u^{\mathfrak{T}_h,\Delta t_h}))_h$
(where $u^{\mathfrak{T}_h,\Delta t_h}$ give rise to L^1_{loc}-bounded discrete gradients family $(\nabla^{\mathfrak{T}_h} u^{\mathfrak{T}_h,\Delta t_h})_h$)
that satisfies discrete evolution equations of the form $(DEvEq)$ with L^1_{loc}-bounded $(\mathcal{F}^{\mathfrak{T}_h,\Delta t_h})_h$
is relatively compact on the space-time cylinder in the L^1_{loc} strong sense.

The proofs are quite straightforward: one follows the arguments of the "continuous" Kruzhkov lemma as given in [Th].

We have used the above lemma for treating various degenerate parabolic problems in [11[1], 11[4], [4]] with M. Bendahmane et al.; but, for the case of the degenerate parabolic-hyperbolic problem $(PbPH)$ of Section 3.4, the lemma is not applicable.

In [10[2]], for problem $(PbPH)$ we have used a direct estimation of L^1 time translates of the discrete solutions, following the variational technique of Alt and Luckhaus [6].

The technique is very well known for L^2 estimates (see in particular Eymard, Gallouët, Herbin and Michel [79]), but if the nonlinearity $\varphi(\cdot)$ in the diffusion term $-\operatorname{div}\varphi(v)$ is not Lipschitz, one needs subtler (L^1) techniques to estimate the time translates of $\varphi(v^{\mathfrak{T}})$. Our tool was a careful use of concave moduli of continuity and their inverse functions, in a way similar to my works [[7], 10[1]] with M. Maliki.

With the two above techniques in hand, I wanted to summarize my understanding of the issue of time compactness for discretized evolution PDEs; beyond the above results, the main impulse was provided by the recent work of Gallouët and Latché [86] where the authors proved a very general discrete version of the Aubin-Lions-Simon compactness argument (see Simon [149])[21]. A discussion with E. Emmrich brought to my attention another variant of

[19] the results of [11[1], [4]] were shown for Euler schemes in time, but it should not be difficult to get versions for higher-order in time schemes

[20] we require some kind of very weak summation-by-parts property (discrete duality is a much stronger property); the boundedness of the operator $\nabla u \mapsto \nabla^{\mathfrak{T}}\mathbb{P}^{\mathfrak{T}}u$ in L^∞; and we require an estimate that says, roughly speaking, that the operator $\nabla^{\mathfrak{T}}$ has its kernel reduced to zero.

[21] The result [86] manages to encompass, in a rather astonishing way, a very wide setting of "moving discrete functional spaces". Indeed, dealing with h-dependent Sobolev and dual Sobolev spaces is the key difficulty, in comparison to the L^1-based discrete Kruzhkov lemma. In most of the practical cases, both results can be applied; the Kruzhkov lemma has the advantage of being "a bit more nonlinear". The Aubin-Lions-Simon lemma is a widely used result, so its discrete version permits to mimic convergence proofs developed for a huge variety of evolution PDEs.

the Simon lemma that has no need of being "discretized": see Emmrich and Thalhammer [75]. Indeed, when time translates are estimated from Slobodetskii space bounds of fractional time derivatives, piecewise constant in time functions can be considered for order $s \in (0, 1/2)$ of derivation.

All the above time compactness results were briefly discussed and illustrated in the note [11[6]]. On the same occasion, I formalized the compactness-from-monotonicity approach described in Section 4.1.2; indeed, this technique allows to get compactness of (order-preserving) finite volume discretizations of degenerate parabolic equations in the situation where all the preceding methods fail, due to the elliptic-parabolic degeneracy of the problem.

The interested reader may refer to [11[6]] for the ideas and results discussed above.

5.4.3 Penalization operators and discretization of nonlinear reaction terms

The particularity of DDFV schemes is that the discrete solution is a two-component one[22]: $u^{\mathfrak{T}}$ is the couple $(u^{\mathfrak{T},\circ}, u^{\mathfrak{T},*})$ consisting of the solution on the primal mesh and the solution on the dual mesh. A typical asymptotic compactness result (see [07[2]]for 2D and [[4]] for 3D) is:

if $u^{\mathfrak{T}_h} \in \mathbb{R}_0^{\mathfrak{T}}$ (i.e. $u^{\mathfrak{T}_h}$ is zero on the boundary) and $(\nabla^{\mathfrak{T}_h} u^{\mathfrak{T}_h})_h$ is L^p bounded, $p > 1$, then there exist $u^\circ, u^* \in W^{1,p}$ such that $u^{\mathfrak{T},\circ} \to u^o$, $u^{\mathfrak{T},*} \to u^*$ strongly in L^p and $\nabla^{\mathfrak{T}_h} u^{\mathfrak{T}_h} \to \nabla(\frac{1}{d}u^\circ + \frac{d-1}{d}u^*)$ weakly in L^p, as $h = size(\mathfrak{T}_h) \to 0$.

Here d is the space dimension. Therefore, the natural limit of $(u_h^{\mathfrak{T}})_h$ is the limit of the averaged function $(\frac{1}{d}u^\circ + \frac{d-1}{d}u^*)$; and it is easy to construct examples where the discrete gradients are bounded and the two components $u^{\mathfrak{T}_h,\circ}, u^{\mathfrak{T}_h,*}$ do converge to *different* limits u°, u^*.

In practical applications, whenever we have uniqueness of the continuous solution and somewhat strong proportionality assumptions on the meshes, and provided we can show the strong convergence of discrete gradients, we eventually deduce that $u^\circ = u^*$: see [07[2]]. But in general, having the two components is a major technical concern for DDFV schemes. Thus for the sake of convergence analysis, in [10[2]] with M. Bendahmane and K.H. Karlsen

we have used a penalization operator that penalizes the discrepancy between $u^{\mathfrak{T},\circ}$ and $u^{\mathfrak{T},*}$.

Adding this operator amounts to adding a diffusion term of the kind $\bar{\bar{o}}_{h\to 0}\Delta u$ to the discrete equations; it does not enlarge the stencil of the DDFV scheme.

Adding the penalization operator to the scheme brings the additional a priori *estimate; it permits to conclude that $u^\circ = u^*$ in the above compactness result.*

Analogous situation happens for the discrete Kruzhkov Lemma in its DDFV version: without penalization, we have "per component compactness" (see [[4]]), and penalization estimate permits to identify the limits of the two components.

Further,

A related difficulty appears when nonlinear reaction terms are discretized with a DDFV scheme.

Whenever we need to pass to the limit in a term $h(u^{\mathfrak{T}})$ where $h(\cdot)$ is a nonlinear function, with a straightforward DDFV discretization $[h(u)]^{\mathfrak{T}} := h(u^{\mathfrak{T}})$ we would find the "wrong" limit function $(\frac{1}{d}h(u^\circ) + \frac{d-1}{d}h(u^*))$ in the place of the expected limit $h(\frac{1}{d}u^\circ + \frac{d-1}{d}u^*)$;

if we are unable to guarantee that u° and u^* coincide, convergence proof cannot be concluded.

[22]In the 3D CeVeFE-DDFV scheme of Coudière and Hubert [57], the solution is even three-component

In the work [11[4]] with M. Bendahmane, K.H. Karlsen and Ch. Pierre,

we modify the discretization of the reaction term: for $[h(u)]^{\mathfrak{T}}$ we take the projection on the DDFV mesh of the function $h(\frac{1}{d}h(u^\circ) + \frac{d-1}{d}h(u^*))$.

This operation resolves the difficulty and it does not increase the stencil of the scheme[23] Another way to prevent problems is to add the penalization operator, but this has the disadvantage of adding a small amount of artificial diffusion to the discretized problem.

5.4.4 Entropy inequalities in DDFV schemes on orthogonal meshes

When one treats a discretized PDE by "variational" techniques, the unknown solution $u^{\mathfrak{T}}$ is used as test function, and the discrete duality property is brought to use. But degenerate convection-diffusion equations should be treated using the methods of entropy solutions; this includes the use of nonlinear test functions of the form $\eta(u^{\mathfrak{T}})$ and the use of *chain rules*[24].

In the discrete setting, chain rules fail, and in order to carry out the convergence analysis for the problem studied in [10[2]] with M. Bendahmane and K.H. Karlsen, we had to replace chain rules with convexity inequalities. Our conclusion was,

Methods of entropy solutions can be used on DDFV discretizations of Leray-Lions operators if, firstly, the operator takes the form $-\mathrm{div}\,\big(k(|\nabla\varphi(u)|)\nabla\varphi(u)\big)$ and, secondly, the primal mesh underlying the DDFV scheme is an orthogonal mesh.

This includes the practically important case of quasilinear isotropic diffusion $-\Delta\varphi(u)$ on simplicial Delanay primal mesh and the Voronoï dual mesh.

In general, discrete entropy inequalities for DDFV schemes may fail; we still do not know how to justify convergence of the scheme to an entropy solution on general meshes.

5.4.5 Monotone two-point schemes for non-Lipschitz convection flux

The above paragraph discusses the part of entropy inequalities that stems from the diffusion terms of convection-diffusion equations. The recipe for convection terms is well known: one uses monotone consistent two-point schemes (see in particular [76, 158]), and the "weak BV inequalities", following Eymard, Gallouët and Herbin [76]. Yet these classical techniques are usually written with Lipschitz continuous convection terms $\mathrm{div}\,(\mathbf{v}f(u))$ of $\mathrm{div}\,\mathfrak{f}(u)$. In [10[2], Section 6.4], with M. Bendahmane and K.H. Karlsen we considered the case of non-Lipschitz convections.

We have formulated and proved the technical lemmas that allow to prove and to use weak BV inequalities for non-Lipschitz convection terms.

To my opinion, this generalization even made clearer the method introduced in [76]. As an illustration of this result, for the case of Hölder continuous flux we may state that

if $\mathfrak{f}$ is locally Hölder continuous of order γ, then the "remainder term" in the discrete entropy inequalities is of order $h^{\frac{\gamma}{\gamma+1}}$,

[23] We mean that, for the applications that make it useful to consider a DDFV scheme and not a simpler (e.g., two-point flux) scheme, the diffusion operator couples already the primal and dual volumes that intersect. Both the penalization operator and the above form of $[h(u)]^{\mathfrak{T}}$ bring a low-order coupling of the same volumes.

[24] the same is true if one is interested in discretization of renormalized solutions of non-degenerate problems

which generalizes the classical $h^{1/2}$ estimate (see e.g. [76]) of the Lipschitz case. The discussion is too technical to give any details here: I refer to the original paper [10[2]], where the result is presented as a succession of lemmas.

In conclusion of this section let me say that, to my opinion, writing down convergence proofs for finite volume schemes is a delicate issue. Suggestive notation may help a lot to guide the reader. Moreover, I tend to present the calculations "as if" we were working in the continuous setting, and separate the general "discrete functional analysis" statements from their concrete applications. This was the philosophy used in the work [10[2]] with M. Bendahmane and K.H. Karlsen, where we had to provide a large number of definitions and tools (many of them have just been presented above) before combining them into a relatively concise convergence proof.

Chapter 6

Miscellaneous problems originating from applications

While in the previous sections I have already presented several applications, they were either general results (finite volume approximation of general Leray-Lions problems, or of general doubly nonlinear convection-diffusion equations) or illustrations of the power of some general, already established tools in concrete situations (vanishing capillarity solutions of Buckley-Leverett equations, hyperbolic road traffic model with flux constraint, particle-in-Burgers model).

In this section, I gather the results devoted to or motivated by very concrete PDEs or systems of PDEs, brought to my attention by different collaborators. The results range from very theoretical developments (formulation of a singular limit model, existence of a global attractor) to numerical tests; but in most of the problems, a convergent finite volume numerical method was obtained for approximating solutions.

In the sequel, I skip completely the theoretical part of analysis of finite volume methods; but it should be understood that several tools discussed in the previous section were developed because we needed them for the below applications.

6.1 A singular limit of the two-phase flow equations in porous medium

Porous medium equations and systems were a long-standing source of inspiration for the mathematical subjects that I described in the previous Chapters: conservation laws, degenerate parabolic-hyperbolic and elliptic-parabolic problems, entropy solutions, finite volume approximation. It was a pleasure for me to do a bit of work on one of porous medium models.

I participated to the conclusion of the work [1] initiated by R. Eymard, M. Ghilani and N. Marhraoui dedicated to the infinite-air-mobility limit of the two-phase flow equations

$$(2phF) \qquad \begin{cases} \partial_t u - \operatorname{div}\left(k_w(u)\nabla p\right) = s_w, \\ \partial_t(1-u) - \operatorname{div}\left(\mu k_a(u)\nabla(p+p_c(u))\right) = s_a. \end{cases}$$

This model is classical (see, e.g., [84] for details): u is the water saturation, p is the water pressure; p_c stands for the capillary pressure function, k_a and k_w are relative mobilities of water and air phases; finally, s_w, s_a are source terms that will later assume a particular form.

It is usual, in the context of hydrogeology, to use the one-phase Richards model[1]

$$(RE) \qquad \begin{cases} \partial_t u - \operatorname{div}(k_w(u)\nabla p) = s_w, \\ u = p_c^{-1}(p_{atm} - p). \end{cases}$$

for description of the water flow in a porous medium. To be precise, the Richards model is adequate in certain regimes, in particular it assumes that there is no air trapping[2]. The more general model is the "quasi-Richards" equation introduced by R. Eymard et al. [81] and in [1]:

$$(qRE) \qquad \begin{cases} \partial_t u - \operatorname{div}(k_w(u)\nabla p) = \overline{s} - \theta \underline{s} 1\!\!1_{[u=1]}, \\ u = 1 \quad \text{or} \quad \nabla(p + p_c(u)) = 0 \quad \text{a.e..} \end{cases}$$

The form of the source term corresponds to the realistic injection/draining regime where fully saturated water phase is injected at intensity $\overline{s}$ and the mixture of the two phases is extracted at the rate $\underline{s}$.

This quasi-Richards model (qRE) is obtained[3] in [1] as as a singular limit of the two-phase flow equations $(2phF)$ with respect to the air mobility parameter $\mu \to \infty$; the uniqueness for this model is investigated in the note [12[3]].

To be specific, we obtain existence by passage to the limit in a sequence of solutions (u_μ, p_μ) of the two-phase flow (see also Eymard, Henry, Hilhorst [81]). Uniqueness is established in absence of source terms, by the method of renormalized solutions (cf. Plouvier and Gagneux [140]). As a consequence,

we prove that in absence of gravity and source terms, the limit of the two-phase flow model $(2phF)$ is the classical Richards model (RE).

It should be stressed that in general, we expect that quasi-Richards model is different from the Richards one; in particular, it should be able to describe the air trapping phenomenon. Investigation of uniqueness for quasi-Richards model with source terms active in fully saturated zone is the interesting question; it cannot not be solved by the renormalization approach.

Besides the above theoretical results, the main focus of [1] was on finite volume approximation of the two-phase flow with large values of μ, and numerical comparison with the Richards model.

We have constructed, analyzed and implemented a specially designed finite volume scheme for the two-phase flow equations $(2phF)$ which is robust wrt the air mobility μ, and compared it to a standard finite volume discretization of the Richards equation (RE).

The scheme is based upon a "1/2-Kirchoff transformation" $\zeta : z \mapsto \int_z^1 \sqrt{k_a(s)}dp_c(s)$ and it analysis is facilitated by a kind of global pressure formulation. The essential point is to obtain

[1] in the Richards equation, the reciprocal function of the capillary pressure function p_c is extended by the value 1 on $\mathbb{R}^-$.

[2] the air pressure is equal to the atmospheric pressure p_{atm} everywhere, whenever all zones with non-zero air saturation are connected to the exterior of the domain

[3] The derivation of the quasi-Richards equation uses a priori lower bound on the saturation which is not possible to achieve unless one considers a horizontal medium without gravity effect. For the general case with gravity, the model of [12[3]] should be further generalized.

a uniform L^2 estimate on the quantity $\sqrt{\mu k_a(u)}\nabla(p + p_c(u))$, which results in the pointwise constraint "$u = 1$ or $\nabla(p + p_c(u)) = 0$" at the limit $\mu \to \infty$. The passage to the limit in the diffusion term is ensured by a uniform estimate of the gradient of $\zeta(u)$. For a brief exposition of the results, we refer to [12[3]].

Finally, a bunch of numerical examples (programmed with MatLab) is given in [[1]]. We illustrate the behaviour of the scheme in one and two space dimensions, exhibit convergence order (to the continuous model) in discretization parameter h and compare the scheme with large but finite mobility parameter μ to the scheme for the Richards equation. It is curious that the rate of convergence of the two-phase model water saturation to the Richards saturation seems to be quite precisely μ^{-1}, in the examples we have implemented in [[1]].

6.2 Analysis and approximation of the bidomain cardiac electric activity model

Several teams worked in the last years on theoretical study and numerical approximation of the so-called bidomain cardiac electrical activity model: this is the degenerate parabolic system

$$(BiDom) \qquad \begin{cases} \partial_t v - \text{div } (\mathbf{M}_i(x)\nabla u_i) + h[v] = I_{ap}, \\ \partial_t v + \text{div } (\mathbf{M}_e(x)\nabla u_e) + h[v] = I_{ap}, \end{cases}$$

where $v = u_i - u_e$ is the "transmembrane potential" obtained as the difference of the intercellular and extracellular potentials u_i, u_e. The bidomain model is constructed "as if" the whole heart domain was occupied *simultaneously* by the cells and by the extracellular medium. This model can be seen as a homogenized limit of microscopic electrostatic equations where the two media form a partition of the domain and the transmembrane potential evolves on the membrane separating the media ([137]).

For the sake of simplicity, we supplement the system with the homogeneous Neumann boundary condition; and the initial condition is prescribed, naturally, only on the transmembrane potential v. The nonlinearity $h[\cdot]$ in the system above is a very delicate "feature", it is considered as being non-local in time and it is modelled with the help of more or less involved and stiff systems of ODEs (FitzHugh-Nagumo, Leo-Rudy, etc.). The mechanism is the current of different ions, among which the calcium ions play the most significant role.

Yet describing precisely the nonlocal in time term $h[v]$ is a "detail" that is not essential neither for the well-posedness issue, nor for space discretization strategies. Because we do not look beyond these two questions, we assumed that $h[v]$ is a local function which is a bistable cubic polynomial. It is known that this very simplified model may partially reproduce important phenomena in electrocardiology such as depolarization fronts.

M. Bendahmane and K.H. Karlsen developed an approach to existence by viscous regularization of the bidomain system, generalizations to nonlinear Leray-Lions diffusions, and a finite volume method with two-point flux approximation on orthogonal meshes. Ch. Pierre et al. developed a more realistic DDFV approach on general 3D meshes. We joined the efforts in the joint work [11[4]]. As a theoretical result,

we presented a new variant of weak variational formulation,
much in the spirit of Alt and Luckhaus [6] formulation for parabolic-elliptic problems,
and illustrated the convenience of this formulation by giving a uniqueness proof.

The key point of the formulation is the $(L^2(0,T;H^1)\cap L^4)-(L^2(0,T;(H^1)^*)+L^{4/3}$ duality[4] and a regularization lemma that permits to "take u_i and u_e as test functions in the first and the second equations, then add the equations". Indeed, the *a priori* regularity allows to give sense to $h(v)v = h(v)(u_i - u_e)$, but not to each of the terms $h(v)u_{i,e}$; this obstacle is, nevertheless, easily circumvented.

Existence is proved along with justification of convergence of a finite volume scheme. To be specific, we considered two DDFV schemes (either the 2D scheme, or the 3D CeVe-DDFV scheme[5] that were developed in [08³] and [12¹] mainly in view of this application). Convergence analysis led to many of the 'discrete functional analysis" results of Section 5.4. The main results are:

– *for the fully implicit DDFV, the 3D CeVe-DDFV discrete solutions exist and converge to the unique solution of (BiDom), under mild proportionality assumptions on the meshes*;
– *for the linearized implicit scheme*[6]*, the discretizations converge under an additional growth restriction on the ionic current nonlinearity* $h(\cdot)$.

While the analysis of the fully implicit scheme followed rather natural guidelines, the linearized implicit case required some restriction on $h(\cdot)$ (almost satisfied in the practical case of a cubic polynomial) and finer tools, including the discrete Kruzhkov lemma of Section 5.4.

It should be stressed that both the convergence analysis and the implementation of the scheme, carried out by Ch. Pierre, used a discrete weak formulation in the spirit of mimetic finite difference schemes[7] While doing numerical approximations, it appeared very advantageous to pre-condition the bidomain discrete system using the simpler "monodomain" model. The numerical examples showed reasonable convergence rates, and also an adequacy of the depolarization front propagation with what was expected from this simplified model. As a conclusion, one may say that

we have validated the DDFV strategy for space discretization of the bidomain model.

The bottleneck of the cardioelectical simulations resides in the time-consuming approximation of the stiff ODE system governing the realistic models for ionic current $h[u]$; yet, whatever be the strategy for time discretization, it is feasible to use the DDFV scheme and the codes developed by Ch. Pierre et al. for approximating realistic bidomain models.

I continue to work on the bidomain model; a next step is the work in progress with Bendahmane, A. Quarteroni and R. Ruiz Baier on electromechanical coupling for heart modelling.

[4] the L^4 in space and time integrability of v stems from the assumption that $h(v)$ is a cubic polynomial: the term $vh(v)$ brings the L^4 *a priori* estimate on v.

[5] Actually the difference between our 3D CeVe-DDFV scheme and the pioneering scheme of Pierre [139, 58] is very thin: the two schemes have the same discrete gradient operator, and the main difference lies in discretization of source terms.

[6] Linearized implicit approximation of the ionic current $h(v)$ at time level n consists in taking for $(h(v))^n$ the product $\frac{h(v^{n-1})}{v^{n-1}}\,v^n$. It has the clear computational advantage but, from the viewpoint of analysis, it yields weaker *a priori* estimates.

[7] In this formulation, "cellwise" discrete equations are replaced by a weak formulation with discrete test function, and the discrete divergence operator is eliminated from the equations using the discrete duality. From this viewpoint, our scheme is very close to mimetic finite difference schemes, except for the fact that the discrete divergence operator is explicit and it has a clear interpretation in terms of finite volumes discretization.

6.3 Miscellaneous reaction-diffusion systems

The universe of reaction-diffusion systems is huge, and I dealt with three particular cases. In the first and the second ones, we were concerned with a usual, with respect to my previous works, study of weak formulation, existence of solutions and convergence of finite volume approximations. The last one is apart, because I had to treat the issue of asymptotic behaviour of solutions; is was also the occasion to acquire some experience in applying the classical linear theory of analytic semigroups.

6.3.1 Analysis and approximation of a class of cross-diffusion systems

For the paper [11[1]], together with M. Bendahmane and R. Ruiz Baier I worked on a class of cross-diffusion systems originating from population dynamics. In basic population dynamics, the populations are governed by a system of ODEs; the classical PDE extension is to add some multiple Laplacian operator to account for the diffusion phenomenon, separately for each population in the system. Yet it was observed that some features of real populations are not captured by such "self-diffusion" PDE model, because it neglects the interaction of the populations within the diffusion process. A general 2×2 *cross-diffusion* system takes the form

$$(CrD) \qquad \partial_t \mathbf{u} - \operatorname{div}\left[\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \nabla \mathbf{u}\right] - \operatorname{div}\left[\mathcal{A}(\mathbf{u})\nabla \mathbf{u}\right] = \begin{pmatrix} F(\mathbf{u}) \\ G(\mathbf{u}) \end{pmatrix},$$

where $\mathbf{u}$ is the vector ${}^t(u,v)$ of population densities, and $\mathcal{A}$ is the cross-diffusion matrix that is zero in the case of a self-diffusion. The reaction term ${}^t(F,G)$ is a standard quadriatic polynomial.

The main difficulty - which we have not faced - is that the resulting diffusion matrix need not, in general, be positive definite. This makes it difficult to use the standard H^1-based formulations, and as a matter of fact, no general theory is available. The work of Chen and Jüngel [48] introduced an approach by entropy estimates, which allows to treat some range of parameters. In [11[1]], we treated a much smaller range, with energy methods.

We have concentrated on the simplest case of positive definite (self+cross)-diffusions, and extracted the assumptions that permit to prove existence with "variational" methods.

The prototype case is the cross-diffusion of the form

$$\mathcal{A}(u,v) = \begin{pmatrix} u+v & u \\ v & u+v \end{pmatrix}.$$

The corresponding system (CrD) with homogeneous Neumann BC and initial conditions $u_0, v_0 \geq 0$ possess "natural estimates" that are[8]: $u, v \geq 0$, $\sqrt{1+u+v}(|\nabla u| + |\nabla v|) \in L^2$. Thus, the cross-diffusion terms do not belong to L^2, and

we put forward a notion of solution based on L^1 integrability[9] of the cross-diffusion terms, and prove existence of non-negative solutions.

Existence of nonnegative solution is achieved by a truncation and penalization approach, with the Kruzhkov lemma to justify compactness of approximate solutions.

[8] These are *not* estimates *a priori*: we lack a proof of uniqueness, and the positivity of u, v is shown only for the solution obtained by a particular construction procedure.

[9] more precisely, under our assumptions these terms are bounded in $L^{1+\varepsilon}$, which brings weak compactness

The next step is construction and analysis of a finite volume scheme. We used orthogonal meshes and two-point flux approximations. The discretization of $\mathcal{A}$ should be very particular[10] if we have to enforce the nonnegativity[11].

Using a variant[12] of the techniques of [76], the new Sobolev embedding inequalities and the discrete Kruzhkov lemma (see Section 5.4), we have shown that

our finite volume scheme converges, up to a subsequence, to a solution of the system.

A numerical study, with a focus on the instability phenomenon and on a comparison with the self-diffusion case, concludes the work [11¹]. Let me point out that it is possible to modify our convergence arguments so that they apply to the case of cross-diffusion systems with entropies, following [48]; this is the subject of an unpublished work with M. Bendahmane.

6.3.2 Approximation of Keller-Segel model with volume-filling effect and degenerate diffusion

The work [11⁵] with M. Bendahmane and M. Saad was devoted to finite volume approximation of the following variant of the celebrated Keller-Segel model for chemoattraction:

$$\begin{cases} \partial_t u - \operatorname{div}\left(a(u)\nabla u - \chi(u)\nabla v\right) = 0, \\ \partial_t v - d\Delta v = g(u,v) \end{cases}$$

with the Neumann (zero-flux) boundary condition and initial conditions $u_0 \in [0,1]$, $v_0 \geq 0$. Here u is the cell (amoebae, etc.) concentration; u is limited to the normalized value 1 (this models the volume-filling effect) thanks to the assumption that $\chi(1) = 0$. We also have $\chi(0) = 0$ and the sign of χ is constant on $(0,1)$; according to this sign, the model describes chemo-attraction or chemo-repulsion, with v representing the chemical. The particular feature of the diffusion we consider is the degeneracy: $a > 0$ on $(0,1)$ but $a(0) = 0 = a(1)$. Numerical examples we provide in [11⁵] demonstrate that there is a considerable difference in qualitative behaviour of Keller-Segel models with degenerate and non-degenerate diffusions.

Concerning well-posedness of the problem, let me mention that

the notion of weak solution seems appropriate; existence is shown in particular in our work, through convergence of the finite volume scheme,

moreover, uniqueness was established, in some situations, using the duality approach. A previous work of Bendahmane et al. [20] established Hölder regularity of solutions.

The focus of the paper [11⁵] is on numerical analysis of the problem.

We construct a finite volume scheme, implement it and prove convergence.

The scheme is implicit except for the reaction term of the equation governing the concentration of v; the explicit reaction term permits to decouple the two equations while solving

[10] the value $\mathcal{A}(\min\{u_K,u_L\},\min\{v_K,v_L\})$ is taken on the interface between volumes K, L

[11] yet in practice, with the straightforward centered approximation we have not observed any negative value.

[12] In [11¹, ¹, 11⁵], we have used the two-point schemes as described and throughly treated by Eymard, Gallouët and Herbin [76]. But we have relaxed the mesh proportionality assumption due to a systematic use of the weakly convergent discrete gradient: in dimension d, we set $\nabla_{K|L} u^{\mathfrak{T}} := d\frac{u_L - u_K}{|x_L - x_K|}\overrightarrow{x_K x_L}$ in the diamond containing the interface $K|L$. The slight simplification of the arguments wrt [76] was one of the reasons why full convergence proofs were given in our papers.

the system numerically. The diffusion term is discretized in the way that now became usual: a chain rule is used to combine $a(u)\nabla u$ into $\nabla A(u)$, then $A(u)$ is used is the main variable. The convection term is approximated using an upwind choice, e.g. with the splitting of χ into the sum $\chi_\uparrow + \chi_\downarrow$ of the increasing and the decreasing parts:

$$(\nabla v^{\mathfrak{T}} \chi(u^{\mathfrak{T}}) \cdot \nu)|_{K|L} = (\chi_\uparrow(u_K) + \chi_\downarrow(u_L))[(\nabla \cdot \nu)_{K|L} v^{\mathfrak{T}}]^+ + (\chi_\uparrow(u_L) + \chi_\downarrow(u_K))[(\nabla \cdot \nu)_{K|L} v^{\mathfrak{T}}]^-.$$

The tools we use for the analysis are mainly those of Eymard, Gallouët and Herbin [76].

For the implementation, we used the Newton method for solving the nonlinear discrete system. The tests show that the mass conservation (or the precise law of mass decay), respected by the continuous equation and by the theoretical numerical scheme, is very well approximated by the fully practical scheme. We illustrate numerically the chemoattraction behaviour predicted by the model, the finite speed of propagation effect induced by the degenerate diffusion, the volume-filling effect, and a smoothing of randomly perturbed data with the tendency of the cells to agglomerate.

6.3.3 Attractors for a class of reaction-diffusion system motivated by hemoglobin oxidation

This new subject was brought to my attention by H. Labani; our collaboration resulted in paper [11[9]]. In the pioneering works of Martin and Pierre [120], an L^p technique was introduced for proving global in time existence of solutions. Further, in the previous works of H. Labani with Ph. Bénilan [27] and S. Amraoui [9], several additional tools for estimating solutions of reaction-diffusion systems were introduced; the goal was to prove existence of attractor in L^∞ for some concrete systems. To be specific, consider the following system that was the motivation of our work:

$$(HbO) \qquad \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = u_3 - u_1 u_2 \\ \partial_t u_2 - d_2 \Delta u_2 = u_3 - u_1 u_2 \\ \partial_t u_3 - d_3 \Delta u_3 = u_1 u_2 - u_3, \end{cases}$$

with the boundary conditions of the following general form:

$$\lambda_i \partial_n u_i + (1-\lambda_i) u_i = \alpha_i \quad \text{on } \Omega, \quad \alpha_i \geq 0, \quad i = 1..3.$$

Here $0 \leq \lambda_i \leq 1$. System (HbO) appears as a model of oxigenation reaction in blood (with u_1,u_2,u_3 representing the concentrations of Hemoglobin Hb, Oxigen O_2 and of HbO_2, respectively).

When $\lambda_1 = \lambda_2 = \lambda_3 < 1$, "estimates of attractor type" were obtained in [9]; the general case was open. Estimates of attractor type (here, in L^∞) are those that give a bounded absorbing set; then compactness of the nonlinear solution semigroup, that is not difficult to establish in the context (HbO), implies existence of a maximal attractor in L^∞. The starting point of the estimate of [9] was, obtain a bound on $\Delta^{-1} u_i$, where Δ^{-1} is the inverse of the Laplace operator with the boundary condition common to all the components u_i, $i = 1, 2, 3$. A related idea of Bénilan and Labani [27] developed for the "Brusselator" system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = u_1^2 u_2 - (B+1) u_1 + A \\ \partial_t u_2 - d_2 \Delta u_2 = -u_1^2 u_2 + B u_2 \end{cases}$$

worked on *different* boundary conditions. Namely, given a bound on $\Delta_1^{-1}\Delta_2$ (where the two Laplace operators now satisfy two different boundary conditions), the authors estimate $\Delta_i^{-1}u_i$ in L^p. The inspiration for the study of $\Delta_1^{-1}\Delta_2$ comes from another work of Martin and Pierre, [121]. The details of the estimates are quite technical. Let me mention that the key arguments are: the maximal regularity property for the operators $\partial u_i + \Delta_i u_i$; the $L^p - L^q$ decay and regularizing effect for the semigroups $e^{-t\Delta_i}$; and the L^p techniques of [120]. These same elements participate to the conclusion of our estimates in the work [11⁹] with H. Labani. But

> *the main ingredient of our work is the idea of a "preconditioning operator" B satisfying a bound on $B^{-1}\Delta_i$, and the additional positivity property $(B^{-1}\Delta_i - I) \geq 0$,*

where Δ_i are the laplacian operators that appear in (HbO). Using this additional tool, the classical techniques of linear analytic semigroups, and the ideas of the previous works (see [11⁹] for details), recasting the problem in an abstract setting,

> *we establish global existence, estimates of attractor type, and existence of a maximal attractor in L^∞ for solutions of abstract 3×3 reaction-diffusion systems $\partial u_i + A_i(u_i - \bar{\alpha}_i) = f_i(u_1, u_2, u_3)$*

under a series of hypotheses of f_i and on operators A_i; here $\bar{\alpha}_i$ stand for a lifting of boundary conditions. As an application,

> *we deduce existence of a maximal attractor in L^∞ for (HbO), for a wide range of parameters λ_i that excludes neverhteless the Neumann case ($\lambda_i = 1$); for the Neumann case, we deduce global in time existence of solutions.*

The preconditioning operator used in practice is $B = -d\Delta$ with $d = \min\{d_1, d_2, d_2\}$ and with the Robin boundary condition corresponding to $\lambda = \max\{\lambda_1, \lambda_2, \lambda_3\}$. Finally, let me mention that

> *along the same guidelines, different systems can be analyzed; as an example, we treat a 5×5 system modelling the coupled reactions $Hb + O_2 \rightleftharpoons HbO_2$, $Hb + CO_2 \rightleftharpoons HbCO_2$.*

Although linear semigroup techniques are essential for the study of [11⁹], the idea of a preconditioning operator may bring useful estimates also for the some nonlinear diffusion operators; a work in this direction has been initiated recently.

Chapter 7

Some research perspectives

Several research directions that are in a direct relation to my previous works were already mentioned in the previous sections. Here, let me indicate several further questions that attracted my attention and that I will investigate in the forthcoming years.

Systems of hyperbolic conservation laws will be one of the main objects of my forthcoming work. With A. Benabdallah and C. Donadello, we started a study of global existence in BV for the viscoelasticity system with memory. This is a well-known model (see [125]) for which the problem of global existence was solved in L^∞, but in BV it is open since several years. We will focus on the case of a bounded domain; the key tools are those of Dafermos and Hsiao [64, 63]. With K.H. Karlsen and N.H. Risebro, we plan to complement and apply the techniques of the works [11^2, ^{10}] in order to study uniqueness for "triangular" systems of conservation laws (cf. [54]). Indeed, in this context, the first equation $\partial_t u + \operatorname{div} \mathfrak{f}(u) = 0$ is decoupled from the second one: $\partial_t v + \operatorname{div} \mathfrak{g}(u, v) = 0$; thus we can consider that $\mathfrak{g}(u, v)$ is of the form $\tilde{\mathfrak{g}}(t, x; v)$, which is the framework of discontinuous-flux problems. This research direction would require a generalization of the ideas of [11^2, ^{10}] to flux with BV or, more realistically, SBV coefficients. Finally, with F. Lagoutière, N. Seguin and T. Takahashi we plan to benefit from the conclusions of the work [10^7, 12^2, 10^7] (the "particle-in-Burgers" model in one space dimension) and to develop a numerical approach to 'particle-in-Euler" problem in space dimension two. Finally, in collaboration with A. Bendabdallah and C. Donadello, in the years to come I hope to master the modern techniques for control of conservation laws and apply them to the viscoelasticity system with memory and other hyperbolic problems.

Singular (1-Laplacian) kind diffusion operators will be a new subject for me; in particular, with N. Igbida and S. Ouaro we will look at $p(x)$ laplacian problems with $1 \leq p(x) < \infty$ (letting p assume the value 1 is important for image restoration problems, see, e.g., [49]). Another direction is, the numerical study of the so-called "relativistic heat equation" ([143, 11]) $\partial_t u - \operatorname{div} \frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} = 0$ that we started with M. Ghilani and N. Marhraoui. For both problems, our analysis approach is based upon the works of F. Andreu, V. Caselles and Mazón (see in particular [10, 11]). For the numerical study, DDFV and co-volume techniques will be applied. The essential feature of the "relativistic heat equation" is the finite speed of propagation of fronts; from this viewpoint, it can be interesting to compare this model with the standard convection-diffusion problems in population dynamics. We plan, at first, to develop a comparative numerical study for different nonlinear diffusion models.

The work on relativistic heat equation will continue the "finite volume" direction in my research; another trend is the work on $p(x)$-laplacian approximation, a sequel to [10[3]]. I am always interested in occasional development of analysis tools for DDFV and co-volume schemes for nonlinear and singular diffusion problems. Efficient numerical approximation of non-local problems may be a new and interesting direction.

I have several other projects motivated by concrete applications. With K.H. Karlsen we have started a numerical study of the elliptic equation for p-harmonic maps on a sphere; we hope to continue this work, using DDFV or co-volume or other gradient schemes. With M. Bendahmane and R. Ruiz Baier, we currently work on the coupling of the bidomain cardioelectrical model with the elastic models for the heart tissue. Our plan is to construct a finite element numerical scheme and prove its convergence; this scheme will be used in the team of A. Quarteroni for numerical modelling of human heart.

Another new direction in my research is, continue the first studies [10[6], 10[9]] on nonlocal operators. With A. Ouédraogo, we are looking at kinetic solution techniques for fractional conservation laws. With E. Emmrich, we look at renormalized solutions of certain nonlocal in time evolution problems, with applications to second-order evolution equations. With N. Alibaud, we will continue to work on nonlocal (fractional) conservation laws in the direction of entropy and renormalized solutions (as a continuation of [10[9], [8]]); in collaboration with Alibaud and E. Jakobsen, we've started a work on fractional problems in a bounded domain. We also want to consider, in the years to come, several nonlocal models related to conservation laws; this subject undergoes a quick development, and we hope for emergence of some general approaches to specific non-local convection-diffusion operators.

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