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Clearing vectors in financial networks

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Systemic risk in financial and economical networks

Résumé

Le risque systémique menancant le système financier est une préoccupation majeure pour les régulateurs.Indicateurs adéquats de risque systémique devraiement les aider à accomplir les lois réglementaires appropriées. la thèse propose un modèle dynamique du systeme bancaire pour calculer un indicateur de risque systémique de deux composantes: la probabilité d'un évenement declencheur qui provient de la baisse des prix des actifs , et les pertes correspondantes dans le système financier.
La thèse prouve egalement l'existence et l'unicité de deux modèles d'equilibre de compensation : le premier avec un modèle de différentes hiérarchie de dette et le second modèle avec plusieurs stratégies de liquidation

Summary

Systemic risk threatening the financial system is a major concern for regulators. Adequate indicators of systemic risk would help them perform appropriate regulatory laws. The thesis proposes a dynamic model of banking system to calculate a systemic risk indicator of two components: The probability of a triggering event originated from external asset price decline, and the corresponding losses through the financial system. The thesis also proves the existence and uniqueness of two clearing equilibrium: the first deals with a model of different debt seniorities, the second with a model of several illiquid asset following a proportional liquidation strategy.

Mots-clés

Risque systemique, chocs communs, faillite,contagion, modeèle dynamique, indicateur de risque systémique, compensation, actifs illiquides, equilibre du système, hiérarchie , vecteur de compensation

Keywords

Systemic risk, common shocks, default, contagion, dynamic model, interlink ages, indicator of systemic risk, clearing systems, illiquid asset, equilibrium, seniority, clearing vector.

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Abstract

Systemic Risk impairing the functioning of the financial system has substantial negative consequence on the society and the macro-economic growth. Regulators and supervisory agencies responsible of the maintenance of the financial system need to dispose adequate indicators of financial stability and correspondingly perform the appropriate policy actions aiming to mitigate the systemic risk. In this thesis we aim to construct new indicator of systemic risk without investigating signs of financial distress. Its importance is that it could in normal times indicate the risk while regulators still have time to act.

Systemic Risk This chapter focuses on the definition, concept and measures of systemic risk as well as its negative consequences on the macro-economy. The essence of systemic risk is that a trigger event takes place and disrupt the functioning of the financial systems. Complexity and opacity of the financial system are fundamental generators of systemic risk. Moral hazard, excessive commercial risky activities, portfolio correlation, and sometimes inefficient regulation contribute to the build up of that risk. The concept of systemic risk is split up into four parts: build up, trigger event, materialization and short and medium term economic consequences. The trigger event could be endogenous or exogenous, could be the failure of a SIFI, or a market crash or it follows a long period a financial stress weakening the financial system slowly. The risk spreading could be through several channels of risk including interbank connections, portfolios correlation, liquidity contagion, fire sales and informational asymmetry. As the financial system is built in confidence the spread of a bad news could have an aggressive contagious negative feedback. We explain the portfolio structure of a bank, assets, liabilities and equity and we end this chapter by reviewing the regulations progress.

On uniqueness of clearing vector reducing the systemic risk In this chapter we study the clearing problem. We review the model proposed by Eisenberg and Noe in their influential paper published in 2001 and the one suggested by Amini, Filipovic, and Minca. The model of Eisenberg and Noe is based on simplified assumptions: limited liability of the firm, and absolute priority of the debt over equity. They proved the existence by a straightforward application of the Knaster–Tarski theorem and got the uniqueness of equilibrium using arguments that rely upon the graph structure generated by the liability

matrix. We extend their model by including several levels of seniority in the interbank debt structure. The proof of existence is easy: the Knaster–Tarski theorem again can be used. and, under a certain condition, the uniqueness of the equilibrium. We consider also the model with crossholdings. It appeared in the paper by Suzuki published in 2002 and was developed further by Elsinger in a preprint of 2009. We clarify some aspects of this model, where the problem is formulated to find an equilibrium consisting of the clearing vector and the equity vector.

We generalize the approach of Amini, Filipovic, and Minca who extended the Eisenberg and Noe model to incorporate an illiquid asset which the institutions will be forced to liquidate when they suffer from interbank losses and not able to cover their liabilities otherwise. The basic assumption is that the market price is determined by an inverse demand function. They proved they existence and uniqueness of the clearing payment vector as well as the asset clearing price. We consider here a similar model but with a more realistic assumption that portfolios are composed by several illiquid assets. We follow a proportional asset liquidation strategy and prove the existence and, under the same conditions, uniqueness of the equilibrium.

We complete the chapter by a succinct discussion of the Fischer model involving derivatives in the context of clearing problem.

Dynamic model of systemic risk and contagion In this chapter we generate an indicator providing a kind of alarm that will not ring directly before the materialization of the crisis but a long time before. The aim is to provide a new approach aiming to improve measurement and management of the systemic risk and maintain the financial stability. We consider a model where the external asset in the balance sheet has a dynamic price following a geometric Brownian motion. We calculate the probability of the price decline sufficient de cause one (or several) bank default (defaults). We identify the weakest bank (or group of banks) and check for possible contagion of losses via interbank linkages and fire sales. This approach combine the two main aspects of systemic risk: market crash and interbank contagion.

Contents

Chapter 1

Systemic Risk

1.1 What is the systemic risk?

Driven by the outbreak of the last 2007 US subprime crisis, preceded by the East-Asian crisis, the Russian crisss and the Brazilian crisss, theoretical and empirical studies of the financial stability has increased substantially. Although "systemic risk" is the fundamental underlying concept of all the research on financial stability, most work has pointed on one or several aspects of that risk and the absence of a clear and unified consensus regarding the concept of "systemic risk" has reminded. The last US subprime crisis painfully illustrated the need of a better understanding of the overall concept of "systemic risk" and the relations between its different aspects, as it was obvious that financial regulators and safety supervisors has underestimated it. The essence of a systemic risk is a risk that an event takes place and misbalances or disrupt the functioning of the entire financial system. On the level of a single institution market risk, credit risk, operational risk, model risk, can be directly assigned but systemic risk can only be assigned indirectly.

A survey by Bisias et al. (2012 [11] revealed that the literature have dealt with different aspects of systemic risk; e.g. unbalances, collapse of confidence, correlated exposures of financial institutions, negative impact on the real economy, information disruption, feedback affects, asset bubbles, contagion and negative externalities. And they suggest that a single consensus measure of systemic risk may neither be possible, nor desirables as it would be a "Maginot strategy". Instead, more than one risk measure will be needed to capture the complexity of real financial systems.

Kaufman and Scott [1995] [37], John B.Taylor [2010] [51] and others divided systemic risk into three parts: First, a systemic event. Second, the spreading of shocks form one financial institution to another through interconnections. Third, significant negative effect on the macro economy. The analysis of the systemic risk has to cover factors contributing to its accumulation, its materialization and its spreading in the financial system. System risk evolves along the development of the financial market innovation, through the creation of alternative investments and entering new domains, which increased the complexity of the financial system. The growth of the scale of securitization, although it has facilitated the availability of liquidity, has enlarged the volume of shadow banking adding to the size of complexity of the financial system. Dow (2000) [19] showed that moral hazard plays an essential role in disrupting the activities of financial institutions, raising the extent of systemic risk. Competition between financial institutions leads them to adopt extremely risky activities, an aggressive organizational practice, a collective failure of bank management, which lead to inertia and failure to respond to changed economic circumstances and also to an increasing level of correlation. We conclude that no individual financial institution has sufficient incentive to limit its risk taking in order to reduce the systemic danger to other financial institution and to the economy as a whole. Appropriate regulations on the institutional level are an important sector of the regulations required to maintain financial stability.

Definition of systemic risk:

In the economy literature, there is not yet an agreed general definition of systemic risk, and because the problem can not be solved unless it is well defined, possible implementations of public policies aiming to mitigate systemic risk are pointless and dangerous. The diversity of systemic risk definitions in the literature and their differences is exemplified by the following sample of definitions:

- Kaufman (1995) [37]: "The probability that cumulative losses will occur from an event that ignites a series of successive losses along a chain of [financial] institutions or market comprising a system".
- Kupiec and Nickerson (2004) [38]: "The potential for a modest economic shock to induce substantial volatility in asset prices, significant reductions in corporate liquidity, potential bankruptcies and efficient losses".
- Mishkin (1995) [43]: "The likelihood of a sudden, usually unexpected, event that disrupts information in financial markets, making them unable to effectively channel funds to those parties with the most productive investment opportunities".
- Bank of international settlements (1994) [8]: "The risk that the failure of a participant to meet its contractual obligations may in turn cause other participants to default with a chain reaction leading to broader financial difficulties".

These definitions have a common factor: It is that a trigger event causes a chain of negative economic outcomes. However, the nature of the trigger event and the negative economic outcomes are different. Studies based on each of these definitions will only succeed to capture a certain aspect of the credit risk.

A more general definition of the systemic risk is proposed by the group of Ten (2001, p.126) [32]: "Systemic risk is the risk that an event will trigger a class of economic value or confidence in it, and attendant increases in uncertainty about, a substantial portion of the financial system that is serious enough to quite probably have significant adverse effects on the real economy. Systemic risk event can be sudden and unexpected, or the likelihood of their occurrence can build up through time in the absence of appropriate policy responses. The adverse real economic effects from systemic problems are generally seen arising from disruption to the payment system, to credit flows, and from the destruction of asset values". This definition covers most of the parts of the systemic risk, i.e. trigger event, information asymmetry, the suffer of a substantial part of the financial system, the need of appropriate policy framework, the impairment of the functions of the financial system and the negative economic consequences. This definition, although it does not describe the spreading of losses, we find it the closest to capture the concept of systemic risk.

On the other hand, Central Banks and a number of authors propose definitions of the financial stability rather than systemic risk. European Central Bank defines financial stability as "a state in which the financial system comprising intermediaries, markets and infrastructure is able to withstand financial intermediaries process".

Another definition by Svensson (2012), [50]:

"Financial stability is the situation in which the financial system can perform the essential functions (payment services, transforming savings into finance/investment and risk management services) resisting and shocks that threaten these functions".

These definitions focus on the resistance of the financial system to shocks and its ability to maintain the performance of its main functions. These definitions give us a description of the situations in which there is no systemic risk. Monitory activities should be able to generate indicators of the systemic risk build up and take adverse actions in order to mitigate it before its materialization.

The concept of systemic risk in the financial system:

The concern of the regulators on financial supervisors to preserve an insure the financial system stability should be pointed to four stages of systemic risk. First, the buildup of the systemic risk in normal times; second, the trigger event or shock representing the first step in the materialization of the systemic risk; third, the spreading the shock through the financial system; fourth, the feedback or negative consequences on the real economy. Their aim is to try mitigating the extent of these four stages. Of course, the most fundamental target is to prevent the financial crisis of taking place, which means to try to monitor and mitigate the accumulation of systemic risk. However, if the financial crisis erupts, efficient crisis management and confinement policy responds becomes of great importance. In what follows we try to present an anatomy of each stage of the financial risk.

The buildup of systemic risk:

Two fundamental symptoms are the signs of the buildup of the systemic risk: the increase of complexity and the decrease of diversity of the financial system. In normal times, many factors participate in the appearances of these two symptoms. The remuneration structure in the major financial agents as well as the competition between them will lead to excessive short term risk taking behavior, shrinking the return differential between risky and less risky asset, reducing the credit standards, increasing the leverage and reducing the coordination between them.

Institutions at the boom phase tends to ignore the systemic risk inherent in their financial activities. For example, when levering up with short-term debt, each investor consider his own risk of not being able to meet its liability at maturity and be forced to sell the assets at fire sale prices. However, derived by myopia, they do not take into account that their selling will depress prices, potentially leading other investors to sell as well, aggravating the fire sale. Financial agents have the incentive to follow similar portfolio strategies increasing exposure concentration in a given sector raising their portfolios correlation. This could lead to a bubble in the asset price, which is normally followed by a sharp decline in the asset value, and the correlation makes them simultaneously prone to the same shock, which becomes in this case a systemic event. The increase in diversification of bank exposures may reduce systemic risk.

The financial system is by its nature a complex adaptive system. Highly connected heterogeneous networks may be robust yet fragile; they may be resilient to small shocks, yet vulnerable to a shock that affects or highly connected nodes, or to aggregate market shocks. In such network, connections that we consider as shock absorbers may act as shock diffusive in crisis time. In normal times stability will increase with connectivity, it in bad times, the information asymmetry will increase the uncertainty about counterparty risk, due to the complicated network structure and the nature of the connections. In this situation, stability declines with connectivity. Financial innovation, particularly securitization, creates additional instability. As CDOs, CDS and other structured credit instruments escalate internationally, they substantially extendes the size and the breath of the crisis. The structure of these instruments increases the opacity of the network, increasing in turn its complexity.

1.2 Triggering events

Trigger event or systemic event in the narrow sense or in the micro dimensional level, is an idiosyncratic shock that affects a given financial institution or a financial market leading to subsequent significant outcomes disrupting one or several financial institutions or market and even causing their failure on crash. Trigger event in the broad sense or in macro dimensional level, is a systematic shock affecting a large number of financial institutions, or financial markets, simultaneously. The distinction between the narrow and the broad sense in the trigger event is important, since crisis management measures, dealing with the source of the problem, might be different in the case of an idiosyncratic that could cause a cascade of defaults to the case of a systemic shock might have a significant broad destabilization effects. Systemic events in the narrow sense and the broad sense are closely related, macro economic downturns might weaken the majority of the financial institutions increasing the probability that a single failure will lead to a contagion of financial institution defaults. Equivalently, if the financial system suffers a contagion of defaults this is very likely to cause disruption of the system at the macro level.

An example of an idiosyncratic shock to the financial system is the failure of systemic important financial institution due to fraud, mismanagement or it can be also caused by the spreading of some bad information about a bank causing panic and a massive demand of deposit withdrawals. In such a case the bank, although it will try to liquidate illiquid investments at a distressed prices, may fail to meet the depositors demand and defaults. An example of systematic shock is general business cycle fluctuations, or a sudden increase in the inflation rate, or a stock market crash affecting the majority of the financial institutions simultaneously. The systemic event could have its source outside the financial system like the disruption of a broad financial market on which the financial system has a significant exposure, or even a terroristic attack affecting the entire system.

Contagion and channels of systemic risk Contagion, or domino effect, describes the mechanism of the spreading of the financial instability (or an initial shock) within the financial system through several channels, leading to a system wide crisis and completing the materialization of systemic risk. Contagion is sequential and materializes in several subsequent rounds of transmission. The initial financial instability that could spread in the financial system is caused by a triggering event. The impact of a triggering event can be non linear and change rapidly, and the normal conditions of the financial system are subject to structural changes. Contagion effect affects all institutions, not only the vulnerable ones but also those well performing that could be affected in further rounds of risk transmission. One of the financial system, main structurally inherent, vulnerability to contagion is its fundamental function of liquidity transformation, e.g. financing long term assets (loans) with short term liabilities (exposures). Factors raising the vulnerability to contagion are: high leverage, complex interbank interconnectedness, shadow banking, risk of confidence loss, the use of aggressive liquidity management strategies. Another factor increasing the risk of contagion is the agent's behavior in an environment of asymmetric information, leading to coordination failure (Diamond and Dybvig, 1983) [18]. Lubbloy(2004) [39] argued that the sources of underestimation of the contagion risk may be due to:

- Ignoring the repo positions.
- Ignoring the systemic effect of cross-holding of shares.
- Ruling out the imported contagion.
- Assumption of dispersed bilateral exposures.
- Definition of default
- Using end of year data.
- Ignoring the off balance sheet items
- Neglecting the risk stemming from the payment and settlements systems.

On the other hand, factors leading to contagion risk overestimation include:

- Ignoring the reaction of central bank.
- Ignoring the netting arguments.
- Neglecting the potential measures of the regulatory authorities.
- Neglecting the potential reactions of banks (withdrawal of interbank exposures, raising capital).

Financial institutions are linked directly to direct interbank liability (exposure), or indirectly (portfolio correlation, influence of the whole financial system conditions). As contagion occurs through different channels of risk during a financial crisis, either as a primary cause or as the result of spillover effects resulting from the critical shock, it is important to identify these channels.

Defaults contagion Default contagion is a contagion mechanism in which the initial default of one or some institutions will lead to a cascade of defaults of other institutions in the financial system. The cascade of default could become wide enough leading to a systemic crisis. Banks are traditionally connected through interbank lending. When a bank borrows from another bank a specified amount of money for a specified period of time, this amount appears on the asset side of the balance sheet of the lender as interbank asset. And on the liability side of the balance sheet of the debtor as interbank liability. Banks are also interconnected through other types of linkages including swaps, derivatives and other types of securitized assets. If a bank becomes insolvent, and not bailed out by a government agency, it will be forced into bankruptcy. Losing its availability to meet its interbank commitments a creditor to such a bank receives a shock on the asset side of its balance sheets shrinking its net worth and sometimes wiping it out and causing its default as well. Such shocks to creditor banks at the time of default of a debtor bank are the channel of default contagion. Without government intervention such a shock could chain together like dominos to create a defaults cascade. Default cascade is most likely to occur when interbank exposures are a high fraction of lending bank's equity. In reality, few banks seem to cause this type of contagion, due to banks bailouts.

Asset correlation: Banks have the tendency to have a herding behavior, investing in similar assets, thus, creating correlation in their portfolios. In boom times, this correlation increases in a fast manner, making the banks susceptible to correlated asset shocks creating a channel of systemic risk. A speculative bubble in some asset class looses its impetus and is followed by price deterioration. If the depression is deep and wide, it urges concern about additional losses, increasing uncertainty and risk aversion. As firms are forced to liquidate many assets to cover margins and reduce exposures, other assets can lose value as well, providing reciprocally reinforcing feedback and making the collapse general. In the last US subprime financial crisis in 2007, major banks around the world were found to hold significant positions in the US subprime mortgage market. The US housing prices decline was deep and protracted in that year, and acted as huge assert shock that affected the asset portfolios of most banks.

Liquidity contagion: A financial institution suffering liquidity shortage could fail to meet future obligations and defaults. It will seek liquidity provision adopting strategies that are considered as constrainging its balance sheet. It will try to access the market sale and repurchase agreements (repo market) which is a major source of funding for financial institutions, and highly liquid in normal times. It will also refuse to rollover short term loans and repo lending to other counterparties. During a financial crisis liquidity provision increases in cost as each financial institution will have larger concerns about its future ability to access liquidity. As a consequence, financial institutions will hoard liquidity as a defensive action. When banks respond to funding illiquidity by curtailing a large fraction of their interbank lending, the resulting funding shocks to other banks are the channels for liquidity contagion in the system.

Market illiquidity and fire sale: In good times, competition and positive perspective of the future of the given market collectively induces banks to increase their investments in given assets. This is done, most of the times by raising leverage levels. This collective behavior pushes up the prices, and creates illusions of even better times, encouraging banks to widen their leverage scales. The reverse is true in depressed times, the tendency of a distressed bank to liquidate apart of their assets, especially, when market does not have enough liquidity or the intention to absorb such offers, will lead to price depression that could induct the contagion mechanism known as an asset fire sale. This mechanism has two guide points. First, the asset sale in a dry market leads to price depreciations. Second, the marketing-to market leads to losses by other banks holding these assets.

Other types of contagion including bad news spreading or informational channels plays important role in impairing the system leading to a mismanagement of the crisis during its materialization. The uncertainty about all the elements of the financial system will lead each agent to perform defensive strategies, like hoarding liquidity by refusing to renew short term loans to other counterparties, due to the inability to assess their counterparties exposures to risk during the crisis. Concerns about some distressed markets lead banks to liquidate their shares despite their needs of liquidity, adding to the prices depressions. Some banks could adopt some shark behaviors by short selling assets. Informational contagion is one of the most critical contagions in the financial system, because the financial system structure is based on confidence. The loss of confidence is the key of many times types of contagion in the financial market.

1.3 Impact on real economy:

Loses due to financial crisis are, in majority, beard by the public (taxpayers). When banks are bailed out by governments and central banks, their debts are taken over and paid by the entire society, leading to a large scale of poverty. Recessions associated with financial crisis tend to be unusually severe, resulting in much larger decline in real economic activity and their recoveries are typically slow. Effective countercyclical monetary policy can help shorten recessions, but its effectiveness is limited during financial crisis. The direct impact of a financial crisis on the real economy is on major market indices, interest rate, the production economy and the level of employment. But the main direct impact on the real economy is the impairment of the fundamental function of the financial system, intermediation. A large number of recoveries from crisis occur in absence of credit growth, but the average growth during these episodes is lower than during normal recoveries.

1.4 Measuring systemic risk:

There are, among others, three approaches that could help assessing the accumulation of systemic risk. The first approach deals with monitoring traditional indicators soundness or stability aiming to appraise the developments in the financial system in the broad sense. The second is concerned with assessing interconnections between financial institutions, and the third focuses on modifications of financial assets prices. In what follows we discuss each approach.

Aggregate indicators of financial stability: Aggregate indicators have been essential for central banks and supervisory agencies in order to measure vulnerabilities in the financial system; given the importance of this approach we try to review it in more details. Many indicators are used in this approach including:

Internet rates and asset prices: Information contained in interest rates and asset prices reflects that market participants view of different risks. This information is continuously available and can be used to generate different indicators of the health of the financial market, such as measures of liquidity premium, risk spread, asset bubbles.

Financial stock and flows: These comprise measures of bank lending volume, capital flow, net issuance of bonds. These indicators are useful because a reduction in lending activity by institutions usually happens during financial stress and loss of confidence in the financial system.

Macroeconomic indicators: Consumer price index; cross domestic product and unemployment figures etc. These indicators are used by regulators to gauge the overall health of the economy.

These indicators are based on data collection which is usually very costly. The usefulness of these indicators is limited by the availability of such information. A second limitation of these indicators is that they are not frequently reported and rely on backward information, and as previous crisis showed systemic risk can materialize very rapidly and previous data becomes useless to capture risks at the present time. The third shortcoming originates from their focus on the macro-changes in the financial system, and they do not provide information on the level of individual financial institutions and their interconnectedness. While the health of major financial institutions play important role in the stability of the financial system, information about interlinkages could determine how an initial shock could spread in the financial system from a given starting point.

Each of these indicators has its advantages and shortcomings. It is better to use a number of measures when attempting the systemic risk assessment. However these indicators have failed to identify in advance some developments that played essential roles in the last financial crisis. These developments include global current account unbalances, decline a real interest rates, the growth of subprime loan markets and the fast growth of the market of structured products which were very opaque and complex.

Measure of conditions of individual institutions: Policy makers aiming to generate better macroprudential indicators of systemic risk cannot ignore the risk arising from the failure of one or a group of financial institutions. This result in the need of indicators of the state of individual institutions. Regulators accessing data from individual institutions could determine those with high vulnerability to risk. An IMF (2009 a) analysis studies the possibility of a set of standard financial stability indicators (FSI) to help identifying which institutions would face difficulties sufficiently sever to call for a government bail-out. Their finding was that measures of leveraged provided useful information for predicting intervention, but on the other hand measures like liquidity ratios, capital adequacy ratios, non-performing loans did not provide such information. This indicates that not all the information in the institutional level is useful. As a consequence, the data collection problem in view of the number of institutions covered and the timelines of the data could be minimized as the number of systemically important institutions is very small compared to the total number of institutions.

Assessing systemic linkages The last 2007 crisis has shed light on the importance of understanding the interconnections between financial institutions, both domestically and internationally. Knowledge about the way the default of one institution would influence other institutions helps regulators to generate appropriate policy responses. IMF (2009b) reviews a number of approaches to assess inter-linkages between financial institutions and discussed four of them:

– The network approach, dealing with financial stress transmission across the banking system through connections in the interbank market.
– The co-risk model uses data on the credit default swaps market to measure how the default risk of a financial institution is affected by the default risk of another institution.
– The distress dependence matrix, which enable analysts to study financial institutions in groups and gouge the probability of distress for a pair of institutions taking into account a set of other institutions.
– The default intensity model, which studies the probability of the failure of a large number of financial institution through interconnections.

The IMF study has three conclusions: First, in principle, a continuous assessment of systemic risk is possible. While each approach, considered alone, is not enough to present a complete view of the systemic risk inherent in interconnections, regulators can use integrated approaches to measure that risk. Second, a huge amount of data is demanded to execute this monitoring exercises. Policy makers require information about the linkages between a large numbers of institutions across the world. Knowing that these linkages are subject to rapid changes during financial instability, this requires the collection and transmission of data to policy makers in short term delays. Third, as the last financial crisis showed, financial models, used by regulators to assess systemic risk, could be inefficient. Regulators are doubtful about the adequacy of the outcomes of the statistical risk monitoring tools and are unwilling to extremely build them up.

1.5 Capital structure of a bank:

The banking industry comprises firms, with different types and sizes. To the side of a large number of traditional banks like retail, commercial and investment banks, the financial system includes a wide diversity of shadow banking institutions like hedge funds, investment and pension funds, saving institutions. As the financial system evolves with time, new elements of the growing production will be added to the system. These different financial firms have a wide range of holdings and liabilities that is very difficult to compress in a brief overview. Balance sheet reports of these firms are quarterly declared to the public and present details on their capital structures that are the values of their aggregated assets and liabilities. These accounting books are updated on a daily basis by banks managers and only regulators have access to any bank books at anytime.

Systemic regulators differentiate between internal (domestic) and external (international) assets and liabilities, despite the wide class of assets existing in the banking industry, their most important characteristics are: maturity, credit quality, interest rate and liquidity. In a similar way, the essential characteristics of different types of liabilities are maturity, interest rate, and seniority. The formal duality between assets and liabilities is an important feature of the capital structure, as any financial institution's asset is another agent's liability.

Bank assets classes: Some of the essential basic forms of banks assets are presented below:

Loan portfolio: The main function of a bank is intermediation the maturity transformation, borrowing at short term (deposits) lending at long term (loans) to different economic sectors (industrial, agricultural, real estate, etc.), though a wide variety of agents ranging from the retail sector, to small and medium enterprises (SMEs), and to major corporate. These loans are of different maturities and their credit quality depends on the health of each counterparty as well as the health of the economic sector in which the counterparty is involved. Banks benefits from interest rate difference between deposits and loans. However, these loans (assets) are illiquid by their nature and when liquidated they are sold below their book values. Increased unprudential real-estate lending was at the core of the last 2007-2008 crisis.

Over-the-counter securities: A large class of transactions including bonds, swap contracts and other derivatives are performed is the OTC markets, usually bilaterally, but increasingly through a central clearing part (CCP). Some of these exposures are subject to rapid fluctuations, both in volume and in sign. Sometimes, these exposures are collateralized in order to diminish counterparty risk. While two financial institutions could be reciprocally exposed to each other, a master netting agreement (MNA) allows them to offset exposures of opposite signs reducing the counterparty risk. Total return swaps

(TRS) constitute an important part of OTC securities; they exchange the random return of an underlying asset for a fixed periodic payment. A credit default swap (CDS) is an example of (TRS), it exchanges fixed quarterly payments for a large payment at the moment the underlying defaults. Thus, a CDS provides some kind of insurance. These contracts have raised the complexity and the opacity of the financial system and should be entirely netted in order to overcome the risk they pose to the financial system.

Cash and market securities: Liquid securities, or liquid assets, are assets that could be sold easily without any decrease in price, and are considered as a sort of cash. Examples of liquid asset are money-market accounts paying the over-night rate, stocks and exchange traded derivatives. A prudential liquidity management, suggested by Based III regulatory framework, requires banks to keep a certain fraction of their portfolios in cash and liquid assets that could be easily transformed into cash when the bank has to meet its short term liabilities.

Reverse repo asset A repo or repurchase agreement is a form of a short term borrowing; the bank sells a security to investors (other banks) and agrees on buying them in the future. For the party selling the security (and agreeing to repurchase it in the future) it is a repo. For its counterparty it is a reverse repo, it is considered an asset, and can in turn be used as collateral for another repo agreement with a third party.

Debt and liabilities:

Deposits: Deposits constitute a large part of the liability side of a traditional bank balance sheet. These deposits are of the two types: deposits made by small retail investor, and deposits made by institutional investors (interbank liabilities). These deposits have different maturities and interest rates and in aggregate they can be thought of as a single asset paying a constant dividend rate. Short-term money-market funds are an important class of institutional deposits. This kind of interbank linkages played an important role in the stress transmission during the last crisis. Small depositors are usually protected by deposit insurance in case the bank defaults, but institutional depositors are not. Instead banks seek protection through collateralization.

Bonds: Bond issuing is an essential banking tool for raising long term capital debt. An institutional bond is characterized by its nominal amount, maturity and coupon rate. Sometimes these bonds have different seniorities. While the most senior bonds have the smallest coupon rates, they are paid in full before recovery of the junior bonds in case of default.

Market securities: When a commercial bank takes short positions in given security markets, these positions appear as a positive amount on the liability side of its balance sheet.

Collateralized loans (Repo) Already explained in the asset section.

Hybrid capital: Hybrid capital is a form of firm's funding that has both debt and equity features. This means that it pays tax-deductible dividends, not interest rates, but in case of default it maintains seniority over equity.

Equity: By definition a financial institutions' equity is the value of the difference between its assets and its liabilities, what it owns minus what it owes. Equity, in publicly owned firms, is divided into shares that are held by shareholders. The share fluctuating market prices are publicly available. For such institutions, the market value equity is calculated by multiplying the total number of shares by the share market price. On the other hand, for private institutions, equity can only be calculated by examination of the institution's accounts. In highly leveraged institutions, equity becomes the difference of large positive numbers, which makes difficult to estimate and this uncertainty about the equity value in high stock price volatility. The principle of "limited liability" applies to almost all publicly held firms. Shareholders are not committed to pay additional payments to cover unpaid liabilities in case of the firm defaults.

1.6 Regulations:

Regulators and supervisory agencies are responsible of the maintenance of the financial stability. As a financial crisis usually results in painfully expensive consequences on the public and wide economy, regulators are stimulated to design rules and laws aiming to prevent or mitigate the impact of financial crises. These regulations constitute the framework of the financial system activities and sometimes inherent undesired and unintended consequences. So rules that may effectively prevent some types of risk, contribute to the creation of new sources of risk. Examples of such rules:

- Strict bank regulations that led to the emergence of the shadow banking system.
- Capital regulations that allowed financial institutions to report healthy capital levels while actually decreasing levels of effective capital.
- Requirements for all banks to be prudently run. Like the Basel II adequacy ratio (CAR), which was a macroprudential rule, but could cause pro-cyclicality and dangerous swings of the credit cycle. In contraction periods, capital buffers of some banks can fall below the regulatory minimum leading to massive asset liquidation and fire sale.

Microprudential rules are imposed on the level of individual institutions, and are concerned by their health one by one, but do not capture the network effects. Macroprudential rules are designed to mitigate systemic in the financial system as a whole, accounting for interconnectivity of risks between banks. Basel III closed the gap, created by Basel II CAR, by relaxing the capital requirements in distressed periods making them countercyclical.

The identification of systemically important financial institutions (SIFIs), whose defaults could threaten the stability of the financial system, is another Basel III macroprudential rule. These SIFIs are subject to capital surcharge decreasing their vulnerabilities. Basel III has also imposed constraints on the level of bank leverage. Other measures addressing liquidity risk are proposed by Basel III including "Liquidity Coverage Ratio" (LCR) and the "Net Stable Funding Ratio". These measures are designed to capture the ability of an institution to survive liquidity pressure, as well as shocks to the long term assets. An important aspect is the duality and interaction between micro- and macroprudential regulations. Maintaining a healthy financial system protects the individual institutions. On the other hand, assuring the health of individual institutions could, at least, mitigate the systemic risk arising from the failure of a SIFI.

Chapter 2

On uniqueness of clearing vector

2.1 Introduction

To explain the clearing problem we start with the simplest example of a financial system with two agents each having in a cash 10 dollars. The first agent gets from the second a credit 1M dollars, the second gets from the first a credit 1M and 1 dollar. Apparently, as a result both agents has a huge liabilities with respect to each other. Of course, the agents can be asked to reduce their liabilities by reimbursing credits partially (e.g., to the levels 0.5M and 0.5M+1 in liabilities and 10 dollars both in cash) or completely, with zero liabilities and cash reserves 11 and 9 dollars respectively. Intuitively, the situation where the liability are reduced (i.e. the system is cleared) seems to be less risky: if one of agent became bankrupt and only the percentage of the huge debt value can be reimbursed, the creditor's losses will be also huge. For complex financial systems involving large numbers of agents with chains of borrowing the clearing problem, that is the reduction of absolute values by reimbursement, looks much more complicated.

In the influential paper [22]published in 2001, Eisenberg and Noe suggested a clearing procedure in the model describing a financial system composed by N banks (under "banks" can be understood various financial institutions); a more general model was introduced independently at the same time by Suzuki, [49]. The assets of the bank are cash and interbank exposures which are, in turn, liabilities for its debtors. The clearing consists in simultaneous paying all debts. Each bank pays to its counterparties the debts *pro rata* of their relative volume using its cash reserve and money collected from the credited banks. The rule is: either all debts are payed in full or the zero level of the equity is attained and the bank defaults. The totals reimbursed by banks form an N-dimensional clearing vector. A remarkable feature is that this vector is a fixed point of a monotone mapping of a complete lattice into itself and its existence follows immediately from the Knaster–Tarski theorem, a beautiful and fairy simple result which proof needs only a few lines of arguments. The uniqueness of the clearing vector is a more delicate result involving the graph structure of the system.

The ideas of the Eisenberg–Noe paper happened to be very fruitful and their model was generalized in many directions having not only financial importance but posing interesting mathematical questions. One of them is the question on uniqueness of clearing vector or equilibrium on financial market.

Theorem 2.4.2 provides a new sufficient condition for the Elsinger model of clearing with debts priority structure. This model is given by a set of liability matrices corresponding to each seniority. The idea of our approach is to use the largest clearing vector which always exists to construct a new liability matrix generating a graph structure with which one can work in a similar way as in the Eisenberg–Noe model. Theorem 2.5.2 deals with the uniqueness of equilibrium in a clearing model with several illiquid assets and a market impact. In the presence of several illiquid assets the banks are faced the choice of asset selling strategies. We use the proportional scheme of selling similar to that in the paper by Cont–Wagalath, [15], leaving game-theoretical versions for future studies. In the case of one illiquid asset our result is close to that of the study by Amini–Filipovic–Minca, [5], but our definition of the equilibrium is different (but equivalent).

The structure of the chapter is as follows. In the introductory Section 2 we discuss briefly the general principle and results in the framework of the Eisenberg–Noe model. To facilitate the comparison with further development we provide also short proofs. We discuss also a modification of the Newton elimination algorithm for nonlinear equations of a special type and its applications to compute clearing vectors. In Section 3 we consider the the equilibrium is Suzuki–Elsinger model with crossholdings. In Section 4 we prove a uniqueness of the clearing vector for the Elsinger model where senior liabilities must be reimbursed before juniors. Section 5 contains the sufficient condition for the uniqueness of the equilibrium in the model where clearing requires selling of the illiquid assets with price impact. Economically speaking, it is oriented to the recovering of the market after fire sales. Section 6 deals with the Fischer model with liability matrices depending on the clearing vector. Finally, we provide in the concluding section a short information about the Knaster–Tarski theorem adapted to our needs.

Notations. We denote by $\geq$ the partial ordering in $\mathbf{R}^n$ and its subsets induced by the cone $\mathbf{R}^n_+$. In other words, the inequality $y \geq x$ is understood componentwise. Also the symbols $x \wedge y$ and $x \vee y$ mean, respectively, the componentwise minimum and maximum, $x^+ := x \vee 0$, $x^- := (-x)^+$. The notation $[x, z]$ is used for the order interval, that is $[x, z] := \{y \in \mathbf{R}^n : \ x \leq y \leq z\}$. If $A \subseteq [x, z]$, then $\inf A$ is the unique element $\underline{y} \in [x, z]$ such that $\underline{y} \leq y$ for all $y \in A$ and for any $\tilde{y}$ such that $\tilde{y} \leq y$ for all $y \in A$ we have that $\tilde{y} \leq \underline{y}$. That is, the component $\underline{y}^i = \inf\{y^i : \ y \in A\}$ for $i \in \mathcal{N} := \{1, \dots, n\}$.

We use the matrix notations where the vectors are columns, $'$ is the symbol of transpose, $\mathbf{1}' := (1, \dots, 1)$ (the dimension of the vector is supposed to be clear from the context). If $D \subset \mathcal{N}$, then $\mathbf{1}_D$ is the vector with the ith component equal to 1 if $i \in D$ and 0 otherwise. The diagonal matrix $\Lambda_D := \operatorname{diag} \mathbf{1}_D$ in the matrix notations is a substitute for the indicator function when vectors on $\mathbf{R}^N$ are interpreted as a function on $\mathcal{N}$. Symbols $|.|_1$ and $|.|_\infty$ denote l^1-norm and l^∞-norm, respectively.

2.2 The Eisenberg–Noe model

2.2.1 Model formulation and existence of clearing vectors

In the paper [22], Eisenberg and Noe investigated the model describing a financial system composed of N banks (under "banks" can be understood various financial institutions). In the aggregate oversimplified form the balance sheet of the bank i can be split into to parts: assets and liabilities. The assets are of two types: interbank assets (exposures) $\tilde{X}^i$ and cash e^i. The liabilities are: interbank debts (liabilities) $\tilde{L}^i$ and the equity C^i (or proper capital reserve) equalizing the two sides of the balance sheet:

$$e^i + \tilde{X}^i = \tilde{L}^i + C^i.$$

All this values are assumed to be greater or equal to zero. The condition that $C^i \geq 0$ means that the bank is solvent.

More detailed balance sheet provides the information on the values of liabilities of the bank i to the bank j, namely, vectors $(L^{i1}, ..., L^{iN})'$ of and $(X^{i1}, ..., X^{iN})$ of exposures. With this we have $\tilde{X}^i = X^{i1} + ... + X^{iN}$ and $\tilde{L}^i = L^{i1} + ... + L^{iN}$.

The matrix $L = (L^{ij})$ with positive entries and zero diagonal defines the total interbank exposures. Since the value of the exposure of i to j is the value of the liability of j to i, we have that the transpose $L' = X$. So, the matrix L and the vector e gives a description of a financial system in this model.

Put

$$\Pi^{ij} := \frac{L^{ij}}{\tilde{L}^i} = \frac{L^{ij}}{\sum_j L^{ij}}, \quad \text{if } \ \tilde{L}^i \neq 0, \ \text{and } \Pi^{ij} := \delta^{ij} \text{ otherwise},$$

where the Kronecker symbol $\delta^{ij} = 0$ for $i \neq j$ and $\delta^{ii} = 1$. Then Π^{ij} describes the proportion of the value debtor i due to the creditor j of the total interbank debt of i; $\Pi = (\Pi^{ij})$ called relative liabilities matrix. Note that in this definition, to get a stochastic matrix Π, we deviate from in [22] where $\Pi^{ii} = 0$ when $L^i = 0$

In general, financial system may have a complicated structure with cyclical interdependences and banks may have large exposures within cycles. To reduce them one can impose a clearing mechanism satisfying several natural requirements: limited liability and proportionality. Formally, this leads to the concept of a *clearing payment vector* $p^* \in \prod_i [0, \tilde{L}^i]$ satisfying the following properties:

a. Limiting liability. For every i,

$$p_i^* \leq e^i + \sum_j \Pi^{ji} p_j^*.$$

b. Absolute priority. For every i, either $p_i^* = \tilde{L}^i$, or

$$p_i^* = e^i + \sum_j \Pi^{ji} p_j^*.$$

One may think that the central clearing authority forces each bank to make a "fair" payment of debts in such a way that, having the total payment p_i^*, the bank i remains solvent and pays to j the fraction $p_i^*\Pi^{ij}$ in such a way that either its total debts are paid or all the resources are exhausted.

Alternatively, the condition $a.$ and $b.$ can be written in the following way:

$$p^* = \min\left\{e + \Pi' p^*, \tilde{L}\right\}, \tag{2.2.1}$$

where the minimum is understood in the componentwise sense, i.e. accordingly to the partial ordering defined by the cone $\mathbf{R}^N_+$.

The main result of Eisenberg and Noe asserts that the set of clearing vectors is non-empty. Moreover, there are the minimal and the maximal clearing vectors, denoted here $\underline{p}$ and $\bar{p}$, respectively. This assertion follows immediately from the Knaster–Tarski fixed point theorem: the monotone mapping $f : p \mapsto (e + \Pi' p) \wedge \tilde{L}$ of a complete lattice $[0, \tilde{L}]$ into itself has the largest and the smallest fixed points, see Section 2.7 for details. The set $[0, \tilde{L}]$ is convex and compact and f is a continuous mapping. So, the existence of its fixed point follows also from the classical Brouwer theorem.

Using the obvious identity $(x - y)^+ = x - x \wedge y$ we can rewrite the equation (2.2.1) in the following equivalent form

$$(e + \Pi' p^* - \tilde{L})^+ = e + \Pi' p^* - p^* \tag{2.2.2}$$

where the left-hand side is the equity vector of the system after clearing.

An important but simple observation: *the equity does not depend on the clearing vector*. Indeed, P being a stochastic matrix, $\mathbf{1}'\Pi' = \mathbf{1}'$. Therefore, multiplying the above representation (2.2.2) from the left by $\mathbf{1}'$ we get that the sum of equities

$$\mathbf{1}'(e + \Pi' p^* - \tilde{L})^+ = \mathbf{1}' e$$

is equal to the sum of the initial cash reserves, that is invariant with respect to the choice of the clearing vector. On the other hand, by monotonicity, we have that

$$(e + \Pi' p^* - \tilde{L})^+ \le (e + \Pi' \bar{p} - \tilde{L})^+.$$

If the both side here are not equal, then $\mathbf{1}'(e+\Pi' p^* - \tilde{L})^+ < \mathbf{1}'(e+\Pi'\bar{p} - \tilde{L})^+$ in contradiction with the invariance of the total of equities.

2.2.2 On uniqueness of the clearing vector

As in [22] we shall assume that $\tilde{L}^i > 0$ for all i.

For a stochastic matrix Π, we say that $I \subseteq \{1, ..., N\}$ is a (Π-)*surplus set* if $\Pi^{ij} = 0$ for all $i \in I$, $j \in I^c$, and $\sum_{j\in I} e^j > 0$.

Recall that j is the creditor of i if $\Pi^{ij} > 0$ (i.e. $\Pi^{ij} > 0$); in this case we shall use, as in the theory of Markov chains or in the graph theory, the notation $i \to j$.

We denote by $o(i)$ *the orbit of* i that is the set of all j for which there is a directed path $i \to i_1 \to i_2 \to \dots \to j$, i.e. $o(i)$ is the set of all direct or indirect creditors of i.

Note that the orbit $o(i)$ with $\sum_{j\in I} e^j > 0$ is a surplus set. Indeed, if $\Pi^{jj'} > 0$ for some $j \in o(i)$, $j' \notin o(i)$, i.e. $j \to j'$, then there is a path $i \to i_1 \to i_2 \to \dots \to j \to j'$.

Lemma 2.2.1. Suppose that the market is cleared by a vector $p^* \in [0, \tilde{L}]$. Let I be a surplus set. Then at least one node of I has strictly positive equity value.

In particular, any orbit $o(i)$ with $\sum_{j\in o(i)} e^j > 0$ has an element with strictly positive equity value.

Proof. Multiplying the identity (2.2.2) by $\mathbf{1}'_I$ and noticing that $(\mathbf{1}'_I\Pi')^i = 1$ for $i \in I$, we obtain that

$$\mathbf{1}'_I(e + \Pi' p^* - \tilde{L})^+ \geq \mathbf{1}'_I e > 0$$

implying the claim. □

A financial system is called *regular* if for every i the orbit $o(i)$ is a surplus set.

Theorem 2.2.2. Suppose that the financial system is regular. Then $\underline{p} = \bar{p}$.

Proof. Suppose that $\underline{p}$ and $\bar{p}$ are not equal, i.e. $\underline{p} \leq \bar{p}$ but for some i we have the strict inequality $\underline{p}^i < \bar{p}^i$. We denote C the vector of equities (it is common for all clearing vectors). By assumption the orbit $o(i)$ is a surplus set and, by Lemma 2.2.1, it contains an element m with the equity value $C^m > 0$. By definition of the orbit, there is a path $i \to i_1 \to \dots \to m$ and we may assume without loss of generality that in this path m is the first node with strictly positive equity value.

First, we prove that we may consider only the case where the path consists of one step, i.e. $i \to m$. To this end, we check that $\underline{p}^{i_1} < \bar{p}^{i_1}$ if $i_1 \neq m$. In other words, the property that $\underline{p}^i \neq \bar{p}^i$ propagates along the path.

Suppose that $\bar{p}^{i_1} < \tilde{L}^{i_1}$. Then also $\underline{p}^{i_1} < \tilde{L}^{i_1}$. In such a case

$$\underline{p}^{i_1} = e^{i_1} + \sum_j \Pi^{ji_1}\underline{p}^j, \qquad \bar{p}^{i_1} = e^{i_1} + \sum_j \Pi^{ji_1}\bar{p}^j,$$

and we have that

$$\bar{p}^{i_1} - \underline{p}^{i_1} = \sum_j \Pi^{ji_1}(\bar{p}^j - \underline{p}^j) > 0$$

because $\Pi^{ii_1} > 0$. That is, $\underline{p}^{i_1} < \bar{p}^{i_1}$. This inequality also holds trivially, if $\bar{p}^{i_1} = \tilde{L}^{i_1}$ but $\underline{p}^{i_1} < \tilde{L}^{i_1}$. The remaining case where $\underline{p}^{i_1} = \bar{p}^{i_1} = \tilde{L}^{i_1}$ is excluded as we suppose that $C^{i_1} = 0$. Indeed, accordingly to (2.2.2), this leads to the equalities

$$e^{i_1} + \sum_j \Pi^{ji_1}\bar{p}^j - \tilde{L}^{i_1} = 0, \qquad e^{i_1} + \sum_j \Pi^{ji_1}\underline{p}^j - \tilde{L}^{i_1} = 0,$$

implying the identity

$$\sum_j \Pi^{ji_1}(\bar{p}^j - \underline{p}^j) = 0$$

which cannot be true since in the above sum the term corresponding to $j = i$ is strictly positive.

So, it is sufficient to consider only one-step case. Since $C^m > 0$ we have the representations

$$C^m = e^m + \sum_j \Pi^{jm} \underline{p}^j - \tilde{L}^m, \qquad C^m = e^m + \sum_j \Pi^{jm} \bar{p}^j - \tilde{L}^m.$$

As above, we again obtain the impossible equality

$$\sum_j \Pi^{jm} (\bar{p}^j - \underline{p}^j) = 0.$$

Therefore, the assumption $\underline{p}^i < \bar{p}^i$ leads to a contradiction. The uniqueness of clearing vector is proven. □

Remark 2.1. The above theorem reveals that the problem to find a clearing vector is ill-posed. Indeed, adding an infinitesimally small amount $\varepsilon > 0$ (say, one cent) to the initial endowments leads to a unique clearing vector. Similar effect will have small a increase in liabilities. One can think that the "true" liability matrix has all elements strictly positive and the in the model matrix zero elements appeared because liabilities are neglected. These phenomena are related to the ill-posedness of the spectral problem for stochastic matrices. Another question is which clearing vector is natural.

The above proof is rather straightforward and is based on the graph-theoretical approach. One can get another one appealing to the contraction property of the mapping $f : p \mapsto (e + \Pi' p) \wedge \tilde{L}$ defined on the set $[0, \tilde{L}]$ equipped with l_1-distance $|p - \tilde{p}|_1$. We give here only a sketch of it.

Proposition 2.2. *For every* $p, \tilde{p} \in [0, \tilde{L}]$

$$|f(p) - f(\tilde{p})|_1 \le |\Pi' p - \tilde{p}|_1 \le |p - \tilde{p}|_1. \tag{2.2.3}$$

Moreover, the first relation above is the equality if and only if the union of two subsets $A := \{i : \ (\Pi' p)^i = (\Pi' \tilde{p})^i\}$ *and* $B := \{i : \ (\Pi' p)^i, (\Pi' \tilde{p})^i \le \tilde{L}^i - e^i\}$ *of the set of indices* $\{1, \dots, N\}$.

Proof. Using the elementary inequality $|a \wedge c - b \wedge c| \le |a - b|$ which holds as the equality if and only if when $a = b$ or $a, b \le c$ we obtain that $|f(p) - f(\tilde{p})|_1 \le |\Pi' p - \Pi' \tilde{p}|_1$ where the equality holds if and only if for every i we have $(\Pi' p)^i = (\Pi' \tilde{p})^i$ or $(\Pi' p)^i, (\Pi' \tilde{p})^i \le \tilde{L}^i - e^i$. Since $|\Pi' y|_1 \le |\Pi'|_1 |y|_1$ and $|\Pi'|_1 = 1$, we have the claim. □

Let us consider the case where the matrix Π is irreducible. Suppose that $\mathbf{1}' e > 0$ and p and $\tilde{p}$ are two different fixed points of the mapping f. According to above proposition

$$\sum_{j \in B} \Pi^{ji} (\bar{p}^j - \underline{p}^j) = p^i - \tilde{p}^i, \quad i \in B.$$

This means that the non-zero vector with the coordinates $p^i - \tilde{p}^i$, $i \in B$, is a left eigenvector of the matrix $(\Pi^{ij})_{i,j\in B}$ corresponding to unit eigenvalue. This is possible only if the latter matrix coincides with Π. Thus, $p = f(p) = e + \Pi' p$. Since $\mathbf{1}'\Pi' p = \mathbf{1}' p$ we get that $\mathbf{1}' e = 0$ which is a contradiction. Using the decomposition of the matrix Π on the irreducible component, we get that that the clearing vector is unique if for any irreducible component there is a node with strictly positive initial endowment.

2.2.3 Computing clearing vectors

There are various procedures of calculations of the clearing vectors. For example, the vector $\bar{p}$ can be obtained by the iterative procedure $p_n = f(p_{n-1})$, $n \geq 1$, which starts from $p_0 = l$. Indeed, since f is monotone we get easily that $\bar{p} \leq p_{n+1} \leq p_n$; the decreasing bounded sequence p_n has a limit point $p_\infty \in [\bar{p}, l]$. The continuity of f implies that $p_\infty = f(p_\infty)$. Since $\bar{p}$ is the largest fixed point, $p_\infty = \bar{p}$. The same procedure but starting from the zero vector provides a sequence converging to $\underline{p}$. The disadvantage of above procedure is that to reach the limit it needs, in general, infinite number of iterations. In [22] ir was introduced Fictitious Default algorithm which allows to obtain the clearing vector (supposed to be unique) at $N+1$ steps at most. In this subsection we describe an algorithm which is a modification pf the classical Gauss elimination algorithm which also allows to compute clearing vectors in finite number of steps. To explain its idea we recall the Gauss elimination algorithm when the linear system is written not in the traditional form as $Ap = e$ but as a fixpoint problem: $p = e + (I - A)p$. To approach our setting we suppose that $I - A = \Pi'$.

Let $D \neq \emptyset$ be a proper subset of $\mathcal{N}$. Changing the numbering we may assume without loss of generality that $D := \{1, \dots, m\}$, $1 \leq m < N$. We introduce the notations $\Pi_D := (\Pi^{ij})_{i,j\in D}$, $\Pi_{D^c} = (\Pi^{ij})_{i,j\in D^c}$, $e_D := (e^i)_{i\in D}$, etc. Supposing that p solves the equation $p = e + \Pi' p$, we rewrite the latter in the form

$$\begin{pmatrix} p_D \\ p_{D^c} \end{pmatrix} = \begin{pmatrix} e_D \\ e_{D^c} \end{pmatrix} + \begin{pmatrix} \Pi'_D & R' \\ T' & \Pi'_{D^c} \end{pmatrix} \begin{pmatrix} p_D \\ p_{D^c} \end{pmatrix}. \tag{2.2.4}$$

Thus, we have that

$$p_D = e_D + \Pi'_D p_D + R' p_{D^c}, \tag{2.2.5}$$

$$p_{D^c} = e_{D^c} + T' p_D + \Pi'_{D^c} p_{D^c}. \tag{2.2.6}$$

Suppose that the matrix $I_m - \Pi_D$ is invertible. Substituting in (2.2.6) the expression for p_D from (2.2.5) we obtain that the vector $\pi_1 := \pi_{D^c} \in \mathbf{R}^{N-m}$ solves the equation

$$p_1 = e_1 + \Pi'_1 p_1, \tag{2.2.7}$$

where

$$e_1 := e_{D^c} + (I_m - \Pi'_D)^{-1} T' e_D, \tag{2.2.8}$$

$$\Pi_1 := R(I_m - \Pi_D)^{-1} T + \Pi_{D^c}. \tag{2.2.9}$$

It is easily seen that Π_1 is a stochastic matrix. The equation (2.2.7) is of the same type as the initial one but of a lower dimension and its solution, via (2.2.5), gives us the solution of the former. Of course, for $m = 1$ the reduction to a lower dimension described above is nothing but the Gauss elimination algorithm for solving linear equation in $\mathbf{R}^N$.

As was observed by Sonin in a different context, namely of the Bellman equations arising in the optimal stopping theory, see [48] and references therein, the elimination algorithm can be modified to solve the fixpoint problem of the type $p = (e + \Pi' p) \wedge \tilde{L}$, even with an arbitrary matrix. Indeed, if the set

$$\{i \in \mathcal{N}:\ e^i + (\Pi'\tilde{L})^i < \tilde{L}^i\} = \emptyset,$$

then the problem is solved with $p = \tilde{L}$. If this set is non-empty take its proper subset D. Without loss of generality we may assume that $D = \{1, \dots, m\}$. In an analogy with (2.2.5), (2.2.6) we get that

$$\begin{aligned} p_D &= (e_D + \Pi'_D p_D + R' p_{D^c}) \wedge \tilde{L}_D, && (2.2.10)\\ p_{D^c} &= (e_{D^c} + T' p_D + \Pi'_{D^c} p_{D^c}) \wedge \tilde{L}_{D^c}. && (2.2.11) \end{aligned}$$

By definition of D the first equation is linear: it is the same as (2.2.5). Thus, if the matrix $I_m - \Pi_D$ is invertible, then $p_1 := p_{D^c}$ solves the equation of lower dimension

$$p_1 = (e_1 + \Pi'_1 p_1) \wedge \tilde{L}_{D^c}, \qquad (2.2.12)$$

with e_1 and Π_1 given by (2.2.8) and (2.2.9).

As in the classical Gauss algorithm we can take $D = \{1\}$ and eliminate p^1 reducing the problem to the search of the vector $(p^2, \dots, p^N)$ satisfying the equation of the same type.

2.3 The Suzuki–Elsinger model with crossholdings

2.3.1 Existence of equilibrium

Now we consider a version of the Suzuki–Elsinger model, [49], [26], with crossholdings defined by a substochastic matrix $\Theta = (\theta^{ij})$ where $\theta^{ij} \in [0, 1]$ is a share of the bank i held by the bank j.

In this model the clearing vector and the equity vectors are interdependent and the problem formulated in the spirit of equilibrium problem, that is as a simultaneous search of both vectors satisfying an equation in $\mathbf{R}^{2N}$. The latter can be presented in several equivalent forms.

We assume as a standing hypothesis that there is no group composed by banks owned completely by banks of this group, that is the condition:

H. There is no subset $A \subseteq \{1, \dots, N\}$ such that $\mathbf{1}'_A \Theta = \mathbf{1}'_A$.

The advantage of the above formulation of the hypothesis playing an essential role is in its economic interpretation. For our purposes it is more convenient its equivalent form: $x \to \Theta' x$ is a contraction in the space $\mathbf{R}^N$ equipped by the l^∞-norm (or max-norm) $|x|_\infty = \max_i |x^i|$, i.e. that $|\Theta'|_\infty < 1$. Equivalently, this means that $|\Theta|_1 < 1$, where the l^1-norm of x is $|x|_1 = \sum_i |x^i|$, i.e. that $|\Theta|_1 < 1$. In turns, the latter property is equivalent to the property that unit is not an eigenvalue of Θ etc.

Lemma 2.3.1. For any $x \in \mathbf{R}^N$ the equations

$$v = (x + \Theta' v)^+, \tag{2.3.13}$$
$$w = x + \Theta' w^+ \tag{2.3.14}$$

have unique solutions $v = v(x) \in \mathbf{R}^N_+$ and $w = w(x) \in \mathbf{R}^N$.

The mappings $x \mapsto v(x)$ and $x \mapsto w(x)$ are Lipschitz and order preserving.

Proof. Using the elementary inequality $|a^+ - b^+| \le |a - b|$ we have:

$$|(x + \Theta' v)^+ - (x + \Theta' \tilde{v})^+|_\infty \le |\Theta'(v - \tilde{v})|_\infty \le |\Theta'|_\infty |v - \tilde{v}|_\infty.$$

Since $|\Theta'|_\infty < 1$, the right-hand side of (2.3.13) defines a contraction mapping in $(\mathbf{R}^N, |.|_\infty)$ (depending on the parameter x) which has a fixed point $v = v(x)$ which is unique. Also,

$$|v(x) - v(y)|_\infty = |(x + \Theta' v(x))^+ - (y + \Theta' v(y))^+|_\infty \le |x - y|_\infty + |\Theta'|_\infty |v(x) - v(y)|_\infty.$$

It follows, that

$$|v(x) - v(y)|_\infty \le (1 - |\Theta'|_\infty)^{-1} |x - y|_\infty.$$

The similar arguments show that $x \mapsto w(x)$ is Lipschitz.

Let $\Delta := w(x + h) - w(x)$ where $h \in \mathbf{R}^N_+$. Put $A := \{i\colon\ \Delta^i < 0\}$ and define the diagonal matrix $\Lambda := \operatorname{diag} \mathbf{1}_A$. The inequality $a < b$ implies that $a^+ - b^+ \ge a - b$, the inequality $a \ge b$ implies that $a^+ - b^+ \ge 0$. Therefore,

$$\Theta'(w^+(x + h) - w^+(x)) \ge \Theta' \Lambda \Delta$$

and

$$\Lambda \Delta = \Lambda h + \Lambda \Theta' \Lambda (w^+(x + h) - w^+(x)) \ge \Lambda h + \Lambda \Theta' \Lambda \Delta.$$

Regrouping terms and summing up the components we get that

$$\mathbf{1}' \Lambda (I - \Theta') \Lambda \Delta \ge \mathbf{1} \Lambda h \ge 0.$$

If $A \ne \emptyset$, we arrive to a contradiction since the lhs above is

$$\sum_{j \in A} \Delta^j - \sum_{j \in A} \Big(\sum_{i \in A} \theta^{ij} \Big) \Delta^j < 0 \tag{2.3.15}$$

in virtue of the hypothesis $\mathbf{H}$: all sums in parentheses are less than unit and at least one should be strictly less. Thus, $A = \emptyset$, i.e. $w(x)$ is order preserving.

Let $h \in \mathbf{R}_+$, $\Delta := v(x+h) - v(x)$, $A = \{i: \ \Delta^i < 0\}$,

$$\begin{aligned} B_1 &:= \{i: \ x^i + (\Theta' v(x))^i > 0\}, \\ B_2 &:= \{i: \ x^i + (\Theta' v(x))^i < 0\}, \\ B_3 &:= \{i: \ x^i + (\Theta' v(x))^i = 0\}. \end{aligned}$$

Define the diagonal matrices $\Lambda := \operatorname{diag} \mathbf{1}_A$ and $\Lambda_k = \operatorname{diag} \mathbf{1}_{B_k}$, $k = 1, 2, 3$.

Note that for $i \in B_2$ we have obviously $v^i(x+h) \geq v^i(x) = 0$. Moreover,

$$\Delta^i = v^i(x+h) = (x^i + h^i(\Theta' v(x+h))^i)^+ = 0$$

when $|h|$ is sufficiently small. For $i \in B_3$ we have $\Delta^i = v^i(x+h) \geq 0$. For $i \in B_1$ we have $x^i + h + (\Theta' v(x+h))^i > 0$ when $|h|$ is sufficiently small and, therefore,

$$\Lambda_1 \Delta = \Lambda_1 (h + \Theta' \Lambda_1 \Delta + \Theta' \Lambda_3 \Delta) \geq \Lambda_1 (h + \Theta' \Lambda_1 \Delta).$$

Since $A \subseteq B_1$ we get that $\Lambda \Delta \geq \Lambda h + \Lambda \Theta' \Lambda \Delta)$ and as above $A = \emptyset$. $\Box$

We consider the following system of equations whose set of solutions will be denoted by $\Gamma_1 \subseteq [0, \tilde{L}] \times \mathbf{R}^N_+$:

$$\begin{aligned} p &= (e + \Pi' p + \Theta' V)^+ \wedge \tilde{L}, && (2.3.16) \\ V &= (e + \Pi' p - p + \Theta' V)^+. && (2.3.17) \end{aligned}$$

For $(p, V) \in \Gamma_1$ the components p and V are called, respectively, *clearing vector* and *equity*.

Accordingly to Lemma 2.3.1 for every p the equation (2.3.17) admits a unique solution, namely, $V(p) := v(e + \Pi' p - p)$ which is Lipschitz in p. Thus, the equation

$$p = (e + \Pi' p + \Theta' V(p))^+ \wedge \tilde{L} \qquad (2.3.18)$$

has a solution in virtue of the Brouwer theorem claiming that a continuous mapping (given by the left-hand side above) of a continuous compact set ($[0, \tilde{L}]$ in our case) has a fixed point. So,

$$\Gamma_1 = \{(p, V(p)) : \ p \text{ solves } (2.3.18)\} \neq \emptyset.$$

We also introduce the systems

$$\begin{aligned} p &= (e + \Pi' p + \Theta' U)^+ \wedge \tilde{L}, && (2.3.19) \\ U &= (e + \Pi' p - \tilde{L} + \Theta' U)^+ && (2.3.20) \end{aligned}$$

with the set of solutions $\Gamma_2 \subseteq [0, \tilde{L}] \times \mathbf{R}^N_+$ and the system

$$\begin{aligned} p &= (e + \Pi' p + \Theta' W^+)^+ \wedge \tilde{L}, && (2.3.21) \\ W &= e + \Pi' p - \tilde{L} + \Theta' W^+ && (2.3.22) \end{aligned}$$

with the set of solutions $\Gamma_3 \subseteq [0, \tilde{L}] \times \mathbf{R}^N$.

Introducing the equation

$$p = (e + \Pi' p + \Theta' U(p))^+ \wedge \tilde{L} \tag{2.3.23}$$

and using Lemma 2.3.1 we can prove that

$$\Gamma_2 = \{(p, U(p)) \colon \ p \text{ solves } (2.3.23)\} \neq \emptyset,$$

where $U(p) := v(e + \Pi' p - \tilde{L})$. Since the latter function is monotone, we can apply to the equation (2.3.23) the Knaster–Tarski theorem providing additional useful information: there exists the smallest $\underline{p}$ and the largest $\bar{p}$ solutions of (2.3.23). The monotonicity of $U(p)$ allows to conclude that Γ_2 has the minimal and maximal elements, namely, $(\underline{p}, U(\underline{p}))$ and $(\bar{p}, U(\bar{p}))$.

In the same way, introducing the equation

$$p = (e + \Pi' p + \Theta' W^+(p))^+ \wedge \tilde{L} \tag{2.3.24}$$

and defining the function $W(p) = w(e + \Pi' p - \tilde{L})$, we prove that the set

$$\Gamma_3 = \{(p, W(p)) \colon \ p \text{ solves } (2.3.24)\}$$

contains the minimal and maximal elements $(\underline{p}, W(\underline{p}))$ and $(\bar{p}, W(\bar{p}))$.

It remains to show that Γ_1 also has the minimal and maximal elements and establish the relations between all these sets.

Let us introduce the function $\varphi : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N \times \mathbf{R}^N_+$ with $\varphi(x, y) := (x, y^+)$.

Lemma 2.3.2. $\Gamma_1 = \Gamma_2 = \varphi(\Gamma_3)$.

Proof. $(\Gamma_1 \subseteq \Gamma_2)$ Let $(p, V(p)) \in \Gamma_1$. If $V^i(p) > 0$, then $(e + \Pi' p - p + \Theta' V(p))^i > 0$. Rewriting the last inequality as $p^i > (e + \Pi' p + \Theta' V(p))^i$ we obtain in view of (2.3.18) that $p^i = \tilde{L}^i$. Thus, for such i we have that

$$V^i(p) = ((e + \Pi' p - \tilde{L} + \Theta' V(p))^i)^+.$$

If $V^i(p) = 0$, then the above equality holds trivially due to (2.3.18). That is, $V(p)$ solves the equation (2.3.20) for U. Hence, due to the uniqueness of solution, $V(p) = U(p)$ and $(p, V(p)) \in \Gamma_2$.

$(\Gamma_2 \subseteq \Gamma_1)$ Let $(p, U(p)) \in \Gamma_2$. If $0 \le p^i < \tilde{L}^i$, then, accordingly to equation (2.3.23), $p^i = ((e + \Pi' p + \Theta' U(p))^i)^+$, implying via (2.3.20) that $U^i(p) = 0$ and

$$U^i(p) = ((e + \Pi' p - p + (\Theta' U(p))^i))^+.$$

If $p^i = \tilde{L}^i$ this equality follows directly from the definition of $U(p)$. Thus, $U(p)$ solves (2.3.17) and, therefore, coincides with $V(p)$. But this means that $(p, U(p)) \in \Gamma_2$.

($\varphi(\Gamma_3) \subseteq \Gamma_2$) Let $(p, W(p)) \in \Gamma_3$. By definition, $W(p)$ satisfies the equation (2.3.22). Taking the positive part of both sides of this equation, we obtain that $W^+(p)$ satisfies the equation (2.3.20). Hence, $(p, W^+(p)) \in \Gamma_2$.

($\Gamma_2 \subseteq \varphi(\Gamma_3)$) Let $(p, U(p)) \in \Gamma_2$. Note that

$$W(p) := e + \Pi' p - \tilde{L} + \Theta' U(p) \leq (e + \Pi' p - \tilde{L} + \Theta' U(p))^+ = U(p)$$

and $W^i(p) = U^i(p)$ if $W^i(p) \geq 0$. Thus, $W^+(p) = U(p)$. It follows that $W(p)$ solves (2.3.22) and $(p, W(p)) \in \Gamma_3$. □

2.3.2 Uniqueness

In the sequel we use the abbreviations: $\underline{V} := V(\underline{p})$, $\bar{V} := V(\bar{p})$,

$$\Delta_p := \bar{p} - \underline{p} \geq 0, \qquad \Delta_V := \bar{V} - \underline{V} \geq 0,$$

$A_p := \{i\colon\ \Delta_p > 0\}$, $A_V := \{i\colon\ \Delta_V > 0\}$. We define the diagonal matrices

$$\Lambda_p := \operatorname{diag} \mathbf{1}_{A_p}, \qquad \Lambda_V := \operatorname{diag} \mathbf{1}_{A_V}, \qquad \Lambda := \operatorname{diag} \mathbf{1}_{A_p \cup A_V}.$$

Lemma 2.3.3. The following identities hold:

$$\Lambda(I - \Theta')\Lambda_V = 0, \qquad \Lambda(\Pi' - I)\Lambda_p = 0. \tag{2.3.25}$$

Proof. Note that $\underline{V} \geq e + \Pi \underline{p} - \underline{p} + \Theta' \underline{V}$ and for each $i \in A_p \cup A_V$ necessarily $\bar{p}^i > 0$ and $\bar{V}^i = e^i + (\Pi' \bar{p})^i - \bar{p}^i + (\Theta' \bar{p})^{\bar{i}}$. Thus,

$$\Lambda \Delta_V \leq \Lambda(\Pi' - I)\Delta_p + \Lambda \Theta' \Delta_V.$$

Taking into account that $\Delta_V = \Lambda \Delta_V$ and $\Delta_p = \Lambda \Delta_p$, we get from here that

$$\mathbf{1}' \Lambda (I - \Theta') \Lambda \Delta_V \leq \mathbf{1}' \Lambda (\Pi' - I) \Lambda \Delta_p.$$

Inspecting the explicit expressions (similar to that in (2.3.15)) we conclude that the left-hand side above is less or equal to zero while the right-hand side is greater or equal to zero. So,

$$\mathbf{1}' \Lambda (I - \Theta') \Lambda \Delta_V = 0, \qquad \mathbf{1}' \Lambda (\Pi' - I) \Lambda \Delta_p = 0. \tag{2.3.26}$$

Since $\Delta_V^i > 0$ on A_V and $\Delta_p^i > 0$ on A_p, these equalities are equivalent to (2.3.25). □

Theorem 2.3.4. Suppose that for any subset of indices $A \neq \emptyset$ there exits $j \in A$ such that

$$\sum_{i \in A} \theta^{ij} < 1, \qquad \sum_{i \in A} \Pi^{ij} < 1.$$

Then $(\underline{p}, V(\underline{p})) = (\bar{p}, V(\bar{p}))$.

Proof. The identities (2.3.26) (equivalent to (2.3.26)) can be written as

$$\begin{aligned}\sum_{j\in A_V}\Delta_V^j-\sum_{j\in A_V}\Big(\sum_{i\in A_V\cup A_p}\theta^{ij}\Big)\Delta_V^j &= 0,\\ \sum_{j\in A_p}\Delta_p^j-\sum_{j\in A_p}\Big(\sum_{i\in A_V\cup A_p}\Pi^{ij}\Big)\Delta_p^j &= 0.\end{aligned}$$

Applying the assumption with $A=A_p\cup A_V$ we get the result. □

Theorem 2.3.5. Suppose that for any subset of indices A such that for all $i\in A$

$$\sum_{j\in A}\theta^{ij}=1 \quad \text{or} \quad \sum_{j\in A}\Pi^{ij}=1$$

it holds that

$$\sum_{i\in A}e^i>\sum_{i\in A}\Big(1-\sum_{j\in A}\Pi^{ij}\Big)\tilde{L}^i.$$

Then the clearing vector is unique. In particular, for the Eisenberg–Noe model where $\Theta=0$, if any subset of indices A such that $\sum_{j\in A}\Pi^{ij}=1$ for all $i\in A$ we have that $\sum_{i\in A}e^i>0$, then the clearing vector is unique.

Proof. We start from the equality

$$\Lambda\bar{V}=\Lambda(e+(\Pi'-I)\bar{p}+\Theta'\bar{V}).$$

Regrouping terms and multiplying from the left by $\mathbf{1}'$ we obtain the identity

$$\mathbf{1}'\Lambda(I-\Theta')\Lambda\bar{V}+\mathbf{1}'\Lambda(I-\Pi')\Lambda\bar{p}=\mathbf{1}'\Lambda e+\mathbf{1}'\Lambda\Pi'(I-\Lambda)\bar{p}+\mathbf{1}'\Lambda\Theta'(I-\Lambda)\bar{V}.$$

Note that $\bar{V}^i=0$ for $i\in A_p\setminus A_V$. Therefore, $(\Lambda-\Lambda_V)\bar{V}=0$. Combining with (2.3.25) we conclude that the first term in the left-hand side of the identity is zero.

If $i\in A_V\setminus A_p$, that is, $\bar{p}^i=\underline{p}^i$ and $\bar{V}^i>\underline{V}^i\geq 0$, then in virtue of definitions, $\bar{V}^i=e^i+(\Pi'\bar{p})^i-\bar{p}^i+(\Theta'V(\bar{p}))^i>0$ implying, via (2.3.18), that $\bar{p}^i=\tilde{L}^i$. Thus,

$$(\Lambda-\Lambda_p)\bar{p}=(\Lambda-\Lambda_p)\tilde{L}. \tag{2.3.27}$$

Using the second relation in (2.3.25) we obtain that the second term in the left-hand side of the identity is equal to $\mathbf{1}'\Lambda(I-\Pi')\Lambda\tilde{L}$. So,

$$\mathbf{1}'\Lambda(I-\Pi')\Lambda l=\mathbf{1}'\Lambda e+\mathbf{1}'\Pi'(I-\Lambda)\bar{p}+\mathbf{1}'\Theta'(I-\Lambda)\bar{V}\geq\mathbf{1}'\Lambda e.$$

That is,

$$\sum_{i\in A_p\cup A_V}\Big(1-\sum_{j\in A_p\cup A_V}\Pi^{ij}\Big)\tilde{L}^i\leq\sum_{i\in A_p\cup A_V}e^i.$$

Applying the assumption with $A=A_p\cup A_V$ we get the result. □

Remark. In the early paper by Suzuki the model was analyzed using a different approach. In the above notations the equations (2), (3) of [49], with the positive part (2), can be written as

$$\begin{aligned} p &= (e + \Pi' p + \Theta' V)^+ \wedge \tilde{L} =: g_1(p, V), \\ V &= (e + \Pi' p - \tilde{L} + \Theta' V)^+ =: g_2(p, V), \end{aligned}$$

where the second equation is the equation for W^+. If $\lambda := |\Pi'|_1 \vee |\Theta'|_1 < 1$, then the mapping $(p, V) \mapsto g(p, V)$ is a contraction in $(\mathbf{R}^{2N}_+, |.|_1)$. Indeed, the elementary inequality

$$|a^+ \wedge c - b^+ \wedge c| + |(a-c)^+ - (b-c)^+| \le |a - b|, \qquad a, b \in \mathbf{R},\ c \in \mathbf{R}_+,$$

implies that

$$\begin{aligned} |g(p, V) - g(\tilde{p}, V)|_1 &\le |\Pi'(p - \tilde{p})|_1 + |\Theta'(V - \tilde{V})|_1 \le |\Pi'|_1 |p - \tilde{p}|_1 + |\Theta'|_1 |V - \tilde{V}|_1 \\ &\le \lambda(|p - \tilde{p}|_1 + |V - \tilde{V}|_1). \end{aligned}$$

With this the existence and uniqueness of equilibrium is obvious.

2.4 The Elsinger model

We consider a version of the Elsinger model where the interbank debts may be senior and junior. In this model the system of N banks is described by the vector of cash reserves and by M matrices $L_1 = (L_1^{ij})$, ..., $L_M = (L_M^{ij})$ representing the hierarchy of liabilities with decreasing seniority. That is, the element L_1^{ij} represents the debt of the bank i to the bank j of the highest seniority etc., $\sum_j L_S^{ij}$ is the total of debts of the bank i of the seniority S.

The relative liabilities are defined by the matrix Π_S with

$$\Pi_S^{ij} = \frac{L_S^{ij}}{\tilde{L}_S^i} = \frac{L_S^{ij}}{\sum_j L_S^{ij}}.$$

The clearing procedure requires the complete reimbursement of the debts starting from the highest priority and, for each seniority level, the distribution is proportional to the volume of debts of this seniority. For the bank i we denote by p_S^i the value distributed to cover the debts of the seniority S. So, the clearing can be described by the set of vectors p_S, $S = 1, \dots, M$, which can be considered as a "long" vector from $(\mathbf{R}^N)^M$ satisfying the system of equations

$$\begin{aligned} p_1^i &= \min\Big\{ e^i + \sum_S \sum_j \Pi_S^{ji} p_S^j, \tilde{L}_1^i \Big\}, \\ p_S^i &= \min\Big\{ \Big(e^i + \sum_S \sum_j \Pi_S^{ji} p_S^j - \sum_{r<S} \tilde{L}_r^i \Big)^+, \tilde{L}_S^i \Big\}, \qquad 1 < S \le M. \end{aligned}$$

In a vector form these equations can be written as follows:

$$p_S = \Big(e + \sum_S \Pi'_S p_S - \sum_{r<S} \tilde{L}_r\Big)^+ \wedge \tilde{L}_S, \qquad S = 1, ..., M. \tag{2.4.28}$$

It is clear that, for the partial ordering in $(\mathbf{R}^N)^M$ induced by the cone $(\mathbf{R}^N_+)^M$, the function

$$(p_1, ..., p_M) \mapsto \left(\Big(e + \sum_S \Pi'_S p^*_S\Big)^+ \wedge \tilde{L}_1, ..., \Big(e + \sum_S \Pi'_S p^*_S - \sum_{r<M} \tilde{L}_r\Big)^+ \wedge L_M\Big\}\right)$$

is a monotone mapping of the order interval $[0, \tilde{L}_1] \times ... \times [0, \tilde{L}_M] \subset (\mathbf{R}^N)^M$ into itself. Thus, according to the Knaster–Tarski theorem the set of fixed points of this mapping, i.e. the solutions of the equation (2.4.28), is non-empty and has the maximal and the minimal elements.

In the case of liabilities of different seniority after clearing by the vector $p \in (\mathbf{R}^N)^M$ the equity vector $C \in \mathbf{R}^N$ has the form:

$$C = \Big(e + \sum_S \Pi'_S p_S - \sum_S \tilde{L}_S\Big)^+.$$

Lemma 2.4.1. The equity vector does not depend on the clearing vector.

Proof. Note that

$$\Big(e + \sum_S \Pi'_S p_S\Big) \wedge \sum_S \tilde{L}^i_S = \sum_S p_S.$$

Therefore,

$$\Big(e + \sum_S \Pi'_S p_S - \sum_S \tilde{L}_S\Big)^+ = e + \sum_S \Pi'_S p_S - \sum_S p_S.$$

With this identity the reasoning is analogous to that with a single seniority class. □

The aim of this section is to provide a sufficient condition for the uniqueness of clearing vector using a specific graph structure induced by the matrices Π_S.

For a given clearing vector p we define the *default index* d^i of the node i as the smallest r such that

$$\bar{p}^i_r = e^i + \sum_S \sum_j \Pi^{ji}_S \bar{p}^j_S - \sum_{r'<r} \tilde{L}^i_{r'}.$$

In another words, d^i is the lowest seniority for which the bank equity after clearing is equal to zero. Define the matrix $\Delta = \Delta(p)$ by putting $\Delta^{ij} = 1$ if $\Pi^{ij}_{d(i)} > 0$, and $\Delta^{ij} = 0$ otherwise. We use the notation $i \rightsquigarrow j$ if $\Delta^{ij} = 1$ and denote by $O(i)$ *the* Δ*-orbit of* i, that is the set of all j for which there is a directed path $i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow ... \rightsquigarrow j$.

Theorem 2.4.2. Suppose that for the clearing vector $\bar{p}$ any Δ-orbit is a surplus set. Then the clearing vector is unique.

Proof. By definition, the default index

$$d^i := \min\Big\{r :\ \bar{p}^i_r = e^i + \sum_S\sum_j \Pi^{ji}_S \bar{p}^j_S - \sum_{r'<r} \tilde{L}^i_{r'}\Big\}.$$

It follows that $\bar{p}^i_r = 0$, hence, $\underline{p}^i_r = 0$ for every $r > d^i$. Suppose that $\underline{p}^i_r < \bar{p}^i_r$ and consider a path

$$i \rightsquigarrow i_1 \rightsquigarrow i_2 \rightsquigarrow \dots \rightsquigarrow m$$

ending up at the node with strictly positive equity value.

First, we show that at least for one seniority $\underline{p}^{i_1}_S < \bar{p}^{i_1}_S$.

Let $r' := d^{i_1}$. By definition we have: $\bar{p}^{i_1}_r = \tilde{L}^{i_1}_r$, $r \le r'$, and $\bar{p}^{i_1}_r = \underline{p}^{i_1}_r = 0$, $r > r'$. The claim holds, if $\underline{p}^{i_1}_r < \tilde{L}^{i_1}_r$ for some $r < r'$. Thus, it remains to consider only the case where $\underline{p}^{i_1}_r = \bar{p}^{i_1}_r = \tilde{L}^{i_1}_r$ for all $r < r'$ and prove that $\underline{p}^{i_1}_{r'} < \bar{p}^{i_1}_{r'}$. We have the alternative: either $\underline{p}^{i_1}_{r'} < \bar{p}^{i_1}_{r'} \le \tilde{L}^{i_1}_r$ (what we need), or $\underline{p}^{i_1}_{r'} = \bar{p}^{i_1}_{r'} \le \tilde{L}^{i_1}_r$. The second case is impossible, since the equalities

$$\begin{aligned}
\bar{p}^{i_1}_{r'} &= e^{i_1} + \sum_S\sum_j \Pi^{ji_1}_S \bar{p}^j_S - \sum_{r<r'} \tilde{L}^{i_1}_r, \\
\underline{p}^{i_1}_{r'} &= e^{i_1} + \sum_S\sum_j \Pi^{ji_1}_S \underline{p}^j_S - \sum_{r<r'} \tilde{L}^{i_1}_r.
\end{aligned}$$

imply that

$$\bar{p}^{i_1}_{r'} - \underline{p}^{i_1}_{r'} = \sum_S\sum_j \Pi^{ji_1}_S(\bar{p}^j_S - \underline{p}^j_S) \ge \Pi^{ii_1}_r(\bar{p}^i_r - \underline{p}^i_r) > 0.$$

This is contradiction.

The above argument reduces the problem to the case $i \rightsquigarrow m$ and the node m has a strictly positive equity. The equity C^m does not depend on the clearing vector. Therefore,

$$\begin{aligned}
C^m &= e^m + \sum_S\sum_j \Pi^{jm}_S \bar{p}^j_S - \sum_S \tilde{L}^m_S, \\
C^m &= e^m + \sum_S\sum_j \Pi^{jm}_S \underline{p}^j_S - \sum_S \tilde{L}^m_S.
\end{aligned}$$

It follows that

$$0 = \sum_S\sum_j \Pi^{jm}_S(\bar{p}^j_S - \underline{p}^j_S) \ge \Pi^{im}_r(\bar{p}^i_r - \underline{p}^i_r) > 0.$$

This contradiction shows that $\underline{p} = \bar{p}$.

2.4.1 Example 1

Let us consider the system consisting of 3 nodes with the initial cash endowments given by the vector $e = (0.1, 0, 0)$ and the liability and the "distribution" matrices corresponding senior and junior debts:

$$L_S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \quad L_J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi_S = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Pi_J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this model the vectors of total liabilities corresponding to the senior and junior debts are, respectively, $\tilde{L}_S = (1, 2, 2)$ and $\tilde{L}_J = (0, 2, 0)$.

The equations for clearing vectors are:

$$\begin{aligned} p_S^1 &= (0.1 + 0.5\, p_S^2) \wedge 1, \\ p_S^2 &= (p_S^1 + p_S^3) \wedge 2, \\ p_S^3 &= (0.5\, p_S^2 + p_J^2) \wedge 2, \\ p_J^1 &= 0, \\ p_J^2 &= (p_S^1 + p_S^3 - 2)^+ \wedge 2, \\ p_J^3 &= 0. \end{aligned}$$

It is not difficult to check that there are infinite set of clearing vectors. Namely, we have that $p_S = (1, 2, 1 + t)$, $p_J = (0, t, 0)$ where $t \in [0, 1]$. The minimal clearing vector corresponds to $t = 0$, the maximal corresponds to $t = 1$.

2.4.2 Example 2

The vector of cash endowments and the matrix of the senior debts is the same as in the Example 1. The junior debts matrix L_J and the corresponding distribution matrix Π_J are now:

$$L_J = \begin{pmatrix} 0 & 0 & 0 \\ 0.4 & 0 & 1.6 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Pi_J = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.8 \\ 0 & 0 & 0 \end{pmatrix}.$$

We are looking for positive solutions of the following equations:

$$\begin{aligned} p_S^1 &= (0.1 + 0.5\, p_S^2 + 0.2\, p_J^2) \wedge 1, \\ p_S^2 &= (p_S^1 + p_S^3) \wedge 2, \\ p_S^3 &= (0.5\, p_S^2 + 0.8\, p_J^2) \wedge 2, \\ p_J^1 &= 0, \\ p_J^2 &= (p_S^1 + p_S^3 - 2)^+ \wedge 2, \\ p_J^3 &= 0. \end{aligned}$$

Note that $p_S^1 \le 1$, $p_S^2 \le 2$, hence, $p_J^2 \le 1$ and the 3rd equation is linear:

$$p_S^3 = 0.5\, p_S^2 + 0.8\, p_J^2. \tag{2.4.29}$$

Substituting into the 2nd equation this expression for p_S^3 and the expression for p_S^1 from the 1st equation we get that

$$p_S^2 = ((0.1 + 0.5\,p_S^2 + 0.2\,p_J^2) \wedge 1 + 0.5\,p_S^2 + 0.8\,p_J^2) \wedge 2$$

The inequality $p_S^1 < 1$ is impossible since in this case $0.1 + 0.5\,p_S^2 + 0.2\,p_J^2 < 1$, implying that

$$p_S^2 = (0.1 + p_S^2 + p_J^2) \wedge 2.$$

For positive values of unknown variables the last equality may hold only if $p_S^2 = 2$ but then the 1st equation tells us that $p_S^1 = 1$.

Thus, we determined that $p_S^1 = 1$.

Combining the 2nd equation with (2.4.29) we obtain the equality

$$p_S^2 = (1 + 0.5\,p_S^2 + 0.8\,p_J^2) \wedge 2$$

implying that $p_S^2 = 2$.

Available information allows us to reduce the 5th equation a simple one of the form $p_J^2 = 0.8(p_J^2)^+ \wedge 2$ having the unique solution $p_J^2 = 0$.

Summarizing, we get that $p_S = (1, 2, 1)$, $p_J = (0, 0, 0)$.

Comment. In the first example the bank 1 has met all liabilities and finished with a positive equity, the bank 2 has payed the senior liabilities but defaulted on the junior debts, the bank 3 has defaulted already at the senior debts; the bank 2 has no junior liabilities with the bank 1. So, the Δ-orbit of the banks 2 and 3 are not surplus sets and there are infinite many clearing vectors. In the second example the bank 2 has a junior debt to bank 1, all Δ-orbits are surplus sets and the clearing vector is unique.

2.5 Models with illiquid assets and a price impact

Let us consider the clearing problem without seniority structure where the bank i owns not only cash e^i but also K illiquid assets, in quantities $y^{i1}, \dots y^{iK}$ represented in the model by the row i of the matrix $Y = (y^{im})$, $i \le N$, $m \le K$. The nominal prices per unit of illiquid assets are strictly positive numbers $Q^1, ..., Q^K$. The clearing might require their partial sale influencing the market price. If the bank sells $u^{im} \in [0, y^{im}]$ units of the m-th assets for the price q_m, its total increase in cash is

$$(Uq)^i = \sum_{m=1}^{K} u^{im} q^m.$$

The price formation is modeled by the inverse demand function $F_0 : \mathbf{R}^K \to \mathbf{R}^K$ assumed to be continuous and monotone decreasing ($F_0(z) \le F_0(x)$ when $z \ge x$ in the sense of partial ordering defined by $\mathbf{R}_+^K$) and such that $F_0(0) = Q$ and $F_0^m(Y'\mathbf{1}) > 0$ for $m = 1, \dots K$. The first condition means that in the absence of supply the prices are just the nominal prices while the second one shows even in the case of total sale the prices of illiquid assets remain strictly positive.

The clearing rules: each bank pays debts in accordance to the matrix of relative liabilities and sell illiquid assets if it has insufficient amount of cash. The result of clearing should be: all debts of the bank are covered or its equity falls down to zero.

In the case of several illiquid assets there is a problem how the banks chose their strategies of selling. In principle, one can imagine the situation that they have full freedom and, acting in the noncooperative way, drop down the market of illiquid assets because of an excessive supply. It seems reasonable that the central authority may impose extra rules on selling illiquid assets. We suppose that this is done by prescribing that the bank i must sell all assets in the same proportion α^i:

$$\alpha^i(q) = \frac{\left(\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j\right)^+}{\sum_k y^{ik} q^k} \wedge 1, \quad i = 1, \ldots, N. \tag{2.5.30}$$

This formula means that for a fixed market price the bank does not sell illiquid assets if its cash reserve together with collected debts covers the liabilities. In the another extreme case where

$$\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j \geq \sum_k y^{ik} q^k = (Yq)^i$$

all illiquid assets have to be sold and the bank defaults. In the intermediate case the bank sells a share $\alpha^i \in]0,1[$ of the mth asset adding to its cash an extra amount

$$\frac{\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j}{\sum_k y^{ik} q^k} y^{im} q_m.$$

The total increase in cash allows to cover the liabilities.

Under such a rule the ith bank sells u^{im} units of the mth asset where

$$u^{im} := u^{im}(p,q) := \frac{y^{im}\left(\tilde{L}^i - e^i - \sum_j \Pi^{ji} p^j\right)^+}{\sum_k y^{ik} q^k} \wedge y^{im}.$$

The total supply of the illiquid assets is given by the vector $\mathbf{1}'U(p,q)$ where $U(p,q)$ is the matrix with entries given by the above formula.

Define the equilibrium vector $(p^*, q^*) \in [0, \tilde{L}] \times [F_0(1Y), Q]$ as the solution of the system of $N + K$ equations written in the matrix form as

$$p = (e + U(p,q)q + \Pi' p) \wedge \tilde{L}, \tag{2.5.31}$$
$$q = F_0(U'(p,q)\mathbf{1}). \tag{2.5.32}$$

The existence of the equilibrium is easy. Indeed, we check that

$$U'(p,q)\mathbf{1} \geq U'(\tilde{p},\tilde{q})\mathbf{1}, \qquad U(p,q)q + \Pi' p \leq U(\tilde{p},\tilde{q})\tilde{q} + \Pi' \tilde{p}$$

when $(\tilde{p}, \tilde{q}) \geq (p, q)$. Denoting $F(p, q)$ the right-hand side of the first equation we obtain that $(p, q) \mapsto (F(p, q), F_0(U'(p, q)\mathbf{1})$ is a monotone mapping of the order interval $[0, \tilde{L}] \times [F_0(1Y), Q]$ into itself. Accordingly to Knaster–Tarski theorem the set of its fixed points is nonempty and contains the minimal and maximal elements $(\underline{p}^*, \underline{q}^*)$ and $(\bar{p}^*, \bar{q}^*)$.

For a fixed q the function $p \to F(p, q)$ is monotone. Thus, by the Knaster–Tarski theorem the set of solutions of the equation (2.5.31) is nonempty and contains, in particular, the maximal element $\bar{p}(q)$.

For any fixed $q \in [F_0(Y), Q]$ the largest solution $\bar{p} = \bar{p}(q)$ of (2.5.31) is given by formula:

$$\bar{p} = \sup\{p \in [0, \tilde{L}] : \; p \leq (e + U(p, q)q + \Pi' p) \wedge \tilde{L}\}$$

implying that $q \mapsto \bar{p}(q)$ is an increasing (and continuous) function on $[F_0(Y), Q]$. It follows that the supply function

$$q \mapsto \zeta(q) := U'(\bar{p}(q), q)\mathbf{1}$$

is decreasing and, therefore, the $q \mapsto F_0(\zeta(q))$ is an increasing (and continuous) mapping of the interval $[F_0(Y), Q]$ into itself and, therefore, it has the minimal and maximal fixed points we shall denote q_1 and q_2.

Lemma 2.5.1. Suppose that the scalar function $x \to x' F_0(x)$ is strictly increasing on $[F_0(Y), Q]$. Then the solution of the equation $q = F_0(\zeta(q))$ is unique, i.e. $q_1 = q_2$.

Proof. Arguing by contradiction, suppose that $q_1 \neq q_2$. Since $q_1 \leq q_2$ and $\zeta(.)$ is decreasing, $\zeta(q_1) \geq \zeta(q_2)$. Moreover, $\zeta(q_1) \neq \zeta(q_2)$ as the values of F_0 at these points are q_1 and q_2. The assumed strict monotonicity implies that

$$\zeta'(q_1)F_0(\zeta(q_1)) > \zeta'(q_2)F_0(\zeta(q_2)).$$

It follows that

$$\zeta'(q_1)q_1 > \zeta'(q_2)q_2.$$

To get a contradiction it is sufficient to show that

$$\Delta := \zeta'(q_2)q_2 - \zeta'(q_1)q_1 \geq 0.$$

Let $\bar{p}_k := \bar{p}(q_k)$ and let

$$D_k := \{i : \; (\tilde{L} - e - \Pi' \bar{p}(q_k))^i \geq (Yq)^i\},$$

i.e. D_k is the set of banks that are forced to sell all their illiquid assets for the price q_k, $k = 1, 2$. Since $\bar{p}(.)$ is increasing, $D_2 \subseteq D_1$. With the notation $\mathbf{1}'_A$ for the row-vector representing the indicator function of the subset $A \subseteq \{1, \dots, N\}$, we have, taking into account that $a^+ = a + a^-$, that

$$\zeta'(q_k)q_k \;\; = \;\; \mathbf{1}'_{D_k} Y q_k + \mathbf{1}'_{D_k^c}(\tilde{L} - e - \Pi' \bar{p}_k) + \mathbf{1}'_{D_k^c}(\tilde{L} - e - \Pi' \bar{p}_k)^-.$$

This formula leads to the representation

$$\begin{aligned}\Delta &= \mathbf{1}'_{D_2}Y(q_2-q_1)-\mathbf{1}'_{D_1\setminus D_2}Yq_1-\mathbf{1}'_{D_1^c}\Pi'(\bar p_2-\bar p_1)+\mathbf{1}'_{D_2^c\setminus D_1^c}(\tilde L-e-\Pi'\bar p_2)\\ &\quad +\mathbf{1}'_{D_1^c}((\tilde L-e-\Pi'\bar p_2)^- -(\tilde L-e-\Pi'\bar p_1)^-)+\mathbf{1}'_{D_2^c\setminus D_1^c}(\tilde L-e-\Pi'\bar p_2)^-.\end{aligned}$$

Since the function $x\to x^-$ (on $\mathbf{R}^N$) is positive and decreasing, the last two terms in the right-hand side are positive. Regrouping the third and the forth terms we get that

$$\Delta\ge \mathbf{1}'_{D_2}Y(q_2-q_1)-\mathbf{1}'_{D_1\setminus D_2}q_1Y-\mathbf{1}'_{D_2^c}\Pi'(\bar p_2-\bar p_1)+\mathbf{1}'_{D_1\setminus D_2}(\tilde L-e-\Pi'\bar p_1). \tag{2.5.33}$$

From the equation (2.5.31) it follows that

$$\begin{aligned}\mathbf{1}'\Pi'(\bar p_2-\bar p_1) &= \mathbf{1}'(\bar p_2-\bar p_1)=\mathbf{1}'_{D_1}(\bar p_2-\bar p_1)\\ &= \mathbf{1}'_{D_2}(q_2u(\bar p_2,q_2)-q_1u(\bar p_1,q_1)+\Pi'(\bar p_2-\bar p_1))\\ &\quad +\mathbf{1}'_{D_1\setminus D_2}(\tilde L-(e+q_1u(\bar p_1,q_1)+\Pi'\bar p_1)).\end{aligned}$$

implying that

$$\begin{aligned}\mathbf{1}'_{D_2^c}\Pi'(\bar p_2-\bar p_1) &= \mathbf{1}'_{D_2}(U(\bar p_2,q_2)q_2-U(\bar p_1,q_1)q_1)-\mathbf{1}'_{D_1\setminus D_2}U(\bar p_1,q_1)q_1\\ &\quad +\mathbf{1}'_{D_1\setminus D_2}(\tilde L-e-\Pi'\bar p_1).\end{aligned}$$

Substituting this expression in (2.5.33), we have

$$\begin{aligned}\Delta &\ge \mathbf{1}'_{D_2}Y(q_2-q_1)-\mathbf{1}'_{D_1\setminus D_2}Yq_1\\ &\quad -\mathbf{1}'_{D_2}(U(\bar p_2,q_2)q_2-U(\bar p_1,q_1)q_1)+\mathbf{1}'_{D_1\setminus D_2}q_1u(\bar p_1,q_1)=0\end{aligned}$$

since the cash increment $(U(\bar p_2,q_2)q_2)^i=(Yq)^i$ for the bank $i\in D_2$ and $(U(\bar p_1,q_1)q_1)^i=(Yq_1)^i$ for $i\in D_1\supseteq D_2$. □

Theorem 2.5.2. Suppose that the scalar function $x\to x'F_0(x)$ is strictly increasing on $[F_0(Y),Q]$. Then there is q^* such that the set of solutions of the system (2.5.31), (2.5.32) is contained in the interval with the extremities $(\underline{p}(q^*),q^*)$ and $(\bar p(q^*),q^*)$. In particular, if for each q the solution of (2.5.31) is a unique, then the solution of the system is also unique.

Proof. Let Γ be the set of q for which (p,q) is a solution of the system (2.5.31), (2.5.32). If $q^*\in\Gamma$, then $(\bar p(q^*),q^*)$ is the solution of (2.5.31), (2.5.32). Accordingly to the above lemma the point q^* is uniquely defined. This implies the result. □

Note that the uniqueness of the solution of (2.5.31) is guarantied if for each i the orbit of i contains an element with positive cash reserve.

Remark. In the paper [5] it was considered a model coinciding with studied above in the case of a single illiquid asset. The difference is that in the cited paper the equilibrium is defined as a vector (p,q) satisfying the system of equations

$$p = \big(e+qy+\Pi'p\big)^+\wedge\tilde L, \tag{2.5.34}$$

$$q = F_0(\mathbf{1}'((q^{-1}(\tilde L-e-\Pi'p)^+)\wedge y)). \tag{2.5.35}$$

To our opinion, the definition of the equilibrium given by the system (2.5.31), (2.5.32), which is in the one liquid asset case has the form

$$p = \big(e + (\tilde{L} - e - \Pi' p)^+ \wedge (qy) + \Pi' p\big) \wedge \tilde{L}, \tag{2.5.36}$$
$$q = F_0(\mathbf{1}'((q^{-1}(\tilde{L} - e - \Pi' p)^+) \wedge y)), \tag{2.5.37}$$

is more natural. In fact, the right-hand sides of (2.5.34) and (2.5.36) as functions $R_1(p,q)$ and $R_2(p,q)$ defined on $[0,\tilde{L}] \times [F_0(1Y), Q]$ coincide. To see this, fix i and consider the three possible cases.

1. Let $e^i + qy + (\Pi' p)^i \le \tilde{L}^i$. Then the expressions for $R_1^i(p,q)$ and $R_2^i(p,q)$ have the same form $e^i + qy + (\Pi' p)^i$.

2. Let $e^i + qy + (\Pi' p)^i > \tilde{L}^i$ and $\tilde{L}^i - e^i - (\Pi' p)^i \ge 0$. Then the values $R_1^i(p,q)$ and $R_2^i(p,q)$ are equal to $\tilde{L}^i$.

3. Let $e^i + qy + (\Pi' p)^i > \tilde{L}^i$ and $\tilde{L}^i - e^i - (\Pi' p)^i < 0$. Then the value of $R_1^i(p,q)$ is $\tilde{L}^i$ and the value of $R_1^2(p,q)$ is $(e^i + (\Pi' p)^i) \wedge \tilde{L}^i = \tilde{L}^i$.

2.6 The Fischer model: clearing with derivatives

In the recent paper [30] Fisher generalized the Elsinger–Suzuki model to cover systems where banks besides of straight debts may have liabilities in terms of derivatives having different seniorities.

Mathematically, this means that matrices L_S may depend on the clearing vectors. The clearing equations for the situation with cross-holdings can be represented as follows:

$$p_S = \Big(e + \Theta' V + \sum_{r \le M} \Pi_r' p_r - \sum_{r<S} \tilde{L}_r(p)\Big)^+ \wedge \tilde{L}_S(p), \qquad S = 1,\dots,M, \tag{2.6.38}$$
$$V = \Big(e + \Theta' V + \sum_{r \le M} \Pi_r' p_r - \sum_{S} p_S\Big)^+. \tag{2.6.39}$$

Economically, Fisher's model is quite different from those previously discussed because now the matrices Π_S are disconnected from $L_S(p)$ and become input parameters of the model.

Theorem 2.6.1. Suppose that the functions $p \mapsto L_S(p)$ are bounded and continuous, $|\Theta| < 1$. Then the system (2.6.38), (2.6.39) has a solution.

Proof. By virtue of Lemma 2.3.1 the equation (2.6.39) has a solution $V(p)$ for any p and this solution is continuous in p. Plugging $V(p)$ into (2.6.38) we obtain in the right-hand side a continuous function which maps into itself the compact convex set $[0,\tilde{L}_1^*] \times \dots \times [0,\tilde{L}_M^*]$ where $\tilde{L}_S^* = \sup_p \tilde{L}_S(p)$. The application of the Brouwer theorem leads to the claim. □

In particular, the above theorem ensures the existence of clearing vector in the model with credit default swaps (CDS) where L_1 is the matrix of the straight debts having the highest priority and

$$l_S^{ij} := \lambda_S^{ij}(\tilde{L}_S - p_S)^+, \qquad S \geq 2,$$

where $\lambda_S^{ij} \geq 0$ are arbitrary constants.

Lemma 2.6.2. The system (2.6.38), (2.6.39) is equivalent to the system

$$p_S = \Big(e + \Theta' V + \sum_{r\leq M} \Pi'_r p_r - \sum_{r<S} \tilde{L}_r(p)\Big)^+ \wedge \tilde{L}_S(p), \qquad S = 1, ..., M. \quad (2.6.40)$$

$$V = \Big(e + \Theta' V + \sum_{r\leq M} \Pi'_r p_r - \sum_{S} \tilde{L}_S(p)\Big)^+. \quad (2.6.41)$$

Proof. If we fix V and take p that satisfies the relations (2.6.38), then the right-hand sides of (2.6.39) and (2.6.41) coincide. □

The paper [30] contains results on the existence and uniqueness of solution of (2.6.40), (2.6.41) without assumption on boundedness of $\tilde{L}_S$ but with a more stringent condition on coefficients, namely, on the matrices Π_S.

Theorem 2.6.3. Suppose that $e \geq 0$, the functions $p \mapsto L_S(p)$ are continuous, and $|\Theta| < 1$, $|\Pi_S| < 1$ for all S. Then the system (2.6.40), (2.6.41) has a solution.

Proof. Put $\Pi_{M+1} := \Theta$, $p_{M+1} := V$, and $\tilde{L}_{M+1} := \infty = (\infty, \dots, \infty)$. Slightly abusing notations we retain the symbol p for the $N(M+1)$-dimensional vector $(p_1, ..., p_M, p_{M+1})$ and write the system (2.6.40), (2.6.41) in a more compact form $p = \Phi(p)$ where

$$\Phi_S(p) := \Big(e + \sum_{r\leq M+1} \Pi'_r p_r - \sum_{r<S} \tilde{L}_r(p)\Big)^+ \wedge \tilde{L}_S(p), \qquad S = 1, ..., M+1.$$

For $y \in \mathbf{R}$, $a_1, \dots, a_M \in \mathbf{R}_+$ we have the identity

$$\sum_{S=1}^{M} \Big(y - \sum_{r<S} a_r\Big)^+ \wedge a_S + \Big(y - \sum_{r\leq M} a_r\Big)^+ = y \quad (2.6.42)$$

easily verified by induction based on the observation that $w^+ \wedge a = w^+ - (w-a)^+$ when $w \in \mathbf{R}$ and $a \in \mathbf{R}_+$.

Using it we infer that for any $p \in \mathbf{R}_+^{N(M+1)}$

$$\sum_{S\leq M+1} \Phi_S(p) = e + \sum_{S\leq M+1} \Pi'_S p_S$$

and

$$\sum_{S\leq M+1} |\Phi_S(p)|_1 = |e|_1 + \sum_{S\leq M+1} |\Pi'_S p_S|_1 \leq |e|_1 + \theta \sum_{S\leq M+1} |p_S|_1$$

where $\theta := \max_{S\le M+1} |\Pi'_S|_1$. In particular, if

$$\sum_{r\le M+1} |p_r|_1 \le \frac{1}{1-\theta}|e|_1, \tag{2.6.43}$$

then also

$$\sum_{S\le M+1} |\Phi_S(p)|_1 \le \frac{1}{1-\theta}|e|_1.$$

So, the continuous function $p \mapsto \Phi(p)$ maps the convex compact set of $p \in \mathbf{R}_+^{N(M+1)}$ satisfying (2.6.43) into itself and, by the Brouwer theorem, has a fixed point, i.e. the equation $p = \Phi(p)$ has a solution.

Note that if $p = \Phi(p)$, then the so-called *accounting equation* is fulfilled

$$\sum_{S\le M+1} p_r = e + \sum_{S\le M+1} \Pi'_S p_S$$

and, therefore,

$$\sum_{S\le M+1} |p_S|_1 = |e|_1 + \sum_{S\le M+1} |\Pi'_S p_S|_1 \le |e|_1 + \sum_{S\le M+1} |\Pi'_S|_1 |p_S|_1$$

implying that

$$(1-\theta)\sum_{S\le M+1} |p_r|_1 \le \sum_{S\le M+1} (1-|\Pi'_S|_1)|p_r|_1 \le |e|_1.$$

Thus, any solution of the equation $p = \Phi(p)$ satisfies (2.6.43).

Remark. It is easily seen that the claim of the theorem holds also in the case where the matrices Π'_S depend on p continuously and $\theta := \sup_p |\Pi'_S(p)|_1 < 1$ for $S = 1, \dots, M+1$.

The uniqueness result of [30] is based on the following elementary statement:

Lemma 2.6.4. Let $a_r, b_r \in \mathbf{R}$ be such that $b_r \ge a_r \ge 0$, $r \ge 1$. Let $A_0 := 0$, $B_0 := 0$, $A_r := \sum_{j\ge r} a_j$, $B_r := \sum_{j\le r} b_j$ for $r \le 1$. If $w, z \in \mathbf{R}$ are such that

$$z - w \ge B_M - A_M, \tag{2.6.44}$$

then

$$z - w = \sum_{r\le M} \left|(z-B_{r-1})^+ \wedge b_r - (w - A_{r-1})^+ \wedge a_r\right| + \left|(z-B_M)^+ - (w-A_M)^+\right|.$$

Proof. Since $b_M - a_M \ge 0$ the inequality (2.6.44) implies that $z - y \ge B_{M-1} - A_{M-1}$. As for (2.6.42) we can use induction arguments but based this time on the identity

$$|v^+ \wedge b - u^+ \wedge a| = |v^+ - u^+| - |(v-b)^+ - (v-a)^+|$$

which holds when $b \ge a \ge 0$ and $v - u \ge b - a$. $\square$

Theorem 2.6.5. In addition to the assumptions of preceding theorem suppose that

$$\tilde{L}_r^i(p) = \psi_r^i\Big(\sum_{r\le M+1}(\Pi'_r p_r)^i\Big)$$

where $\psi_r^i : \mathbf{R}_+ \mapsto \mathbf{R}_+$ are increasing functions such that for any $u, v \in \mathbf{R}_+$ such that $v \ge u$ we have the bound

$$v - u \ge \sum_{r\le M}\big(\psi_r^i(v) - \psi_r^i(u)\big), \quad i = 1, \dots, N.$$

Then the system (2.6.40), (2.6.41) has a unique solution.

Proof. We check that Φ is a contraction mapping in the space $\mathbf{R}_+^{N(M+1)}$ in the metric induced by the l^1-norm. Let p and $\tilde{p}$ be two clearing vectors. Define

$$x^i := \sum_{r\le M+1}(\Pi'_r p_r)^i, \qquad y^i := e^i + x^i, \qquad \Sigma_r^i := \sum_{j\le r}\psi_r^i(x^i),$$

and $\tilde{x}^i$, $\tilde{y}^i$, $\tilde{\Sigma}_r^i$ similarly; put also $\psi_{M+1}^i(x^i) = \psi_{M+1}^i(x^i) = \infty$. With these definitions

$$|\Phi(p) - \Phi(\tilde{p})|_1 \;=\; \sum_{i\le N}\sum_{r\le M+1}\big|(y^i - \Sigma_{r-1}^i)^+ \wedge \psi_r^i(x^i) - (\tilde{y}^i - \tilde{\Sigma}_{r-1}^i)^+ \wedge \psi_r^i(\tilde{x}^i)\big|.$$

The hypothesis of the theorem allows us to apply Lemma 2.6.4, choosing a correspondence with its notations in dependence of the sign of the difference $x^i - \tilde{x}^i$, and conclude that the interior sum is equal to $|y^i - \tilde{y}^i| = |x^i - \tilde{x}^i|$. Thus,

$$\begin{aligned}|\Phi(p) - \Phi(\tilde{p})|_1 &= \sum_{i\le N}\left|\sum_{r\le M+1}(\Pi'_r(p_r - \tilde{p}_r))^i\right| \le \sum_{r\le M+1}|\Pi'_r(p_r - \tilde{p}_r)|_1 \\ &\le \sum_{r\le M+1}|\Pi'_r|_1|p_r - \tilde{p}_r|_1 \le \theta|p - \tilde{p}|_1,\end{aligned}$$

where $\theta := \max_{S\le M+1}|\Pi'_S|_1 < 1$. □

2.7 Knaster–Tarski fixpoint theorem

Let X be a set with a partial ordering $\ge$ and let A be its nonempty subset. By definition, $\sup A$ is an element $\bar{x}$ such that $\bar{x} \ge x$ for all $x \in A$ and if y is such that $y \ge x$ for all $x \in A$ then $y \ge \bar{x}$. The definition of $\inf A$ follows the same pattern but with the dual ordering $\le$. A partially ordered set X is *complete lattice* if for any its nonempty subset A there exist $\inf A$ and $\sup A$. In particular, the order interval $[a, b] \subset \mathbf{R}^d$ with the component-wise ordering is complete lattice.

Theorem 2.7.1. Let X be a complete lattice and let $f : X \mapsto X$ be an order-preserving mapping, $L := \{x : \ f(x) \leq x\}$, $U := \{x : \ f(x) \geq x\}$. The set $L \cap U$ of fixed points of f is non-empty and has the smallest and the largest fixed points which are, respectively, $\underline{x} := \inf L$ and $\bar{x} := \sup U$.

Proof. Note that $L \neq \emptyset$ since it contains the element $\sup X$. Take arbitrary $x \in L$. Then $\underline{x} \leq x$ implying that $f(\underline{x}) \leq f(x) \leq x$. Thus, $f(\underline{x}) \leq \underline{x}$ as $\underline{x}$ is $\inf L$. So, $\underline{x} \in L$. Since $f(L) \subseteq L$, also $f(\underline{x}) \in L$, hence, $\underline{x} \leq f(\underline{x})$, i.e. $\underline{x} = f(\underline{x})$. All fixed points belong to L and, therefore, $\underline{x}$ is the smallest one.

The proof of the statement for the largest fixed point is analogous. □

Corollary 2.7.2. Let f_i, $i = 1, 2$, be two order-preserving mappings of a complete lattice $(X, \geq)$ into itself such that $f_2 \leq f_1$. Let $\underline{x}_i := \inf L_i$ and $\bar{x}_i$ be their smallest and largest fixed points. Then $\underline{x}_2 \geq \underline{x}_1$ and $\bar{x}_2 \geq \bar{x}_1$.

The claim is obvious because $L_1 = \{x : \ f_1(x \leq x\} \supseteq \{x : \ f_2(x) \leq x\} = L_2$ and $U_1 = \{x : \ f_1(x \geq x\} \subseteq \{x : \ f_2(x) \geq x\} = U_2$, see [42].

Chapter 3

Dynamic Models of Systemic Risk and Contagion

3.1 Inroduction

In interbank market, systemic risk is a risk threatening the entire system and leading to a potential financial crisis, resulting in high economical and social costs. Understanding financial stability and assessing financial risk is a major concern of central banks and financial regulators. The rapid growth of financial innovation and integration as well as a complicated network of claims and obligations linking the balance sheets of banks raises the challenge for the analysis of systemic risk. This kind of risk is highly dynamic, building up almost un-noticeably during periods of stability and rapidly rising during crises and spreading through the network. On the other hand, the connections between banks lead to enhance the liquidity and increase the risk sharing among the financial institutions.

One of the aim of theoretical studies is to provide regulators comprehensive indicators allowing to monitor the risk of contagion, understood as a cascade of defaults that may lead to a serious consequences and even to the collapse of the whole economy. To the moment, there is a substantial progress in understanding of various phenomena causing the contagion on the basis of modeling using random graphs. Network models became the mainstream of current researches in the field, see the book by T. Hurd [36] and the references wherein.

Recent crisis revealed that the systemic risk might take various forms. One form of the systemic risk is an interbank contagion process when, due to the interconnectedness of banks through interbank loans, the default of one bank leads to losses and subsequent defaults of other banks. This kind of risk is usually combined with a risk related to a correlation externality between banksĂ portfolios that consists in the phenomena that a common shock, due to common asset holdings, affects many banks at once.

Bandt *et al.* (2009), [17] provide a categorization of systemic risks, distinguishing between those understood in a broad and in a narrow sense: contagion effects pose a systemic risk in the narrow sense while in the broad sense it is a common shock that affects many nodes and once. Same in Gai and Kapadia (2010), [33], who model two channels of contagion in financial system that can trigger further rounds of defaults: contagion due to the direct interbank claims and obligations as well as due to the common shocks on the asset side of the balance sheet, especially, when the market for key financial system assets is illiquid.

Deposits could affect the financial system stability as a large sudden withdrawal triggered by depositors panic could lead to a collapse of the system. However, we do not consider this as one of the major sources of system risk as its impact can be minimized and controlled by the central bank intervention forcing an appropriate withdrawal limit.

A large part of literature have focused on the analysis of the contagion effect due to the interbank market while only a few authors studied the impact of the correlated defaults which is of great importance related to the magnitude of correlation between the banks balance sheets, to the amount of external investments and to the appropriate assessment of the risk embedded in these external assets. Acharya and Yorulmazer (2008), [3] proved that banks are motivated to increase the correlation between their investments amplifying by this the risk of a common shock. In their analysis, Elsinger *et al.* (2006), [27] combine the two major sources of systemic risk and find that the correlation in investments is far more important than financial linkages.

We can also consider a subordinate source of risk due to the fire sale of external assets of defaulting banks which will lead to other banks default because of the price depreciation. This is the why some banks have an interest to bailout other peers in order to minimize the default cost of the system and to prevent fire sale and the writing down of their own external assets.

While the interbank risk is concerned, Gai and Kapadia (2008) show that the risk of systemic crises is reduced with increasing connectivity while the amplitude of the systemic crises is increasing at the same time. Higher connectivity simply creates more channels of contact through which default could spread, increasing the potential or probability for contagion. However, in the financial system setup, greater connectivity allows counterparties risk sharing as exposures are distributed over a wider set of banks, especially, in periods of stability. In times of crisis, however, the same interconnections can amplify shocks that spread through the system.

Allen and Gale (2000), [4] demonstrate that the spread of contagion depends from the network structure of the financial system and highly interconnected banking systems are less affected by the systemic risk. They also point out that the assumption that the agents have complete information on their environment is not realistic. Acharya and Bisin (2014), [2], compare over-the-counter (OTC) and centralized clearing markets in a general equilibrium model. They show that the untransparency of OTC markets is ex-ante inefficient and will lead to underpricing of counterparty risk.

The counterparty risk makes it clear that the network structure of financial system plays an important role when assessing systemic risk.

Empirical analyses of the interbank network structure exist for a number of countries. It shows that the interbank network has a scale free topology. This means that there are a few large banks with many interconnections and many small banks with a few connections. In contrast, other authors argue that the untransparency of real data makes the random network more valid to capture the hidden links. More formally, the terminology "scale free network" means that, at least, when the number of nodes increases to infinity the number k of connections ("in" or "out") attributed to each node decays as $k^{-\gamma}$, $\gamma > 1$.

Georg (2013) [34] proposes a dynamic model of cascading banking defaults: at each stage of the cascade, each bank collects all his exposures, pays all his liabilities, adjust the price of its external assets and, when remains solvent, it optimizes a portfolio of risky and risk-free assets and initiates other interconnections within the banking system.

On the other hand, Gai and Kapadia (2009) highlight that in normal times, developed country banks are robust and minor variations in their default probabilities do not affect lending decisions on the interbank market. But in crises, as illustrated by the sudden failures of Lehman Brothers, contagion may spread rapidly with banks having little time to alter their behavior before they are affected. Thus, the almost static behavior of the system during crisis is best captured by the static model as also applied in our paper.

It seems that the majority of existing literature deals with "homogeneous" models, like Erdös–Renyi model where the network graph is generated by a matrix whose the non-diagonal entries are identically distributed independent Bernoulli random variables, see [33], or even models where all nodes has the same number of connections, [41]. Though such models are convenient for theoretical studies, they look to be too far from the reality and in the present paper we investigate the behavior of the systemic risk indicator using networks with a structure obtained by a preference attachment algorithm leading to a scale free network.

Under the Basel II accord, improving the quality of default models is the key risk-management priority. Many researchers have studied the loss or impact of the systemic risk once a crisis or shock is in place. However, there is a need to predict and prevent the defaults of banks before it happens. To the date, the major part of research papers concentrates on studies of static or stationary models. In this note we suggest an approach influenced by the structural model of defaultable securities, see [9]. Namely, we suppose that the cascade of default is triggered in a natural way when the value of a portfolio process of some bank falls below a certain level. Financial market react negatively to such an event. Prices of the external assets drop down and contagion propagates not only to interconnected banks but also via correlation. Assuming that the matrix of exposures as well as the vector of the investments into external assets is known, the regulators, having a model for the dynamic of the "reference portfolio" can compute with moving time horizons two "alert indicators": the probability that the default happens during the planning period and the total losses incurred when the default happens. The total losses are the aggregation of the losses due to the external asset price depreciation (correlation) and the losses due to the interbank linkages (contagion). To simplify our calculation, we assume that there is a single external risky asset common to all banks in the system and the difference is only in the size of portfolios. A model where each bank has its own portfolio structure can be treated in a similar way. Our approach is rather flexible and can be combined with existing methods of reconstructing of the exposure matrix.

Thus, the main novelty of our approach, in contrast to the majority of existing studies concentrated on static or stationary models, we are interested here in a dynamic model of financial system before the crisis in combination with a static contagion model for the crisis. The model is described by a graph which nodes are banks (or other financial institutions). The directed graph structure arises from the matrix of liabilities/exposures. Each bank is characterized by a stylized balance sheet. On the asset side there are exposures (due to the interbank landing) and liquid assets, risky (stocks) and non-risky (cash). The liability side is composed by the received interbank loans and the net worth, the quantity, equating both sides of the balance sheet. The dynamic is introduced via random fluctuations of the value of the risky asset. Decreasing of its price means that the net worth is decreasing. We suppose that the risky asset is unique for all banks. One may think of this asset as a "benchmark (or reference) portfolio". Taking into account that banks try to mimic behavior of each other ("herding effect"), we believe that this assumption may suit to our highly stylized model but for practical applications it can be relaxed. Of course, there is a need to introduce dynamic in other parts of the balance sheet but we prefer not to do this in this paper.

The paper contains some numerical experient. Unfortunately, the liability matrix of a financial system is not publicly available (with rare exceptions). By this reason we test applicability of our model on simulated data. In numerical experiments we use a construction of the scale-free network using a preferential attachment algorithm, see [6]. We populate the model by balance sheets and compute the alert indicators. Our experiments show that the alert indicators can be used as a tool to support regulator's decision to increase the stability of the financial system by withdrawal of the license of the bank having low reliability.

The structure of the article is as follows. In Section 3.2 we describe the general network approach to contagion. Section 3.3 gives the model description and the definition of the alert indicators. Section **??** is devoted to simulation results.

3.2 Network approach

3.2.1 General principles

The basic ideas are very simple and can be described as follows. The set $G = \{1, ..., N\}$ stands for the banking system involving N financial institutions described by an $N \times N$ matrix $L = (L^{ij})$ with non-negative entries vanishing on the diagonal ($L^{ii} = 0$) and a vector $C \in \mathbf{R}^N$ with non-negative components.

The entry L^{ij} represents the *liability* of the ith bank to the jth bank, i.e. the debts of i to j or, in other words, the total amount of credit provided by j to i. By the reciprocity, for the ith bank the value L^{ji} is its *exposure* to the bank j. By this reason, in the literature the model quite often is described by the matrix of the liabilities $X = (X^{ij})$, $X = L'$ where $'$ is used to denote the transpose. Let $B^{ij} = I_{\{L^{ij}>0\}}$. The matrix B (whose entries are zeros and units) defines the graph structure on the set of N points in a usual way (as is done in the theory of Markov chains): there is a flesh $i \to j$ if $B^{ij} > 0$, showing that the ith bank is indebted to the jth bank (attention: in some papers the direction of fleshes can be opposite). With this observation, one can use the standard terminology of the network theory and identify banks with the nodes of the (weighted) oriented graph.

The component C^i of the vector C represents the proper capital reserve the ith bank; it is *solvent* if the net worth

$$NW^i := \sum_{j \in G} L^{ji} - \sum_{j \in G} L^{ij} + C^i \geq 0. \tag{3.2.1}$$

If the above solvency condition does not hold, the bank *defaults*.

It is important to note that the definitions "exposure", "liability", "default" appeal to a common sense rather having a precise meaning. Their understanding varies from paper to paper. In practice, the balance sheet of a bank has a much more complicated structure. E.g., the exposure may include overnight credits as well as long term loans, the debts are of different seniority, and so on. The "standard" highly stylized balance sheet, i.e. the equality $Assets = Liabilities$ presented as a table, containg on the assets sides the interbank exposures (loans) and external assets (that can be split in liquid and illiquid, risky and non-risky) while on the liability side there are interbank borrowings, deposits and, to equate the both side, the net worth (called also capital reserve or equity) — in the case that the bank is solvent.

3.2.2 Defaults

In the literature, the typical description of the contagion process and defaults "in cascade" is given as follows, see [36] and references there in. Let us denote by $I_{out}(i)$ (respectively, by $I_{in}(i)$) the set of banks to which the bank i has a liability (respectively, an exposure). That is, $I_{out}(i)$ is the set of nodes terminal for the fleshes outgoing from the node i while $I_{in}(i)$ is the set of nodes with fleshes ending at this node. We denote by $n_{out}(i)$ and $n_{in}(i)$ the cardinality of the corresponding sets, i.e. the numbers of outgoing and ingoing fleshes. Clearly, $n_{out}(i) = \sum_j B^{ij}$, $n_{in}(i) = \sum_j B^{ji}$.

The default of the bank i triggers the following procedure. The bank is excluded from the network. Debts are collected from debtors at liquidation. Creditors loose a fraction $(1 - R)$ of their exposures to i, where the parameter R is referred to as the recovery rate. Formally, one can think that the matrix L is replaced by the matrix $\bar{L}$ obtained by replacing the elements of the ith row and ith column by zeros. The transformed vector of capital reserves $\bar{C}$ has the components $\bar{C}^j = C^j + RL^{ij} - L^{ji}$, $j \neq i$, $\bar{C}^i = 0$. Put $D_0(i) := \{i\}$ and skip further the argument i here and in further definitions (depending also on R). For some j (different from i) the solvency condition

$$\sum_{k \in G \setminus D_0} \bar{L}^{kj} - \sum_{k \in G \setminus D_0} \bar{L}^{jk} + \bar{C}^j \geq 0, \tag{3.2.2}$$

which can be written also as

$$\sum_{k \in G} L^{kj} - \sum_{j \in G} L^{jk} + C^j - (1-R)L^{ij} \geq 0, \tag{3.2.3}$$

may fail. We denote by $D_1 := D_1(i)$ the set of such indices, corresponding to the first-order defaults in the cascade of the defaults triggered by the default of i. In the same way, the contagion is propagated further, to the set of banks $D_2 = D_2(i)$ which is a subset of indices j outside of the union D_0^1 of D_0 and D_1 and such that the solvency condition becomes negative:

$$\sum_{k \in G}^{N} L^{kj} - \sum_{j \in G} L^{jk} + C^j - (1-R) \sum_{k \in D_0^1} L^{jj} < 0.$$

Continue in the same way, for the set D_0^n, we put $D_0^{n+1} := D_0^n \cup D_{n+1}$ where D_{n+1} is the set of indices j in the complement of D_0^n such that

$$\sum_{k \in G}^{N} L^{kj} - \sum_{j \in G} L^{jk} + C^j - (1-R) \sum_{k \in D_0^n} L^{jk} < 0.$$

The process stops if $D_{n+1} = \emptyset$. One can consider the value

$$L(i) := (1-R) \sum_{n=0}^{N} \sum_{j \in D_{n+1}} \sum_{k \in D_0^n} L^{jj}$$

as the total losses incurred by the cascade of defaults triggered by the default of the ith bank.

It is not difficult to extend the above definitions to obtain expressions for losses triggered by simultaneous defaults of a group of banks.

3.2.3 Practical aspects and difficulties

On the first sight, the above formulae are of great help for the researchers in financial systemic risk providing them N functions of the recovery rate which allows to classify banks accordingly to their systemic importance. The described procedure also can be used to find the most vulnerable banks, sensitive to defaults of others. However, the practical implementation is not so straightforward. Indeed, in the majority of cases the exposure matrix X (having one million entries for a system with $N = 1000$) is not publicly available though a certain subset of its entries may be known. Usually, only the sums of elements along each line and each column can be extracted from the balance sheets. If only this information is available, one cannot recover the matrix L in a unique way: one needs to solve the system of $2N$ equations

$$\sum_{j \in G} L^{ji} = a^i, \qquad \sum_{j \in G} L^{ij} = b^i, \qquad 1 \leq i, j \leq N, \tag{3.2.4}$$

with $N^2 - N$ unknown $L^{ij} \geq 0$ and all $L^{ii} = 0$.

Obviously, the system (3.2.4) has the non-negative solution $x^{ij} = a^j b^i / \sum_i b^i$ (note that $\sum_i b^i = \sum_j a^j$). But this is not the needed solution since not all $x^{ii} = 0$. In the literature, see [44], it is recommended to take as the matrix X the solution of the entropy minimization problem:

$$\sum_{ij} \ln \frac{L^{ji}}{x^{ji}} \to \min,$$

under constraints (3.2.4), $L^{ij} \geq 0$ and $L^{ii} = 0$ for all i, j.

This approach is criticized since it leads to a matrix generating a complete graph and the overestimation of stability of financial system. On the other hand, in some cases, a part of the matrix L is known, e.g., the absence of connections between some nodes can be a plausible hypothesis. The entropy minimization method can be easily adapted to such cases leading to a rather realistic recovery of the exposure matrix.

3.2.4 Probabilistic modeling

Due to the lack of the information on the real structure of the financial system, there is an interest to generate numerically models which have at least, basic features of such models.

Apparently, the liability matrix L and the reserve vector R are random and evolve as stochastic processes. Due to the high dimensionality of the problem the modeling is extremely complicated and simplifying assumptions are unavoidable. The majority of available studies consider static models or stationary models and start modeling with the description of the incidence matrix B.

The simplest model is based on the hypothesis that the non-diagonal elements of the incidence matrix B are independent identically distributed Bernoulli random variables, see, e.g., [46] where low-parameter models are suggested to evaluate the impact of various factors on the financial stability. In addition to N and $p = P(B^{ij} = 1)$, there are three more parameters: the total value of assets A, the value of external assets C, and the net worth as the percentage of the total value of assets γ. These parameters are used to generate the balance sheets. In our notations, the interbank exposures and liabilities for the ith bank are defined as follows:

$$a^i = (A - C)\frac{n_{out}(i)}{|B|}, \qquad b^i = (A - C)\frac{n_{in}(i)}{|B|},$$

where $|B| := \sum_{ij} B^{ij}$. The value of external assets of the bank are defined by the formula

$$C^i = (b^i - a^i) I_{\{a^i < b^i\}} + \frac{1}{N}\Big(C - \sum_j (b^j - a^j) I_{\{a^j < b^j\}}\Big).$$

If the second term is positive, then all banks in the system are solvent. Since a^j and b^j are random, one should have a sufficiently high ratio C/A (in [46] it was always taken greater then 0.3). The quantity $\gamma(a^i - b^i + C^i)$ models the net worth while $(1-\gamma)\gamma(a^i - b^i + C^i)$ stands for the consumer deposits.

3.3 Dynamic models and alert indicators

3.3.1 Structural model

The aim of the model is to provide regulators two functions on the current state of the system which can be used to calculate the alert indicators. The first one is the probability that the system will suffer a cascade of defaults before a specified time horizon. The second indicator is the total losses incurred by the cascade of defaults, if it happens.

We suppose that at time zero the regulators dispose the liability matrix L or its transpose the exposure matrix X (in reality, this is a public information in rare countries, like Brasil, but can be available to central banks) and the vector of capital reserves C which is decomposed into non-risky reserve c (say, Treasury bonds) and investments y into a single risky asset, interpreted as a market portfolio. In our very stylized model all these values are fixed up to the time horizon T. Of course, in reality the banks trade and portfolios are composed in many assets. Nevertheless, quite often banks mimic the behavior of each other and one may guess that a typical portfolio value has the same evolution as a certain reference portfolio. We describe its dynamics by a geometric Brownian motion:

$$dS_t/S_t = \mu dt + \sigma dW_t.$$

That is,

$$S_t = S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t}.$$

At time zero all banks supposed to be solvent.

The default cascade will be triggered at the instant when one of the solvency conditions will be violated.

The solvency condition for the ith bank has the form:

$$V_t + y_0^i S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t} \geq 0. \tag{3.3.5}$$

where

$$V_t := \sum_{j \in G} L_t^{ji} - \sum_{j \in G} L_t^{ij} + c_t^i$$

$$b_t^i := \sum_{j \in G} L_t^{ji}, \qquad a_t^i := \sum_{j \in G} L_t^{ij}.$$

Hypothesis: $V_t = V$ for all $t \in [0, T]$.

The above assumption allows us to provide the regulators some easily calculated indicators of the system stability. Without any doubts, in the present oversimplified form they can be criticized. For example, we assume that the interbank operations to a large extend are balanced by liquid assets. In favor of this are evidences that interbank lending is not the main activity of banks. We also assume a rigidity of the investment portfolio. Again, econometric studies confirm that banks have a tendency to follow similar behavior. The benchmark portfolio process may have various dynamics and various theoretical and statistical models can be used for its description.

Put

$$\lambda^i := \frac{1}{\sigma} \ln \frac{V^i}{y^i S_0}$$

with a convention that $\lambda^i := -\infty$, if $V^i \le 0$. Let i_0 be the index corresponding to the largest of values of λ^i. We may assume, with very minor loss of generality, that all finite values of λ^i are different (the coincidence is not expected in the present context) and that λ^{i_0} is finite (otherwise there will be no defaults).

Let us introduce the stopping time

$$\tau := \inf\{t \ge 0 : \; w_t + \beta t \le \lambda^{i_0}\}$$

where $\beta := \mu/\sigma - \sigma/2$. If $\tau \le T$, the system will have a default and it happens with the node i_0; the price of the market portfolio at this date will be $S_0 e^{\sigma \lambda^{i_0}}$. The distribution of τ is the well-known inverse Gaussian distribution (see [13]) and we have that

$$P(\tau \le T) = \Phi(h_1(T)) + e^{2\beta\lambda^{i_0}} \Phi(h_2(T)),$$

where Φ is the standard Gaussian distribution function and

$$h_1(T) := \frac{\lambda^{i_0} - \beta T}{\sqrt{T}}, \qquad h_2(T) := \frac{\lambda^{i_0} + \beta T}{\sqrt{T}}.$$

The default of the bank i_0 generates a cascade of the defaults. It seems reasonable to suppose that the market reacts to such an event and the risky asset may loss a certain percentage of its value. With this assumption the set $D_1 = D_1(i_0)$ of first order defaults of the banks correspond to the indices j such that

$$\sum_{k \in G} L^{kj} - \sum_{j \in G} L^{jk} + c^j + \alpha y^j S_0 e^{\lambda^{i_0}} - (1-R) L^{i j_0} < 0, \tag{3.3.6}$$

$D_0^1(i_0) := D_0 \cup D_1$ etc. The parameter $\alpha \in]0,1]$ represents the default impact on the price of the reference portfolio.

The second alert indicator is the amount of total losses

$$L(i_0) := (1-R) \sum_{n=0}^{N} \sum_{j \in D_{n+1}} \sum_{k \in D_0^n} L^{jk}.$$

In the considered setting it can be augmented, e.g., by the losses of non-defaulted banks due to a depreciation of their portfolios:

$$\tilde{L}(i_0) := (1-\alpha) \sum_{j \in G \setminus D_0^N} y^j S_0 e^{\lambda^{i_0}}.$$

3.3.2 Discussion

The model introduced above has an advantage of its simplicity. It combines structural approach to defaultable securities with ideas of modern theory of financial networks. The alert indicators have a simple and comprehensive meaning. They can be easily computed at the monitoring dates t_m (when the new balance sheets are communicated) for the moving time horizons $t_m + T$. This allows regulator to see dangerous trends in the evolution of the system. It is worth noting that the model combines two channels of contagions: via the network as well as via the correlation due to common source of randomness.

Surely, the model is highly stylized. How serious are the weak points and how the model can be improved?

1. It is assumed that the investment in the single risky asset are static though in reality there is an intensive trading. For a fixed input there is only the bank triggering the default is uniquely determined.

To our mind, these objections should be examined carefully. Due to extreme complexity of financial systems (recall that they may contain hundreds of banks) and complexity of individual balance sheets, for more sophisticated models one can have an accumulation of various factors: misspecification errors, calibration errors, data aggregation errors etc. That is why simplifying hypotheses seems to be inevitable. It seems that we can accept that banks investment portfolios are close to the most performant one.

Of course, the predetermined bank triggering of the default cascade is not intuitive. However, as we know from the literature the matrix L is rarely known and should be reconstructed from the aggregated exposure of the banks. It is not difficult to implement a random reconstruction procedure, for each realized reconstruction one can compute conditional alert indicators, and take the average.

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