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PROBLÈMES DE NON ARBITRAGE, DE RECOUVREMENT ET D'OPTIMISATION DE CONSOMMATION DANS UN MARCHÉ FINANCIER AVEC COÛTS DE TRANSACTIONS.

par

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Résumé

Ces 20 dernières années, la mondialisation de l'économie a nécessité la recherche active de solutions mathématiques aux enjeux financiers. Nos travaux de thèse se sont portés sur plusieurs problèmes non résolus dans les marchés avec coûts de transaction.

Coûts de transactions proportionnels : la plupart des modèles célèbres en économie choisissent d'ignorer les coûts de transaction malgré la réalité économique où ceux-ci sont fréquents. La raison de l'omission de ces coûts de transaction est simple : cela permet de travailler sur $\mathbb{R}$, en dimension 1. Dès lors que l'on modélise des coûts de transactions, on est contraint de travailler dans $\mathbb{R}^d$, avec des outils issus de la géométrie et de l'analyse convexe. Tous nos travaux se sont effectués dans ce contexte.

Non arbitrage : en finance, le mot "arbitrage" désigne la création d'argent à coup sûr à partir de rien. La traduction en terme de probabilité est évidente : si le processus V_t modélise un portefeuille autofinancé avec $V_0 = 0$, on dira qu'il y a une opportunité d'arbitrage si $\mathbb{P}(V_t \geq 0) = 1$ et $\mathbb{P}(V_t > 0) > 0$. Considérant que les arbitrages doivent être évités, on étudie les conditions sur les modèles afin que $V_t \geq 0$ p.s. implique $V_t = 0$. Le premier chapitre de cette thèse fournit un résultat dans la situation où l'agent ne dispose pas de toutes les informations du marché.

Recouvrement d'options américaines : Pour modéliser une option de type option américaine, c'est-à-dire un contrat qui peut être exécuté à tout instant, on se donne un processus $(U_t)_{t\in\mathbb{R}^+}$. Notre but est de déterminer les investissements initiaux $x \in \mathbb{R}^d$ tels qu'il existe un portefeuille autofinancé partant de x qui permet de couvrir le processus U_t. Autrement dit, on s'intéresse à l'ensemble $\Gamma_U = \{x \in \mathbb{R}^d : \; \exists V \in \mathcal{V}_b^x, V \succeq_G U\}$ où $\mathcal{V}_b^x$ désigne l'ensemble des portefeuilles autofinancés admissibles. Le deuxième chapitre de cette thèse propose une représentation duale de cet ensemble, en introduisant un système de prix dit "cohérent".

Consommation et investissement : appelé également "problème de Merton", le problème de consommation-investissement consiste à maximiser l'utilité d'une consommation. Notre travail s'est concentré sur la situation où le processus des prix est contrôlé par un processus de Lévy. Nos avons montré que ce maximum est en fait (l'unique) solution au sens des viscosités d'une équation intégro différentielle, dite équation d'Hamilton-Jacobi-Belman. Notre approche se distingue des travaux précédents par le fait que nous autorisons les positions négatives tant que nous restons dans le cône de solvabilité.

Extended abstract

During the last 20 years, the globalization of the economy needed a very active research in financial mathematic and engineering. Our work focused on some unsolved problems in markets with transaction costs.

Proportional transaction costs : most of the economical models choose to ignore transactions costs, although in "real life" costs have to be paid. The reason why they are ignored is simple : it allows to work on $\mathbb{R}$, in dimension 1. As soon as you choose to work with transaction costs, you need to stay in $\mathbb{R}^d$, which leads to some geometrical and convex analysis problems. Our work is always in that context.

No arbitrage : in finance, basically, an "arbitrage" is the opportunity to make money out of nothing without any risks. In mathematical terms, if V_t models a self-financing portfolio with $V_0 = 0$, we will say that an arbitrage opportunity exists if $\mathbb{P}(V_t \geq 0) = 1$ and $\mathbb{P}(V_t > 0) > 0$. Considering that arbitrages should be avoided, we studied the conditions such that $V_t \geq 0$ implies $V_t = 0$ a.s. The first chapter of our thesis gives a condition of no arbitrage in the specific case of incomplete information.

Hedging of American options : To model an American option-like asset, that is to say a contract that can be executed at any time by the buyer, we give a process $(U_t)_{t \in \mathbb{R}^+}$ in continuous time. Our goal is to characterize the initial investments $x \in \mathbb{R}^d$ such that there is a self financing portfolio starting from x which allows to cover the process U_t. Thus, we study the set $\Gamma_U = \{x \in \mathbb{R}^d : \ \exists V \in \mathcal{V}_b^x, V \succeq_G U\}$ where $\mathcal{V}_b^x$ denotes the self financing and admissible portfolios. Using an advanced model, we obtained in the chapter 2 a dual characterization of this set thanks to what we called "coherent price system".

Consumption-investment : also known as "Merton's problem", the goal is to maximize the utility of a consumption, starting from x. We focus on the case where the prices are driven by an exponential Lévy process. Using a dynamic programming principle, we prove that the goal functional, the "Bellman's function", is (the unique) solution in the viscosity sense to a second order integro-differential equation, the Hamilton-Jacobi-Bellman's equation. Our approach differs from previous ones because we allow short positions as soon as we stay in the solvency cones.

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Introduction

Initiées au début du XXe siècle avec Louis Bachelier et sa thèse sur la théorie de la spéculation, les mathématiques financières prennent leur essor dans les années 70 avec la mondialisation et la multiplication des échanges financiers. Les progrès en théorie des probabilités, en particulier la compréhension du mouvement brownien et des divers outils de la théorie des processus ont permis le développement de modèles complexes et relativement réalistes. On ne pourrait parler de mathématiques financières sans citer la célèbre formule dite de Black et Scholes, mais les travaux actuels dépassent largement le cadre du modèle assez simple employé par Merton, Black et Scholes dans leurs travaux.

Marchés avec coûts de transaction : Nos travaux se sont orientés sur un domaine peu exploré des mathématiques financières : les marchés avec coûts de transaction. Il s'agit de développer des modèles rendant compte d'une réalité sur les marchés : les frais supplémentaires associés aux opérations effectuées. Il existe différents types de coûts de transaction, mais les plus étudiés sont les coûts de type convexes. Notons que ceux-ci sont parfois utilisés pour rendre compte des problèmes de liquidité : la notion de convexité entraine en effet des coûts d'autant plus grands que la somme transférée est importante, ce qui peut traduire le problème de liquidités limitées sur les marchés.

Notre étude concerne un cas particulier de coûts de transaction convexes : les coûts de transaction proportionnels. Ainsi, les frais occasionnés par un transfert seront supposés proportionnels à la somme transférée. A l'échelle de la banque pour particulier, c'est typiquement ce genre de frais qui sont appliqués lors d'un échange de devises. L'étude des marchés avec coûts de transaction a connu des progrès importants avec, entres autres, les travaux précurseurs de Elyes Jouini et Hédi Kallal ([29]). La difficulté du travail avec coûts de transaction est liée à l'impossibilité de se ramener facilement à $\mathbb{R}$: il est par exemple inutile de faire la somme des valeurs des différents actifs afin d'en déduire la solvabilité de l'agent. En effet, les coûts de transactions impliqueront éventuellement un état de banqueroute alors que la simple somme était positive.

Exemple 1. *Supposons que la valeur initiale du portefeuille soit donnée par* $V_0 = (-1000, 503, 504)$*, chaque coordonnée correspondant à la valeur investie dans un actif, et que les coûts de transaction proportionnels sont donnés par la* *matrice* $\begin{bmatrix} 0 & 0.01 & 0.01 \\ 0.01 & 0 & 0.02 \\ 0.01 & 0.03 & 0 \end{bmatrix}$ *où* $\lambda_{i,j}$ *est le coût de transaction proportionnel de transfert de* i *vers* j*. On pourrait par exemple renflouer la position de l'actif* 1 *en prenant* 500 *de l'actif* 2 *et* 500 *de l'actif* 3*. En payant les coûts de transaction, notre portefeuille devient* $V_1 = (0, -2, -1)$*, ce qui est clairement une position débitrice.*

En fait, ce portefeuille V_0 *n'est pas un portefeuille solvable, bien que la somme brute des composantes donne une valeur totale positive. Quels que soient les ordres donnés, le fait de payer les coûts de transactions mènera toujours à une position négative pour au moins l'un des actifs...*

En conséquence de cette difficulté, de nouveaux modèles sont apparus. Celui que nous avons utilisé pendant ces 4 années de recherche dans une version discrète, puis dans sa version continue sera présenté en détail dans les 3 chapitres constitutifs de cette thèse. Il est introduit en 1999 par Kabanov dans [30]. Le concept fondamental est l'espace de solvabilité : dans le cas proportionnel, c'est un cône positif de $\mathbb{R}^d$ où d est le nombre d'actifs constituant le portefeuille. Être dans ce cône signifie que l'ont peut liquider notre portefeuille avec un gain positif ou nul, en incluant les coûts de transaction. C'est ce concept qui est la clé de tous les travaux sur lesquels nous nous sommes appuyés car il propose une grande généralité et s'avère très maniable. L'originalité de l'approche initiée dans [30] est que le portefeuille n'est plus nécessairement décrit en terme de valeur investie dans chacun des actifs, mais en terme de quantité d'actifs. Cela rend alors les modèles de portefeuille beaucoup plus simples à décrire, puisque les variations en quantités investies ne sont le résultat que des actions de l'agent. Dans le cas où on parle en terme de valeur, il fallait prendre en considération également la variation des prix.

Évidemment, cette modification n'est pas sans conséquences sur les cônes de solvabilités. Dans le cas où les portefeuilles sont décrits en terme de valeur, le cône de solvabilité à l'instant t, noté K_t, ne dépend que des coûts de transactions à l'instant t. En revanche, si on travaille en terme de quantité, le cône de solvabilité, habituellement noté $\hat{K}_t$, dépend à la fois des coûts de transactions et des prix. Néanmoins, cet effet négatif est largement contrebalancé par les découvertes fondamentales que cette approche a permis.

Problème d'arbitrage : Un concept fondamentale en économie et en mathématique financière est le concept de non arbitrage. Une opportunité d'arbitrage est une stratégie Π permettant, à partir d'un portefeuille $V_0 = 0$ d'obtenir à l'instant terminal T un portefeuille V_T tel que $V_T \geq 0$ presque sûrement, et $\mathbb{P}(V_t > 0) > 0$. Autrement dit, un arbitrage permet à partir de 0 de créer "sans risque" une richesse. Nous noterons l'absence d'opportunité d'arbitrage sous le sigle NA. Le résultat fondamental, obtenu dans [20], est que, dans un marché sans coûts de transaction, NA est vraie si et seulement si il existe une probabilité $\mathbb{Q}$ équivalente à la probabilité $\mathbb{P}$ initiale telle que le processus des prix est une martingale.

Ce résultat dans le cas discret sans coûts de transaction porte le nom de théorème de Dalang–Morton–Wellinger, et de nombreuses preuves différentes existent. Dans [34], Youri Kabanov et Christophe Stricker proposent une preuve globale, là où la plupart des preuves procédaient pas par pas en remarquant que l'absence d'arbitrage à l'instant final T équivaut à une absence d'arbitrage entre l'instant t et l'instant $t + 1$. La preuve présentée dans [34] n'utilise pas de telle technique, et cela permet une généralisation extrêmement aisée au cas de l'information partielle (voir [35]) : il s'agit de la situation où les prix, les coûts de transaction ou les deux ne sont pas complètement connus de l'agent effectuant les opérations. C'est le cas par exemple s'il y a un délais entre l'ordre et son exécution.

Néanmoins, le passage à la situation avec coûts de transaction est difficile : quel sens donner à la loi de martingale présente dans le théorème de Dalang–Morton–Wellinger ? S'il est clair que l'existence d'une telle loi entraîne effectivement NA, l'équivalence n'est pas vérifiée (voir [29] ou encore [38]). L'effervescence autour de ce sujet a donné lieu à de nombreux travaux et à l'introduction de deux variantes du non arbitrage : le non arbitrage strict (NA^s), introduit dans [38], et le non arbitrage robuste (NA^r), introduit dans [46] par W. Schachermayer. Finalement, un résultat fondamental est apparu dans [46] : l'absence d'opportunité d'arbitrage robuste est équivalente à l'existence d'un processus de prix qui est une martingale évoluant dans l'intérieur du dual des cônes de solvabilité. Un tel processus est appelé "Système de prix strictement consistant" (SCPS).

Nos premiers travaux ont consisté en une étude approfondie du problème de non arbitrage dans la situation d'information partielle avec coûts de transaction proportionnels. Combiner information partielle et coûts de transaction n'est pas aussi facile que dans le cas sans coûts de transaction, et cette difficulté a été relevée par Bruno Bouchard dans [12] : la modélisation des portefeuilles employée dans [30] ne peut rendre compte d'ordres pourtant élémentaires. En généralisant l'idée présente dans [12], nous proposons donc d'utiliser un autre modèle, plus complexe, et de travailler cette fois dans un espace de taille $d \times d$ voire $2d \times d$, puis d'appliquer un opérateur linéaire rendant compte de l'impact de l'ordre donné. La difficulté est que cet opérateur n'est pas nécessairement mesurable par rapport à la filtration de l'investisseur. C'est l'objet du premier chapitre de notre thèse où, en nous appuyant sur les travaux de B. Bouchard, nous avons déterminé la condition nécessaire et suffisante à l'absence d'arbitrage robuste pour ce modèle, en utilisant une variante des systèmes de prix consistant.

Recouvrement d'option américaine, surréplication : A l'issue de cette recherche sur le non arbitrage, la suite logique est de s'intéresser au problème de recouvrement d'option. Il s'agit de déterminer les portefeuilles initiaux permettant de couvrir le paiement d'une option de type européenne ou américaine. Le cas de l'option européenne est traité dans de nombreux ouvrages, et la version pour le cas à information partielle est présente dans ce mémoire à la fin du chapitre 1. Le cas de l'option américaine est plus complexe et n'était pas résolu complètement dans le cas coûts de transaction proportionnels dans un modèle à temps continu.

Le modèle à temps continu repose sur une généralisation du modèle à temps discret, et on peut le voir introduit dans [37]. Néanmoins, son utilisation était limitée au cas où les prix suivaient une évolution continue. Un contre exemple, fourni par Miklos Rasonyi, montre qu'un tel modèle ne peut pas être utilisé à des fins de théorème de surréplication dans le cas où les processus de prix ont des sauts (voir [44]). Néanmoins, en introduisant une condition supplémentaire et nécessaire, mais dont la signification financière fait encore débat, Campi et Schachermayer dans [16] proposent une version d'un théorème de surréplication pour le cas de l'option de type européen.

Pour le cas de l'option américaine, la première difficulté est apparue dès le cas discret. Comme l'ont remarqué Chalasani et Jha dans [17], les systèmes de prix consistant ne permettent pas de caractériser les positions initiales permettant de couvrir une option de type américain. Dans leur article, ces auteurs proposent de passer par des temps d'arrêt randomisés, mais nous avons choisi de nous inspirer du cas à temps discret proposé par Bouchard et Temam dans [14]. En utilisant le modèle de portefeuille élaboré dans [16], nous avons introduit une nouvelle famille de processus, intitulés "système de prix cohérent", qui généralise le concept de "système de prix consistent" et qui permet de tester si un investissement initial permet d'honorer un contrat de type option américaine. Le théorème obtenu est l'objet du chapitre 2 de ce mémoire.

Il est intéressant ici de noter que le modèle utilisé dans [16] diffère du modèle présent dans [37] sur deux points : les portefeuilles y sont supposés prévisibles (alors qu'ils sont seulement adaptés dans [37]) et ne sont pas càdlàg (continue à droite et limité à gauche), mais seulement làdlàg (limité à droite et à gauche). C'est la combinaison de ces deux propriétés qui permet d'avancer dans la résolution du problème : la prévisibilité garantie le fait que les sauts du portefeuille issus des actions de l'agent et ceux du processus des prix ne peuvent avoir lieu simultanément. Autrement dit, si V est le processus modélisant le portefeuille et S celui modélisant les prix, le processus $\Delta V \Delta S$ est indistinguable du processus nul. Enfin, le "double saut" permis par le fait que le portefeuille est làdlàg nous assurera la possibilité de couvrir l'option à tout instant, ce qui est essentiel pour une option de type américaine. Intuitivement, ce double saut correspond à une réactivité de l'agent qui peut agir à l'instant $t-$, puis à l'instant t.

Problème de Merton : optimisation d'utilité de consommation Afin d'obtenir un résultat très général, le chapitre 2 propose un modèle où les prix ne sont pas nécessairement des processus continus. Nous nous sommes alors intéressés aux modèles utilisés classiquement dans la littérature, en particulier aux systèmes de prix dirigés par un processus de Levy. Des travaux préliminaires nous ont alors amené à nous intéresser aux problèmes d'optimisation de consommation, parfois appelé problème de Merton, toujours sous la contrainte de coûts de transaction proportionnels. Le problème consiste à maximiser l'utilité d'une consommation effectuée à partir d'une position initiale donnée. L'idée était que le modèle de portefeuille introduit dans le chapitre 2 de ce mémoire permettrait de résoudre le problème de consommation avec coûts de transaction proportionnels et processus de prix de type Levy. Ce problème, bien que traité dans plusieurs articles, dont [11] et [10] qui nous ont servi de base de recherche, n'a jamais véritablement été traité en toute généralité, c'est à dire avec un modèle rendant compte des cônes de solvabilité tels qu'introduits généralement. En effet, les articles de Benth et al. considèrent des portefeuilles sans aucune position négative. Dans leur modèle, la ruine de l'agent ne peut avoir lieu que du fait de ses actions. En autorisant les positions négatives ("short positions") tout en restant dans le cône de solvabilité, la ruine peut intervenir à cause de la variation des prix.

Etant donné que la ruine peut intervenir à la fois à cause des sauts et des variations de prix, il est naturel de penser qu'un modèle comme celui de [16] nous permette de mieux résoudre le problème de Merton dans ce cas. Le canevas de résolution ne varie pas beaucoup par rapport à la littérature : nous travaillons avec une équation intégro différentielle elliptique dont la "fonction but" (appelée aussi fonction de Bellman) est solution au sens des viscosités. En effet, la fonction de Bellman n'est pas nécessairement de classe $\mathcal{C}^2$: elle n'est donc pas nécessairement solution de l'équation au sens classique. Les outils utilisés sont donc les outils classiques de la théorie des solutions de viscosités et nous démontrerons le principe de programmation dynamique dans le cas particulier que nous considérons. Malheureusement, nous n'avons pris connaissance que tardivement de l'article de Bruno Bouchard et Nizar Touzi sur le sujet (voir [15]), ce qui ne nous a pas permis d'utiliser les résultats démontrés dans leur travail.

Une première difficulté dans le cas que nous considérons est que l'opérateur intégro-différentiel est définit globalement, et non plus localement. Appliquer mécaniquement la formule d'Itô n'est plus possible, même en supposant que la fonction de Bellman est lisse. Pourtant, d'autres difficultés du travail avec coûts de transactions et processus de prix avec sauts se cachent dans des étapes a priori simples. Par exemple, nous verrons dans le chapitre 3 que la fonction de Bellman n'est pas aussi trivialement convexe que dans le cas avec processus de prix continu. Cette convexité était pourtant fort pratique pour montrer au moins la continuité de cette fonction ! Par chance, on peut montrer que cette fonction reste continue si on ajoute des hypothèses d'homogénéité de la fonction d'utilité.

Le chapitre 3 offre ainsi des résultats dans un contexte plus général que celui traité par Benth et al., avec des différences significatives quant au traitement du problème à cause des exigences de notre modèle. Nous donnons ainsi un théorème de vérification, en utilisant le principe de programmation dynamique, suivi du résultat d'unicité de la solution sous certaines hypothèses associées à la mesure des sauts du processus de Lévy.

Chapter 1

No-Arbitrage Criteria for Financial Markets with Transaction Costs and Incomplete Information

1.1 Introduction

In the classical arbitrage theory it is usually assumed that the investor makes his decisions using all market information and the majority of no-arbitrage criteria are developed in this framework. However, even though there is a vast amount of information available, an investor may base his decision only on a part of this information. On the other hand, mathematically, such an important feature as partial information used in the investor's decisions can be easily modeled, namely, by a subfiltration $\mathbf{G} = (G_t)$ of the main filtration $\mathbf{F} = (\mathcal{F}_t)$ describing the information flow. What are consequences of such modeling for the arbitrage theory?

Until recently, the only result in this more general framework was an extension of the Dalang–Morton–Willinger theorem for the model of the frictionless financial market in discrete-time given in the paper [35]. It happens that the no arbitrage property (shortly, NA-property) for the price process S holds if and only if there is a bounded strictly positive $\mathbf{F}$-martingale ρ such that the optional projection $(\rho S)^o$ is a $\mathbf{G}$-martingale, i.e. $\tilde{E}(S_{t+1} - S_t|\mathcal{G}_t) = 0$ where $\tilde{P} := \rho_T P$. On the other hand, the "global" (multi-step) NA-property is no longer equivalent to the NA-properties for all one-step sub-models. This explains why such a natural generalization was not obtained earlier: all proofs (of the"only if" part) except that given in [34] use a reduction to the one-step case.

The study of no-arbitrage properties for markets with friction was initiated by Jouini and Kallal [29] for a model with bid-ask spread and developed further in a number of papers: [38], [46], [32], [25] and others. There are several concepts of the no-arbitrage property. Equivalent conditions for them can be formulated in terms of the existence of martingales evolving in the duals to solvency cones (in the space used to represent the investor's positions in physical units) or in the interiors of these duals.

It is natural to consider as an arbitrage opportunity a self-financing portfolio strategy (with zero initial capital) yielding a positive outcome on a set of positive probability with no losses elsewhere. The absence of such "strict" arbitrage opportunities, i.e. the relation $\widehat{R}_T \cap L^0(\mathbf{R}^d_+) = \{0\}$ where $\widehat{R}_T$ is the set of the terminal values of portfolios, is called *weak no-arbitrage property* (shortly, NA^w-property). For the case of finite Ω the criterion for the NA^w-property was obtained in [38]: the latter holds if and only if there is a martingale evolving in the duals to solvency cones. For general Ω this equivalence holds only for the two-asset model, see [25]. An evaluation of the portfolio results without taking into account the transaction costs (as could be done by auditors) leads to a larger set of weak arbitrage opportunities. Their absence is referred to as *strict no-arbitrage property*, NA^s. In the case of arbitrary Ω and "efficient friction", i.e. non-emptiness of the interiors of dual cones, NA^s is equivalent to the existence of a martingale evolving in these interiors, see [32]. Without further assumptions, as was shown first in [46], the existence of a martingale evolving in relative interiors of duals to the solvency cones is equivalent to the so-called *robust no-arbitrage property*, NA^r. The latter means that there are no-arbitrage opportunities in strict sense even for smaller transaction costs.

The setting of market models with friction where the investor's information may be different from that given by the main filtration was investigated by Bruno Bouchard [12] who discovered some new phenomena. He showed that models with transaction costs and partial information not only necessitate important changes in the description of value processes but also appropriate modifications of the basic concepts. In particular, one cannot work on the level of portfolio positions, represented by a point in $\mathbf{R}^d$, but has to remain on the primary level, of the investor's decisions (orders), i.e. in a space of much higher dimension. In the model with partial information there is a difference between the investor's orders "exchange 1000 dollars for euros" and "exchange dollars to increase the holding in euros by 1000 euros": they are of a different nature. For the first type of orders the investor controls the decrease of the dollar holdings (hence, his debts), while for the second type, due to limited information, he may have no idea what is the resulting value of the eventually short position in dollars.

The model of [12] can be classified as that of a barter market but it covers also the case of the model with a numéraire by introducing auxiliary "fictive" assets. Bouchard suggested the coding of orders by real-valued $d \times d$-matrices (with zero diagonal) where the sign of each entry serves as in indicator of the order type ("to send" or "to get" an increment). His main result is the criterion for the NA^r-property of the market for a partially informed investor. It is necessary to recall that in models with full information there is no difference between "barter markets" and "financial markets". In the theory of markets with transaction costs it happens that it is much easier to analyze models where holdings are expressed in terms of physical units rather than in units of a numéraire. In the development of this idea in some recent papers, e.g., [46] and [25] the initial set-up is that of a "barter market", i.e. "conversion" matrices (π_t^{ij}) are specified. This is by no means a restriction: in the models with full information one can always construct prices S_t and matrices λ_t^{ij} of transaction costs coefficients (of course, not uniquely). However, the setting based on prices and transaction costs coefficients may lead to an information structure which seems not to be covered by models based on conversion matrices.

The aim of the present work is to simplify and extend the approach of [12] to include explicitly models with a numéraire. To this end we use an alternative coding of the investor's order and enjoy from the very beginning the linear structure of the problem which leads to a more transparent presentation. Our results include a criterion for the NA^w-property for the case of finite Ω (extending the criterion of [38]) and the criteria for the NA^r-property which is a generalization of those of [46] and [32]. We conclude with a version of the hedging theorem for the situation with partial information.

One should take into account that we are dealing here with a highly stylized mathematical model where orders should be executed, independently of the realized price movements. This means that we are working within the framework of linear control system. The practical situations might be much more complicated and depend on a market microstructure. Certain financial markets are organized as auctions where investors indicate reservation prices, when selling, and limit prices, when buying. A trading system equilibrates the supply and demand, generating asset prices. For example, during the following trading cycle, an order to buy may not be executed or executed partially if the price goes up above the limit price. Of course, an analysis of models incorporating such features as liquidity constraints, constrained orders etc. is of great interest and could be a subject of further studies.

Comment on notations: as usual $L^0(\mathbf{R}_+, \mathcal{F}_t)$ is the set of positive $\mathcal{F}_t$-measurable random variables (note that we prefer to say "positive" rather than "nonnegative") and, consistently, $\mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$ stands for the set of martingales with strictly positive components; $\mathbf{1} := \sum e_i = (1, ..., 1)$.

1.2 Examples and mathematical framework

Example 1. Let us consider the barter market which is described by an $\mathbf{F}$-measurable conversion ("bid-ask") process $\Pi = (\pi^{ij}_t)$ taking values in the set of strictly positive $d \times d$ matrices such that $\pi^{ij}_t \pi^{ji}_t \geq 1$. The entry π^{ij}_t stands for a number of units of the ith asset needed to exchange, at time t, for one unit of the jth asset. The above inequality means that exchanging one unit of the ith asset for $1/\pi^{ij}_t$ units of the jth asset with simultaneous exchange back of the latter quantity results in decreasing of the ith position.

In the case of fully informed investor, the portfolio process is generated by an $\mathbf{F}$-adapted process (η_t^{ij}) with values in the set $\mathbf{M}_+^d$ of positive $d \times d$ matrices; the entry $\eta_t^{ij} \geq 0$ is the investor's order to increase the position j on η_t^{ij} units by converting a certain number of units of the ith asset. The investor has a precise idea about this "certain number": it is $\pi_t^{ij}\eta_t^{ij}$. The situation is radically different when the information available is given by a smaller filtration $\mathbf{G}$, i.e. η_t^{ij} is only $\mathcal{G}_t$-measurable. The decrease of the i-th asset implied by such an order, being $\mathcal{F}_t$-measurable, is unknown to the investor. However, one can easily imagine a situation where the latter is willing to control the lower level of investments in some assets in his portfolio. This can be done by using the $\mathbf{G}$-adapted order process $(\tilde{\eta}_t^{ij})$ with the element $\tilde{\eta}_t^{ij}$ representing the number of units of the ith asset to be exchanged for the jth asset – the result of this transaction yields an increase of the jth position in $\tilde{\eta}_t^{ij}/\pi_t^{ij}$ units and, in general, now this quantity is unknown to the investor at time t. Of course, orders of both types, "to get", "to send", can be used simultaneously. In other words, the investor's orders form a $\mathbf{G}$-adapted process $[(\eta_t^{ij}), (\tilde{\eta}_t^{ij})]$ taking values in the set of positive rectangular matrices $\mathbf{M}_+^{d\times 2d} = \mathbf{M}_+^d \times \mathbf{M}_+^d$. The dynamics of the portfolio processes is given by the formula

$$\Delta\widehat{V}_t = \widehat{\Delta B}_t^1 + \widehat{\Delta B}_t^2, \tag{1.1}$$

where the coordinates of $\widehat{\Delta B}_t^1$ and $\widehat{\Delta B}_t^2$ are

$$\begin{aligned}\widehat{\Delta B}_t^{1,i} &:= \sum_{j=1}^d [\eta_t^{ji} - \pi_t^{ij}\eta_t^{ij}],\\ \widehat{\Delta B}_t^{2,i} &:= \sum_{j=1}^d [\tilde{\eta}_t^{ji}/\pi_t^{ji} - \tilde{\eta}_t^{ij}].\end{aligned}$$

Let $(e^{ij}) \in \mathbf{M}_+^d$ be a matrix with all zero entries except the entry (i,j) which is equal to unity. The union of the elementary orders $[(e^{ij}), 0]$ and $[0, (e^{ji})]$ forms a basis in $\mathbf{M}^{d\times 2d}$. The execution of the order $[(e^{ij}), (e^{ji})]$ (buying a unit of the jth asset in exchange for the ith asset and then exchanging it back) leads to a certain loss in the ith position while others remain unchanged, i.e. $\Delta\widehat{V}_t^i \leq 0$, $\Delta\widehat{V}_t^j = 0$, $j \neq i$. This observation will be used further, in the analysis of the NA^r property.

Example 2. Let us turn back to our basic model which is defined by a price process $S = (S_t)$ (describing the evolution of prices of units of assets in terms of some numéraire, e.g., the euro) and an $\mathbf{M}^d_+$-valued process $\Lambda = (\lambda^{ij}_t)$ of transaction costs coefficients. This model admits a formulation in terms of portfolio positions in physical units: one can introduce the matrix Π by setting

$$\pi^{ij}_t = (1 + \lambda^{ij}_t) S^j_t / S^i_t, \qquad 1 \le i, j \le d.$$

In the full information case the difference between two models is only in parametrizations: one can introduce in the barter market "money" by taking as the price process S an arbitrary one evolving in the duals to the solvency cones and non-vanishing and defining λ^{ij}_t from the above relations. On the other hand, from the perspective of partial information, the setting based on price quotes is more flexible and provides a wider range of possible generalizations.

Again, assume that the investor's information is described by a smaller filtration $\mathbf{G}$ while S and Λ are $\mathbf{F}$-adapted (note that these processes may be adapted with respect to different filtrations).

In contrast to the barter market, the investor now may communicate orders of four types: in addition to the orders (η^{ij}_t) and $(\tilde{\eta}^{ij}_t)$ one can imagine also similar orders, "to get", "to send", but expressed in units of the numéraire and given by $\mathbf{G}$-adapted matrix-valued processes (α^{ij}_t) and $(\tilde{\alpha}^{ij}_t)$ with positive components. The entry α^{ij}_t is the increment of value in the position j due to diminishing the position i, while the entry $\tilde{\alpha}^{ij}_t$ is a value of the ith asset ordered to be exchanged for the jth asset.

The dynamics of value processes in such a model, in physical units, is given by the formula

$$\Delta \widehat{V}_t = \widehat{\Delta B}^1_t + \widehat{\Delta B}^2_t + \widehat{\Delta B}^3_t + \widehat{\Delta B}^4_t, \tag{1.2}$$

where $\widehat{\Delta B}^{3,i}_t := \Delta B^{3,i}_t / S^i_t$, $\widehat{\Delta B}^{4,i}_t := \Delta B^{4,i}_t / S^i_t$ with

$$\begin{aligned} \Delta B^{3,i}_t &:= \sum_{j=1}^d \alpha^{ji}_t - \sum_{j=1}^d (1 + \lambda^{ij}_t) \alpha^{ij}_t, \\ \Delta B^{4,i}_t &:= \sum_{j=1}^d \frac{\tilde{\alpha}^{ji}_t}{1 + \lambda^{ji}_t} - \sum_{j=1}^d \tilde{\alpha}^{ij}_t. \end{aligned}$$

Of course, in this case the dynamics can be expressed also in values, that is in units of the numéraire (using the relation $X^i = \widehat{X}^i S^i$).

Thus, in both cases the set of "results" (for portfolios with zero initial endowments) consists of the d-dimensional random variables

$$\xi = \sum_{t=0}^{T} \mathcal{L}_t \zeta_t, \qquad \zeta_t \in O_t := L^0(\mathbf{M}_+^{d\times m}, \mathcal{G}_t), \tag{1.3}$$

where m is either $2d$ or $4d$ and $\mathcal{L}_{\omega,t} : \mathbf{M}^{d\times m} \to \mathbf{R}^d$ are linear operators such that the mappings $\omega \mapsto \mathcal{L}_{\omega,t}$ are measurable with respect to the σ-algebra $\mathcal{F}_t$. We shall denote this set $\widehat{R}_T$ or, when needed, $\widehat{R}_T(\mathcal{L})$ to show the dependence on the defining operator-valued random process. As usual, we define the set of hedgeable claims $\widehat{A}_T(\mathcal{L}) := \widehat{R}_T(\mathcal{L}) - L^0(\mathbf{R}^d_+)$.

Let us associate with the random linear operator $\mathcal{L}_t$ (acting on elements of $\mathbf{M}^{d\times m}$) the linear operator $\mathbf{L}_t$ acting on $\mathbf{M}^{d\times m}$-valued random variables, $\mathbf{L}_t : L^0(\mathbf{M}^{d\times m}, \mathcal{G}_t) \to L^0(\mathbf{R}^d, \mathcal{F}_t)$, by setting $(\mathbf{L}_t\zeta)(\omega) = \mathcal{L}_{\omega,t}\zeta(\omega)$. With this notation,

$$\widehat{R}_T = \sum_{t=0}^{T} \mathbf{L}_t(O_t).$$

Sometimes, it is convenient to view $\mathbf{M}^{d\times m}$ as the set of linear operators defined by the corresponding matrices.

Unlike the case of a frictionless market the set $\widehat{R}_T$, in general, is not closed even for models with full information: see Example 1.3 in [25] (due to M. Ràsonyi) where the set $\widehat{R}_1 = \widehat{A}_1$ is not closed though the NA^w-condition is satisfied. However, as in the case of models with full information, we have the following result. We comment on its proof in the subsequent remark.

Proposition 1.2.1. *The sets $\mathbf{L}_t(O_t)$ are closed in probability.*

Proof. The arguments being standard, we only sketch them. In a slightly more general setting, consider a sequence of random vectors $\zeta^n = \sum_{i=1}^N c_i^n g_i$ in a finite-dimensional Euclidean space where g_i are $\mathcal{G}$-measurable random vectors and $c_i^n \in L^0_+(\mathcal{G})$. Let $\mathcal{L}$ be an $\mathcal{F}$-measurable random linear operator. Knowing that the sequence $\xi^n = \mathcal{L}\zeta^n$ converges to ξ, we want to show that $\xi = \mathcal{L}\zeta$ for some $\zeta = \sum_{i=1}^N c_i g_i$. Supposing that the result holds for $N-1$ (for $N = 1$ it is obvious), we extend it to N. Indeed, it is easy to see, recalling the lemma on random subsequences[1], that we may assume without loss of generality that

[1] For any sequence of $\mathbf{R}^d$-valued random variables $\{\eta_n\}$ with $\liminf_n |\eta_n| < \infty$ one can find a sequence of random variables $\{\eta'_n\}$ such that $\{\eta'_n(\omega)\}$ is a convergent subsequence of $\{\eta_n(\omega)\}$ for almost all ω, see [34].

all sequences c_i^n converge to infinity and, moreover, the normalized sequences $\tilde{c}_i^n := c_i^n/|c^n|$, where $|c^n|$ is the sum of c_i^n, converge to some $\mathcal{G}$-measurable random variables $\tilde{c}_i$. For the random vector $\tilde{\zeta} := \sum_{i=1}^N \tilde{c}_i g_i$ we have that $\mathcal{L}\tilde{\zeta} = 0$. Put $\alpha^n := \min_i\{c_i^n/\tilde{c}_i : \ \tilde{c}_i > 0\}$. Note that $\bar{c}_i^n := c_i^n - \alpha^n \tilde{c}_i \geq 0$ and, for each ω, at least one of $\bar{c}_i^n(\omega)$ vanishes. For $\bar{\zeta}^n = \sum_{i=1}^N \bar{c}_i^n g_i$ we have that $\mathcal{L}\bar{\zeta}^n$ also tends to ξ. Considering the partition of Ω by disjoint $\mathcal{G}$-measurable subsets Γ_i constructed from the covering of Ω by sets $\{\liminf_n \bar{c}_i^n = 0\}$ and replacing on Γ_i the coefficients $\bar{c}_i^n$ by zero (without affecting the limit ξ), we obtain a reduction to the case with $N-1$ generators. $\square$

Remark 1.2.2. *We give the above assertion by methodological reasons, as a case study explaining the basic ideas and techniques. Though, formally, this result of independent interest will not be used in the sequel we recommend to the reader to make efforts to understand its proof. Its first idea is that we can consider a $\mathcal{G}$-measurable partition of Ω and prove the result separately for each elements of the partition. That is why we start with a two-element partition Ω_0, Ω_0^c such that on Ω_0 the result is obvious because, by virtue of the lemma on subsequences we can replace the initial sequence c^n by a convergent one defining the required representation for the limit. On Ω_0^c we can normalize the sequence and, using again the lemma on subsequences, obtain an identity which allows us to reduce the dimensionality of the problem (the number of generators in the considered case). The dimension reduction, resembling the Gauss algorithm of solving linear systems, can be done separately on elements of a subpartition. This type of reasoning, explained in details in [34], was used repeatedly in many proofs, and became standard. For multiperiod results the Gauss-type algorithm is imbedded in an induction in the number of periods and looks more involved but the principle remains the same. That is why we opt to present it in the case of a one-step assertion.*

1.3 No Arbitrage Criteria: Finite Ω

The definition of the NA^w-property remains the same as in the model with full information: $\widehat{R}_T \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$ or $\widehat{A}_T \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$.

As always, criteria in the case of finite Ω are easy to establish using the finite-dimensional separation theorem.

Proposition 1.3.1. *Let Ω be finite. The following conditions are equivalent:*

(a) NA^w;

(b) there exists $Z \in \mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) \leq 0$ for any $\zeta \in O_t$.

Proof. $(a) \Rightarrow (b)$ Note that $\widehat{A}_T$ is a finite-dimensional polyhedral (thus, closed) cone containing $-L^0(\mathbf{R}^d_+)$. The NA^w-property implies that non-zero elements of $L^0(\mathbf{R}^d_+)$ can be separated from $\widehat{A}_T$ in a strict sense. Using a classical argument, we construct an $\mathbf{F}$-martingale $Z = (Z_t)$ with strictly positive components such that $EZ_T\xi \leq 0$ for every $\xi \in \widehat{A}_T$. Namely, we can take Z_T equal to the sum of functionals negative on $\widehat{A}_T$ and strictly positive on $e_i I_\Gamma$ with the summation index Γ running through the family of atoms of $\mathcal{F}_T$ and $i = 1, 2, ..., d$. It follows that $E(Z_t \mathcal{L}_t \zeta_t) \leq 0$ for any $\zeta_t \in O_t$, implying the assertion.

$(b) \Rightarrow (a)$ This implication is obvious because for ζ admitting the representation (1.3) we have that

$$EZ_T\xi = \sum_{t=0}^{T} E[E(Z_t \mathcal{L}_t \zeta_t | \mathcal{G}_t)] \leq 0$$

and, therefore, ξ cannot be an element of $L^0(\mathbf{R}^d_+, \mathcal{F}_T)$ other than zero.

As we know, even in the case of full information, a straightforward generalization of the above criterion to an arbitrary Ω fails to be true, see [46], [25]. To get "satisfactory" theorems one needs either to impose extra assumptions, or to modify the concept of absence of arbitrage. We investigate here an analog of the NA^r-condition starting from the simple case when Ω is finite.

First, we establish a simple lemma which holds in a "very abstract" setting where the word "premodel" instead of "model" means that we do not suggest any particular properties of $(\mathcal{L}_t)$.

Fix a subset $\mathcal{I}_t$ of O_t. The elements of $\mathcal{I}_t$ will be interpreted later, in a more specific "financial" framework, as the reversible orders.

We say that the premodel has the *NA^r-property* if the NA^w-property holds for the premodel based on an $\mathbf{F}$-adapted process $\mathcal{L}' = (\mathcal{L}'_t)$ such that

(i) $\mathcal{L}'_t\zeta \geq \mathcal{L}_t\zeta$ componentwise for every $\zeta \in O_t$;

(ii) $\mathbf{1}\mathcal{L}'_t\zeta \neq \mathbf{1}\mathcal{L}_t\zeta$ if $\zeta \in O_t \setminus \mathcal{I}_t$ (i.e. the above inequality is not identity).

Lemma 1.3.2. *Let Ω be finite. If a premodel has the NA^r-property, then there is a process $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ for every $\zeta \in O_t$ and, if $\zeta \in O_t \setminus \mathcal{I}_t$,*

$$\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \in \mathcal{I}_t. \tag{1.4}$$

Proof. According to Proposition 1.3.1 applied to the premodel based on the process $\mathcal{L}'$ from the definition of NA^r there exists $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}'_t\zeta|\mathcal{G}_t) \le 0$ for any $\zeta \in O_t$. Hence, $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ by virtue of (i). Again by (i) we have, for $\zeta \in O_t \setminus \mathcal{I}_t$, that

$$Z_t\mathcal{L}'_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \ge Z_t\mathcal{L}_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}}.$$

If the order $\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}}$ is not in $\mathcal{I}_t$, this inequality is strict on a non-null set. Thus, taking the expectation, we obtain

$$EZ_t\mathcal{L}'_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} > 0$$

which is contradiction. □

Now we give a precise meaning to the word "model" by imposing an **assumption** on the generating process (fulfilled in both our examples) and specifying the sets $\mathcal{I}_t$.

Namely, we suppose that in $\mathbf{M}^{d\times m}$ there is a basis formed by the union of two families of vectors $\{f_i\}$ and $\{\tilde{f}_i\}$, $1 \le i \le md/2$, belonging to $\mathbf{M}^{d\times m}_+$ and such that componentwise

$$\mathcal{L}_t f_i + \mathcal{L}_t\tilde{f}_i \le 0, \tag{1.5}$$

while $\mathcal{I}_t$ is the cone of (matrix-valued) random variables having the form $\sum_i(\eta_i f_i + \tilde{\eta}_i\tilde{f}_i)$ with $\eta_i, \tilde{\eta}_i \in L^0_+(\mathcal{G}_t)$ and such that $\mathcal{L}_t\sum_i(\eta_i + \tilde{\eta}_i)(f_i + \tilde{f}_i) = 0$.

Note that the latter equality implies that $\mathbf{L}_t(\mathcal{I}_t) \subseteq \mathbf{L}_t(O_t) \cap (-\mathbf{L}_t(O_t))$. It is clear that the set $\mathcal{I}_t$ is stable under multiplication by elements of $L^0(\mathbf{R}_+, \mathcal{G}_t)$. This implies that the equality (1.4) for $\zeta \in \mathcal{I}_t$ always holds (cf. the formulations of Lemma 1.3.2 and the theorems below).

The inequality (1.5) means that the elementary transfers in opposite directions cannot lead to gains. The orders from $\mathcal{I}_t$, even symmetrized, do not incur losses.

For the models, in the definition of the NA^r the words "premodel" are replaced by "models", i.e. we require that the property (1.5) should hold also for the dominating process $\mathcal{L}'$.

Theorem 1.3.3. *Let Ω be finite. Then the following properties of the model are equivalent:*

(a) NA^r;

(b) there is $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ for every $\zeta \in O_t$ and, if $\zeta \in O_t$,

$$\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \in \mathcal{I}_t.$$

Proof. To check the remaining implication $(b) \Rightarrow (a)$ we put $\mathcal{L}'_t\zeta := \mathcal{L}_t\zeta - \bar{\mathcal{L}}_t\zeta$ defining the action of $\bar{\mathcal{L}}_t$ on the element $\zeta = \sum_i(\eta_i f_i + \tilde{\eta}_i\tilde{f}_i)$ by the formula $\bar{\mathcal{L}}_t\zeta := \sum_i(\eta_i + \tilde{\eta}_i)\theta_i$ where $\theta_i = \theta_i(t)$ has the components

$$\theta_i^k := \max\left\{\frac{1}{2}[\mathcal{L}_t(f_i + \tilde{f}_i)]^k, \frac{1}{d}\frac{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)}{E(Z_t^k|\mathcal{G}_t)}, \frac{1}{d}\frac{E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t)}{E(Z_t^k|\mathcal{G}_t)}\right\}.$$

The values $\theta_i^k(t)$ being negative, the condition (i) holds. The inequality (1.5) for $\mathcal{L}'_t$ is obviously fulfilled due to the first term in the definition of $\theta_i^k(t)$. Now let ζ be an element of $O_t \setminus \mathcal{I}_t$. This means that for some k and i the set

$$\Gamma := \{(\eta_i + \tilde{\eta}_i)[\mathcal{L}_t(f_i + \tilde{f}_i)]^k < 0\} = \{(\eta_i + \tilde{\eta}_i)Z_t^k[\mathcal{L}_t(f_i + \tilde{f}_i)]^k < 0\}$$

is non-null. From elementary properties of conditional expectations it follows that $(\eta_i + \tilde{\eta}_i)E(Z_t^k[\mathcal{L}_t(f_i + \tilde{f}_i)]^k|\mathcal{G}_t) < 0$ on Γ. The property (ii) holds because on Γ both $E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)$ and $E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t)$ are strictly negative as follows from the coincidence of sets

$$\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t) < 0\} = \{E(Z_t\mathcal{L}_t\tilde{f}_i)|\mathcal{G}_t) < 0\} = \{E(Z_t\mathcal{L}_t(f_i + \tilde{f}_i)|\mathcal{G}_t) < 0\}$$

which can be established easily. Indeed, $f_i I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}} \in \mathcal{I}_t$ and, by definition of $\mathcal{I}_t$,

$$I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}}\mathcal{L}_t\tilde{f}_i = -I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}}\mathcal{L}_t f_i.$$

Multiplying this identity by Z_t and taking the conditional expectation with respect to $\mathcal{G}_t$ we get that

$$I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}}E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t) = 0.$$

Similarly,

$$I_{\{E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t)=0\}}E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t) = 0.$$

These two equalities imply the coincidence of sets where the conditional expectations (always negative) are zero, i.e. the required assertion.

Finally, we check the NA^w-property of $(\mathcal{L}'_t)$ using Proposition 1.3.1. For any $\zeta = \sum_i (\eta_i f_i + \tilde{\eta}_i \tilde{f}_i)$ from O_t we have:

$$
\begin{aligned}
E(Z_t \mathcal{L}'_t \zeta | \mathcal{G}_t) &= E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) - E\Big(\sum_i (\eta_i + \tilde{\eta}_i) \sum_{k=1}^{d} Z_t^k \theta_i^k \Big| \mathcal{G}_t \Big) \\
&\leq E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) - \sum_i \eta_i E(Z_t \mathcal{L}_t f_i | \mathcal{G}_t) - \sum_i \tilde{\eta}_i E(Z_t \mathcal{L}_t \tilde{f}_i | \mathcal{G}_t) = 0.
\end{aligned}
$$

It follows that $EZ_T \xi \leq 0$ for every $\xi \in \widehat{R}_T(\mathcal{L}') \cap L^0(\mathbf{R}^d_+)$, excluding arbitrage opportunities for the model based on $\mathcal{L}'$.

The theorem is proven. □

Remark 1.3.4. *One might find it convenient to view $\mathbf{M}^{d\times m}$ as the set of linear operators defined by corresponding matrices and consider the adjoint operators $\mathcal{L}^*_{\omega,t} : \mathbf{R}^d \to (\mathbf{M}^{d\times m})^*$. This gives a certain flexibility of notations, e.g., the property "$E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) \leq 0$ for every $\zeta \in O_t$" can be formulated as "the operator $E(\mathcal{L}^*_t Z_t | \mathcal{G}_t)$ is negative" (in the sense of partial ordering induced by $\mathbf{M}^{d\times m}_+$), the inclusion $f_i \in \operatorname{Ker} E(\mathcal{L}^*_t Z_t | \mathcal{G}_t)$ can be written instead of the equality $E(Z_t \mathcal{L}_t f_i | \mathcal{G}_t) = 0$ and so on. However, the current notation has the advantage of being easier adjustable for more general situation where $\mathcal{L}_t$ is a concave positive homogeneous mapping from $\mathbf{M}^{d\times m}_+$ into $L^0(\mathbf{R}^d, \mathcal{F}_t)$.*

Remark 1.3.5. *The hypothesis on the structure of invertible claims may not be fulfilled for Examples 1 and 2. For the investor having access to full information, the set of all assets can be split into classes of equivalence within which one can do frictionless transfers though not necessary in one step. Our assumption means that all transfers within each class are frictionless, a hypothesis which, as was noted in [46], does not lead to a loss of generality as a fully informed "intelligent" investor will not lose money making charged transfers within an equivalence class. However, in the context of restricted information it seems that such an assumption means that the information on equivalence classes is available to the investor.*

1.4 No Arbitrage Criteria: Arbitrary Ω

In the general case the assertion of Proposition 1.3.1 fails to be true though with a suitable modification its condition (b) remains sufficient for the NA^w-property. Namely, we have:

Proposition 1.4.1. *The NA^w-property holds if there exists $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that all conditional expectations $E(|Z_t||\mathcal{L}_t f_i||\mathcal{G}_t)$ and $E(|Z_t||\mathcal{L}_t \tilde{f}_i||\mathcal{G}_t)$ are finite and $E(Z_t \mathcal{L}_t \zeta|\mathcal{G}_t) \le 0$ for any $\zeta \in O_t$.*

This result is an obvious corollary of the following technical lemma dealing with integration issues.

Lemma 1.4.2. *Let $\Sigma_T = Z_T \sum_{t=0}^T \xi_t$ with $Z \in \mathcal{M}(\mathbf{R}^d_+, \mathbf{F})$ and $\xi_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $E(|Z_t||\xi_t||\mathcal{G}_t) < \infty$ and $E(Z_t\xi_t|\mathcal{G}_t) \le 0$. Put $\bar{\Sigma}_T := E(\Sigma_T|\mathcal{G}_T)$. If $\bar{\Sigma}_T^- \in L^1$, then $\bar{\Sigma}_T \in L^1$ and $E\bar{\Sigma}_T \le 0$.*

Proof. We proceed by induction. The claim is obvious for $T = 0$. Suppose that it holds for $T-1$. Clearly,

$$Z_T \sum_{t=0}^{T-1} \xi_t = \Sigma_T - Z_T\xi_T.$$

By the martingale property $E(Z^i_T|\xi_t||\mathcal{G}_t) = E(Z^i_t|\xi_t||\mathcal{G}_t) < \infty$ implying that $E(|Z_T||\xi_t||\mathcal{G}_t) < \infty$ for any $t \le T$. Thus, $\bar{\Sigma}_T$ is well-defined and finite. Taking the conditional expectation with respect to $\mathcal{G}_T$ in the above identity we get, using the martingale property, that

$$E(\Sigma_{T-1}|\mathcal{G}_T) = E\left(Z_T \sum_{t=0}^{T-1} \xi_t \middle| \mathcal{G}_T \right) = \bar{\Sigma}_T - E(Z_T\xi_T|\mathcal{G}_T) \ge \bar{\Sigma}_T.$$

Therefore, the negative part of $E(\Sigma_{T-1}|\mathcal{G}_T)$ is dominated by the negative part of $\bar{\Sigma}_T$ which is integrable. Using Jensen's inequality we have:

$$\begin{aligned} \bar{\Sigma}^-_{T-1} &= [E(E(\Sigma_{T-1}|\mathcal{G}_T)|\mathcal{G}_{T-1})]^- \\ &\le E([E(\Sigma_{T-1}|\mathcal{G}_T)]^-|\mathcal{G}_{T-1}) \\ &\le E(\bar{\Sigma}^-_T|\mathcal{G}_{T-1}). \end{aligned}$$

Thus, $\bar{\Sigma}^-_{T-1} \in L^1$ and, by virtue of the induction hypothesis, $\bar{\Sigma}_{T-1} \in L^1$ and $E\bar{\Sigma}_{T-1} \le 0$. In the representation $\bar{\Sigma}_T = E(\bar{\Sigma}_{T-1}|\mathcal{G}_T) + E(\bar{\Sigma}^-_T|\mathcal{G}_{T-1})$ the first term is integrable and has negative expectation while the second is negative. Thus, $E\bar{\Sigma}_T \le 0$ and, automatically, $E\bar{\Sigma}^+_T < \infty$. □

The NA^r-criterion, suitably modified, remains true without any restriction on the probability space. Of course, in its formulation one needs to take care about the existence of the involved conditional expectations. This can be done as in the next result.

Theorem 1.4.3. *The following conditions are equivalent:*

(a) *NA^r;*

(b) *there is $Z \in \mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$ such that all random variables $E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)$, $E(Z_t\mathcal{L}_t \tilde{f}_i|\mathcal{G}_t)$ are finite, $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ for every $\zeta \in O_t$ and, if $\zeta \in O_t$,*

$$\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \in \mathcal{I}_t. \tag{1.6}$$

We have no trouble with the implication $(b) \Rightarrow (a)$: an inspection of the arguments given in the case of finite Ω shows that they work well until the concluding step which now can be done just by reference to Lemma 1.4.2.

The proof of the "difficult" implication $(a) \Rightarrow (b)$ follows the same line of ideas as in the case of full information.

Lemma 1.4.4. *Suppose that the equality*

$$\sum_{t=0}^{T} \mathcal{L}_t\tilde{\zeta}_t - \tilde{r} = 0 \tag{1.7}$$

with $\tilde{\zeta}_t \in O_t$ and $\tilde{r}_t \in L^0(\mathbf{R}^d_+)$ holds only if $\tilde{\zeta}_t \in \mathcal{I}_t$ and $\tilde{r} = 0$. Then $\widehat{A}_T$ is closed in probability.

Proof. For $T = 0$ the arguments are exactly the same as were used for Proposition 1.2.1 with obvious changes caused by the extra term describing the funds withdrawals. Namely, the difference is that for the limiting normalized order $\tilde{\zeta} := \sum_{i=1}^N \tilde{c}_i g_i$ we get the equality $\mathcal{L}\tilde{\zeta} - \tilde{r} = 0$ where $\tilde{r} \in L^0(\mathbf{R}^d_+, \mathcal{F}_T)$ is the limit of normalized funds withdrawals. By hypothesis, $\tilde{r} = 0$ and we can complete the proof using the same Gauss-type reduction procedure.

Arguing by induction, we suppose that $\widehat{A}_{T-1}$ is closed and consider the sequence of order processes $(\zeta^n_t)_{t\le T}$ such that $\sum_{t=0}^T \mathcal{L}_t\zeta^n_t - r^n \to \eta$. There is an obvious reduction to the case where at least one of "elementary" orders at time zero tends to infinity. Normalizing and using the induction hypothesis we obtain that there exists an order process $(\tilde{\zeta}_t)_{t\le T}$ with nontrivial $\tilde{\zeta}_0$ such that $\sum_{t=0}^{T}\mathcal{L}_t\tilde{\zeta}_t-\tilde{r}=0$ and we can use the assumption of the lemma. It ensures that $\tilde{r}=0$ and there are $\zeta'_t\in O_t$ such that $\mathcal{L}_t\zeta'_t=-\mathcal{L}_t\tilde{\zeta}_t$. This allows us to reduce a number of non-zero coefficients (i.e. "elementary" orders) at the initial order by putting, $\bar{\zeta}^n_0=\zeta^n_0-\alpha^n\tilde{\zeta}_0$, as in the proof of Proposition 1.2.1, and $\bar{\zeta}^n_t=\zeta^n_t+\alpha^n\zeta'_t$ for $t\ge 1$. □

Lemma 1.4.5. *The NA^r-condition implies the hypothesis of the above lemma.*

Proof. Of course, $\tilde{r}=0$ (otherwise, $(\tilde{\zeta}_t)$ is an arbitrage opportunity, i.e. even NA^w is violated). For the process $(\mathcal{L}'_t)$, from definition of NA^r we have that componentwise

$$\sum_{t=0}^{T}\mathcal{L}'_t\tilde{\zeta}_t\ge\sum_{t=0}^{T}\mathcal{L}_t\tilde{\zeta}_t=0$$

and $\mathbf{1}\sum_{t=0}^{T}\mathcal{L}'_t\tilde{\zeta}_t>0$ with strictly positive probability if at least one of $\tilde{\zeta}_t$ does not belong to $\mathcal{I}_t$. This means that $(\tilde{\zeta}_t)$ is an arbitrage opportunity for the model based on $(\mathcal{L}'_t)$. □

Lemma 1.4.6. *Assume that the hypothesis of Lemma 1.4.4 holds. Then for any "elementary" order f and every $t\le T$ one can find a bounded process $Z=Z^{(t,f)}\in\mathcal{M}(\operatorname{int}\mathbf{R}^d_+,\mathbf{F})$ such that:*

1) *$E(|Z_s||\mathcal{L}_s g|)<\infty$ and $E(Z_s\mathcal{L}_s g|\mathcal{G}_s)\le 0$ for all $s\le T$ and all "elementary" orders g,*

2) *$fI_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}}\in\mathcal{I}_t$.*

Proof. We may assume without loss of generality that all portfolio increments $\mathcal{L}_s g$ corresponding to the elementary orders g are integrable (otherwise we can pass to an equivalent measure P' with the bounded density ρ, find the process Z' with the needed properties under P' and take $Z=\rho\tilde{Z}'$).

Let $\mathcal{Z}$ be the set of all bounded processes $Z\in\mathcal{M}(\mathbf{R}^d_+,\mathbf{F})$ such that $EZ_T\xi\le 0$ whenever is $\xi\in\widehat{A}^1_T:=\widehat{A}_T\cap L^1$. Let

$$c_t:=\sup_{Z\in\mathcal{Z}}P(E(Z_t\mathcal{L}_t f|\mathcal{G}_t)<0). \tag{1.8}$$

Let Z be an element for which the supremum is attained (one can take as Z a countable convex combination of any uniformly bounded sequence along which the supremum is attained).

If 2) fails, then the random vector $\mathcal{L}_t(f+\tilde{f})I_{\{E(Z_t\mathcal{L}_tf|\mathcal{G}_t)=0\}}$ (all components of which are negative) is not zero. This implies that the element $-\mathcal{L}_t\tilde{f}I_{\{E(Z_t\mathcal{L}_tf|\mathcal{G}_t)=0\}}$ does not belong to $\widehat{A}^1_T$. Indeed, in the opposite case we would have the identity

$$\sum_{s=0}^{T}\mathcal{L}_s\zeta_s = -\mathcal{L}_t\tilde{f}I_{\{E(Z_t\mathcal{L}_tf|\mathcal{G}_t)=0\}}.$$

The assumption of Lemma 1.4.4 ensures that the order $\tilde{f}I_{\{E(Z_t\mathcal{L}_tf|\mathcal{G}_t)=0\}}+\zeta_t$ is in $\mathcal{I}_t$. Thus, for the symmetrized order we have that

$$\mathcal{L}_t(f+\tilde{f})I_{\{E(Z_t\mathcal{L}_tf|\mathcal{G}_t)=0\}}+\mathcal{L}_t(\zeta+\tilde{\zeta})=0.$$

Since the second term is also negative componentwise, both should be equal to zero and we get a contradiction.

By the Hahn–Banach theorem one can separate $\varphi := -\mathcal{L}_t\tilde{f}I_{\{E(Z_t\mathcal{L}_tf|\mathcal{G}_t)=0\}}$ and $\widehat{A}^1_T$: that is we may find $\eta\in L^\infty(\mathbf{R}^d)$ such that

$$\sup_{\xi\in\widehat{A}^1_T} E\eta\xi < E\eta\varphi.$$

Since $\widehat{A}^1_T$ is a cone containing $-L^1(\mathbf{R}^d_+)$ the supremum above is equal to zero, $\eta\in L^1(\mathbf{R}^d_+)$ and $E\eta\varphi>0$. The latter inequality implies that for $Z^\eta_t = E(\eta|\mathcal{G}_t)$ we have $EE(Z^\eta_t\mathcal{L}_tf|\mathcal{G}_t)I_{\{E(Z_t\mathcal{L}_tf|\mathcal{G}_t)=0\}}<0$. Therefore, for the martingale $Z' := Z+Z^\eta$ we have that

$$P\big(E(Z'_t\mathcal{L}_tf|\mathcal{G}_t)<0\big) > P\big(E(Z_t\mathcal{L}_tf|\mathcal{G}_t)<0\big)=c_t.$$

This contradiction shows that 2) holds.

The process Z constructed in this way may be not in $\mathcal{M}(\operatorname{int}\mathbf{R}^d_+,\mathbf{F})$. However, it can be easily "improved" to meet the latter property. To this end, fix $i\le d$ and consider, in the subset of $\mathcal{Z}$ on which the supremum c_t in (1.8) is attained, a process Z with maximal probability $P(Z^i_T>0)$ (such process does exist). Then $P(\bar{Z}^i_T>0)=1$. Indeed, in the opposite case, the element $e_iI_{\{Z^i_T=0\}}\in L^1(\mathbf{R}^d_+)$ is not zero and, therefore, does not belong to $\widehat{A}^1_T$. So it can be separated from the latter set. The separating functional generates a martingale $Z'\in\mathcal{Z}$. Since $P(\bar{Z}_T+Z'_T>0)>P(\bar{Z}_T>0)$, we arrive to a contradiction with the definition of $\bar{Z}$. The set of $Z\in\mathcal{Z}$ satisfying 1) and 2) is convex and, hence, a convex combination of d processes obtained in this way for each coordinate has the required properties. □

The implication $(a) \Rightarrow (b)$ of the theorem follows from the lemmas above. Indeed, by virtue of Lemmas 1.4.5 – 1.4.6, NA^r ensures the existence of processes $Z^{(t,f)}$ satisfying 1) and 2) of Lemma 1.4.6. One can take as a required martingale Z the process $Z := \sum_{t,f} Z^{(t,f)}$ where $t = 0, 1, ..., T$ and f runs through the set of "elementary" orders. An arbitrary order $\zeta \in O_t$ is a linear combination of elementary orders with positive $\mathcal{G}_t$ measurable coefficients. The condition $E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) \le 0$ follows from the property 1) of Lemma 1.4.6. To prove the inclusion (1.6) we note that $I_{\{\Sigma \xi_i = 0\}} = \prod I_{\{\xi_i = 0\}}$ when $\xi_i \le 0$. With this observation the required inclusion is an easy corollary of the property 2) of Lemma 1.4.6 and the stability of $\mathcal{I}_t$ under multiplication by positive $\mathcal{G}_t$-measurable random variables.

Remark 1.4.7. *In the above proof we get from NA^r a condition which looks stronger than (b), with bounded Z and integrable random variables $|Z_t||\mathcal{L}_t f|$, but, in fact, it is equivalent to (b).*

1.5 Hedging Theorem

Thanks to the previous development, hedging theorems in the model with partial information do not require new ideas. For the case of finite Ω the result can be formulated in our "very abstract" setting without additional assumptions on the structure of the sets $\mathcal{I}_t$.

We fix a d-dimensional random variable $\widehat{C}$, the contingent claim expressed in physical units. Define the set

$$\Gamma = \{v \in \mathbf{R}^d : \ \widehat{C} \in v + \widehat{A}_T\}.$$

Let $\mathcal{Z}$ be the set of martingales $Z \in \mathcal{M}_T(\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t \mathcal{L}_t \zeta_t | \mathcal{G}_t) \le 0$ for every $\zeta_t \in O_t$. Put

$$D := \left\{ v \in \mathbf{R}^d : \ \sup_{Z \in \mathcal{Z}} E(Z_T \widehat{C} - Z_0 v) \le 0 \right\}.$$

Proposition 1.5.1. *Let Ω be finite and $\mathcal{Z} \ne \emptyset$. Then $\Gamma = D$.*

In this theorem the inclusion $\Gamma \subseteq D$ is obvious while the reverse inclusion is an easy exercise on the finite-dimensional separation theorem. We leave it to the reader.

In the case of general Ω we should take care about integrability and closedness of the set $\widehat{A}_T$. To this end we shall work with the model in the "narrow" sense of the preceding sections assuming the NA^r-property. Now $\mathcal{Z}$ is the set of bounded martingales $Z \in \mathcal{M}_T(\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)$, $E(Z_t\mathcal{L}_t \tilde{f}_i|\mathcal{G}_t)$ are finite, $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ and $E(Z_T\widehat{C})^- < \infty$. The definitions of the sets Γ and D remain the same.

Theorem 1.5.2. *Suppose that NA^r holds. Then $\Gamma = D$.*

Proof. The inclusion $\Gamma \subseteq D$ follows from the inequality

$$Z_T(\widehat{C} - v) \le Z_T \sum_{t=0}^{T} \mathcal{L}_t\zeta_t, \qquad \zeta_t \in O_t,$$

and Lemma 1.4.2.

To check the inclusion $D \subseteq \Gamma$ we take a point $v \notin \Gamma$ and show that $v \notin D$. It is sufficient to find $Z \in \mathcal{Z}$ such that $Z_0 v < EZ_T\widehat{C}$. Consider a measure $\tilde{P} \sim P$ with bounded density ρ such that $\widehat{C}$, and all $|\mathcal{L}_t||f_i|$ and $|\mathcal{L}_t||f_i|$ belong to $L^1(\tilde{P})$. Under NA^r the convex set $\tilde{A}^1 := A^T_0 \cap L^1(\tilde{P})$ is closed and does not contain the point $\widehat{C} - v$. Thus, we can separate the latter by a functional η from L^∞. This means that

$$\sup_{\xi \in \tilde{A}^1} E\rho\eta\xi < E\eta\rho(\widehat{C} - v).$$

It is clear, that the bounded martingale $Z_t := E(\rho\eta|\mathcal{F}_t)$ satisfies the required properties. $\square$

Chapter 2

Hedging of American Options under Transaction Costs

2.1 Introduction

A classical result of the theory of frictionless market asserts that the set of initial capitals needed to hedge a European option ξ with the maturity(=exercise) date T is a semi-infinite closed interval $[x_*, \infty[$ whose left extremity $x_* = \sup_\rho E\rho_T\xi$ where $\rho = (\rho_t)$ runs through the set of martingale densities for the price process S. Recall that "to hedge" means to dominate the random variable ξ by the terminal value of a self-financing portfolio. Basically, the assertion remains the same for the case of American-type option having as the pay-off function an adapted càdlàg stochastic process $f = (f_t)_{t\le T}$. In this case, $x_* = \sup_{\rho,\tau} E\rho_\tau f_\tau$ where τ (an exercise date) runs through the set of stopping times dominated by T. "To hedge" means here to dominate, on the whole time interval, the pay-off process by a portfolio process. In both cases, as was shown by Dmitri Kramkov [40], the results can be deduced from the optional decomposition theorem applied to a corresponding Snell envelope.

We deliberately formulated the statements above (omitting assumptions) in terms of density processes rather than in terms of martingale measures to facilitate the comparison with the corresponding theorems for models with market friction.

In the theory of markets with transaction costs hedging theorems for European options are already available for discrete-time as well as for continuous-time models. Mathematically, in discrete-time, the model is given by an adapted cone-valued process $G = (G_t)_{t=0,1,\dots,T}$ in $\mathbf{R}^d$. The portfolio (value) process X is adapted and its increments $\Delta X_t = X_t - X_{t-1}$ are selectors of the random cones $-G_t$. The contingent claim C is a random vector. The hedging problem is to describe the set Γ of initial values x for which one can find a value process X such that $x + X_T$ dominates C in the sense of the partial ordering induced by the cone G_T. It happens that, under appropriate assumptions,

$$\Gamma = \{x \in \mathbf{R}^d : \ Z_0 x \ge EZ_T C \quad \forall Z \in \mathcal{M}_0^T(G^*)\}$$

where $\mathcal{M}_0^T(G^*)$ is the set of martingales evolving in the (positive) duals G_t^* of the cones G_t. In the financial context, G_t are solvency cones $\widehat{K}_t$, "hat" means that the assets are measured in physical units (the notation K_t is used for the solvency cones when values of assets are expressed in units of a numéraire), and the elements of $\mathcal{M}_0^T(\widehat{K}^*)$ are called consistent price systems. The correspondence with the frictionless case is simple: $\xi = CS_T$ and $Z = \rho S$. For the continuous-time model the description remains the same but the theorem becomes rather delicate. The reason for this is that the model formulation is more involved and even the basic definition of value processes has several versions. Moreover, one needs assumptions on the regularity of the cone-valued process, see the development and extended discussion of financial aspects in [19], [30], [37], [39], [16].

The hedging problem for the vector-valued American option $U = (U_t)$ in the discrete-time framework with transaction costs was investigated in the paper [14] by Bruno Bouchard and Emmanuel Temam (see also the earlier article [17] where the two-asset case for finite Ω was studied). It happens that one needs a richer set of "dual variables" to describe the set Γ formed by the initial values of self-financing portfolios dominating, in the sense of partial ordering, the vector-valued adapted pay-off process U. Bouchard and Temam proved the identity

$$\Gamma = \left\{ x \in \mathbf{R}^d : \ \bar{Z}_0 x \geq E \sum_{t=0}^{N} Z_t U_t \ \ \forall Z \in \mathcal{Z}_d(G^*, P) \right\}$$

where $\mathcal{Z}_d(G^*, P)$ is the set of discrete-time adapted process $Z = (Z_t)$ such that the random variables $Z_t, \bar{Z}_t \in L^1(G_t^*)$ for all $t \leq T$ with the notation $\bar{Z}_t := \sum_{s=t}^{T} E(Z_s|\mathcal{F}_t)$. Note that the inclusion $\mathcal{M}_0^T(G^*) \subseteq \mathcal{Z}_d(G^*, P)$ is obvious.

In the theory of financial markets with transaction costs modelling of portfolio processes is rather involved. It is quite convenient to consider right-continuous portfolio processes and work in the standard framework of stochastic calculus. This approach leads to satisfactory hedging theorems, e.g., for a model with constant transaction costs and a continuous price process, see [30], [37], [39]. However, as was shown by Miklós Rásonyi in [44], such a definition is not appropriate when the price process is discontinuous: in general, the natural formulation of the hedging theorem (for European options) fails to be true. Luciano Campi and Walter Schachermayer in [16] suggested a more complicated definition of the portfolio processes for which the natural formulation of hedging theorem can be preserved.

In this chapter, we investigate the hedging problem using the approach of Campi and Schachermayer in a slightly more general mathematical framework, which is described in the next section. In Section 3 we recall the definition of portfolio processes together with some known results adjusted to our purposes and accompanied by explicative comments. Section 4 contains the formulation of the main theorem preceding by a discussion of objects involved. Financial interpretation is given in the concluding Section 5.

2.2 Basic Concepts

Standing hypotheses. We shall work from the very beginning in a slightly more general and more transparent "abstract" setting where we are given two cone-valued processes $G = (G_t)_{t\in[0,T]}$ and $G^* = (G^*_t)_{t\in[0,T]}$ in duality, i.e. $G^*_t(\omega)$ is the positive dual of the cone $G_t(\omega)$ for each ω and t. We suppose that $G_t = \operatorname{cone}\{\xi^k_t : \ k \in \mathbf{N}\}$ where the generating processes are càdlàg, adapted, and for each ω only a finite number of vectors $\xi^k_t(\omega)$, $\xi^k_{t-}(\omega)$ are different from zero, i.e. the cones $G_t(\omega)$ and $G_{t-}(\omega) := \operatorname{cone}\{\xi^k_{t-}(\omega) : \ k \in \mathbf{N}\}$ are polyhedral, hence, closed.

Throughout the paper we assume that all cones G_t contain $\mathbf{R}^d_+$ and are proper, i.e. $G_t \cap (-G_t) = \{0\}$ or, equivalently, $\operatorname{int} G^*_t \neq \emptyset$; moreover, we assume that the cones G_{t-} are also proper.

In a more specific financial setting (see [39], [16]) the cones G_t are the solvency cones $\widehat{K}_t$ provided that the portfolio positions are expressed in physical units[1]. The hypothesis that the cones G_t are proper means that there is **efficient friction**.

It is important to note that, in general, the continuity of generators does not imply the continuity of the cone-valued processes. The following simple example in $\mathbf{R}^2$ gives an idea: the process $G_t = \operatorname{cone}\{\xi^1_t, \xi^2_t\}$ where $\xi^1_t = e_1$, $\xi^1_t = (t-1)^+ e_2$ is not right-continuous though the generators are continuous.

To formulate the needed regularity properties of G we introduce some notation. Let $G_{s,t}(\omega)$ denote the closure of $\operatorname{cone}\{G_r(\omega) : \ s \le r < t\}$ and let

$$G_{s,t+} := \cap_{\varepsilon>0} G_{s,t+\varepsilon}, \quad G_{s-,t} := \cap_{\varepsilon>0} G_{s-\varepsilon,t}, \quad G_{s-,t+} := \cap_{\varepsilon>0} G_{s-\varepsilon,t+\varepsilon}$$

with an obvious change when $s = 0$.

We assume that $G_{t,t+} = G_t$, $G_{t-,t} = G_{t-}$, and $G_{t-,t+} = \operatorname{cone}\{G_{t-}, G_t\}$ for all t.

It is easy to see that these regularity conditions are fulfilled for the case where the cones G_t and G_{t-} are proper and generated by a finite number of generators of unit length. Indeed, let $G_t = \operatorname{cone}\{\xi^k_t : \ k \le n\}$ with $|\xi^k_t| = 1$ for all t. Since the dependence on ω here is not important we may argue for the deterministic case. Let $x \notin G_t$. The proper closed convex cones $\mathbf{R}_+ x$ and G_t intersect each other only at the origin, so the intersections of the interiors of

[1]The notation K_t is reserved for the solvency cones when the portfolio positions are expressed in terms of a numéraire.

$(-\mathbf{R}_+x)^*$ and G_t^* is non-empty (this is a corollary of the Stiemke lemma as it is given in the appendix in [36]). It follows that there is $y \in \mathbf{R}^d$ such that x belongs to the open half-space $\{z: \ yz < 0\}$ while the balls $\{z: \ |z - \xi_t^k| < \delta\}$ for sufficiently small $\delta > 0$ lay in the complementary half-space. Since ξ^k are right-continuous, the cones $G_{s,t+\varepsilon}$ for all sufficiently small $\varepsilon > 0$ also lay in the latter. Thus, $x \notin G_{t,t+}$ and $G_{t,t+} \subseteq G_t$. The opposite inclusion is obvious. In the same way we get other two identities.

Example. Let us consider a financial market with constant proportional transaction costs given by a matrix $\Lambda = (\lambda^{ij})$ defining the proper solvency cone K (in terms of a numéraire). Suppose that the components of the positive càdlàg price process S are such that $\inf_t S_t^i > 0$, $i = 1, ..., d$. Let us consider the mapping

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d).$$

The generators of the cone $G_t = \widehat{K} = \phi_t K$ are vectors $\phi_t x_i$, where x_i, $i \leq N$, are generators of the polyhedral cone K (they can be written explicitly in terms of Λ). The generators of the cone $G_t^* = \widehat{K}^* = \phi_t^{-1} K^*$ are vectors $\phi_t^{-1} z_i$, where z_i, $i \leq M$, are generators of the polyhedral cone K^*.

All above hypotheses are fulfilled for this model. Moreover, if S admits an equivalent martingale measure with the density process ρ and if $w \in \operatorname{int} K^*$, then the process Z with the components $Z_t^i = w^i S_t^i \rho_t$ is a martingale such that $Z_t \in \operatorname{int} G^*$ and $Z_{t-} \in \operatorname{int}(G_{t-})^* = \operatorname{int} G_{t-}^*$ for all t. Existence of such a martingale is the major assumption of the hedging theorem.

Remark. The argument above shows that the regularity assumptions hold for the model specified as in [16], i.e. for the case where $G_t = \widehat{K}(\Pi_t)$ and $G_{t-} = \widehat{K}(\Pi_{t-})$ are proper cones generated by the bid-ask process $\Pi = (\Pi_t)$.

Comment on notation. As usual, $L^0(G_t, \mathcal{F}_t)$ is the set of $\mathcal{F}_t$-measurable selectors of G_t, $\mathcal{M}_0^T(G^*)$ stands for the set of martingales $M = (M_t)_{t \leq T}$ with trajectories evolving in G^*; $\mathbf{1} := \sum e_i = (1, ..., 1)$; $||Y||_t$ is the total variation of the function Y on the interval $[0, t]$.

Let B be a càdlàg adapted process of bounded variation. We shall denote by $\dot{B}$ the optional version of the Radon–Nikodym derivative $dB/d||B||$ with respect to the total variation process $||B||$. In particular, this notation will be used for $B = Y_+$ where $Y_+ = (Y_{t+})$.

We denote by $\mathcal{D} = \mathcal{D}(G)$ the subset of $\mathcal{M}_0^T(\operatorname{int} G^*)$ formed by martingales Z such that not only $Z_t \in L^0(\operatorname{int} G_t^*, \mathcal{F}_t)$ but also $Z_{t-} \in L^0(\operatorname{int}(G_{t-})^*, \mathcal{F}_t)$ for all $t \in [0,T]$. In the financial context the elements of $\mathcal{D}$ are called *consistent price systems*.

Coherent price systems. Let ν be a finite measure on the interval $[0,T]$ and let $\mathcal{N}$ denote the set of all such measures. For an $\mathbf{R}^d_+$-valued process Z we denote by $\bar{Z}^\nu$ the optional projection of the process $\int_{[t,T]} Z_s \nu(ds)$, i.e. an optional process such that for every stopping time $\tau \le T$ we have

$$\bar{Z}^\nu_\tau = E\left(\int_{[\tau,T]} Z_s \nu(ds) \Big| \mathcal{F}_\tau\right).$$

The process $\bar{Z}^\nu$ can be represented as a difference of a martingale and a left-continuous process whose components are increasing:

$$\bar{Z}^\nu_t = E\left(\int_{[0,T]} Z_s \nu(ds) \Big| \mathcal{F}_t\right) - \int_{[0,t[} Z_s \nu(ds).$$

We associate with ν the product-measure $P^\nu(d\omega, dt) = P(d\omega)\nu(dt)$ on the space $(\Omega \times [0,T], \mathcal{F} \times \mathcal{B}_{[0,T]})$; the average with respect to this measure is denoted by E^ν.

Let $\mathcal{Z}(G^*, P, \nu)$ denote the set of adapted càdlàg processes $Z \in L^1(P^\nu)$ such that $Z_t, \bar{Z}^\nu_t \in L^0(G^*_t, \mathcal{F}_t)$ for all $t \le T$. We call the elements of this set *coherent price systems*. In the case where Z is a martingale, $\bar{Z}^\nu_\tau = \nu([\tau,T])Z_\tau$ and, hence, $\mathcal{M}^T_0(G^*) \subseteq \mathcal{Z}(G^*, P, \nu)$.

2.3 The Model and Prerequisites

We define the portfolio processes following the paper [16]. For the reader convenience, we give also full proofs of the basic properties.

Let Y be a d-dimensional predictable process of bounded variation starting from zero and having trajectories with left and right limits (French abbreviation: *làdlàg*). Put $\Delta Y := Y - Y_-$, as usual, and $\Delta^+ Y := Y_+ - Y$ where $Y_+ = (Y_{t+})$. Define the right-continuous processes

$$Y^d_t = \sum_{s \le t} \Delta Y_s, \qquad Y^{d,+}_t = \sum_{s \le t} \Delta^+ Y_s$$

(the first is predictable while the second is only adapted) and, at last, the continuous one:

$$Y^c := Y - Y^d - Y^{d,+}_-.$$

Recall that $\dot{Y}^c$ denotes the optional version of the Radon–Nikodym derivative $dY^c/d||Y^c||$.

Let $\mathcal{Y}$ be the set of such processes Y satisfying the following conditions:

1) $\dot{Y}^c \in -G\ dP\, d||Y^c||$-a.e.;
2) $\Delta^+ Y_\tau \in -G_\tau$ a.s. for all stopping times $\tau \le T$;
3) $\Delta Y_\sigma \in -G_{\sigma-}$ a.s. for all predictable[2] stopping times $\sigma \le T$.

Let $\mathcal{Y}^x := x + \mathcal{Y}$, $x \in \mathbf{R}^d$. We denote by $\mathcal{Y}^x_b$ the subset of $\mathcal{Y}^x$ formed by the processes Y bounded from below in the sense of partial ordering, i.e. such that $Y_t + \kappa_Y \mathbf{1} \in L^0(G_t, \mathcal{F}_t)$, $t \le T$, for some $\kappa_Y \in \mathbf{R}$. In the financial context (where $G = \widehat{K}$) the elements of $\mathcal{Y}^x_b$ are the *admissible* portfolio processes.

To use classical stochastic calculus we shall operate with the following right-continuous adapted process of bounded variation

$$Y_+ := Y^c + Y^d + Y^{d,+},$$

and use the relation $Y_+ = Y + \Delta^+ Y$. Since the generators are right-continuous, the process Y_+ inherits the boundedness from below of Y (by the same constant process $\kappa_Y \mathbf{1}$). Note that $||Y_+||_t = ||Y||_{t-} + |\Delta Y_t + \Delta^+ Y_t|$.

In the sequel we shall use a larger set portfolio processes depending on $Z \in \mathcal{M}^T_0(G^*)$, namely,

$$\mathcal{Y}^x_b(Z) := \{Y \in \mathcal{Y}^x :\ \text{there is a scalar martingale } M \text{ such that } ZY \ge M\}.$$

Lemma 2.3.1. *If $Z \in \mathcal{M}^T_0(G^*)$ and $Y \in \mathcal{Y}^x_b(Z)$, then both processes ZY_+ and ZY are supermartingales and*

$$E\left(- Z\dot{Y}^c \cdot ||Y^c||_T - \sum_{s \le T} Z_{s-} \Delta Y_s - \sum_{s<T} Z_s \Delta^+ Y_s \right) \le Z_0 x - E Z_T Y_T. \quad (2.1)$$

Proof. With the right-continuous process Y_+ (having the same left limits as Y) the standard product formula is readily applied:

$$Z_t Y_{t+} = Z_0 x + Y_- \cdot Z_t + Z\dot{Y}^c \cdot ||Y^c||_t + \sum_{s \le t} Z_s \Delta Y_s + \sum_{s \le t} Z_s \Delta^+ Y_s.$$

[2]Since Y is a predictable process, the set $\{\Delta Y \ne 0\}$ can be represented as a disjoint union of graphs of predictable stopping times. Hence, 3) implies that $\Delta Y_\tau \in -G_{\tau-}$ a.s. for all stopping times $\tau \le T$.

Taking into account that $Y = Y_{-} + \Delta Y$, we rewrite this identity as

$$Z_t Y_{t+} = Z_0 x + Y \cdot Z_t + Z\dot{Y}^c \cdot ||Y^c||_t + \sum_{s \le t} Z_{s-} \Delta Y_s + \sum_{s \le t} Z_s \Delta^+ Y_s.$$

Since $Y_+ = Y + \Delta^+ Y$, we obtain from here the product formula for ZY (which is "non-standard" since Y may not be càdlàg):

$$Z_t Y_t = Z_0 x + Y \cdot Z_t + Z\dot{Y}^c \cdot ||Y^c||_t + \sum_{s \le t} Z_{s-} \Delta Y_s + \sum_{s < t} Z_s \Delta^+ Y_s.$$

By virtue of requirements on Y the stochastic integral $Y \cdot Z$ is a local martingale while the last three terms define decreasing processes (by our standing assumption $Z_{s-} \in (G_{s-})^*$). Recalling that the process ZY is bounded from below by a martingale, we deduce from here that the local martingale $Y \cdot Z$ is bounded from below by a martingale and, hence, is a supermartingale, hence integrable. It follows that the terminal values of the mentioned decreasing processes are integrable. Therefore, ZY is a supermartingale. By the Fatou lemma its right-continuous limit, i.e. the process ZY_+ is a supermartingale. Finally, taking the expectation of the last identity above and using the inequality $Y \cdot Z_T \le 0$, we get the required bound (2.1). □

Lemma 2.3.2. *Suppose that $Y^n \in \mathcal{Y}$ and for all ω (except of a null set) $\lim_n Y_t^n(\omega) = Y_t(\omega)$ for all $t \in [0,T]$, where Y is a process of bounded variation. Then the process Y belongs to $\mathcal{Y}$.*

This assertion follows immediately from the alternative description of $\mathcal{Y}$ given in the lemma below.

Lemma 2.3.3. *Let Y be a predictable process of bounded variation. Then*

$$Y \in \mathcal{Y} \quad \Leftrightarrow \quad Y_\sigma - Y_\tau \in L^0(G_{\sigma,\tau}) \text{ for all stopping times } \sigma, \tau,\ \sigma \le \tau \le T.$$

Proof. ($\Rightarrow$) Follows obviously from the representation

$$Y_\tau - Y_\sigma = \int_\sigma^\tau \dot{Y}_r^c d||Y^c||_r + \sum_{\sigma < r \le \tau} \Delta Y_r + \sum_{\sigma \le r < \tau} \Delta^+ Y_r.$$

($\Leftarrow$) First, we provide an "explicit" formula for $\dot{Y}^c$ using the classical approach due to Doob, see [24], V.5.58. For the reader's convenience we recall the idea. Put $t_k = t_k^n = k2^{-n}T$, fix ω (omitted in the notation) and consider the sequence of functions

$$X_n(t) = \sum_k \frac{Y_{t^n_{k+1}+} - Y_{t^n_k+}}{||Y_+||_{t^n_{k+1}} - ||Y_+||_{t^n_k}} I_{]t^n_k, t^n_{k+1}]}(t), \qquad [0,T].$$

This sequence is a bounded martingale with respect to the dyadic filtration on $[0,T]$ and the finite measure $d||Y_+||$. So, it converges (almost everywhere with respect to this measure) to a limit X_∞ which is the Radon–Nikodym derivative $dY_+/d||Y_+||$ and which may serve also as the Radon–Nikodym derivative $dY^c/d||Y^c||$.

Thus,

$$\dot{Y}^c = \limsup_n \sum_k \frac{Y_{t^n_{k+1}+} - Y_{t^n_k+}}{||Y_+||_{t^n_{k+1}} - ||Y_+||_{t^n_k}} I_{]t^n_k, t^n_{k+1}]} \qquad dP\, d||Y^c||\text{-a.e.}$$

It follows that $-\dot{Y}^c_t \in G_{t-,t+}$ (a.e.). By assumption, $G_{t-,t+} = \operatorname{cone}\{G_{t-}, G_t\}$. But for each ω the set $\{t : \; G_t(\omega) \neq G_{t-}(\omega)\}$ is at most countable and the property 1) in the definition of $\mathcal{Y}$ is fulfilled.

For a stopping time τ we put $\tau^n := \tau + 1/n$. Then $\tau_n \downarrow \tau$

$$\Delta^+ Y_\tau = \lim_n (Y_{\tau^n} - Y_\tau) \in -G_{\tau,\tau+} = -G_\tau.$$

For a predictable stopping time σ one can find an announcing sequence of stopping times $\sigma^n \uparrow \sigma$ with $\sigma^n < \sigma$ on the set $\{\sigma > 0\}$. Thus, on this set

$$\Delta Y_\sigma = \lim_n (Y_\sigma - Y_{\sigma^n}) \in -G_{\sigma-,\sigma} = -G_{\sigma-}.$$

The lemma is proven. □

Lemma 2.3.4. *Let $Z \in \mathcal{D}$. Let A be a subset of $\mathcal{Y}^0_b(Z)$ for which there is a constant κ such that $Y_T + \kappa\mathbf{1} \in L^0(G_T, \mathcal{F}_T)$ for all $Y \in A$. Then there exists a probability measure $Q \sim P$ such that* $\sup_{Y\in A} E_Q||Y||_T < \infty$.

Proof. Fix $Z \in \mathcal{D}$ and consider the random variable

$$\alpha := \inf_{t\le T} \inf_{x\in G_t,\ |x|=1} Z_t x = \inf_{t\le T} Z_t x_t,$$

where $x_t = x_t(\omega)$ is the point on the unit sphere at which the interior infimum is attained. If $t_n \downarrow t_0$ and the sequence x_{t_n} tends to some x_0, then the point

$x_0 \in \cap_{\varepsilon>0} G_{t_0,t_0+\varepsilon} = G_{t_0,t_0+}$. By our assumption, $G_{t_0,t_0+} = G_{t_0}$. If $t_n \uparrow t_0$ and the sequence x_{t_n} tends to some x_0, then $x_0 \in G_{t_0-}$ by virtue of a similar argument. On various ω the infimum in t can be obtained either on a decreasing sequence of t_n (in this case, $\alpha = Z_{t_0} x_{t_0}$) or on a increasing one (in this case, $\alpha = Z_{t_0-} x_{t_0}$). The assumption on Z guaranties that in both cases the values of α are strictly positive.

It is easily seen that the left-hand side of (2.1) dominates

$$E\alpha\left(|\dot{Y}^c|\cdot||Y^c||_T + \sum_{s\le T}|\Delta Y_s| + \sum_{s<T}|\Delta^+ Y_s|\right) = E\alpha||Y||_T$$

and, therefore,

$$E\alpha e^{-\alpha}||Y||_T \le E\alpha||Y||_T \le Z_0 x - EZ_T Y_T \le Z_0 x + \kappa E Z_T \mathbf{1}.$$

It follows that the measure Q with the density $dQ/dP = \alpha e^{-\alpha}/(E\alpha e^{-\alpha})$ is the required one. $\square$

Recall that a sequence a^n is *Césaro convergent* if $\bar{a}^n := n^{-1}\sum_{k=1}^n a^k$ converges. The Komlós theorem asserts that if ξ^n are random variables with $\sup_n E|\xi^n| < \infty$ then there exist $\xi \in L^1$ and a subsequence $\xi^{n'}$ such that all its subsequences are Césaro convergent to ξ a.s.

Lemma 2.3.5. *Let A^n be a sequence of predictable increasing processes starting from zero and with $\sup_n EA^n_T < \infty$. Then there is an increasing process A with $A_T \in L^1$ and a subsequence $A^{n'}$ which is Césaro convergent to A pointwise at every point of $[0,T]$ for all ω except a P-null set.*

Proof. Let $\mathbf{T} := \{k2^{-n}T : \ k = 0,...,2^n, \ n \in \mathbf{N}\}$. Using the Komlós theorem and the diagonal procedure we find a subsequence such that $A^{n'}_r$ is Cesaro convergent a.s. to $A^o_r \in L^1$ (with all further subsequences) for all $r \in \mathbf{T}$. We can always choose a common null set Ω_0 and assume that $A^o_r(\omega)$ is increasing in r for each $\omega \notin \Omega_0$. Let us consider its left-continuous envelope, defined on the whole interval, i.e. the process $A_t := \liminf_{r\uparrow t} A^o_t$ ($r \in \mathbf{T}$ and $r < t$). By the same argument as in the theory of weak convergence of probability distribution functions, we conclude that if the sequence of functions $A^{n'}(\omega)$ converges at all points of $\mathbf{T}$ in Césaro sense to $A^o(\omega)$, then it converges, in the same sense, to the function $A(\omega)$ at all points of continuity of the latter. The crucial observation is that one can sacrifice the left-continuity of A but "improve" the convergence property. To this aim let us consider a sequence of stopping times τ_k exhausting the jumps of the process A (i.e. such that $\{\Delta^+ A > 0\} \subseteq \cup_k [\tau_k]$ where $[\tau_k]$ is the graph of τ_k). Refining the subsequence of $A^{n'}$ we may assume that each sequence of random variables $A^{n'}_{\tau_k}$ also converges in Césaro sense. Replacing A_{τ_k} by these limiting values we obtain the required process which is a pointwise Césaro limit of a certain subsequence of A^n (thus, predictable). □

Remark. The above lemma from [16] is the key element of our proof and it merits to be well-understood. It is worthy to make a look at its deterministic counterpart which is just a version of the Helly theorem. The latter is usually formulated for left-continuous (or, more frequently, for right-continuous) monotone functions. The proof is easy: combining the Bolzano–Weierstrass theorem and the diagonal procedure one defines a monotone function A^o on $\mathbf{T}$ and a subsequence $A^{n'}$ convergent to A^o on $\mathbf{T}$. Let A be the left envelope of A^o. Due to monotonicity, the same subsequence will converge to all points of $[0, T]$ where A is continuous and this gives the standard version of the Helly theorem. Of course, the convergence may fail at the denumerable set where A is discontinuous. Repeating the arguments, one can find a further subsequence having limits also at each point of discontinuity of A. Replacing the values of A by these limits, we get an increasing function approximated by the refined subsequence at all points of the interval. The proof in the stochastic setting follows the same lines with the classical compactness argument replaced by a reference to the Komlós theorem.

2.4 Hedging of American options

Let $U = (U_t)_{t\in[0,T]}$ be an $\mathbf{R}^d$-valued càdlàg process for which there is a constant κ such that $U_t + \kappa\mathbf{1} \in L^0(G_t, \mathcal{F}_t)$ for all t. In the context of financial modelling such a process is interpreted as (the pay-off of) an American option. Let $A_0^T(.)$ be the set of American options which can be dominated by a portfolio with zero initial capital, i.e. by an element of $\mathcal{Y}_b^0$.

Define the convex set

$$\Gamma := \{x \in \mathbf{R}^d :\ \exists Y \in \mathcal{Y}_b^x \text{ such that } Y \succeq_G U\} = \{x \in \mathbf{R}^d :\ U - x \in A_0^T(.)\},$$

and the closed convex set

$$D := D(P) := \{x \in \mathbf{R}^d :\ \bar{Z}_0^\nu x \geq E^\nu Z U \quad \forall Z \in \mathcal{Z}(G^*, P, \nu),\ \forall \nu \in \mathcal{N}\}.$$

It is easy to check that $D(P) = D(\tilde{P})$ if $\tilde{P} \sim P$. Indeed, let $x \in D(P)$, $\nu \in \mathcal{N}$, and $\tilde{Z} \in \mathcal{Z}(G^*, \tilde{P}, \nu)$. Define $\rho_t = E(d\tilde{P}/dP|\mathcal{F}_t)$ and consider the process $Z_t = \rho_t \tilde{Z}_t$. It is in $\mathcal{Z}(G^*, P, \nu)$ and

$$E\rho_T \int_{[0,T]} \tilde{Z}_t U_t \nu(dt) = E\int_{[0,T]} \rho_t \tilde{Z}_t U_t \nu(dt) = E\int_{[0,T]} Z_t U_t \nu(dt) \leq x E^\nu Z$$

Since $E^\nu Z = \tilde{E}^\nu \tilde{Z}$, it follows that $x \in D(\tilde{P})$.

Proposition 2.4.1. $\Gamma \subseteq D$.

Proof. Let $x \in \Gamma$. Then there exists Y in $\mathcal{Y}_b^0$ such that the process $x + Y$ dominates U, i.e. $x + Y_t - U_t \in L^0(G_t, \mathcal{F}_t)$ for all $t \in [0,T]$. It follows that $x + Y_{t+} - U_t \in L^0(G_t, \mathcal{F}_t)$. By duality, for any $Z \in \mathcal{Z}(G^*, P, \nu)$ and $\nu \in \mathcal{N}$ we have that

$$E\int_{[0,T]} Z_t x \nu(dt) + E\int_{[0,T]} Z_t Y_{t+} \nu(dt) \geq E\int_{[0,T]} Z_t U_t \nu(dt).$$

It remains to verify that $E^\nu Z Y_+ \leq 0$. Using the Fubini theorem and the property of the optional projection given by Th.VI.2.57 in [24] we have:

$$\begin{aligned} E\int_{[0,T]} Z_t Y_{t+} \nu(dt) &= E\int_{[0,T]} Z_t \left(\int_{[0,t]} \dot{Y}_{+s} d||Y_+||_s\right) \nu(dt) \\ &= E\int_{[0,T]} \dot{Y}_{+s} \left(\int_{[s,T]} Z_t \nu(dt)\right) d||Y_+||_s \\ &= E\int_{[0,T]} \dot{Y}_{+s} \bar{Z}_s^\nu d||Y_+||_s. \end{aligned}$$

It is easy to see that

$$\int_{[0,T]} \dot{Y}_{+s} \bar{Z}_s^\nu d||Y_+||_s = \int_{[0,T]} \dot{Y}_s^c \bar{Z}_s^\nu d||Y^c||_s + \sum_{s \leq T} \bar{Z}_s^\nu \Delta Y_s + \sum_{s \leq T} \bar{Z}_s^\nu \Delta^+ Y_s.$$

Since $\dot{Y}_s^c$ and $\Delta^+ Y_s$ take values in the cone $-G_s$ and $\bar{Z}_s^\nu$ takes values in G_s^*, the first and the third terms of the above identity are negative. The increment ΔY_s takes values in $-G_{s-}$. If $G_{s-} = G_s$ for all s, the second term is also negative and we conclude. As we do not assume the continuity of the process G, the proof requires a bit more work.

Let us suppose for a moment that the random variable $||Y^d||_T$ is bounded. To get the needed inequality $E^\nu ZY_+ \le 0$ it is sufficient to check that the expectation of the second term is negative. We proceed as follows. Recall that $\bar{Z}^\nu = M^\nu - R$ where M^ν is a martingale and the process

$$R_t = \int_{[0,t[} Z_u \nu(du)$$

is left-continuous. The last property implies that $\Delta \bar{Z}^\nu = \Delta M^\nu$. It follows that

$$\sum_{s \le T} \bar{Z}^\nu_s \Delta Y_s = \sum_{s \le T} \bar{Z}^\nu_{s-} \Delta Y_s + \sum_{s \le T} \Delta M^\nu_s \Delta Y_s.$$

The first sum in the right-hand side is obviously negative while the expectation of the second one is zero. This follows from the classical property (see, e.g. [26], Lemma I.3.12): if M is a positive martingale and B is a predictable increasing process starting from zero, then

$$EM_T B_T = E\int_{[0,T]} M_s dB_s = E\int_{[0,T]} M_{s-} dB_s.$$

We can easily remove the condition of the boundedness of $||Y^d||$. Indeed, a finite predictable increasing process is locally bounded, see [24], Ch. VIII.11. Hence, there is a sequence of stopping times τ^n increasing stationary to T (i.e. with $P(\tau_n = T) \to 1$) such that $||Y^d||_{\tau^n} \le C_n$. Let U^n be a process coinciding with U on $[0, \tau^n[$ and taking the value $x + Y_{\tau^n}$ on $[\tau^n, T]$. It follows from the above arguments that $\bar{Z}^\nu x \ge E^\nu Z U^n$ and the result follows from the Fatou lemma. $\square$

Theorem 2.4.2. *Suppose that $\mathcal{D} \neq \emptyset$. Then $\Gamma = D$.*

Proof. We fix $\tilde{Z} \in \mathcal{D}$ and define the set of hedging endowments corresponding to portfolios with the "relaxed" admissibility property, namely, we put

$$\Gamma(\tilde{Z}) := \{x \in \mathbf{R}^d : \ \exists Y \in \mathcal{Y}^x_b(\tilde{Z}) \text{ such that } Y \succeq_G U\}.$$

Since $\mathcal{Y}^x_b(\tilde{Z}) \supseteq \mathcal{Y}^x_b$, this set is larger than Γ. On the other hand, if a portfolio Y dominate U, it is bounded from below. Hence, $\Gamma(\tilde{Z}) = \Gamma$.

Let $\mathbf{T}^m := \{t_k = t^m_k : \ t_k = k2^{-m}T, \ k = 0, ..., 2^m\}$; then $\mathbf{T} = \cup_{m\ge1}\mathbf{T}^m$. Define the convex set $A_{\mathbf{T}^m}(.)$ of American options W which can be hedged at the dates from $\mathbf{T}^m$ by a portfolio belonging to the class $\mathcal{Y}^0_b(\tilde{Z})$, i.e. such that

$Y_t - W_t \in G_t$, $t \in \mathbf{T}^m$, for some $Y \in \mathcal{Y}_b^0(\tilde{Z})$. Let us consider $A_{\mathbf{T}^m}(.)$ as a subset of the space $L^0(P \otimes \nu^m) := L^0(\Omega \times [0,T], \mathcal{F} \times \mathcal{B}_{[0,T]}, P \otimes \nu^m)$ where the probability measure ν^m is the uniform distribution on $\mathbf{T}^m$, i.e. it charges only the points of $\mathbf{T}^m$ with weights $1/(2^m+1)$. From the point of view of this space, W is just the random vector $(W_0, W_{1/2^m}, ..., W_T)$ (the components of the latter are d-dimensional). For such random vectors (with fixed $m \geq 1$) we extend the concept of the Fatou-convergence in the same spirit as was developed in the problem of hedging of European options. Note that $A_{\mathbf{T}^m}(.)$, in general, depends on $\tilde{Z}$.

We say that a sequence W^n is *Fatou-convergent in* $L^0(P \otimes \nu^m)$ to W if there is a constant κ such that $W_r^n + \kappa \mathbf{1} \in L^0(G_r, \mathcal{F}_r)$ (i.e., $W_r^n \succeq_{G_r} -\kappa\mathbf{1}$) for all $r \in \mathbf{T}^m$, $n \geq 1$, and $W_r^n \to W_r$ a.s., $n \to \infty$, for all $r \in \mathbf{T}^m$. The subsequent definitions *Fatou-closed* and *Fatou-dense* are obvious.

Lemma 2.4.3. *The set $A_{\mathbf{T}^m}(.)$ is Fatou-closed in $L^0(P \otimes \nu^m)$.*

Proof. Let $W^n \in A_{\mathbf{T}^m}(.)$ be a sequence Fatou-converging to W and let Y^n be a corresponding sequence of dominating elements from $\mathcal{Y}_b^0(\tilde{Z})$. Our aim is to show that W also can be dominated by some element $\mathcal{Y}_b^0(\tilde{Z})$ at the points of $\mathbf{T}^m$. Using the preceding results (see Lemmas 2.3.4 and 2.3.5) we can replace W^n and Y^n by appropriate sequences of arithmetic means and suppose without loss of generality that Y^n converges to some predictable process Y of bounded variation almost surely at each point $t \in [0,T]$. Using Lemma 2.3.2 we conclude that $Y \in \mathcal{Y}^0$. It remains to check that $\tilde{Z}Y$ dominates a martingale. By virtue of Lemma 2.3.1 the prelimit processes $\tilde{Z}Y^n$ are supermartingales. Since Y_T^n dominates $W_T \succeq_{G_T} -\kappa\mathbf{1}$, we have that $\tilde{Z}_T Y_T^n \geq -\kappa \tilde{Z}_T$. It follows that the supermartingale $\tilde{Z}Y^n$ dominates the martingale $-\kappa\tilde{Z}$ and so does the supermartingale $\tilde{Z}Y$. □

Lemma 2.4.4. *The set $A_{\mathbf{T}^m}(.) \cap L^\infty(P^{\nu^m})$ is Fatou-dense in $A_{\mathbf{T}^m}(.)$.*

Proof. Let $W \in A_{\mathbf{T}^m}(.)$ be dominated at the points of $\mathbf{T}^m$ by a portfolio $Y \in \mathcal{Y}_b^0(\tilde{Z})$. Let κ be a constant such that $W_t + \kappa\mathbf{1} \in G_t$ for $t \in \mathbf{T}^m$. Put

$$W^n := WI_{\{|W| \leq n\}} - \kappa I_{\{|W|>n\}}.$$

Then $W^n \in L^\infty(P \otimes \nu^m)$ and tends to W as $n \to \infty$. Since for all $t \in \mathbf{T}^m$, $Y_t - W_t^n = (Y_t - W_t)I_{\{|W_t| \leq n\}} + (Y_t + \kappa\mathbf{1})I_{\{|W_t|>n\}} \in G_t$, we have $W^n \in A_{\mathbf{T}^m}(.)$. □

Let $L_b^0(P \otimes \nu^m)$ be the cone in $L^0(P \otimes \nu^m)$ formed by the elements W (interpreted as random vectors) which are adapted and bounded from below in the sense of partial ordering, i.e. such that $W_r + c\mathbf{1} \in L^0(G_r, \mathcal{F}_r)$ for all $r \in \mathbf{T}^m$. The notation $L^1(G^*, P \otimes \nu^m)$ has an obvious meaning.

The following lemma is Theorem 4.3 from [39] formulated in the notation adjusted to the considered situation (where one take $W_0 = 0$).

Lemma 2.4.5. *Let A be a convex subset in $L_b^0(P \otimes \nu^m)$ which is Fatou-closed and such that the set $A^\infty := A \cap L^\infty(\mathbf{R}^d, P \otimes \nu^m)$ is Fatou-dense in A. Suppose that there is $W_0 \in A^\infty$ such that $W_0 - L^\infty(G, P \otimes \nu^m) \subseteq A^\infty$. Then*

$$A = \left\{ W \in L_b^0(P \otimes \nu^m) : \; E^{\nu^m} ZW \le f(Z) \; \forall Z \in L^1(G^*, P \otimes \nu^m) \right\} \qquad (2.2)$$

where $f(Z) = \sup_{Y \in A} E^{\nu^m} ZY$.

With the above preliminaries we can complete the proof of Theorem 2.4.2 by establishing the remaining inclusion $D \subseteq \Gamma = \Gamma(\tilde{Z})$. Indeed, take a point $x \in D$. Suppose that $U - x \notin A_{\mathbf{T^m}}(.)$ for some m. By virtue of Lemma 2.4.5 there exists $Z \in L^1(G^*, P \otimes \nu^m)$ such that $E^{\nu^m} Z(U - x) > f(Z)$. But $f(Z) = 0$ as $A_{\mathbf{T^m}}(.)$ is a cone. We can identify Z with a right-continuous adapted process taking value Z_{t_k} at the points t_k. Since $E^{\nu^m} ZY \le 0$ for all $Y \in A_{\mathbf{T^m}}(.)$, the process $Z \in \mathcal{Z}(G^*, P, \nu^m)$. Thus, $x \notin D$, a contradiction. This means that $U - x \in A_{\mathbf{T}^m}(.)$ for all m, i.e. there exist admissible portfolio processes Y^n dominating $U - x$ at the points of $\mathbf{T}^n$. In particular, the sequence Y_T^n is bounded from below by a constant vector and, by virtue of Lemma 2.3.4, the total variations $||Y^n||_T$ are bounded in a certain $L^1(Q)$ with $Q \sim P$. Using Lemma 2.3.5, we may assume without loss of generality that the sequence Y^n converges to some predictable process Y of bounded variation almost surely at each point $t \in [0, T]$. Recall that $U_t + \kappa\mathbf{1} \in G_t$. The limiting process Y dominates $U - x$ at all points from $\mathbf{T}$. Using the right continuity of the processes, we obtain that Y_+ dominates $U - x$ on the whole interval and so does the "larger" process Y. So, $x \in \Gamma$. □

Remark. Theorem 2.4.2 implies as a corollary a hedging theorem for càdlàg portfolio processes under assumption that the cone-valued process G is continuous. Indeed, let $\mathcal{X}^0$ be the set of all càdlàg processes X of bounded variation with $X_0 = 0$ and such that $dX/d||X|| \in -G \; dP \, d||X||$-a.e. The notations $\mathcal{X}^x$ and $\mathcal{X}_b^x$ are obvious. Let

$$\Gamma_X := \{x \in \mathbf{R}^d : \; \exists \, X \in \mathcal{X}_b^x \text{ such that } X \succeq_G U\}.$$

Similar (but simpler) arguments than those used in the proof of Proposition 2.4.1 show that $\Gamma_X \subseteq D$.

Suppose that all generators ξ^k of G are continuous processes. It is easy to check that if the process $Y \in \mathcal{Y}^0$ then $Y_+ \in \mathcal{X}^0$. Thus, $\Gamma \subseteq \Gamma_X$ and Theorem 2.4.2 implies that if $\mathcal{D}(G) \neq \emptyset$, then $\Gamma_X = D$.

2.5 Financial Interpretation: Coherent Price Systems

In the final section of this note we want to attract the reader's attention to the financial interpretation of the obtained result. In the hedging theorems for European options the important concept is a *consistent price system* which replaces the notion of the martingale density of the classical theory sometimes referred to as "stochastic deflator" or "state-price density". The words "price system" mean that it is a process evolving in the duals $\widehat{K}_t^*$ to the solvency cones $\widehat{K}_t$ while "consistent" alludes that this process is a martingale. Hedging theorems are results claiming that a contingent claim ξ (in physical units) can be super-replicated starting from an initial endowment x by a self-financing portfolio if and only if the "value" $Z_0 x$ of this initial endowment is not less than the expected "value" of the contingent claim $EZ_T\xi$ for any consistent price system Z (we write the word "value" in quotation marks to emphasize its particular meaning in the present context). In other words, consistent price systems allow the option seller to relate benefits from possessing x at time $t = 0$ and the liabilities ξ at time $t = T$ and provide information whether there is a portfolio ending up on the safe side.

The situation with the American option is different. As it was observed by Chalasani and Jha, already in the simplest discrete-time models consistent price systems form a class which is too narrow to evaluate American claims correctly. The phenomenon appears because one cannot prohibit the option buyer to toss a coin and take a decision, to exercise or not, in dependence of the outcome. A financial intuition suggests that the expected "value" of an American claim is an expectation of the weighted average of "values" of assets obtained by the option holder for a variety of exercise dates. This expected "value" should be compared with the "value" of the initial endowment. The main question is: what is the class of price systems which should be involved to compute the "values" to be compared? Our result shows, that in a rather general continuous-time model, the comparison can be done with the systems for which the expected weighted average of future prices knowing the past is again a price system. The structure of such a price system is coherent with the option buyer actions and we propose to call it *coherent price system* and use the abbreviation CoPS.

It is well-known that without transaction costs the rational exercise strategy of the buyer is the optimal solution of a stopping problem which exists in the class of pure stopping times. This explains why in the models of frictionless markets there is no need to go beyond the class of consistent price systems. For markets with transaction costs the rational exercise strategies of the option buyer is an open problem.

A reader acquainted with set-valued analysis may ask a question why we limit ourselves by considering a rather particular cone-valued process defined via a countable family of generators. Indeed, it seems that the natural mathematical framework is a model given by a general cone-valued process G satisfying certain continuity conditions. A possible generalizations of this kind and a development of the theory of set-valued processes are of interest and can be subjects of further studies.

Remark. To the present, the pay-off of American options was usually modelled by a right-continuous (or left-continuous) process. Though we believe that this class is sufficient for financial applications, the problem of the dual description of the set of hedging endowments for the processes only admitting right and left limits is mathematically interesting. It is solved in the preprint [13] by Bouchard and Chassagneux which appeared when our paper was under refereeing. Their dual variables are different from those introduced here and the relations between the two descriptions are left by the authors of [13] as a subject of further studies.

Chapter 3

Consumption-Investment Problem with Transaction Costs for Lévy-driven Price Processes

3.1 Introduction

In this chapter we study the classical consumption-investment model with infinite horizon in the presence of transaction costs. Our aim is to extend the results of [31] to the case where the price processes are geometric Lévy process. Namely, we show that the Bellman function is a viscosity solution of the corresponding Hamilton–Jacobi–Bellman equation. We also prove a uniqueness theorem for the latter.

Mathematically, the consumption-investment problem with transaction costs is a regular-singular control problem for a linear stochastic equation in a cone. Its specificity is that the Bellman function is not smooth and, therefore, one cannot use the verification theorem (at least, in its traditional form) because the Itô formula cannot be applied. Nevertheless, one can show that the Bellman function is a solution of the HJB equation in viscosity sense. Though the general line of arguments is common, one needs to re-examine each step of the proof. In particular, for the considered jump-diffusion model, the HJB equation contains an integro-differential operator and the test functions involved in the definition of the viscosity solution must be "globally" defined. It seems that already in 1986 H.M. Soner noticed that the control problems with jump parts can be considered in the framework of the theory of viscosity solutions, [49], [50].

There is a growing literature on extension of the concept of viscosity solutions to equations with integro-differential operators, see, e.g., [45], [2], [43], [8], [7], [3], [4]. There are several variants of the definition of viscosity solution. Our choice is intended to serve the model with a positive utility function.

A rather detailed study of consumption-investment problems under transaction costs when the the prices follows exponential Lévy processes and the investor is constrained to keep long positions in all assets was undertaken in papers by Benth et *al.* [11] and [10]. Our geometric approach seems to be more general than that of the mentioned papers where the authors consider a "parametric" version of the stock market with transactions always involve money (i.e. either "buy stock" or "sell stock"). A more important difference is that in our setting the investor may take short positions as was always assumed in the classical papers [42], [21], [48]. If short positions are admitted, the ruin may happen due to a jump of the price process. That is why the classical setting we consider here leads to a different HJB equation of a more complicated structure. Following the ideas from the paper [31] we derive the Dynamic Programming Principle splitted into two separate assertions. Though it is the principal tool which allows to check that the Bellman function is a viscosity solution of the HJB equation, it is rarely discussed in the literature (and even taken as granted, see, e.g., in [1], [48], [11]).

The main result of the paper is a uniqueness theorem for the Dirichlet problem arising in the model. We formulate it in terms of the Lyapunov function.

The structure of the problem is the following. In Sections 2 and 3 we introduce the model dynamics and describe the goal functional providing comments on the concavity of the Bellman function W. In Section 4 we show that if the utility function is homogeneous of order $\gamma < 1$ and the Bellman function is finite then the latter is continuous in the interior of the solvency. In Section 5 we give a formal description of the HJB equation. Sections 6 and 7 contain a short account of basic facts on viscosity solutions for integro-differential operators. In Section 8 we explain the role of classical supersolutions to the HJB equations. Section 9 is devoted to the Dynamic Programming Principle. In Section 10 we use it to show that the Bellman function is the solution of our HJB equation. Section 11 contains a uniqueness theorem formulated in terms of a Lyapunov function. In Section 12 we provide examples of Lyapunov functions and classical supersolutions.

3.2 The Model

Our setting is more general than that of the standard model of financial market under constant proportional transaction costs. In particular, the cone K is not supposed to be polyhedral. We assume that the asset prices are geometric Lévy processes. Our framework appeals to a theory of viscosity solutions for non-local integro-differential operators.

Let $Y = (Y_t)$ be an $\mathbf{R}^d$-valued semimartingale on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, P)$ with the trivial initial σ-algebra. Let K and $\mathcal{C}$ be **proper** cones in $\mathbf{R}^d$ such that $\mathcal{C} \subseteq \operatorname{int} K \neq \emptyset$. Define the set $\mathcal{A}$ of controls $\pi = (B, C)$ as the set of **predictable** càdlàg processes of bounded variation such that, up to an evanescent set,

$$\dot{B} \in -K, \qquad \dot{C} \in \mathcal{C}. \tag{3.1}$$

Let $\mathcal{A}_a$ be the set of controls with absolutely continuous C and $\Delta C_0 = 0$. For the elements of $\mathcal{A}_a$ we have $c := dC/dt \in \mathcal{C}$.

The controlled process $V = V^{x,\pi}$ is the solution of the linear system

$$dV_t^i = V_{t-}^i dY_t^i + dB_t^i - dC_t^i, \quad V_{0-}^i = x^i, \quad i = 1, ..., d. \tag{3.2}$$

The solution of (3.2) can be expressed explicitly using the Doléans-Dade exponentials

$$\mathcal{E}_t(Y^i) = e^{Y_t^i - (1/2)\langle Y^{ic}\rangle_t} \prod_{s \le t} (1 + \Delta Y_s^i) e^{-\Delta Y_s^i}. \tag{3.3}$$

Namely,

$$V_t^i = \mathcal{E}_t(Y^i) x^i + \mathcal{E}_t(Y^i) \int_{[0,t]} \mathcal{E}_{s-}^{-1}(Y^i)(dB_s^i - dC_s^i), \quad i = 1, ..., d. \tag{3.4}$$

We introduce the stopping time

$$\theta = \theta^{x,\pi} := \inf\{t : \; V_t^{x,\pi} \notin \operatorname{int} K\}.$$

For $x \in \operatorname{int} K$ we consider the subsets $\mathcal{A}^x$ and $\mathcal{A}_a^x$ of "admissible" controls for which $\pi = I_{[0,\theta^{x,\pi}]}\pi$ **and** $\{V_- + \Delta B \in \operatorname{int} K\} = \{V_- \in \operatorname{int} K\}$. This means that for an admissible control the process $V^{x,\pi}$ stops at the moment when it leaves the interior of the solvency cone and there is no more consumption. Moreover, the process V does not leave the interior of K due to a jump of B: the investor is reasonable enough not to ruin himself by making too expensive portfolio revision.

The important hypothesis that the cone K is proper, i.e. $K \cap (-K) = \{0\}$, or equivalently, $\operatorname{int} K^* \neq \emptyset$, corresponds to the model of financial market with *efficient friction*. In a financial context K (usually containing $\mathbf{R}_+^d$) is interpreted as the solvency region and $C = (C_t)$ as the consumption process; the process $B = (B_t)$ describes accumulated fund transfers. In the "standard" model with proportional transaction costs (sometimes referred to as the model of currency market)

$$K = \operatorname{cone}\{(1 + \lambda^{ij})e_i - e_j, \; e_i, \; 1 \le i, j \le d\}$$

where $\lambda^{ij} \ge 0$ are transaction costs coefficients.

The process Y represents the relative price movements. If S^i is the price process of the ith asset, then $dY_t^i = dS_t^i / S_{t-}^i$ and $S_t^i = S_0^i \mathcal{E}_t(Y^i)$. Without loss of generality we assume that $S_0^i = 1$ for all i. Therefore, the formula (3.4) can be re-written as follows:

$$V_t^i = S_t^i x^i + S_t^i \int_{[0,t]} \frac{1}{S_{s-}^i}(dB_s^i - dC_s^i), \quad i = 1, ..., d. \tag{3.5}$$

We shall work assuming that

$$dY_t = \mu dt + \Xi dw_t + \int z(p(dz,dt) - q(dz,dt)) \tag{3.6}$$

where $\mu \in \mathbf{R}^d$, w is a m-dimensional standard Wiener process and $p(dz,dt)$ is a Poisson random measure with the compensator $q(dz,dt) = \Pi(dz)dt$ such that $\Pi(dz)$ is a measure concentrated on $]-1,\infty[^d$. For the $m \times d$-dimensional matrix Ξ we put $A = \Xi\Xi^*$. We assume that

$$\int (|z|^2 \wedge |z|)\Pi(dz) < \infty. \tag{3.7}$$

It is important to note that the jumps of Y and B cannot occur simultaneously. More precisely, the process $|\Delta B||\Delta Y|$ is indistinguishable of zero. Indeed, for any $\varepsilon > 0$ we have, using the predictability of the process $\Delta B = B - B_-$, that

$$\begin{aligned} E\sum_{s\geq 0} |\Delta B_s||\Delta Y_s| I_{\{|\Delta Y_s|>\varepsilon\}} &= E\int_0^\infty \int |\Delta B_s| I_{\{|z|>\varepsilon\}} |z| p(dz,ds) \\ &= E\int_0^\infty \int |\Delta B_s||z| I_{\{|z|>\varepsilon\}} \Pi(dz)ds = 0 \end{aligned}$$

because for each ω the set $\{s : \Delta B_s(\omega) \neq 0\}$ is at most countable and its Lebesgue measure is equal to zero. Thus, the process $|\Delta B||\Delta Y| I_{\{|\Delta Y|>\varepsilon\}}$ is indistinguishable of zero and so is the process $|\Delta B||\Delta Y|$.

It follows that $\Delta B_\theta = 0$. Since the predictable process $I_{\{V_- \in \partial K\}} I_{[0,\theta]}$ has at most countable number of jumps, the same reasoning as above leads to the conclusion that $I_{\{V_- \in \partial K\}} |\Delta Y| I_{[0,\theta]}$ is indistinguishable of zero. This means that θ is the first moment when either V or V_- leaves $\operatorname{int} K$. This property will be used in the proof that W is lower semicontinuous on $\operatorname{int} K$.

In our proof of the dynamic programming principle (needed to derive the HJB equation) we shall assume that the stochastic basis is a canonical one, that is the space of càdlàg functions under which the coordinate process is the Lévy process.

We denote by $C_\rho(K)$ the subspace of the space of continuous functions f on K such that $\sup_{x\in K} |f(x)|(1 + |x|^\rho) < \infty$.

Let $f \in C_1(K) \cap C^2(\operatorname{int} K)$. Using the abbreviation

$$I(z,x) := I_{\{z:\ x+\operatorname{diag} xz \in \operatorname{int} K\}} = I_{\operatorname{int} K}(x + \operatorname{diag} xz)$$

we introduce the function

$$\mathcal{I}(f,x) := \int (f(x+\operatorname{diag} xz) - f(x) - \operatorname{diag} xzf'(x))I(z,x)\Pi(dz), \quad x \in \operatorname{int} K.$$

It is well-defined and continuous in x. Indeed, let $\varepsilon > 0$ be such that the ball $\mathcal{O}_\varepsilon(x) \subset K$ and $\delta := \varepsilon/(2|x|)$. Then, using the Taylor formula, we have the bound

$$|f(x+\operatorname{diag} xz) - f(x) - \operatorname{diag} xzf'(x))| \le \kappa_1|z|^2 I_{\mathcal{O}_{\varepsilon/2}(x)}(z) + \kappa_2|z|I_{K\setminus\mathcal{O}_{\varepsilon/2}(x)}(z)$$

which right-hand side is integrable with respect to Π.

3.3 Goal Functionals and Concavity of the Bellman Function

Let $U : \mathcal{C} \to \mathbf{R}_+$ be a concave function such that $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. With every $\pi = (B,C) \in \mathcal{A}_a^x$ we associate the "utility process"

$$J_t^\pi := \int_0^t e^{-\beta s} U(c_s)\, ds\,, \quad t \ge 0\,,$$

where $\beta > 0$. We consider the infinite horizon maximization problem with the *goal functional* EJ_∞^π and define its *Bellman function* W by

$$W(x) := \sup_{\pi\in\mathcal{A}_a^x} EJ_\infty^\pi\,, \quad x \in \operatorname{int} K\,. \tag{3.8}$$

Since $A_a^{x_1} \subseteq A_a^{x_2}$ when $x_2 - x_1 \in K$, the function W is **increasing** with respect to the partial ordering $\ge_K$ generated by the cone K.

If π_i, $i = 1,2$, are admissible strategies for the initial points x_i, then the strategy $\lambda\pi_1 + (1-\lambda)\pi_2$ is an admissible strategy for the initial point $\lambda x_1 + (1-\lambda)x_2$, $\lambda \in [0,1]$, laying on the interval connecting x_1 and x_2. In the case where the relative price process Y is continuous, the corresponding ruin time for the process

$$V^{\lambda x_1+(1-\lambda)x_2,\lambda\pi_1+(1-\lambda)\pi_2} = \lambda V^{x_1,\pi_1} + (1-\lambda)V^{x_2,\pi_2} \tag{3.9}$$

dominates the maximum of the ruin times for processes V^{x_i,π_i}. The concavity of u implies that

$$J_t^{\lambda\pi_1+(1-\lambda)\pi_2} \ge \lambda J_t^{\pi_1} + (1-\lambda)J_t^{\pi_2}. \tag{3.10}$$

and, hence, the function W is **concave** on $\operatorname{int} K$.

Unfortunately, in our main case of interest, where Y has jumps, the ruin times cannot be related in such a simple way. One can easily imagine a situation where $\theta^{x_1,\pi_1} = \theta^{\lambda x_1+(1-\lambda)x_2,\lambda\pi_1+(1-\lambda)\pi_2} < \infty$ while $\theta^{x_2,\pi_2} = \infty$ and the relations (3.9) and (3.10) do not hold. Therefore, we **cannot guarantee**, by the above argument, that the Bellman function is concave. Of course, these considerations show only that the concavity of W cannot be obtained in a straightforward way as claimed in some publications. It is not excluded. Moreover, the concavity is rather plausible because one may guess that for the optimal strategies there are no short positions in the risky assets and the ruin by jumps is impossible.

The concavity of the Bellman function W is not a property just interesting *per se*. The classical definition of viscosity solution, as was given by the famous "User's guide" [18], requires the continuity. On the other hand, a concave function is continuous in the interior of its domain (and even locally Lipschitz), see, e.g., [5]. Of course, the model must contains a provision which ensures that W is finite. But the latter property in the case of continuous price processes implies that W is continuous on $\operatorname{int} K$. In the case of processes with jumps one needs to analyze the continuity of W using other arguments.

In the next section we show that the finiteness of W still guarantees its continuity in the interior of K. We do this using the following assertion.

Lemma 3.3.1. *Suppose that W is a finite function. Let $x \in \operatorname{int} K$. Then the function $\lambda \mapsto W(\lambda x)$ is right-continuous on $\mathbf{R}_+$.*

Proof. Let $\lambda > 0$. Then $\lambda\pi \in \mathcal{A}_a^{\lambda x}$ if and only if $\pi \in \mathcal{A}_a^x$. For a concave function U with $U(0) = 0$ we have, for any $\varepsilon > 0$ the inequality $U(c) \geq (1+\varepsilon)^{-1}U((1+\varepsilon)c)$. Hence, for an arbitrary strategy $\pi \in \mathcal{A}_a^x$ we have that

$$
\begin{aligned}
J_\infty^{(1+\varepsilon)\pi} - J_\infty^\pi &= E\int_0^\infty e^{-\beta t}\big(U((1+\varepsilon)c_t) - U(c_t)\big)dt \\
&\leq \varepsilon E\int_0^\infty e^{-\beta t}U(c_t))dt \leq \varepsilon W(x).
\end{aligned}
$$

It follows that $W((1+\varepsilon)x) \leq (1+\varepsilon)W(x)$. Since $W(x) \leq W((1+\varepsilon)x)$, we infer from here that $\lambda \mapsto W(\lambda x)$ is right-continuous at the point $\lambda = 1$. Replacing x by λx we obtain the claim. $\square$

Note that if U is a homogeneous function of order γ with $\gamma \in]0,1[$, i.e. $U(\lambda x) = \lambda^\gamma U(x)$ for all $\lambda > 0$, $x \in K$, then $W(\lambda x) = \lambda^\gamma W(x)$. Thus, the function $\lambda \mapsto W(\lambda x)$ is concave and, therefore, continuous if finite.

Remark 1. In financial models usually $\mathcal{C} = \mathbf{R}_+ e_1$ and $\sigma^0 = 0$, i.e. the only first (non-risky) asset is consumed. Our presentation in this section is oriented to the scalar power utility function $u(c) = c^\gamma/\gamma$, $\gamma \in]0,1[$.

Remark 2. We consider here a model with mixed "regular-singular" controls. In fact, the assumption that the consumption process has an intensity $c = (c_t)$ and the agent's utility depends on this intensity is not very satisfactory from the economical point of view. One can consider models with an intertemporal substitution and the consumption by "gulps", i.e. dealing with "singular" controls of the class $\mathcal{A}^x$ and the goal functionals like

$$J_t^\pi := \int_0^t e^{-\beta s} U(\bar{C}_s) ds \, ,$$

where

$$\bar{C}_s = \int_0^s K(s,r) dC_r$$

with a suitable kernel $K(s,r)$ (the exponential kernel $e^{-\gamma(s-r)}$ is the common choice).

3.4 Continuity of the Bellman Function

Proposition 3.4.1. *Suppose that $W(x) < \infty$ for all $x \in \operatorname{int} K$. Then W is continuous on* $\operatorname{int} K$.

Proof. First, we show that the function W is upper semicontinuous on $\operatorname{int} K$. Suppose that this is not the case and there is a sequence x_n converging to some $x_0 \in \operatorname{int} K$ such that $\limsup_n W(x_n) > W(x_0)$. Without loss of generality we way assume that the sequence $W(x_n)$ converges. The points $\tilde{x}_k = (1+1/k)x_0$, $k \geq 1$, belong to the ray $\mathbf{R}_+ x_0$ and converges to x_0. We find a subsequence x_{n_k} such that $\tilde{x}_k \geq_K x_{n_k}$ for all $k \geq 1$. Indeed, since

$$(1+1/k)x_0 \in x_0 + \operatorname{int} K,$$

there exists $\varepsilon_k > 0$ such that

$$(1+1/k)x_0 + \mathcal{O}_{\varepsilon_k}(0) \in x_0 + \operatorname{int} K.$$

It follows that

$$(1+1/k)x_0 + (x_n - x_0) + \mathcal{O}_{\varepsilon_k}(0) \in x_n + \operatorname{int} K$$

and, therefore, $(1+1/k)x_0 \in x_n + \operatorname{int} K$ for all n such that $|x_n - x_0| < \varepsilon_k$. Any strictly increasing sequence of indices n_k for which $|x_{n_k} - x_0| < \varepsilon_k$ gives us in a subsequence of points x_{n_k} with the needed property. The function W is increasing with respect to the partial ordering $\geq_K$. Thus,

$$\lim_k W(\tilde{x}_k) \geq \lim_k W(x_{n_k}) > W(x_0).$$

On the other hand, the function $\lambda \mapsto W(\lambda x_0)$ is right-continuous at $\lambda = 1$ and, hence, $\lim_k W(\tilde{x}_k) = W(x_0)$. This contradiction shows that W is upper semicontinuous on $\operatorname{int} K$.

Let us show now that $\liminf_n W(x_n) \geq W(x_0)$ as $x_n \to x_0$, i.e. W is lower semicontinuous on $\operatorname{int} K$.

Fix $\varepsilon > 0$. Due to the finiteness of the Bellman function there are a strategy π and $T \in \mathbf{R}_+$ such that for $\theta = \theta^{x_0,\pi}$ we have the bound

$$E \int_0^{T\wedge\theta} e^{-\beta s} U(c_s) ds \geq W(x_0) - \varepsilon.$$

It remains to show that

$$\liminf_n (\theta_n \wedge T) \geq \theta \wedge T \qquad \text{a.s.}, \tag{3.11}$$

where we use the abbreviation $\theta_n := \theta^{x_n,\pi}$. Indeed, with this bound we get, using the Fatou lemma, that

$$\begin{aligned} \liminf_n W(x_n) &\geq \liminf_n E \int_0^{\theta_n\wedge T} e^{-\beta s} U(c_s) ds \geq E \liminf_n \int_0^{\theta\wedge T} e^{-\beta s} U(c_s) ds \\ &\geq E \int_0^{\theta\wedge T} e^{-\beta s} U(c_s) ds \geq W(x_0) - \varepsilon \end{aligned}$$

and the claim follows since ε is arbitrarily small.

To prove (3.11), we observe that on $[0, \theta_n \wedge \theta \wedge T]$ we have the representation

$$V_t^{x_n,\pi} - V_t^{x_0,\pi} = \operatorname{diag}(x_n - x_0) S_t$$

implying that

$$\sup_{t \leq \theta_n\wedge\theta\wedge T} |V_t^{x_n,\pi} - V_t^{x_0,\pi}| \leq S_T^* |x_n - x_0|,$$

where $S_T^* := \sup_{t\leq T} |S_t|$. Fix arbitrary, "small", $\delta > 0$. For almost all ω the distance $\rho(\omega)$ of a trajectory $V_t^{x_0,\pi}(\omega)$ on the interval $[0, (\theta \wedge T) - \delta]$ is strictly positive. The above bound shows that for sufficiently large n the $V_t^{x_n,\pi}(\omega)$ does not deviate from $V_t^{x_0,\pi}(\omega)$ more than on $\rho(\omega)/2$ on the interval $[0, \theta_n(\omega) \wedge \theta(\omega) \wedge T]$. It follows that $\theta_n(\omega) \geq \theta(\omega) \wedge T - \delta$. Thus,

$$\liminf_n (\theta_n \wedge T) \geq (\theta \wedge T) - \delta \qquad \text{a.s.}$$

and (3.11) holds. □

3.5 The Hamilton–Jacobi–Bellman Equation

Let $G := (-K) \cap \partial \mathcal{O}_1(0)$ where $\mathcal{O}_r(y) := \{x \in \mathbf{R}^d : \ |x-y| < r\}$. The set G is a compact and $-K = \operatorname{cone} G$. We denote by Σ_G the *support function* of G, given by the relation $\Sigma_G(p) = \sup_{x \in G} px$. The convex function $U^*(.)$ is the Fenchel dual of the convex function $-U(-.)$ which domain is $-\mathcal{C}$, i.e.

$$U^*(p) = \sup_{x \in \mathcal{C}} (U(x) - px).$$

We introduce a function of five variables by putting

$$F(X, p, \mathcal{I}(f, x), W, x) := \max\{F_0(X, p, \mathcal{I}(f, x), W, x) + U^*(p), \Sigma_G(p)\},$$

where X belongs to $\mathcal{S}_d$, the set of $d \times d$ symmetric matrices, $p, x \in \mathbf{R}^d$, $W \in \mathbf{R}$, $f \in C_1(K) \cap C^2(x)$ and the function F_0 is given by

$$F_0(X, p, \mathcal{I}(f, x), W, x) := \frac{1}{2} \operatorname{tr} A(x)X + \mu(x)p + \mathcal{I}(f, x) - \beta W$$

where $A(x)$ is the matrix with $A^{ij}(x) := a^{ij}x^i x^j$, $\mu^i(x) := \mu^i x^i$, $1 \leq i, j \leq d$.

In a more detailed form we have that

$$F_0(X, p, \mathcal{I}(f, x), W, x) = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij} x^i x^j X^{ij} + \sum_{i=1}^{d} \mu^i x^i p^i + \mathcal{I}(f, x) - \beta W.$$

Note that F_0 is increasing in the argument f in the same sense as $\mathcal{I}$.

If ϕ is a smooth function, we put

$$\mathcal{L}\phi(x) := F(\phi''(x), \phi'(x), \mathcal{I}(\phi, x), \phi(x), x).$$

In a similar way, $\mathcal{L}_0$ corresponds to the function F_0.

We show, under mild hypotheses, that W is the unique viscosity solution of the Dirichlet problem for the HJB equation

$$\begin{aligned} F(W''(x), W'(x), \mathcal{I}(W,x), W(x), x) &= 0, \qquad x \in \operatorname{int} K, \qquad (3.12)\\ W(x) &= 0, \qquad x \in \partial K, \qquad (3.13) \end{aligned}$$

with the boundary condition understood in the usual classical sense.

3.6 Viscosity Solutions for Integro-Differential Operators

Since, in general, W may have no derivatives at some points $x \in \operatorname{int} K$ (and this is, indeed, the case for the model considered here), the notation (3.12) needs to be interpreted. The idea of viscosity solutions is to substitute W in F by suitable test functions. Formal definitions (adapted to the case we are interested in) are as follows.

A function $v \in C(K)$ is called *viscosity supersolution* of (3.12) if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \geq f$ the inequality $\mathcal{L}f(x) \leq 0$ holds.

A function $v \in C(K)$ is called *viscosity subsolution* of (3.12) if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \leq f$ the inequality $\mathcal{L}f(x) \geq 0$ holds.

A function $v \in C(K)$ is a *viscosity solution* of (3.12) if v is simultaneously a viscosity super- and subsolution.

At last, a function $v \in C_1(K) \cap C^2(\operatorname{int} K)$ is called *classical supersolution* of (3.12) if $\mathcal{L}v \leq 0$ on $\operatorname{int} K$. We add the adjective *strict* when $\mathcal{L}v < 0$ on the set $\operatorname{int} K$.

For the sake of simplicity and having in mind the specific case we shall work on, we incorporated in the definitions the requirement that the viscosity super- and subsolutions are continuous on K including the boundary. For other cases this might be too restrictive and more general and flexible formulations can be used.

Lemma 3.6.1. *Suppose that the function v is a viscosity solution of (3.12). If v is twice differentiable at $x_0 \in \operatorname{int} K$, then it satisfies (3.12) at this point in the classical sense.*

Proof. One needs to be more precise with definitions since it is not assumed that v' is defined at every point of a neighborhood of x_0. "Twice differentiable" means here that the Taylor formula at x_0 holds:

$$v(x) = P_2(x-x_0) + (x-x_0)^2 h(|x-x_0|)$$

where

$$P_2(x-x_0) := v(x_0) + \langle v'(x_0), x-x_0\rangle + \frac{1}{2}\langle v''(x_0)(x-x_0), x-x_0\rangle$$

and $h(r) \to 0$ as $r \downarrow 0$.

We introduce the notation $\Gamma_r := \{z \in \mathbf{R}^d : \ |\operatorname{diag} x_0 z| \le r\}$, $r \ge 0$.

Let $\varepsilon > 0$. We choose $\delta_0 \in]0,1[$ such that $|h(s)| \le \varepsilon$ for $s \le \delta_0$ and define $\delta := \delta_0/(1+|x_0|)$. Take $\Delta \in]\delta, 1[$ sufficiently close to δ to insure that $x_0 + \mathcal{O}_\Delta(0) \subset K$, $\Pi(\mathcal{O}_\Delta(0) \setminus \mathcal{O}_\delta(0)) \le \varepsilon$, and $\Pi(\Gamma_\Delta \setminus \Gamma_\delta) \le \varepsilon$.

We define the function $f_\varepsilon \in C_1(K) \cap C^2(x_0)$ by the formula

$$f_\varepsilon(x) = \begin{cases} P_2(x-x_0) + \varepsilon(x-x_0)^2, & x \in x_0 + \mathcal{O}_\delta(0), \\ g(x) \vee v(x), & x \in x_0 + \mathcal{O}_\Delta(0) \setminus \mathcal{O}_\delta(0), \\ v(x), & x \in x_0 + \mathcal{O}^c_\Delta(0), \end{cases}$$

where

$$g(x) := P_2\left(\delta \frac{x-x_0}{|x-x_0|}\right) + \varepsilon\delta + \frac{\delta - |x-x_0|}{\Delta - |x-x_0|}.$$

Clearly, $f_\varepsilon(x_0) = v(x_0)$ and $f_\varepsilon \ge v$. Since v is a viscosity subsolution, $\mathcal{L} f_\varepsilon(x_0) \ge 0$. Note that,

$$|\mathcal{L} f_\varepsilon(x_0) - \mathcal{L} v(x_0)| \le \varepsilon \sum_{i=1}^n a^{ii}(x_0^i)^2 + |\mathcal{I}(f_\varepsilon - v, x_0)|.$$

It is not difficult to show that $|\mathcal{I}(f_\varepsilon - v, x_0)|$ is also proportional to ε. Indeed,

$$|\mathcal{I}((f_\varepsilon - v) I_{\mathcal{O}_\delta(0)}, x_0)| \le \varepsilon |x_0|^2 \int_{\mathcal{O}_1(0)} z^2 \Pi(dz).$$

Due to the choice of Δ we have the bound

$$|\mathcal{I}((f_\varepsilon - v) I_{\mathcal{O}_\Delta(0)\setminus\mathcal{O}_\delta(0)}, x_0)| \le 2M\Pi(\mathcal{O}_\Delta(0) \setminus \mathcal{O}_\delta(0)) \le 2M\varepsilon,$$

where M is the supremum of v on the ball $x_0 + \mathcal{O}_{|x_0|}(0)$.

Since

$$|f_\varepsilon(x_0+\operatorname{diag}x_0z)-v(x_0+\operatorname{diag}x_0z)|\le\varepsilon|x_0|^2|z|^2,\quad z\in\Gamma_\delta\setminus\Gamma_0,$$

we get that

$$|\mathcal{I}((f_\varepsilon-v)I_{\mathcal{O}^c_\Delta(0)\cap\Gamma_\delta\setminus\Gamma_0},x_0)|\le\varepsilon|x_0|^2\int_{\mathcal{O}_1(0)}|z|^2\Pi(dz).$$

Also we have

$$|\mathcal{I}((f_\varepsilon-v)I_{\mathcal{O}^c_\Delta(0)\cap\Gamma_\Delta\setminus\Gamma_\delta},x_0)|\le(2m+\varepsilon)\Pi(\Gamma_\Delta\setminus\Gamma_\delta)\le(2m+\varepsilon)\varepsilon,$$

where m is the supremum of v on the ball $x_0+\mathcal{O}_1(0)$. Letting ε tend to zero, we obtain that $\mathcal{L}v(x_0)\ge0$. Arguing in the same way with $\varepsilon<0$, we get the opposite inequality. □

3.7 Jets

Let f and g be functions defined in a neighborhood of zero. We shall write $f(.)\lessapprox g(.)$ if $f(h)\le g(h)+o(|h|^2)$ as $|h|\to0$. The notations $f(.)\gtrapprox g(.)$ and $f(.)\approx g(.)$ have the obvious meaning.

For $p\in\mathbf{R}^d$ and $X\in\mathcal{S}_d$ we consider the quadratic function

$$Q_{p,X}(z):=pz+(1/2)\langle Xz,z\rangle\,,\quad z\in\mathbf{R}^d\,,$$

and define the *super*- and *subjets* of a function v at the point x:

$$\begin{aligned}J^+v(x)&:=\{(p,X):\ v(x+.)\lessapprox v(x)+Q_{p,X}(.)\},\\ J^-v(x)&:=\{(p,X):\ v(x+.)\gtrapprox v(x)+Q_{p,X}(.)\}.\end{aligned}$$

In other words, $J^+v(x)$ (resp. $J^-v(x)$) is the family of coefficients of quadratic functions $v(x)+Q_{p,X}(y-.)$ dominating the function $v(.)$ (resp., dominated by this function) in a neighborhood of the point x with precision up to the second order included and coinciding with $v(.)$ at this point.

In the classical theory developed for differential equations the notion of viscosity solutions admits an equivalent formulation in terms of super- and subjets. Since the latter are "local" concepts, such a characterization is not possible for integro-differential operators. Nevertheless, one can prove the following useful result.

Lemma 3.7.1. *Let v be a viscosity supersolution of the HJB equation and let $x \in \operatorname{int} K$. Let $(p, X) \in J^{-}v(x)$. Then there is a function $f \in C_1(K) \cap C^2(x)$ such that $f'(x) = p$, $f''(x) = X$, $f(x) = v(x)$, $f \geq v$ on K and, hence,*

$$F(X, p, \mathcal{I}(f, x), W(x), x) \leq 0.$$

Moreover, this function f can be chosen equal to v outside an arbitrary small neighborhood of x.

Proof. Take $r > 0$ such that the ball $\mathcal{O}_{2r}(x) = \{y : \ |y - x| \leq 2r\}$ lays in the interior of K. By definition,

$$v(x + h) - v(x) - Q_{p,X}(h) \geq |h|^2 \varphi(|h|),$$

where $\varphi(u) \to 0$ as $u \downarrow 0$. We consider on $]0, r[$ the function

$$\delta(u) := \sup_{\{h:\ |h| \leq u\}} \frac{1}{|h|^2} (v(x + h) - v(x) - Q_{p,X}(h))^{-} \leq \sup_{\{y:\ 0 \leq y \leq u\}} \varphi^{-}(y).$$

Obviously, δ is continuous, increasing and $\delta(u) \to 0$ as $u \downarrow 0$. The function

$$\Delta(u) := \frac{2}{3} \int_{u}^{2u} \int_{\eta}^{2\eta} \delta(\xi) d\xi d\eta$$

vanishes at zero with its two right derivatives; $u^2 \delta(u) \leq \Delta(u) \leq u^2 \delta(4u)$. It follows that the function $x \mapsto \Delta(|x|)$ belongs to $C^2(\mathcal{O}_r(0))$, its Hessian vanishes at zero, and

$$v(x + h) - v(x) - Q_{p,X}(h) \geq -|h|^2 \delta(|h|) \geq -\Delta(|h|).$$

Thus, $f(y) := v(x) + Q_{p,X}(y - x) - \Delta(|y - x|)$ is dominated by $v(y)$ in the ball $\mathcal{O}_r(x) = \{y : \ |y - x| \leq r\}$. We put $f(y) = v(y)$ outside of the ball $\mathcal{O}_{2r}(x)$. We can extend f continuously to the remaining set $\mathcal{O}_{2r}(x) \setminus \mathcal{O}_r(x)$ preserving the inequality $f \leq v$. $\square$

For subsolutions we have a similar result with the inverse inequalities.

3.8 Supersolutions and Properties of the Bellman Function

3.8.1 When is the Bellman Function W Finite on K?

First, we present sufficient conditions ensuring that the Bellman function W of the considered maximization problem is finite.

Functions we are interested in are defined in the solvency cone K while the process V which may jump out of the latter. In order to be able to apply the Itô formula we stop $V = V^{x,\pi}$ at the moment immediately preceding the ruin and define the process

$$\tilde{V} = V^{\theta-} = V I_{[0,\theta[} + V_{\sigma-} I_{[\theta,\infty[},$$

where θ is the exit time of V from the interior of the solvency cone K. This process coincides with V on $[0,\theta[$ but, in contrast to the latter, either always remains in K (due to the stopping at θ if $V_{\theta-} \in \operatorname{int} K$) or exits to the boundary in a continuous way and stops on it.

It follows from the definitions (3.2) and (3.6) that

$$\begin{aligned} \tilde{V}_t &= v + \int_0^t I_{[0,\theta]}(s) \operatorname{diag} \tilde{V}_s (\mu_s ds + \Xi dw_s) \\ &\quad + \int_0^t \int \operatorname{diag} \tilde{V}_{s-} z I(\tilde{V}_{s-}, z)(p(dz, ds) - q(dz, ds)) + B_t - C_t. \end{aligned}$$

Let Φ be the set of continuous functions $f : K \to \mathbf{R}_+$ increasing with respect to the partial ordering $\geq_K$ and such that for every $x \in \operatorname{int} K$ and $\pi \in \mathcal{A}_a^x$ the positive process $X^f = X^{f,x,\pi}$ given by the formula

$$X_t^f := e^{-\beta t} f(\tilde{V}_t) + J_t^\pi, \tag{3.14}$$

where $V = V^{x,\pi}$, is a supermartingale.

The set Φ of f with this property is convex and stable under the operation $\wedge$ (recall that the minimum of two supermartingales is a supermartingale). Any continuous function which is a monotone limit (increasing or decreasing) of functions from Φ also belongs to Φ.

Lemma 3.8.1. (a) *If $f \in \Phi$, then $W \leq f$;*

(b) *if a point $y \in \partial K$ is such that there exists $f \in \Phi$ such that $f(y) = 0$, then W is continuous at y.*

Proof. (a) Using the positivity of f, the supermartingale property of X^f, and, finally, the monotonicity of f we get the following chain of inequalities leading to the required property:

$$E J_t^\pi \leq E X_t^f \leq f(\tilde{V}_0) = f(V_0) \leq f(V_{0-}) = f(x).$$

(b) The continuity of W at the point $y \in \partial K$ follows from the inequalities $0 \le W \le f$. $\Box$

Remark. Recall that a concave function is locally Lipschitz continuous on the interior of its domain, i.e. on the interior of the set where it is finite. Thus, if W is concave function and Φ is not empty, then W is continuous (and even locally Lipschitz continuous) on $\operatorname{int} K$. The concavity of W holds in the case where the price process has no jumps.

Lemma 3.8.2. *Let $f : K \to \mathbf{R}_+$ be a function in $C_1(K) \cap C^2(\operatorname{int} K)$. If f is a classical supersolution of (3.12), then $f \in \Phi$, i.e. X^f is a supermartingale.*

Proof. First, notice that a classical supersolution is increasing with respect to the partial ordering $\ge_K$. Indeed, by the finite increments formula we have that for any $x, h \in \operatorname{int} K$

$$f(x+h) - f(x) = f'(x + \vartheta h)h$$

for some $\vartheta \in [0,1]$. The right-hand side is greater or equal to zero because for the supersolution f we have the inequality $\Sigma_G(f'(y)) \le 0$ whatever is $y \in \operatorname{int} K$, or, equivalently, $f'(y)h \ge 0$ for every $h \in K$, just by the definition of the support function Σ_G and the choice of G as a generator of the cone $-K$. By continuity, $f(x+h) - f(x) \ge 0$ for every $x, h \in K$.

Applying the "standard" Itô formula to $e^{-\beta t} f(\tilde V_t)$ we obtain that

$$\begin{aligned} e^{-\beta t} f(\tilde V_t) \;\; = \;\; & f(x) + \int_0^t e^{-\beta s} f'(\tilde V_{s-}) d\tilde V_s - \beta \int_0^t e^{-\beta s} f(\tilde V_{s-}) ds \\ & + \frac{1}{2} \int_0^t e^{-\beta s} \operatorname{tr} A(\tilde V_{s-}) f''(\tilde V_{s-}) ds \\ & + \sum_{s \le t} e^{-\beta s} [f(\tilde V_{s-} + \Delta \tilde V_s) - f(\tilde V_{s-}) - f'(\tilde V_{s-}) \Delta \tilde V_s]. \end{aligned}$$

Note also that

$$\begin{aligned} & \sum_{s \le t} e^{-\beta s} [f(\tilde V_{s-} + \Delta \tilde V_s) - f(\tilde V_{s-}) - f'(\tilde V_{s-}) \Delta \tilde V_s] I_{\{\Delta B_s = 0\}} \\ = \;\; & \int_0^t \int e^{-\beta s} [...] I(\tilde V_{s-}, z) I_{\{\Delta B_s = 0\}} I_{[0,\theta]}(s) p(dz, ds) \\ = \;\; & \int_0^t \int e^{-\beta s} [...] I(\tilde V_{s-}, z) I_{[0,\theta]}(s) \Pi(dz) ds \\ & + \int_0^t \int e^{-\beta s} [...] I(\tilde V_{s-}, z) I_{\{\Delta B_s = 0\}} I_{[0,\theta]}(s) (p(dz, ds) - \Pi(dz) ds), \end{aligned}$$

where we replace in the integrals by dots the lengthy expression

$$f(\tilde{V}_{s-} + \operatorname{diag} \tilde{V}_{s-} z) - f(\tilde{V}_{s-}) - f'(\tilde{V}_{s-}) \operatorname{diag} \tilde{V}_{s-} z.$$

Using the above formulae we obtain after regrouping terms the following representation for $X^f = e^{-\beta t} f(\tilde{V}_t)$:

$$X_t^f = f(x) + \int_0^{t\wedge\theta} e^{-\beta s} [\mathcal{L}_0 f(\tilde{V}_s) - c_s f'(\tilde{V}_s) + U(c_s)] ds + R_t + m_t, \qquad (3.15)$$

where

$$R_t := \int_0^{t\wedge\theta} e^{-\beta s} f'(V_{s-}) dB_s^c + \sum_{s\le t} e^{-\beta s} [f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})] \qquad (3.16)$$

and m is the local martingale

$$\begin{aligned} m_t &= \int_0^{t\wedge\theta} e^{-\beta s} f'(\tilde{V}_{s-}) \operatorname{diag} \tilde{V}_s \Xi dw_s \\ &\quad + \int_0^{t\wedge\theta} \int e^{-\beta s} [f(\tilde{V}_{s-} + \operatorname{diag} \tilde{V}_{s-} z) - f(\tilde{V}_{s-})] I(\tilde{V}_{s-}, z)(p(dz, ds) - \Pi(dz)ds). \end{aligned}$$

By definition of a supersolution, for any $x \in \operatorname{int} K$,

$$\mathcal{L}_0 f(x) \le -U^*(f'(x)) \le c f'(x) - U(c) \quad \forall\, c \in \mathcal{C}.$$

Thus, the integral in (3.15) is a decreasing process. The process R is also decreasing because the terms of the sum in (3.16) are less or equal to zero by monotonicity of f while the integral is negative since

$$f'(V_{s-}) dB_s^c = I_{\{\Delta B_s = 0\}} f'(V_{s-}) \dot{B}_s d||B||_s$$

where $f'(V_{s-})\dot{B}_s \le 0$ since $\dot{B}$ takes values in K. Let σ_n be a localizing sequence for m. Taking into account that $X^f \ge 0$, we obtain from (3.15) that for each n the negative decreasing process $R_{t\wedge\sigma_n}$ dominates an integrable process and so it is integrable. The same conclusion holds for the stopped integral. Being a sum of an integrable decreasing process and a martingale, the process $X^f_{t\wedge\sigma_n}$ is a positive supermartingale and, hence, by the Fatou lemma, X^f is a supermartingale as well. □

Lemma 3.8.2 implies that the existence of a smooth positive supersolution f of (3.12) ensures the finiteness of W on K. Sometimes, e.g., in the case of power utility function, it is possible to find such a function in a rather explicit form.

Remark. Let $\bar{\mathcal{O}}$ be the closure of an open subset $\mathcal{O}$ of K and let $f : \bar{\mathcal{O}} \to \mathbf{R}_+$ be a classical supersolution in $\bar{\mathcal{O}}$. Let $x \in \mathcal{O}$ and let τ be the exit time of the process $V^{x,\pi}$ from $\bar{\mathcal{O}}$. The above arguments imply that the process $X^f_{t\wedge\tau}$ is a supermartingale and, therefore,

$$E[e^{-\beta(t\wedge\tau)} f(\tilde{V}_{t\wedge\tau}) + J^\pi_{t\wedge\tau}] \le f(x). \tag{3.17}$$

3.8.2 Strict Local Supersolutions

For the strict supersolution we can get a more precise result which will play the crucial role in deducing from the Dynamic Programming Principle the property of W to be a subsolution of the HJB equation.

We fix a ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$ such that the larger ball $\bar{\mathcal{O}}_{2r}(x) \subseteq \operatorname{int} K$ and define $\tau^\pi = \tau^\pi_r$ as the exit time of $V^{\pi,x}$ from $\mathcal{O}_r(x)$, i.e.

$$\tau^\pi := \inf\{t \ge 0 : \ |V^{\pi,x}_t - x| \ge r\}.$$

Lemma 3.8.3. *Let $f \in C_1(K) \cap C^2(\mathcal{O}_{2r}(x))$ be such that $\mathcal{L}f \le -\varepsilon < 0$ on $\bar{\mathcal{O}}_r(x)$. Then there exist a constant $\eta > 0$ and an interval $]0, t_0]$ such that*

$$\sup_{\pi \in \mathcal{A}^x_a} E X^{f,x,\pi}_{t\wedge\tau^\pi} \le f(x) - \eta t \qquad \forall\, t \in]0, t_0].$$

Proof. We fix a strategy π and omit its symbol in the notations below. In what follows, only the behavior of the processes on $[0, \tau]$ does matter. Note that $|V_\tau - x| \ge r$ on the set $\{\tau < \infty\}$. As in the proof of Lemma 3.8.2, we apply the Itô formula and obtain the representation

$$\begin{aligned} X^f_{t\wedge\tau} &= f(x) + \int_0^{t\wedge\theta\wedge\tau} e^{-\beta s} \mathcal{L} f(\tilde{V}_s) ds \\ &\quad - \int_0^{t\wedge\theta\wedge\tau} e^{-\beta s} [U^*(V_s) + c_s f'(\tilde{V}_s) - U(c_s)] ds + R_{t\wedge\tau} + m_{t\wedge\tau}. \end{aligned}$$

Due to the monotonicity of f we may assume without loss of generality that on the interval $[0, \tau]$ the increment ΔB_t does not exceed the distance of V_s to the boundary of $\mathcal{O}_r(x)$. In other words, if the exit from the ball is due to an action (and not because of a jump of the price process), we can replace this action by a less expensive one, with the jump to the same direction but ending on the boundary.

By assumption, for $y \in \mathcal{O}_r(x)$ we have the bounds $\mathcal{L}f(y) \le -\varepsilon$ (helpful to estimate the first integral in the right-hand side) and $\Sigma_G(f'(y)) \le -\varepsilon$. The latter the latter inequality means that $kf'(y) \le -\varepsilon|k|$ for $k \in -K$ (hence, $f'(\bar{\mathcal{O}}_r(x)) \subset \operatorname{int} K^*$). In particular, for $s \in [0,\tau]$

$$f'(V_{s-})\dot{B}_s \le -\varepsilon|\dot{B}_s|, \qquad [f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})] \le -\varepsilon|\Delta B_s|.$$

Since $|\tilde{V}_{s-} - x| \le r$ for $s \in [0,\tau]$, we obtain, using the finite increment formula and the linear growth of f the bound

$$[f(\tilde{V}_{s-} + \operatorname{diag}\tilde{V}_{s-}z) - f(\tilde{V}_{s-})]I(\tilde{V}_{s-}, z) \le \kappa_1|z|^2 I_{\{|z|\le 1/2\}} + \kappa_2|z|I_{\{|z|>1/2\}}.$$

It follows that the local martingale $(m_{t\wedge\tau})$ is a martingale with $m_{t\wedge\tau} = 0$.

The above observations imply the inequality

$$EX^{f,x}_{t\wedge\tau} \le f(x) - e^{-\beta t}EN_t,$$

where

$$N_t := \varepsilon(t\wedge\tau) + \int_0^{t\wedge\tau} H(c_s, f'(V_s))ds + \varepsilon\int_0^{t\wedge\tau} |\dot{B}_s| d||B||_s$$

with $H(c,p) := U^*(p) + pc - U(c) \ge 0$. It remains to verify that EN_t dominates, on a certain interval $]0,t_0]$, a strictly increasing linear function which is independent of π.

The process N_t looks a bit complicated but we can replace it by another one of a simpler structure. To this end, note that there is a constant κ ("large", for convenience, $\kappa \ge 1$) such that

$$\inf_{p\in f'(\bar{\mathcal{O}}_r(x))} H(c,p) \ge \kappa^{-1}|c|, \qquad \forall\, c \in \mathcal{C}, \quad |c| \ge \kappa.$$

Indeed, being the image of a closed ball under continuous mapping, the set $f'(\bar{\mathcal{O}}_r(x))$ is a compact in $\operatorname{int} K^*$. The lower bound of the continuous function U^* on $f'(\bar{\mathcal{O}}_r(x))$ is finite. For any p from $f'(\bar{\mathcal{O}}_r(x))$ and $c \in \mathcal{C} \subseteq K$ we have the inequality $(c/|c|)p \ge \varepsilon$. At last, $U(c)/|c| \to 0$ as $c \to \infty$. Combining these facts we infer the claimed inequality. Thus, for the first integral in the definition of N_t we have the bound

$$\int_0^{t\wedge\tau} H(c_s, f'(V_s))ds \ge \kappa^{-1}\int_0^{t\wedge\tau} I_{\{|c_s|\ge\kappa\}}|c_s|ds.$$

The second integral in the definition dominates $\tilde\kappa||B||_{t\wedge\tau}$ for some $\tilde\kappa > 0$. To see this, let us consider the absolute norm $|.|_1$ in $\mathbf{R}^d$. The total variation of B with respect to this norm is $\sum_i \operatorname{Var} B^i$ and

$$|\dot B|_1 = \sum_i |\dot B^i| = \sum_i \left|\frac{dB^i}{d||B||}\right| = \sum_i \left|\frac{dB^i}{d\operatorname{Var} B^i}\right| \frac{d\operatorname{Var} B^i}{d||B||} = \frac{d\sum_i \operatorname{Var} B^i}{d||B||}.$$

But all the norms in $\mathbf{R}^d$ are equivalent, i.e. $\tilde\kappa^{-1}|.| \le |.|_1 \le \tilde\kappa|.|$ for some strictly positive constant $\tilde\kappa$ and the same inequalities relate the corresponding total variation processes.

Summarizing, we conclude that it is sufficient to check the domination property for $E\tilde N_t$ with the simpler processes

$$\tilde N_t := t\wedge\tau + \int_0^{t\wedge\tau} I_{\{|c_s|\ge\kappa\}}|c_s|ds + ||B||_{t\wedge\tau}. \tag{3.18}$$

The idea of the concluding reasoning is very simple: on a certain set of strictly positive probability, where one may neglect the random fluctuations, either τ is "large", or the total variation of the control is "large".

The formal arguments are as follows. Using the stochastic Cauchy formula (3.4) and the fact that $\mathcal{E}_{0+}(Y^i) = \mathcal{E}_0(Y^i) = 1$, we get immediately that there exist a number $t_0 > 0$ and a measurable set Γ with $P(\Gamma) > 0$ on which

$$|V^{x,\pi} - x| \le r/2 + \delta(||B|| + ||C||) \quad \text{on } [0, t_0]$$

whatever is the control $\pi = (B, C)$. Of course, diminishing t_0, we may assume without loss of generality that $\kappa t_0 \le r/(4\delta)$. For any $t \le t_0$ we have on the set $\Gamma \cap \{\tau \le t\}$ the inequality $||B||_\tau + ||C||_\tau \ge r/(2\delta)$ and, hence,

$$\tilde N_t \ge ||B||_\tau + ||C||_\tau - \int_0^\tau I_{\{|c_s|<\kappa\}}|c_s|ds \ge \frac{r}{2\delta} - \kappa t_0 \ge \kappa t_0 \ge t_0 \ge t.$$

On the set $\Gamma \cap \{\tau > t\}$ the inequality $\tilde N_t \ge t$ is obvious. Thus, $E\tilde N_t \ge tP(\Gamma)$ on $[0, t_0]$ and the result is proven. $\square$

3.9 Dynamic Programming Principle

The aim of this section is to establish the following two assertions needed to derive the HJB equation for the Bellman function.

Lemma 3.9.1. *Let $\mathcal{T}_f$ be the sets of finite stopping times. Then*

$$W(x) \le \sup_{\pi\in\mathcal{A}_a^x} \inf_{\tau\in\mathcal{T}_f} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi}) I_{\{\tau<\theta\}}\right). \tag{3.19}$$

Lemma 3.9.2. *Suppose that W is continuous on* $\operatorname{int} K$*. Then for any $\tau \in \mathcal{T}_f$*

$$W(x) \ge \sup_{\pi\in\mathcal{A}_a^x} E\left(J_\tau^\pi + e^{-\beta\tau} W(V_{\tau-}^{x,\pi}) I_{\{\tau\le\theta\}}\right). \tag{3.20}$$

We work on the canonical filtered space of càdlàg functions equipped with the measure P which is the distribution of the driving Lévy process. The generic point $\omega = \omega_.$ of this space is a d-dimensional càdlàg function on $\mathbf{R}_+$, zero at the origin. Let $\mathcal{F}_t^\circ := \sigma\{\omega_s,\ s \le t\}$ and $\mathcal{F}_t := \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$. We add the superscript P to denote σ-algebras augmented by all P-null sets from Ω. Recall that $\mathcal{F}_t^{\circ,P}$ coincides with $\mathcal{F}_t^P$ (this assertion follows easily from the predictable representation theorem). The Skorohod metric makes Ω a Polish space and its Borel σ-algebra coincides with $\mathcal{F}_\infty$, for details see [26].

Since elements of Ω are paths, we can define such operators as the stopping $\omega_. \mapsto \omega_.^s$, $s \ge 0$, where $\omega_.^s = \omega_{s\wedge .}$ and the translation $\omega_. \mapsto \omega_{s+.} - \omega_s$. Taking Doob's theorem into account, one can describe $\mathcal{F}_s^\circ$-measurable random variables as those of the form $g(w_.) = g(w_.^s)$ where g is a measurable function on Ω.

We define also the "concatenation" operator as the measurable mapping

$$g : \mathbf{R}_+ \times \Omega \times \Omega \to \Omega$$

with $g_t(s, \omega_., \tilde{\omega}_.) = \omega_t I_{[0,s[}(t) + (\tilde{\omega}_{t-s} + \omega_s) I_{[s,\infty[}(t)$.

Notice that

$$g_t(s, \omega_.^s, \omega_{.+s} - \omega_s) = \omega_t.$$

Thus, $\pi(\omega) = \pi(g(s, \omega_.^s, \omega_{.+s} - \omega_s))$.

Let π be a fixed strategy from $\mathcal{A}_a^x$ and let $\theta = \theta^{x,\pi}$ be the exit time from $\operatorname{int} K$ for the process $V^{x,\pi}$.

Recall the following general fact on regular conditional distributions.

Let ξ and η be two random variables taking values in Polish spaces X and Y equipped with their Borel σ-algebras $\mathcal{X}$ and $\mathcal{Y}$. Then ξ admits a regular conditional distribution given $\eta = y$ which we shall denote by $p_{\xi|\eta}(\Gamma, y)$. This means that $p_{\xi|\eta}(., y)$ is a probability measure on $\mathcal{X}$, $p_{\xi|\eta}(\Gamma, .)$ is a $\mathcal{Y}$-measurable function, and

$$E(f(\xi,\eta)|\eta) = \int f(x,y) p_{\xi|\eta}(dx, y)\Big|_{y=\eta} \quad (a.s.)$$

for any $\mathcal{X}\times\mathcal{Y}$-measurable function $f(x,y)\geq 0$.

We shall apply the above relation to the random variables $\xi=(\omega_{.+\tau}-\omega_\tau)$ and $\eta=(\tau,\omega^\tau)$. It is well-known that the Lévy process starts afresh at stopping times, i.e. one can take as the conditional distribution $p_{\xi|\eta}(\Gamma,y)$ the measure P.

At last, for fixed s and w^s, the shifted control $\pi_{.+s}(g(s,\omega^s_.,\tilde\omega.))$ is admissible for the initial condition $V^{x,\pi}_{s-}(\omega)$. Here we denote by $\tilde\omega.$ a generic point of the canonical space.

Proof of Lemma 3.9.1. For arbitrary $\pi\in\mathcal{A}^x_a$ and $\mathcal{T}_f$ we have that

$$\begin{aligned} EJ^\pi_\infty &= EJ^\pi_\tau+Ee^{-\beta\tau}I_{\{\tau<\theta\}}\int_0^\infty e^{-\beta r}u(c_{r+\tau})dr \\ &= EJ^\pi_\tau+Ee^{-\beta\tau}I_{\{\tau<\theta\}}E\Big(\int_0^\infty e^{-\beta r}u(c_{r+\tau})dr\Big|(\tau,\omega^\tau)\Big). \end{aligned}$$

According to the above discussion we can rewrite the second term of the right-hand side as

$$Ee^{-\beta\tau}I_{\{\tau<\theta\}}\int\left(\int_0^\infty e^{-\beta r}u(c_{r+\tau}(g(\tau,\omega^\tau,\tilde\omega)))dr\right)P(d\tilde\omega)$$

and dominate it by $Ee^{-\beta\tau}I_{\{\tau<\theta\}}W(V^{x,\pi}_{\tau-})$. Thus,

$$EJ^\pi_\infty\leq EJ^\pi_\tau+Ee^{-\beta\tau}I_{\{\tau<\theta\}}W(V^{x,\pi}_{\tau-}).$$

This bound leads directly to the first announced inequality. $\square$

Proof of Lemma 3.9.2. Fix $\varepsilon>0$. By hypothesis, the function W is continuous on $\operatorname{int}K$. For each $x\in\operatorname{int}K$ we can find an open ball $\mathcal{O}_r(x)=x+\mathcal{O}_r(0)$ with $r=r(\varepsilon,x)<\varepsilon$ contained in the open set $\{y\in\operatorname{int}K:\ |W(y)-W(x)|<\varepsilon\}$. Moreover, we can find a smaller ball $\mathcal{O}_{\tilde r}(x)$ contained in the set $y(x)+K$ with $y(x)\in\mathcal{O}_r(x)$. Indeed, take a ball $x_0+\mathcal{O}_\delta(0)\subseteq K$. Since K is a cone,

$$x+\mathcal{O}_{\lambda\delta}(0)\subseteq x-\lambda x_0+K$$

for every $\lambda>0$. Clearly, the requirement is met for $y(x)=x-\lambda x_0$ and $\tilde r=\lambda\delta$ when $\lambda|x_0|<\varepsilon$ and $\lambda\delta<r$. The family of sets $\mathcal{O}_{\tilde r(x)}(x)$, $x\in\operatorname{int}K$, is an open covering of $\operatorname{int}K$. But any open covering of a separable metric space contains a countable subcovering (this is the Lindelöf property; in our case, where $\operatorname{int}K$ is a countable union of compacts, it is obvious). Take a countable subcovering indexed by points x_n. For simplicity, we shall denote its elements by $\mathcal{O}_n$ and

$y(x_n)$ by y_n. Put $A_1 := \mathcal{O}_1$, and $A_n = \mathcal{O}_n \setminus \cap_{k<n}\mathcal{O}_k$. The sets A_n are disjoint and their union is $\operatorname{int} K$.

Let $\pi^n = (B^n, C^n) \in \mathcal{A}_a^{y_n}$ be an ε-optimal strategy for the initial point y_n, i.e. such that

$$EJ^{\pi_n} \geq W(y_n) - \varepsilon.$$

Let $\pi \in \mathcal{A}_a^x$ be an arbitrary strategy. We consider the strategy $\tilde{\pi} \in \mathcal{A}_a^x$ defined by the relation

$$\tilde{\pi} = \pi I_{[0,\tau[} + \sum_{n=1}^{\infty}[(y_n - V_{\tau-}^{x,\pi}, 0) + \bar{\pi}^n]I_{[\tau,\infty[}I_{A_n}(V_{\tau-}^{x,\pi})I_{\{\tau\leq\theta\}}$$

where $\bar{\pi}^n$ is the translation of the strategy π^n: namely, for a point $\omega_{.}$ with $\tau(\omega) = s < \infty$ we have

$$\bar{\pi}_t^n(\omega_{.}) := \pi_{t-s}^n(\omega_{.+s} - \omega_s).$$

In other words, the strategy $d\tilde{\pi}$ coincides with π on $[0,\tau[$ and with the shift of π^n on $]\tau,\infty[$ when $V_{\tau-}^{x,\pi}$ is a subset of A_n; the correction term guarantees that in the latter case the trajectory of the control system corresponding to the control $\tilde{\pi}$ passes at time τ through the point y_n.

Now, using the same considerations as in the previous lemma, we have:

$$\begin{aligned}
W(x) \geq EJ_{\infty}^{\tilde{\pi}} &= EJ_{\tau}^{\pi} + \sum_{n=1}^{\infty} EI_{A_n}(V_{\tau-}^{x,\pi})I_{\{\tau\leq\theta\}}\int_{\tau}^{\infty} e^{-\beta s}u(\bar{c}_s^n)ds \\
&\geq EJ_{\tau}^{\pi} + \sum_{n=1}^{\infty} EI_{A_n}(V_{\tau-}^{x,\pi})I_{\{\tau\leq\theta\}}e^{-\beta\tau}(W(y_n) - \varepsilon) \\
&\geq EJ_{\tau}^{\pi} + Ee^{-\beta\tau}W(V_{\tau-}^{x,\pi})I_{\{\tau\leq\theta\}} - 2\varepsilon.
\end{aligned}$$

Since π and ε are arbitrary, the result follows. □

Remark. The previous lemmas implies the identity

$$W(x) = \sup_{\pi\in\mathcal{A}_a^x}\inf_{\tau\in\mathcal{T}_f} E\left(J_{\tau}^{\pi} + e^{-\beta\tau}W(V_{\tau-}^{x,\pi})I_{\{\tau\leq\theta\}}\right).$$

It can be considered as another form of the dynamic programming principle but, seemingly, not sufficient for our derivation of the HJB equation.

3.10 The Bellman Function and the HJB Equation

Theorem 3.10.1. *Assume that the Bellman function W is in $C(K)$. Then W is a viscosity solution of (3.12).*

Proof. The claim follows from the two lemmas below. □

Lemma 3.10.2. *If (3.20) holds then W is a viscosity supersolution of (3.12).*

Proof. Let $x \in \operatorname{int} K$. Choose a test function $\phi \in C^1(K) \cap C^2(x)$ such that $\phi(x) = W(x)$ and $W \geq \phi$. Take $r \in]0,1]$ small enough to ensure that the ball $\bar{\mathcal{O}}_{2r}(x) \subset K$ and ϕ is smooth on $O_{2r}(x)$.

At first, we fix $m \in K$. Let $\varepsilon > 0$ be such that ensure that $x - \varepsilon m \in \mathcal{O}_r(x)$. The function W is increasing with respect to the partial ordering generated by K. Thus,

$$\phi(x) = W(x) \geq W(x - \varepsilon m) \geq \phi(x - \varepsilon m).$$

Taking a limit as $\varepsilon \to 0$, we easily obtain that $-m\phi'(x) \leq 0$ and, therefore, $\Sigma_G(\phi'(x)) \leq 0$.

Take now π with $B_t = 0$ and $c_t = c \in \mathcal{C}$. Let $\tau_r = \tau_r^\pi \leq \theta$ be the exit time of the process $V = V^{x,\pi}$ from the ball $\mathcal{O}_r(x)$; obviously, $\tau_r \leq \theta$. The properties of the test function and the inequality (3.20) imply that

$$\begin{aligned}
\phi(x) = W(x) \quad &\geq \quad E\left(J^\pi_{t\wedge\tau_r} + e^{-\beta(t\wedge\tau_r)} W(V_{t\wedge\tau_r-})\right) \\
&\geq \quad E\left(J^\pi_{t\wedge\tau_r} + e^{-\beta(t\wedge\tau_r)} \phi(V_{t\wedge\tau_r-})\right).
\end{aligned}$$

We get from here using the Itô formula (3.15), that

$$\begin{aligned}
0 \quad &\geq \quad E\left(\int_0^{t\wedge\tau_r} e^{-\beta s} U(c_s)ds + e^{-\beta(t\wedge\tau_r)}\phi(V_{t\wedge\tau_r-})\right) - \phi(x) \\
&\geq \quad E\int_0^{t\wedge\tau_r} e^{-\beta s}[\mathcal{L}_0\phi(V_s) - c\phi'(V_s) + U(c)]ds \\
&\geq \quad \min_{y\in\bar{\mathcal{O}}_r(x)} [\mathcal{L}_0\phi(y) - c\phi'(y) + U(c)] E\left[\frac{1}{\beta}\left(1 - e^{-\beta(t\wedge\tau_r)}\right)\right].
\end{aligned}$$

Dividing the resulting inequality by t and taking successively the limits as t and r converge to zero we infer that $\mathcal{L}_0\phi(x) - c\phi'(x) + U(c) \leq 0$. Maximizing over $c \in \mathcal{C}$ yields the bound $\mathcal{L}_0\phi(x) + U^*(\phi'(x)) \leq 0$ and, therefore, W is a supersolution of the HJB equation. □

Lemma 3.10.3. *If (3.19) holds then W is a viscosity subsolution of (3.12).*

Proof. Let $x \in \operatorname{int} K$ and let $\phi \in C^1(K) \cap C^2(x)$ be a function such that $\phi(x) = W(x)$ and $W \le \phi$ on $\mathcal{O}$. Assume that the subsolution inequality for ϕ fails at x. Thus, there exists $\varepsilon > 0$ such that $\mathcal{L}\phi \le -\varepsilon$ on some ball $\bar{\mathcal{O}}_r(x) \subset \operatorname{int} K$. By virtue of Lemma 3.8.3 (applied to the function ϕ) there are $t_0 > 0$ and $\eta > 0$ such that on the interval $]0, t_0]$ for any strategy $\pi \in \mathcal{A}_a^x$

$$E\left(J^{\pi}_{t\wedge\tau^{\pi}} + e^{-\beta\tau^{\pi}}\phi(V^{x,\pi}_{t\wedge\tau^{\pi}})\right) \le \phi(x) - \eta t,$$

where τ^{π} is the exit time of the process $V^{x,\pi}$ from the ball $\mathcal{O}_r(x)$. Fix arbitrary $t \in]0, t_0]$. By the second claim of Lemma 3.9.1) there exists $\pi \in \mathcal{A}_a^x$ such that

$$W(x) \le E\left(J^{\pi}_{t\wedge\tau} + e^{-\beta\tau}W(V^{x,\pi}_{t\wedge\tau})\right) + \frac{1}{2}\eta t,$$

for every stopping time τ, in particular for τ^{π}.

Using the inequality $W \le \phi$ and applying Lemma 3.8.3 we obtain from the above relations that $W(x) \le \phi(x) - (1/2)\eta t$. This is a contradiction because at the point x the values of W and ϕ are the same. □

3.11 Uniqueness Theorem

Before formulating the uniqueness theorem we recall the Ishii lemma.

Lemma 3.11.1. *Let v and $\tilde{v}$ be two continuous functions on an open subset $\mathcal{O} \subseteq \mathbf{R}^d$. Consider the function $\Delta(x,y) := v(x) - \tilde{v}(y) - \frac{1}{2}n|x-y|^2$ with $n > 0$. Suppose that Δ attains a local maximum at $(\widehat{x}, \widehat{y})$. Then there are symmetric matrices X and Y such that*

$$(n(\widehat{x} - \widehat{y}), X) \in \bar{J}^{+}v(\widehat{x}), \qquad (n(\widehat{x} - \widehat{y}), Y) \in \bar{J}^{-}\tilde{v}(\widehat{y}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.21}$$

In this statement I is the identity matrix and $\bar{J}^{+}v(x)$ and $\bar{J}^{-}v(x)$ are values of the set-valued mappings whose graphs are closures of graphs of the set-value mappings $J^{+}v$ and $J^{-}v$, respectively.

Of course, if v is smooth, the claim follows directly from the necessary conditions of a local maximum (with $X = v''(\widehat{x})$, $Y = \tilde{v}''(\widehat{y})$ and the constant 1 instead of 3 in inequality (3.21)).

The inequality (3.21) implies the bound

$$\operatorname{tr}(A(x)X - A(y)Y) \le 3n|A|^{1/2}|x-y|^2 \tag{3.22}$$

which will be used in the sequel (for the proof see, e.g. Section 4.2 in [33]).

The following concept plays a crucial role in the proof of the purely analytic result on the uniqueness of the viscosity solution which we establish by a classical method of doubling variables using the Ishii lemma.

Definition. We say that a positive function $\ell \in C_1(K) \cap C^2(\operatorname{int} K)$ is the *Lyapunov function* if the following properties are satisfied:

1) $\ell'(x) \in \operatorname{int} K^*$ and $\mathcal{L}_0\ell(x) \le 0$ for all $x \in \operatorname{int} K$,

2) $\ell(x) \to \infty$ as $|x| \to \infty$.

Theorem 3.11.2. *Assume that the jump measure* Π *does not charge* $(d-1)$*-dimensional surfaces. Suppose that there exists a Lyapunov function* ℓ*. Then the Dirichlet problem (3.12), (3.13) has at most one viscosity solution in the class of continuous functions satisfying the growth condition*

$$W(x)/\ell(x) \to 0, \quad |x| \to \infty. \tag{3.23}$$

Proof. Let W and $\tilde{W}$ be two viscosity solutions of (3.12) coinciding on the boundary ∂K. Suppose that $W(z) > \tilde{W}(z)$ for some $z \in K$. Take $\varepsilon > 0$ such that

$$W(z) - \tilde{W}(z) - 2\varepsilon\ell(z) > 0.$$

We introduce a family of continuous functions $\Delta_n : K \times K \to \mathbf{R}$ by putting

$$\Delta_n(x,y) := W(x) - \tilde{W}(y) - \frac{1}{2}n|x-y|^2 - \varepsilon[\ell(x) + \ell(y)], \qquad n \ge 0.$$

Note that $\Delta_n(x,x) = \Delta_0(x,x)$ for all $x \in K$ and $\Delta_0(x,x) \le 0$ when $x \in \partial K$. From the assumption that the function l has a higher growth rate than W we deduce that $\Delta_n(x,y) \to -\infty$ as $|x| + |y| \to \infty$. It follows that the level sets $\{\Delta_n \ge a\}$ are compacts and the function Δ_n attains its maximum. That is, there exists $(x_n, y_n) \in K \times K$ such that

$$\Delta_n(x_n, y_n) = \bar{\Delta}_n := \sup_{(x,y)\in K\times K} \Delta_n(x,y) \ge \bar{\Delta} := \sup_{x\in K} \Delta_0(x,x) > 0.$$

All (x_n, y_n) belong to the compact set $\{(x,y): \ \Delta_0(x,y) \geq 0\}$. It follows that the sequence $n|x_n - y_n|^2$ is bounded. We continue to argue (without introducing new notations) with a subsequence along which (x_n, y_n) converge to some limit $(\widehat{x}, \widehat{x})$. Necessarily, $n|x_n - y_n|^2 \to 0$ (otherwise we would have $\Delta_0(\widehat{x}, \widehat{x}) > \bar{\Delta}$). It is easily seen that $\bar{\Delta}_n \to \Delta_0(\widehat{x}, \widehat{x}) = \bar{\Delta}$. Thus, $\widehat{x}$ is an interior point of K and so are x_n and y_n for sufficiently large n.

By virtue of the Ishii lemma applied to the functions $v := W - \varepsilon\ell$ and $\tilde{v} := \tilde{W} + \varepsilon\ell$ at the point (x_n, y_n) there exist matrices X^n and Y^n satisfying (3.21) such that

$$(n(x_n - y_n), X^n) \in \bar{J}^+ v(x_n), \qquad (n(x_n - y_n), Y^n) \in \bar{J}^- \tilde{v}(y_n).$$

Using the notations $p_n := n(x_n - y_n) + \varepsilon\ell'(x_n)$, $q_n := n(x_n - y_n) - \varepsilon\ell'(y_n)$, $X_n := X^n + \varepsilon\ell''(x_n)$, $Y_n := Y^n - \varepsilon\ell''(y_n)$, we may rewrite the last relations in the following equivalent form:

$$(p_n, X_n) \in \bar{J}^+ W(x_n), \qquad (q_n, Y_n) \in \bar{J}^- \tilde{W}(y_n). \tag{3.24}$$

Since W and $\tilde{W}$ are viscosity sub- and supersolutions, one can find, according to Lemma 3.7.1 the functions $f_n \in C_1(K) \cap C^2(x_n)$ and $\tilde{f}_n \in C_1(K) \cap C^2(y_n)$ such that $f_n'(x_n) = p_n$, $f_n''(x_n) = X_n$, $f_n(x_n) = W(x_n)$, $f_n \leq W$ on K, and $\tilde{f}_n'(y_n) = q_n$, $\tilde{f}_n''(y_n) = Y_n$, $\tilde{f}_n(y_n) = \tilde{W}(y_n)$, $\tilde{f}_n \geq \tilde{W}$ on K,

$$F(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) \geq 0 \geq F(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n).$$

The second inequality implies that $mq_n \leq 0$ for each $m \in G = (-K) \cap \partial\mathcal{O}_1(0)$. But for the Lyapunov function $\ell'(x) \in \operatorname{int} K^*$ when $x \in \operatorname{int} K$ and, therefore,

$$mp_n = mq_n + \varepsilon m(\ell'(x_n) + \ell'(y_n)) < 0.$$

Since G is a compact, $\Sigma_G(p_n) < 0$. It follows that

$$\begin{aligned} F_0(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) + U^*(p_n) &\geq 0, \\ F_0(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n) + U^*(q_n) &\leq 0. \end{aligned}$$

Recall that U^* is decreasing with respect to the partial ordering generated by $\mathcal{C}^*$ hence also by K^*. Thus, $U^*(p_n) \leq U^*(q_n)$ and we obtain the inequality

$$b_n := F_0(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n) - F_0(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n) \geq 0.$$

Clearly,

$$
\begin{aligned}
b_n &= \frac{1}{2}\sum_{i,j=1}^{d}(a^{ij}x_n^i x_n^j X_{ij}^n - a^{ij}y_n^i y_n^j Y_{ij}^n) + n\sum_{i=1}^{d}\mu^i(x_n^i - y_n^i)^2 \\
&\quad -\frac{1}{2}\beta n|x_n - y_n|^2 - \beta\Delta_n(x_n, y_n) + \mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n) \\
&\quad +\varepsilon(\mathcal{L}_0\ell(x_n) + \mathcal{L}_0\ell(y_n)).
\end{aligned}
$$

By virtue of (3.22) the first term in the right-hand is dominated by a constant multiplied by $n|x_n - y_n|^2$; a similar bound for the second sum is obvious; the last term is negative according to the definition of the Lyapunov function. To complete the proof, it remains to show that

$$\limsup_n \left(\mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n)\right) \le 0. \tag{3.25}$$

Indeed, with this we have that $\limsup b_n \le -\beta\bar{\Delta} < 0$, i.e. a contradiction arising from the assumption $W(z) > \tilde{W}(z)$.

Let

$$
\begin{aligned}
F_n(z) &:= \big[(f_n - \varepsilon\ell)(x_n + \operatorname{diag} x_n z) - (f_n - \varepsilon\ell)(x_n) \\
&\qquad -\operatorname{diag} x_n z(f_n' - \varepsilon\ell')(x_n)\big] I(z, x_n), \\
\tilde{F}_n(z) &:= \big[(\tilde{f}_n + \varepsilon\ell)(y_n + \operatorname{diag} y_n z) - (\tilde{f}_n + \varepsilon\ell)(y_n) \\
&\qquad -\operatorname{diag} y_n z(\tilde{f}_n' + \varepsilon\ell')(y_n)\big] I(z, y_n).
\end{aligned}
$$

and $H_n(z) := F_n(z) - \tilde{F}_n(z)$ With this notation

$$\mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n) = \int H_n(z)\Pi(dz)$$

and the inequality (3.25) will follow from the Fatou lemma if we show that there is a constant C such that for all sufficiently large n

$$H_n(z) \le C(|z| \wedge |z|^2) \quad \text{for all } z \in K \tag{3.26}$$

and

$$\limsup_n H_n(z) \le 0 \quad \Pi\text{-a.s.} \tag{3.27}$$

Using the properties of f_n we get the bound:

$$
\begin{aligned}
F_n(z) &\le \big[(W - \varepsilon\ell)(x_n + \operatorname{diag} x_n z) - (W - \varepsilon\ell)(x_n) \\
&\qquad -\operatorname{diag} x_n z n(x_n - y_n)\big] I(z, x_n)
\end{aligned}
$$

Since the continuous function W and l are of sublinear growth and the sequences x_n and $n(x_n - y_n)$ are converging (hence bounded), absolute value of the function in the right-hand side of this inequality is dominated by a function $c(1+|z|)$. The arguments for $-\tilde{F}_n(z)$ are similar. So, the function H_n is of sublinear growth.

We have the following identity:

$$\begin{aligned} H_n(z) \quad = \quad & (\Delta_n(x_n + \operatorname{diag} x_n z, y_n + \operatorname{diag} y_n z) - \Delta_n(x_n, y_n) \\ & +(1/2)n|\operatorname{diag}(x_n - y_n)z|^2)I(z, x_n)I(z, y_n) \\ & +(f_n(x_n + \operatorname{diag} x_n z) - W(x_n + \operatorname{diag} x_n z))I(z, x_n)I(z, y_n) \\ & -(\tilde{f}_n(y_n + \operatorname{diag} y_n z) - \tilde{W}(y_n + \operatorname{diag} y_n z))I(z, x_n)I(z, y_n) \\ & +F_n(z)(1 - I(z, y_n)) - \tilde{F}_n(z)(1 - I(z, x_n)). \end{aligned}$$

The function $\Delta(x, y)$ attains its maximum at (x_n, y_n) and $f_n \le W$, $\tilde{f}_n \ge \tilde{W}$. It follows that

$$H_n(z) \le (1/2)n|x_n - y_n|^2|z|^2 + F_n(z)(1 - I(z, y_n)) - \tilde{F}_n(z)(1 - I(z, x_n)).$$

Let $\delta > 0$ be the distance of the point $\widehat{x}$ from the boundary ∂K. Then $x_n, y_n \in \mathcal{O}_{\delta/2}(\widehat{x})$ for all sufficiently large n and, hence, the second and the third terms in the right-hand side above are functions vanishing on $\mathcal{O}_1(0)$. It follows that for such n the function H_n is dominated from above on $\mathcal{O}_1(0)$ by $c_n|z|^2$ where $c_n := (1/2)n|x_n - y_n|^2 \to 0$ as $n \to \infty$. Therefore, (3.26) holds. The relation (3.26) also holds because the second and the first terms tends to zero (stationarily) for all z except the set $\{z :\ \widehat{x} + \operatorname{diag} \widehat{x} z \in \partial K\}$. The coordinates of points of $\partial K \setminus \{0\}$ are non-zero. So this set is empty if $\widehat{x}$ has a zero coordinate. If all components $\widehat{x}$ are nonzero, the operator $\widehat{x}$ is non-degenerated and the set in question is of zero measure Π in virtue of our assumption. $\square$

Remark 1. In the case where the cone K is polyhedral, the hypothesis of the theorem can be slightly relaxed. Namely, one can complete the proof using the assumption that the measure Π does not charge hyperplanes.

Remark 2. Note that the definition of the Lyapunov function does not depend on U and hence the uniqueness holds for any utility function U for which U^* is decreasing with respect to the partial ordering induced by K^*. However, to apply the uniqueness theorem one needs to determine the growth rate of W and provide a Lyapunov with a faster growth.

3.12 Existence of Lyapunov Functions and Classical Supersolutions

In this section we extend results of [31] on the existence of the Lyapunov function to the considered case.

Let $u \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus \{0\})$ be an increasing strictly concave function with $u(0) = 0$ and $u(\infty) = \infty$. Introduce the function $R := -u'^2/(u''u)$. Assume that $\bar{R} := \sup_{z>0} R(z) < \infty$.

For $p \in K^* \setminus \{0\}$ we define the function $f(x) = f_p(x) := u(px)$ on K. If $y \in K$ and $x \neq 0$, then $yf'(x) = (py)u'(px) \leq 0$. The inequality is strict when $p \in \operatorname{int} K^*$.

Recall that $A(x)$ is the matrix with $A^{ij}(x) = A^{ij}x^i x^j$ and the vector $\mu(x)$ has the components $\mu^i x^i$. Suppose that $\langle A(x)p, p\rangle \neq 0$. Isolating the full square we obtain the identity

$$\begin{aligned} \mathcal{L}_0 f(x) \;&=\; \frac{1}{2}\left[\langle A(x)p,p\rangle u''(px) + 2\langle \mu(x), p\rangle u'(px) + \frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}\frac{u'^2(px)}{u''(px)}\right] \\ &\quad + \frac{1}{2}\frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle} R(px)u(px) + \mathcal{I}(f,x) - \beta u(px). \end{aligned} \tag{3.28}$$

Note that for $x, \operatorname{diag} xz \in \operatorname{int} K$ we have by the Taylor formula that

$$(f(x + \operatorname{diag} xz) - f(x) - \operatorname{diag} xz f'(x)) = \frac{1}{2}u''(x + \vartheta \operatorname{diag} xz)(p\operatorname{diag} xzx)^2,$$

where $\vartheta \in [0,1]$. Since $u'' \leq 0$, the expression in the square brackets is negative and so is the whole right-hand side of the above formula if $\beta \geq \eta(p)\bar{R}$ where

$$\eta(p) := \frac{1}{2}\sup_{x \in K}\frac{\langle \mu(x),p\rangle^2}{\langle A(x)p,p\rangle}.$$

Of course, if $\langle A(x)p,p\rangle = 0$ we cannot argue in this way, but if in such a case also $\langle \mu(x), p\rangle = 0$, then $\mathcal{L}_0 f(x) = -\beta u(px) \leq 0$ for any $\beta \geq 0$.

This simple observations lead us to the following existence result for Lyapunov functions:

Proposition 3.12.1. *Let $p \in \operatorname{int} K^*$. Suppose that $\langle \mu(x), p\rangle$ vanishes on the set $\{x \in \operatorname{int} K : \langle A(x)p,p\rangle = 0\}$. If $\beta \geq \eta(p)\bar{R}$, then f_p is a Lyapunov function.*

Let $\bar{\eta} := \sup_{p \in K^*} \eta(p)$. Note that $\eta(p) = \eta(p/|p|)$. Continuity considerations show that $\bar{\eta}$ is finite if $\langle A(x)p, p\rangle \neq 0$ for all $x \in K \setminus \{0\}$ and $p \in K^* \setminus \{0\}$. Obviously, if $\beta \geq \bar{\eta}\bar{R}$, then f_p is a Lyapunov function for $p \in \operatorname{int} K^*$.

The representation (3.28) is useful also in the search of classical supersolutions for the operator $\mathcal{L}$. Since $\mathcal{L}f = \mathcal{L}_0 f + U^*(f')$, it is natural to choose u related to U. For a particular case, where $\mathcal{C} = \mathbf{R}^d_+$ and $U(c) = u(e_1 c)$, with u satisfying the postulated properties (except, maybe, unboundedness) and assuming, moreover, that the inequality

$$u^*(au'(z)) \leq g(a)u(z) \tag{3.29}$$

holds, we get, using the homogeneity of $\mathcal{L}_0$, the following result.

Proposition 3.12.2. *Assume $\langle A(x)p, p\rangle \neq 0$ for all $x \in \operatorname{int} K$ and $p \in K^* \setminus \{0\}$. Suppose that (3.29) holds for every $a, z > 0$ with $g(a) = o(a)$ as $a \to \infty$. If $\beta > \bar{\eta}\bar{R}$, then there exists a_0 such that for every $a \geq a_0$ the function af_p is a classical supersolution of (3.12), whatever is $p \in K^*$ with $p^1 \neq 0$. Moreover, if $p \in \operatorname{int} K^*$, then af_p is a strict supersolution on any compact subset of $\operatorname{int} K$.*

For the power utility function $u(z) = z^\gamma/\gamma$, $\gamma \in]0,1[$, we have

$$R(z) = \gamma/(1-\gamma) = \bar{R},$$

and $u^*(au'(z)) = (1-\gamma)a^{\gamma/(\gamma-1)}u(z)$. Therefore, the inequality (3.29) holds with $g(a) = o(a)$, $a \to 0$.

If Y satisfies $\mathbf{H}_2$ with $\sigma^1 = 0$, $\mu^1 = 0$ (i.e. the first asset is the *numéraire*) and $\sigma^i \neq 0$ for $i \neq 1$, then, by the Cauchy–Schwarz inequality applied to $\langle \mu(x), p\rangle$,

$$\eta(p) \leq \frac{1}{2}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2.$$

The inequality

$$\beta > \frac{1}{2}\frac{\gamma}{1-\gamma}\sum_{i=2}^{d}\left(\frac{\mu^i}{\sigma^i}\right)^2 \tag{3.30}$$

(implying the relation $\beta > \bar{\eta}\bar{R}$) is a standing assumption in many studies on the consumption-investment problem under transaction costs, see Akian et al. [1] and Davis and Norman [21].

In particular, for the model with only one risky asset and the power utility function, by virtue of the above computations, we have, for the function $f(x) = au(px)$ given by $p \in K^*$ with $p^1 = 1$, that

$$\mathcal{L}_0 f(x) + U^*(f'(x)) = [...] + \left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{\mu^2}{\sigma^2} - \beta + (1-\gamma)a^{1/(\gamma-1)}\right) f(x)$$

where $[...] \leq 0$. This implies the following conclusion.

Proposition 3.12.3. *Suppose that in the two-asset model with the power utility function the Merton parameter*

$$\kappa_M := \frac{1}{1-\gamma}\left(\beta - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{\mu^2}{\sigma^2}\right) > 0.$$

Then the function

$$f(x) = \frac{1}{\gamma}\kappa_M^{\gamma-1}(px)^\gamma = \mathbf{m}(px)^\gamma \tag{3.31}$$

is a classical supersolution of the HJB equation whatever is $p \in K^*$ *with* $p^1 = 1$.

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Problèmes de non arbitrage, de recouvrement et d'optimisation de consommation dans un marché financier avec coûts de transactions.

Résumé

Cette thèse propose une étude de trois grands problèmes de mathématiques financières dans les marchés financiers avec coûts de transactions proportionnels. La première partie est consacrée à l'étude des conditions de non arbitrage dans un marché avec information incomplete. La seconde partie résoud le problème de recouvrement d'option américaine dans le cas continue et introduit le concept de système de prix cohérent. Enfin, la troisième partie traite du problème de consommation - investissement de Merton dans un marché où le processus des prix est dirigé par un processus de Lévy.

Mots-clés probabilité, mathématiques financières, coûts de transaction proportionnels, recouvrement, arbitrage, optimisation, problème de Merton, option américaine.

Abstract

This thesis deals with three problems of financial mathematics in the markets with proportional transaction costs. The first part is devoted to the conditions of no-arbitrage in a market with partial information. The second solves the hedging problem for the american options in a continuous time setting and introduces the concept of coherent price system. Finally, the third part deals with Merton's consumption-investment problem in a market where the price process is driven by a Levy process.

Keywords probability, financial mathematics, proportional transaction costs, hedging, arbitrage, optimization, Merton's problem, American option.

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