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Mémoire d'habilitation à diriger des recherches, présenté par

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Contributions to noncommutative harmonic analysis and duality problems

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Remerciements

When starting the preparation of this thesis, I did not realize that this would be such a personal issue. Going back to the results of the last 9 years means returning to my history, with all its hopes and disappointments, failures and achievements.

I am much obliged to Carine Molitor Braun who hired me for a postdoc in Luxembourg. Four years spent in this sweet country in a young research-oriented university was a precious experience for me; a start of an international career; a possibility to devote myself full-time to the research. I am grateful to my colleagues of this time for mathematical discussions and personal exchanges.

Speaking of France is easier and harder because it has become my second home since then, with its language, and its culture, and its mathematical community. I came as a stranger and I feel being at my own place here by now. I am grateful to all my colleauges, many of them friends, who made this transition possible.

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Contents

Publications included in the thesis

The present thesis includes:

Published

[1] E. Abakoumov, Yu. Kuznetsova, Density of translates in weighted L^p spaces on locally compact groups, *Monatshefte Math.* 183(3), 397–413 (2017).

[2] Yu. Kuznetsova, J. Roydor, Homomorphisms with small bound between Fourier algebras, *Israel J. Math.* 217 (2017), Iss. 1, 283–301.

[3] Yu. Kuznetsova, A duality of locally compact groups which does not involve the Haar measure. *Math. Scand.* 116 (2015), 250–286.

[4] Yu. Kuznetsova, A duality for Moore groups. *J. Oper. Theory* 69 Iss.2 (2013), 571–600.

[5] Yu. Kuznetsova, On continuity of measurable group representations and homomorphisms. *Stud. Math.* 210 no. 3 (2012), 197–208.

[6] Yu. Kuznetsova, C. Molitor-Braun: Harmonic analysis of weighted L^p-algebras. *Expo. Math.* 30 (2012), 124–153.

[7] Yu. Kuznetsova, An example of a weighted algebra $L_p^w(G)$ on uncountable group. *J. Math. Anal. Appl.* 353, Iss. 2 (2009), 660–665.

Prepublications

[8] Yu. Kuznetsova, Duals of quantum semigroups with involution, 22 p. arXiv: 1611.04830 [math.OA] (submitted, under revision).

[9] M. A. Aukhadiev, Yu. Kuznetsova, Quantum semigroups generated by locally compact semigroups, 14 p. ArXiv:1504.00407 [math.FA] (submitted, revised).

Earlier publications, included in my PhD thesis, are listed in the general reference list under [89], [122], [90], [91] and [92].

Chapter 1

Introduction

Mes recherches appartiennent à l'analyse fonctionnelle et sont consacrées aux algèbres de groupes dans un sens large. Elles se divisent en trois axes: algèbres ou espaces à poids sur les groupes localement compacts, dualité d'algèbres de groupes non-commutatifs et quantiques, et des questions topologiques sur des groupes localement compacts et leurs algèbres. Cette division correspond à l'organisation du présent mémoire en trois chapitres.

Dans cette introduction, je présente brièvement le contenu de la suite. Le texte principal est écrit en anglais et donne plus d'informations, sans pourtant descendre dans tous les détails. Les travaux inclus dans le mémoire sont listés sur la page précédente avec des liens vers leurs texte original.

Algèbres et espaces L^p à poids

Les algèbres de ce type formaient le sujet de ma thèse de doctorat «Algèbres à poids sur les groupes localement compacts» élaborée sous la direction du Professeur Alexandre Helemskii à l'Université Lomonosov de Moscou [89], [122], [90], [91], [92]. Plus tard, je suis retourné à cette thématique dans les trois travaux résumés ci-dessous.

Soit G un groupe localement compact, $p \geqslant 1$, et soit ω une fonction positive mesurable sur G. L'espace à poids $L_p^\omega(G)$ est défini comme l'ensemble de f telles que $f\omega \in L_p(G)$, c'est-à-dire que $\int_G |f(t)\omega(t)|^p dt < \infty$. Dans le cas $p = 1$, ces algèbres sont appelées les algèbres de Beurling d'après A. Beurling qui était le premier à les considérer en 1938.

Pour certains poids, $L_p^\omega(G)$ est une algèbre avec la convolution habituelle: $(f * g)(t) = \int_G f(s)g(s^{-1}t)ds$. Dans le cas $p = 1$, la condition suffisante est facile: $\omega(st) \leqslant \omega(s)\omega(t)$ pour tous $s, t \in G$. Pour $p > 1$, une condition analogue, trouvée par J. Wermer, consiste en l'inégalité $\omega^{-q} * \omega^{-q} \leqslant C\omega^{-q}$, où $1/p + 1/q = 1$.

Dans ma thèse, j'ai travaillé surtour dans le cas $p > 1$ et j'ai démontré des résultats sur l'existence de ces algèbres, des conditions nécessaires sur leurs poids, et sur l'analyse harmonique des algèbres en question, notamment la régularité dans le cas commutatif.

[7]. *Yu. Kuznetsova*, **An example of a weighted algebra $L_p^\omega(G)$ on uncountable group**. *J. Math. Anal. Appl.* 353, Iss. 2 (2009), 660–665.

Dans cet article j'ai résolu une question laissée ouverte dans ma thèse: j'ai montre qu'il existe des algèbres $L_p^\omega(G)$, $p > 1$, sur des groupes non-σ-compacts; dans le cas discret cela veut dire que le groupe peut être non-dénombrable. Il faut noter qu'un groupe abélien qui admet des algèbres de ce type est forcément σ-compact, comme démontré dans ma thèse [91].

L'article présente alors un exemple d'un poids ω sur un groupe libre discret $\mathbb{F}_{\mathfrak{m}}$ de cardinalité quelconque $\mathfrak{m}$, tel que $\ell^p(\mathbb{F}_{\mathfrak{m}}, \omega)$ est une algèbre à convolution avec $1 < p \leqslant 2$. Dans le cas $p > 2$ il y est pourtant démontré que $\ell^p(\mathbb{F}_{\mathfrak{m}}, \omega)$ n'est pas une algèbre avec aucun poids.

[6]. *Yu. Kuznetsova, C. Molitor-Braun*, **Harmonic analysis of weighted L^p-algebras**. *Expo. Math.* **30** (2012), 124–153.

Nous avons traité dans le cas non-commutatif certaines propriétés d'analyse harmonique des algèbres $L_p^\omega(G)$, notamment leur régularité. Pour une algèbre de Banach commutative A avec le spectre Ω, A est dite régulière si pour tout fermé $F \subset \Omega$ et tout $t \notin F$ il existe $a \in A$ tel que $\widehat{a}(t) = 1$ et $\widehat{a}|_F \equiv 0$. En particulier, il y a «assez» de fonctions à support compact parmi les transformées de Gelfand des éléments de A.

Beurling a démontré [25] que pour $p = 1$, l'algèbre $L_1^\omega(\mathbb{R})$ est régulière si et seulement si

$$\int_{-\infty}^{\infty} \frac{\ln^+ \omega(t)\, dt}{1+t^2} < \infty,$$

où $\ln^+ x = \max(0, \ln x)$. Cette condition est équivalente à la convergence de la série

$$\sum_{n=1}^{\infty} \frac{\ln^+ \omega(nx)}{n^2} \tag{BD}$$

pour tous $x \in \mathbb{R}$. Domar [53] a généralisé ce critère dans la forme (BD) sur le cas de $L_1^\omega(G)$ où G est un groupe localement compact abélien. Dans ma thèse [92], je l'ai étendu au cas $p > 1$ et G abélien, à condition que $L_p^\omega(G)$ soit invariante sous les translations. Si $L_p^\omega(G)$ n'est pas invariante, elle peut être régulière sans vérifier la condition (BD). Dans le cas $p = 1$ cette différence n'apparait pas car les algèbres $L_1^\omega(G)$ sont toujours invariantes.

Dans le cas d'une $*$-algèbre A non-commutative, au lieu de l'espace Ω des idéaux minimaux on considère l'espace des (noyaux des) $*$-representations topologiquement irréductibles. L'algèbre $L_1(G)$ est régulière seulement pour certaines classes de groupes non-commutatifs, notamment pour les groupes à croissance polynômiale. Dans l'article [6], nous nous sommes limités alors au cas des groupes à croissance polynômiale ou la situation non-pondérée est bien comprise. En présence d'un poids, même dans le cas $p = 1$ les méthodes existantes ne permettent pas de déduire la régularité de la la condition (BD). Le meilleur résultat connu pour $p = 1$ de Dziubanski, Ludwig, Molitor-Braun [54] affirme que si

$$\sum_{n=3}^{\infty} \frac{\ln\ln n\ \ln^+ \omega(nx)}{n^2} < \infty \tag{BD'}$$

pour chaque $x \in G$, alors $L_1^\omega(G)$ est régulière. Evidemment il n'y a pas d'implication inverse. Dans l'article [6], nous avons démontré que cette dernière condition implique la régularité aussi dans le cas $p > 1$, sous la condition additionnelle que ω soit sous-multiplicative.

Nous avons étudié aussi d'autres propriétés non-commutatives, telles que la symmétrie, l'existence du calcul fonctionnel, la propriété de Wiener et de Wiener faible, l'existence d'idéaux minimaux de la coque donnée. Pour certaines d'elles, des différences entre le cas $p = 1$ et le cas $p > 1$ se manifestent.

[1]. *E. Abakoumov, Yu. Kuznetsova*, **Density of translates in weighted L^p spaces on locally compact groups**, *Monatshefte Math.* 183(3), 397–413 (2017).

Cet article considère des espaces $L_p(G,\omega)$ qui ne sont pas forcément des algèbres et s'intéresse à l'action du groupe G sur ces espaces par les translations. Pour que cette question ait du sens, le poids doit vérifier la condition

$$\|T_s\| = \operatorname*{ess\,sup}_{t\in G} \frac{\omega(st)}{\omega(t)} < \infty$$

pour $s \in G$. On peut aussi considérer l'action d'un semigroupe $S \subset G$, et dans ce cas-là il suffit que la norme $\|T_s\|$ soit finie pour $s \in S$ seulement.

Nous avons obtenu un critère très général de l'existence de fonctions S-denses, c'est-à-dire de $f \in L_p(G,\omega)$ dont l'orbite $Orb_S(f) = \{T_s f : s \in S\}$ est dense dans $L_p(G,\omega)$. De fonctions S-denses existent alors si et seulement si pour toute suite croissante $(F_n)_{n\geqslant 1}$ de parties compactes de G et tous $\delta_n > 0$, il existe une suite $(s_n)_{n\geqslant 1} \subset S$ et des compacts $K_n \subset F_n$ tels que les ensembles $s_n^{-1}F_n$ sont deux-à-deux disjoints, $|F_n \setminus K_n| < \delta_n$ et, avec $s_0 = e$, $K_0 = \emptyset$,

$$\sum_{n,k\geqslant 0: n\neq k} \|\omega\|^p_{p,s_n s_k^{-1}K_k} < \infty.$$

Pour un groupe G discret abélien, notre critère se simplifie et prend la forme suivante: de fonctions S-denses existent si et seulement si

$$\inf_{s\in S} \max\big(\omega(s),\omega(s^{-1})\big) = 0.$$

Cela correspond au résultat de H. Salas [119] dans le cas $G = \mathbb{Z}$.

Il est montré à l'aide d'exemples qu'en général, la formulation du critère ne peut pas être simplifiée, et en particulier on ne peut ni éviter d'avoir les produits $s_n s_k^{-1}$ ni garantir que $F_n = K_n$.

Théorèmes non-commutatifs de dualité

Rappelons le classique théorème de dualité de Pontryagin: pour un groupe G commutatif localement compact, soit son dual $\widehat{G}$ le groupe des caractères unitaires continus de G, alors G est canoniquement isomorphe à son groupe bidual $\widehat{\widehat{G}}$. Dans le cas non-commutatif, il est naturel de définir le dual comme l'espace des représentations unitaires irréductibles de G. Or cet espace ne possède pas de structure naturelle de groupe, ce qui fait qu'on ne retrouve pas la même symétrie comme dans le cas commutatif.

Dans la théorie des groupes quantiques localement compacts, il n'y a plus de groupes dans le sens direct, mais on travaille avec des algèbres d'opérateurs à leur place. Pour tout algèbre A de cette catégorie, on démontre que $A \simeq \widehat{\widehat{A}}$. Pour un groupe classique G, les algèbres en dualité sont l'algèbre $C_0(G)$ des fonctions continues qui s'annulent à l'infini et $C_r^*(G)$, la C^*-algèbre réduite de G. Cette approche permet d'utiliser les mêmes outils pour les algèbres et leurs duaux.

La définition de l'algèbre duale et le théorème de dualité s'appuient sur l'existence du poids de Haar (une fonctionnelle linéaire, à priori non-bornée) qui est une généralisation de la mesure de Haar. Des remarques similaires pourraient être faites par rapport à une autre notion existante d'un groupe quantique topologique, par les unitaires multiplicatives de Woronowicz.

Il est important de noter que sur un groupe localement compact, l'existence de sa mesure invariante est un théorème qui provient de ses propriétés topologiques et algébriques de groupe, et il était toujours questionné si on peut exclure l'existence du poids de Haar des axiomes d'un groupe quantique et ensuite la démontrer en tant que théorème.

Cette question est ouverte, mais il existe des résultats qui permettent de construire le dual sans utiliser le poids de Haar. Dans ce cas-là, on arrive naturellement à un dual «universel» qui prend en compte toutes les représentations de l'algèbre donnée et non seulement la représentation régulière. Le premier travail dans cette direction est la thèse de E. Kirchberg [83] qui n'est jamais parue en revue. Il travaille dans la catégorie $\mathcal{H}$ des algèbres coinvolutives de Hopf-von Neumann; cela inclut tous les algèbres de Kac, mais pas tous les groupes quantiques. J. Kustermans [87], en reprenant des idées de Kirchberg, a défini le dual universel $\widehat{M}_u$ d'un groupe quantique localement compact M. P.M. Sołtan et S.L. Woronowicz [127] en construisent l'analogue pour tout unitaire multiplicatif. Les duaux universels trouvent souvent leurs applications dans la théorie des groupes localement compacts [82, 49].

L'isomorphisme $A \simeq \widehat{\widehat{A}}$ n'a pas de sens dans les approches de Kustermans et Sołtan–Woronowicz car le dual universel n'est pas un groupe quantique localement compact (unitaire multiplicatif respectivement) et alors le second dual n'est pas défini. Dans l'approche de Kirchberg, on a bien cette égalité pour les duaux des algèbres de Kac. Le foncteur de Kirchberg n'a pourtant pas résulté en une nouvelle définition d'un groupe quantique. Cela s'explique probablement par le fait que l'égalité $A \simeq \widehat{\widehat{A}}$ n'implique pas que l'algèbre A correspond à un groupe, même si elle est commutative; A peut correspondre à un semigroupe sans qu'il soit un groupe.

Dans les articles [4, 3, 8] j'ai construit des dualités, sur trois catégories différentes, basées sur un autre principe que les approches ci-citées. Étant donné une bigèbre A, l'idée est de considérer son dual linéaire A^* (l'espace des fonctionnelles linéaires), montrer qu'il est une algèbre et définir $\widehat{A}$ comme l'algèbre enveloppante de A^* par rapport à une certaine topologie. Les précisions topologiques sont différentes dans tous les trois cas.

[4]. *Yu. Kuznetsova*, **A duality for Moore groups**. *J. Oper. Theory* **69** Iss.2 (2013), 571–600.

Dans cette article, la catégorie est assez limitée. Ses objets sont des pro-C^*-algèbres (munies de comultiplication et d'antipode), qui sont alors des algèbres topologiques non-Banach. La catégorie inclut les algèbres $C(G)$ des fonctions continues sur tous les groupes de Moore G, c'est-à-dire les groupes dont toutes les représentations irréductibles sont de dimension finie. Cette classe de groupes inclut, en particulier, tous les groupes abéliens et compacts. Les méthodes sont similaires à l'article [11] qui considère une dualité d'une classe de groupes complexes de Lie.

L'ntérêt de cette catégorie est dans le fait que ses objets sont des algèbres de Hopf dans un sens stricte. Aves un produit tensoriel topologique, naturel pour les pro-C^*-algèbres, elles vérifient tous les axiomes algèbriques d'une algèbre de Hopf, sans modifications qui sont habituellement faites dans l'analyse.

[3]. *Yu. Kuznetsova*, **A duality of locally compact groups which does not involve the Haar measure**. *Math. Scand.* **116** (2015), 250–286.

Cette construction est faite dans la catégorie des algèbres coinvolutives de Hopf–von Neumann qui sont des algèbres de von Neumann munies de comultiplication et d'antipode borné (et non pas de mesure de Haar). En particulier, $\mathcal{H}$ contient les algèbres $L^\infty(G)$ et $VN(G)$ pour tout groupe localement compact G. Pour toute algèbre M dans $\mathcal{H}$, je définis son algèbre duale $\widehat{M}$, en considérant ses corepresentations unitaires. Je ne détaille ici pas la construction.

Le dual de $L^\infty(G)$ est $W^*(G)$, l'algèbre d'Ernest du groupe G. C'est au même temps l'algèbre de von Neumann enveloppante de $C^*(G)$. Le dual de $VN(G)$ et de $W^*(G)$ est $C_0(G)^{**}$, l'algèbre de von Neumann enveloppante de $C_0(G)$. Le passage aux algèbres de von Neumann est plutôt technique; je construis aussi une version C^* qui met en dualité les algèbres $C_0(G)$ et $C^*(G)$.

Pour une algèbre de Kac, on retrouve le dual universel défini dans la thése de E. Kirchberg [83]; en général, ces deux constructions sont différentes. Contrairement à [83], toute algèbre duale commutative $\widehat{M}$ dans [3] se décrit comme $C_0(G)$ pour un groupe localement compact G, ainsi que toute algèbre duale co-commutative est isomorphe à $C^*(G)$ pour un certain groupe localement compact G.

[8]. *Yu. Kuznetsova*, **Duals of quantum semigroups with involution**, 22 p., prépublication, ArXiv:1611.04830 [math.OA].

Dans cet article, j'étend l'approche précédent à une catégorie plus large, qui inclut tous les groupes quantiques localement compacts.

Soit M un semigroupe quantique à l'involution, c'est-à-dire une algèbre de von Neumann munie de comultiplication et d'un antipode $S : D(S) \to M$ dont le domaine $D(S)$ est dense dans M. (Quelques conditions techniques sur S sont omises.) Dans ces hypothèses, l'espace prédual M_* est une algèbre de Banach avec la multiplication $\mu * \nu = (\mu \otimes \nu)\Delta$, $\mu, \nu \in M_*$; l'antipode S induit une involution sur M_*, a priori non définie partout. Notons $\mathcal{C}$ la catégorie des algèbres décrites ci-dessus.

Mon but était de construire le dual de toute algèbre $M \in \mathcal{C}$ de sorte que dans le cas où M est un groupe quantique localement compact, son dual soit le dual universel $\widehat{M}_u$ dans le sens de Kustermans, et que le dual de $\widehat{M}_u$ soit M_u.

Le point central dans ce travail est de considérer l'idéal M_*^0 dans M_* égal au noyau commun des représentations unitaires *irréductibles*. Il s'avère que cet idéal contient toute l'information sur l'agèbre duale de M; en particulier, dans le cas de $M = L^\infty(G)$, la C^*-algèbre enveloppante de M_*^0 est égale à $C^*(G)$, et si $M = VN(G)$ (l'algèbre de von Neumann de G), alors $C^*(M_*^0) = C_0(G)$.

Dans le cas général, je pose alors $\widehat{M} = C^*(M_*^0)^{**}$. Par définition, les représentations irreductibles de cette algèbre sont en bijection avec les coreprésentations unitaires irréductibles de M; grâce à la désintégration, la bijection existe également entre les repésentations réductibles. On retrouve alors la dualité universelle de Kustermans, et entre autres l'égalité $\widehat{\widehat{\widehat{M}}} = \widehat{M}$ dans le cas d'un groupe quantique localement compact M.

Les algèbres duales (co-)commutatives se décrivent toujours, comme dans [3], en tant qu'algèbres de groupes classiques. En général, les algèbres duales vérifient l'axiome de l'antipode dans une certaine forme, ce qui suggère qu'elles sont des groupes quantiques et non seulement des semigroupes.

Homomorphismes entre les groupes localement compacts et leurs algèbres

[5]. *Yu. Kuznetsova*, **On continuity of measurable group representations and homomorphisms**. *Stud. Math.* **210** no. 3 (2012), 197–208.

Les questions de dualité m'ont amené à des études d'homomorphismes entre les groupes localement compacts et leurs algèbres.

Dans cet article, je démontre que chaque représentation mesurable d'un groupe localement compact est automatiquement continue. La mesurabilité est comprise ici dans le sens suivant. Soit $\pi : G \to B(H)$, où H est un espace de Hilbert, une représentation de

G. Munissons $B(H)$ de la topologie faible. Alors π est mesurable si pour chaque ouvert U dans $B(H)$, $\pi^{-1}(U)$ est mesurable au sens de Haar dans G.

Cette condition de mesurabilité est plus forte que l'hypothèse que les fonctions coefficients $g \mapsto \langle \pi(g)x, y\rangle$ soient mesurables pour tous $x, y \in H$. En effet, considérons par exemple la représentation régulière du groupe non-discret $\mathbb{T} = \{z : |z| = 1\}$ sur $\ell^2(\mathbb{T}_d)$ où $\mathbb{T}_d$ est la discrétisation de $\mathbb{T}$. Cette représentation est évidemment discontinue; or toutes ses fonctions coefficients ont un support dénombrable et sont donc mesurables.

Dans le cas où H est séparable, il était connu depuis des années 1950s que même la mesurabilité des coefficients suffit pour que π soit continue. Dans le cas où H est non-séparable, le résultat est nontrivial et plutôt inattendu.

Considérons maintenant π comme un homomorphisme de G au groupe $U(H)$ des opérateurs unitaires sur H. C'est un groupe topologique dans la topologie faible. Posons une question plus large: est-ce que chaque homomorphisme mesurable $\pi : G \to H$ est continu, où H est un groupe topologique?

Cette question attire l'attention de plusieurs chercheurs et il n'existe pas de réponse générale. Dans mon article, j'ai démontré que la réponse est positive si G est localement compact et H est un groupe topologique quelconque, or il a fallu assumer l'axiome de Martin, qui est indépendant des axiomes habituels de la théorie des ensembles (ZFC).

[2]. *Yu. Kuznetsova, J. Roydor*, **Homomorphisms with small bound between Fourier algebras**, *Israel J. Math.* 217 (2017), Iss. 1, 283–301.

En analyse harmonique, les groupes topologiques sont systématiquement étudiés à travers leurs algèbres, telles que $L^1(G)$ ou $C^*(G)$. Une question réciproque se pose naturellement: l'algèbre de groupe contient-elle assez d'informations pour reconstruire le groupe?

Par un théorème classique de Wendel [133], deux groupes localement compacts G, H sont isomorphes si et seulement s'il existe un isomorphisme contractant d'algèbres entre leur algèbres de groupes $L^1(G)$ et $L^1(H)$. Ce résultat a été ensuite généralisé aux algèbres de mesures par Johnson [76] et Rigelhof [113] et, avec la condition plus forte que l'isomorphisme soit isométrique, aux groupes quantiques localement compacts par Daws et Pham [50].

La version duale de ce théorème est le théorème de Walter [132] qui affirme qu'un isomorphisme isométrique entre les algèbres de Fourier $A(G)$ et $A(H)$ entraîne un isomorphisme des groupes G et H.

Les résultats isométriques dans l'analyse fonctionnelle sont naturellement accompagnés par des questions sur sa stabilité : le résultat reste-il vrai si la norme de l'isomorphisme concerné n'est pas égal à 1 mais seulement proche de 1? Si oui, quel est le degré de proximité nécessaire? L'un des résultat classiques de ce genre est le théorème d'Amir–Cambern [16, 37] : étant donnés des espaces localement compacts X, Y, s'il existe un isomorphisme $f : C_0(X) \to C_0(Y)$ de norme inférieure à 2, alors X et Y sont homéomorphes.

Pour les algèbres de groupes, un résultat de N. Kalton et G. Wood [80] affaiblissant les assomptions du théorème de Wendel montre que deux groupes localement compacts sont isomorphes si et seulement s'il existe un isomorphisme d'algèbres $T : L_1(G) \to L_1(H)$ de norme $\|T\| < \gamma \approx 1.246$. Ils ont également montré que si tous les deux groupes sont abéliens, la constante optimale vaut $\sqrt{2}$.

Dans notre article, nous avons généralisé le résultat de Walter au cas non-isometrique et avons obtenu donc un résultat dual à celui de Kalton et Wood. Soient G, H des groupes localement compacts. Si $T : A(G) \to A(H)$ est un isomorphisme d'algèbres de Fourier tel que $\|T\|_{cb}\|T^{-1}\|_{cb} < \sqrt{3/2}$ ($\|\cdot\|_{cb}$ est la norme dans le sens d'espaces d'opérateurs), alors les groupes sont isomorphes. De plus, T s'exprime explicitement par cet isomorphisme de groupes.

Un résultat similaire est valable pour les algèbres de Fourier–Stieltjes $B(G)$, $B(H)$ de groupes, par rapport à la norme usuelle d'opérateurs mais avec une constante non-explicite: il existe une constante universelle $\varepsilon_0 > 0$ telle que si $T : B(G) \to B(H)$ est un isomorphisme d'algèbres tel que $\|T\|\|T^{-1}\| < 1 + \varepsilon_0$, alors les groupes sont isomorphes.

[9]. *M. A. Aukhadiev, Yu. Kuznetsova*, **Quantum semigroups generated by locally compact semigroups**, 14 p. ArXiv : 1504.00407 [math.FA] (2015). Soumis, révisé.

La notion d'un semigroupe quantique, comme une C^*-algèbre ou algèbre von Neumann munie de comultiplication, est apparue même avant celle d'un groupe quantique localement compact. Mais c'est surtout ces dernières années que d'exemples nontriviaux de semigroupes quantiques sont construits, dont notamment les familles de transformations d'espaces quantiques finis de P. M. Sołtan [125], semigroupes quantiques de permutations partielles de T. Banica et A. Skalski [20], fonctionelles quantiques presque périodiques de M. Daws [48], compactifications quantiques de Bohr de P. Salmi [120] et P. Sołtan [126].

Dans l'article [9], nous construisons une famille assez classique de semigroupes quantiques compacts, qui sont associés à des sous-semigroupes de groupes localement compacts. Le cas discret a été considéré par X. Li [94] et nous utilisons certaines de ses techniques. L'intérêt dans ces objets s'explique par le fait qu'ils sont des C^*-bigèbres co-commutatives mais ne sont pourtant pas duales à des algèbres de fonctions. A rappeler, cet effet ne se produit pas dans la classe des groupes quantiques: tout groupe quantiques commutatif ou co-commutatif se décrit respectivement comme une algèbre de fonctions, telle que $C_0(G)$ ou $L^\infty(G)$, ou bien est le dual d'une algèbre fonctionnelle, comme par exemple $VN(G)$ qui est le dual de l'agèbre de Fourier $A(G)$.

Soit alors G un groupe localement compact, et soit S son sous-semigroupe tel que $S^{-1}S = G$. Notons par L_a, $a \in G$, l'opérateur de translation à gauche sur $L^2(G)$. Soit E_S la projection orthogonale de $L^2(G)$ sur l'espace $H_S = \{f \in L^2(G) : \operatorname{supp} f \subset S\}$, et soit J_S l'inverse à droite de E_S, de sorte que $E_S J_S = \mathrm{Id}_{H_S}$. Nous définissons $C^*_\delta(S)$ comme la C^*-algèbre engendrée dans $B(H_S)$ par les operateurs $T_a = E_S L_a J_S$, avec $a \in S$. Nous montrons ensuite que $C^*_\delta(S)$ admet une comultiplication Δ telle que $\Delta(T_a) = T_a \otimes T_a$ pour $a \in S$.

Même dans le cas d'un semigroupe commutatif, l'algèbre obtenue peut être non-commutative; si le semigroupe était non-commutatif, on obtient alors des algèbres véritablement quantiques.

Chapter 2

Weighted L^p–spaces

Weights and weighted function spaces play an important role in mathematics. In essence, a weight makes it possible to study the behaviour of functions around a certain point, ignoring their oscillations at infinity, or on the contrary, to amplify the asymptotic behaviour of a function. More precisely, introducing a weight means modelling in a quantitative manner the decay of the functions to be studied. This has numerous applications in numerical mathematics and is often used for concrete applications (signal theory, Gabor analysis, sampling theory; see [71, 45, 70, 61]).

On the other hand, weights appear naturally in analysis, for example in inequalities relating the norm of a function to the norm of its derivatives, or in extension theorems (see, e.g., a survey of Kudryavtsev and Nikol'sky [86]). One of the areas where weighted spaces are applied most intensively is the theory of boundary value problems for partial differential equations (see the surveys [86], [68]).

In the representation theory weights occur for instance in the following way. If G denotes a locally compact group and (T, V) is a continuous representation of G on a Banach space V, then the map $\omega : x \mapsto \|T(x)\|$ is a weight which is submultiplicative:

$$\omega(xy) \leqslant \omega(x)\omega(y). \tag{2.1}$$

This condition makes the weighted space

$$L^1(G, \omega) := \{f : G \to \mathbb{C} \mid f \text{ measurable and } \int_G |f(x)|\omega(x)dx < +\infty\}$$

a convolution algebra, and the map

$$f \mapsto T(f) := \int_G f(x)T(x)dx$$

is a representation of $L^1(G, \omega)$.

The first to consider weighted convolution algebras was Beurling [26] and independently Gel'fand [66, 67], for whom they provided new examples in the emerging theory of Banach algebras. Later Beurling [25] worked on harmonic analysis properties of these algebras, and along other results showed that $L^1(\mathbb{R}, \omega)$ is regular, that is contains functions $f \in L^1(\mathbb{R}, \omega)$ with compactly supported Fourier transform, if and only if

$$\int_{\mathbb{R}} \frac{\ln^+ \omega(x)}{1 + x^2} dx < +\infty.$$

If on the contrary for a weight ω this integral diverges, then for the Fourier transform of $L^1(\mathbb{R},\omega)$ unicity theorem holds, and this function space is called quasianalytic.

Weighted convolution algebras $L^1(G,\omega)$ on locally compact groups have become a popular object of study since then, in particular in the radical case [19]. It would be too long to give a survey of this theory here.

On the other hand, the importance of L^p-spaces of the form $L^p(G)$ or $L^p(G,\omega)$, $1 < p < +\infty$, is well known in functional analysis. It would be attractive to extend the theory of convolution algebras to the L^p-case, because L^p spaces are reflexive, what is not a common property among Banach algebras. However, if G is not compact and if $p \neq 1$, $L^p(G)$ is not an algebra for convolution. This so-called L^p-conjecture was proved in the abelian case by Żelasko [140] and in general, after subsequent generalizations by other authors, by Saeki [118]. Nevertheless, for appropriate groups G and weights ω, the weighted L^p-spaces

$$L^p(G,\omega) := \{f : G \to \mathbb{C} \mid f \text{ measurable and } \Big(\int_G |f(x)|^p \omega(x)^p\Big)^{\frac{1}{p}} < +\infty\}$$

may be algebras. There is no exact criterion, but it sufficient [135, 90] that

$$\omega^{-q} * \omega^{-q} \leq C\omega^{-q} \tag{LpAlg}$$

for some constant $C > 0$, where $\frac{1}{p} + \frac{1}{q} = 1$. For the existence of such weights, a sufficient condition on the group G is that G is σ-compact. In that case, there are even a lot of such weighted L^p-algebras with all kinds of different growth behaviours.

In the context of weighted L^p-algebras, let us mention the works of Wermer [135], Nikol'ski [102], Feichtinger [60], Kuznetsova ([90]-[7]). The best-studied case is $L^p(\mathbb{Z},\omega)$, see, e.g., a long paper of El-Fallah, Nikol'ski and Zarrabi [56]. This is mainly for the reason that in the problems of weighted approximation by polynomials, as initiated by S. N. Bernstein [24], $L^p(\mathbb{Z},\omega)$ algebras play a distinguished role [102]. Other applications include interpolation theory and in questions of factorization [27].

2.1 An example of a weighted algebra $L_p^w(G)$ on uncountable group

On the real line, it is straightforward that the weight $\omega(x) = (1+x^2)^d$ satisfies (LpAlg) for every $d \geqslant 1$ and $p > 1$. By the same principle, one can construct [60] weights on any locally compact group G of polynomial growth. Namely, let $G = \cup_{n\in\mathbb{N}} U^n$ with a compact neighbourhood of identity U. Recall that G is of polynomial growth if the Haar measure of U^n is bounded by a polynomial of n. Set

$$|x| := \min\{n \in \mathbb{N} : x \in U^n\} \ \text{ for } x \neq e \text{ and } |e| = 1.$$

If needed, one can indicate U by denoting the same value by $|x|_U$.

Now, the weight $\omega(x) = (1+|x|)^D$ satisfies (LpAlg) for D sufficiently large [60], so $L^p(G,\omega)$ is an algebra.

Moreover, one can show [6] that $\omega(x) = e^{|x|^\gamma}(1+|x|)^D$ with $0 < \gamma \leqslant 1$ also satisfies (LpAlg) for all D sufficiently large (bound depending on $p > 1$).

If the weight is submultiplicative, as it is usually assumed, then it can grow at most exponentially. Note that however $\omega(x) = e^{|x|}$ is not an L^p-algebra weight, what is not difficult to check.

Subexponential weights like the ones introduced above appear in the context of nilpotent Lie groups. In fact, let G be a connected, nilpotent Lie group. Let G_1 be the derived group of G, i. e. the closed subgroup generated by the elements of the form $[x, y] = x^{-1}y^{-1}xy,\ x, y \in G$. Let U be a generating neighbourhood of the identity e in G and $V = U \cap G_1$ the corresponding neighbourhood of e in G_1. Then it is shown in [15], that for any submultiplicative weight ω,

$$\omega|_{G_1}(x) \leq e^{C|x|_V^{\frac{1}{2}}},\ \forall x \in G_1,$$

for some constant C.

The condition (LpAlg) involves integrals of strictly positive functions, and such functions exist only on σ-compact groups (covered by countably many compact sets). One would like to understand of course whether conversely, a group must be σ-compact to admit weighted L^p-algebras with $p > 1$ (a kind of weighted L^p-conjecture).

A discrete group is σ-compact if and only if it is countable, and it is proved in my thesis [91] that an abelian discrete group must be indeed countable in order that L^p-algebra weights exist.

Later I showed that in general, this condition is not necessary. Namely, in [7], a weight ω is constructed on the free group $\mathbb{F}_{\mathfrak{m}}$ of arbitrary cardinality $\mathfrak{m}$ such that with this weight, $\ell^p(\mathbb{F}_{\mathfrak{m}}, \omega)$ is a convolution algebra for $1 < p \leqslant 2$. In the case $p > 2$, it is shown that the space $\ell^p(\mathbb{F}_{\mathfrak{m}}, \omega)$ is not an algebra with any weight.

For $1 < p \leqslant 2$, the weight is constructed as follows. Denote by A the set of generators of $\mathbb{F}_{\mathfrak{m}}$. We consider elements of $\mathbb{F}_{\mathfrak{m}}$ as reduced words in the alphabet $A \cup A^{-1}$. Let F_n be the set of words of lengths n (in their reduced form). Now, we define

$$\omega|_{F_n} = (n + 1)^3,$$

though in fact any number > 2 may be taken instead of 3. In this example, the condition (LpAlg) necessarily fails, and it is proved by explicit estimates that $\ell^p(\mathbb{F}_{\mathfrak{m}}, \omega)$ is closed under convolution.

2.2 Harmonic analysis of weighted L^p–algebras

In the paper [6] with Carine Molitor-Braun, after giving a series of examples of L^p-algebras, we worked on their harmonic analysis properties. For the reasons discused below, all the groups are supposed locally compact of polynomial growth (compactly generated, $G = \cup U^n$, with polynomial growth of the volume of U^n). A finitely generated discrete group has polynomial growth if and only if it is a finite extension of a nilpotent group, by a theorem of Gromov [72]. Among Lie groups, this class includes all nilpotent groups as for example the Heisenberg group, but contains also many non-nilpotent solvable groups [32].

On a Banach algebra, there is a much richer structure and more means of study in the presence of involution, and this is also true for weighted convolution algebras. As the groups of polynomial growth are unimodular, the natural involution is given by the formula $f^*(t) = \bar{f}(t^{-1})$ (not involving the modular function of the group). It is well defined for all $f \in L^p(G, \omega)$ if $\omega(t) = \omega(t^{-1})$.

2.2.1 Symmetry

It is most natural to work with $*$-representations of a Banach $*$-algebra if it is *symmetric*: such that the spectrum $\sigma(a)$ is real for every self-adjoint element $a = a^*$. Then, as shown by Leptin [93], every algebraically irreducible representation is a $*$-representation with respect to a suitable inner product, what is not true in general.

Any C^*-algebra is symmetric (see [99]). But for the algebra $L^1(G)$, this depends on the group. If $G = \mathbb{R}^n$, $\mathbb{Z}^n$ or $\mathbb{T}^n$, then $L^1(G)$ is symmetric. The same is true for any connected, simply connected, nilpotent Lie group. All these results, though known separately for some time, are particular cases of a general theorem by Losert [96]: If G is of polynomial growth, then $L^1(G)$ is symmetric.

In the weighted case, this property depends also on the weight. It has been proved already by Gel'fand, Raikov and Shilov [101] that $\ell^1(\mathbb{Z}, \omega)$ is symmetric if and only if $\omega(n)^{1/n} \to 1$, $n \to \pm\infty$. In fact, an abelian Banach $*$-algebra is symmetric iff all its characters are unitary, and for the case of $\ell^1(\mathbb{Z}, \omega)$ this means that the space of maximal ideals would be the unit circle, as in the non-weighted case; this corresponds exactly to the growth rate of ω cited above. In the non-abelian case, the general result is due to Fendler, Gröchenig and Leinert [61]. They proved that on a group G of polynomial growth, $L^1(G, \omega)$ is symmetric if and only if ω satisfies the following GRS-condition (after Gel'fand–Raĭkov–Shilov):

Definition 2.2.1. A weight ω on G is said to satisfy the GRS-condition if

$$\lim_{n\to+\infty} \omega(x^n)^{\frac{1}{n}} = 1, \quad \forall x \in G. \tag{GRS}$$

For $p > 1$, we prove the following [6]. Let G be a group of polynomial growth and ω a submultiplicative weight satisfying (LpAlg).

If G is abelian and ω satisfies (GRS), then $L^p(G, \omega)$ is a symmetric Banach $*$-algebra. For example: $L^2(\mathbb{R}, e^{\sqrt{|x|}})$ is symmetric.

In the non-abelian case the situation is more complicated. Before the results of Fendler, Gröchenig and Leinert [61], Pytlik [110] proved that $L^1(G, \omega)$ is symmetric if

$$\omega(xy) \le C(\omega(x) + \omega(y)), \quad \forall x, y \in G$$

for some positive constant C and in addition $\omega^{-1} \in L^q(G)$ for some $0 < q < +\infty$.

We will call such a weight *polynomial in the sense of Pytlik*. In particular, weights of the form

$$\omega(x) = (1 + |x|)^D \tag{2.2}$$

for some positive D, where

$$|x| := \min\{n \mid x \in U^n\},$$

satisfy this condition [109], and moreover, every such weight is dominated by a weight of the form (2.2) [110].

In [6], we prove that for $p > 1$ and under assumptions on (G, ω) as above, if ω is polynomial in the sense of Pytlik then $L^p(G, \omega)$ is a symmetric Banach $*$-algebra.

The case of a general weight is still open.

2.2.2 Regularity

Informally speaking, an abelian Banach algebra is regular is it contains enough functions with compactly supported Gelfand transform. More precisely, this property is defined as follows by Šilov [123]: Let $\mathcal{A}$ be an abelian Banach algebra and let $\Omega(\mathcal{A})$ denote the space of characters of $\mathcal{A}$. Then $\mathcal{A}$ is said to be regular if, given any $\varphi \in \Omega(\mathcal{A})$ and any closed set $F \subset \Omega(\mathcal{A})$ not containing φ, there exists $x \in \mathcal{A}$ such that $\widehat{x}(\varphi) = \varphi(x) = 1$ and $\widehat{x}|_F \equiv 0$, where $\widehat{x}$ denotes the Gelfand transform of x.

If G is a locally compact abelian group, then $L^1(G)$ is regular. This is in particular the case for $L^1(\mathbb{R})$: By Paley-Wiener theorem, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ extends to an entire function of exponential growth, then $\widehat{f}$ is compactly supported. By multiplying f by a complex exponential and/or rescaling, we may make the support fit in any given interval. A simple example of such a function f is given by $f(x) = (e^{ix} + e^{-ix} - 2)/x^2$. For the proof of regularity of $L^1(G)$ for a general locally compact abelian group, see for instance [81].

For abelian regular semi-simple Banach algebras one can say a lot about their ideals and about spectral synthesis as initiated by Šhilov [123].

In the case of a non-abelian Banach $*$-algebra, $\Omega(\mathcal{A})$ should be replaced by the space $\mathrm{Prim}_*\mathcal{A}$, the set of all kernels of topologically irreducible $*$-representations of $\mathcal{A}$ equipped with the hull-kernel topology. The algebra $L^1(G)$ is not regular for every locally compact group G. Since first results of Boidol and Leptin [29], there has been much work on this question. It is known [29, 97] that if G is of polynomial growth, then $L^1(G)$ is symmetric; if G is an exponential Lie group, then $L^1(G)$ is regular iff it is symmetric [107].

In the weighted case, Domar [53] proves that for an abelian group G, the algebra $L^1(G, \omega)$ is regular if and only if the weight ω satisfies

$$\sum_{n=1}^{\infty} \frac{\ln \omega(x^n)}{n^2} < +\infty, \quad \forall x \in G.$$

In the case $p > 1$, the same result is valid [92] if ω is submultiplicative and satisfies (LpAlg).

For a non-abelian group G, there is no criterion, and the best positive result is as follows. Let as before $G = \cup U^n$, where U is a compact neighbourhood of identity. Set

$$s(n) = \sup_{x \in U^n} \omega(x), \qquad \forall n \in \mathbb{N}$$

and $s(0) = 1$. A weight ω is said to satisfy the non-abelian Beurling-Domar condition (BDna) if

$$\sum_{n \in \mathbb{N}, n \geq e^e} \frac{\ln(\ln n) \ln\big(s(n)\big)}{1 + n^2} < +\infty.$$

This condition is independent of the choice of the generating neighbourhood U. Now, if ω satisfies (BDna), then $L^1(G, \omega)$ is regular [54].

For $p > 1$, we show in [6] that $L^p(G, \omega)$ is regular under the same condition (BDna), for a submultiplicative weight ω satisfying (LpAlg). The main tool is, as in [54], functional calculus on a total subset of the algebra $L^p(G, \omega)$ as discussed below.

2.2.3 Functional calculus

Standard tools of functional analysis include holomorphic calculus in Banach algebras and continuous calculus in C^*-algebras. In certain cases, however, one can apply some non-holomorphic continuous functions to elements of non-C^* Banach algebras. This method goes back to Dixmier [52] and works as follows.

Suppose first that the given Banach algebra $\mathcal{A}$ is unital. For a smooth enough function $\psi : [0, 2\pi] \to \mathbb{R}$ we can write

$$\psi(t) = \sum_{n \in \mathbb{Z}} \widehat{\psi}(n) e^{int}.$$

This suggests that we set, for $f = f^* \in \mathcal{A}$,

$$\psi\{f\} = \sum_{n\in\mathbb{Z}} \widehat{\psi}(n)e^{inf}, \tag{2.3}$$

defining e^{inf} as $\sum_{k=0}^{\infty} \frac{(inf)^k}{k!}$. We should guarantee the convergence of the series (2.3) by a rapid enough decay of $\widehat{\psi}(n)$ with respect to the growth of $\|e^{inf}\|$, so in other words for a given f we need to determine the class of ψ for which (2.3) has sense. This question is solved differently on every algebra $\mathcal{A}$.

In a non-unital algebra we replace e^{inf} by

$$u(nf) = \sum_{k=1}^{\infty} \frac{(inf)^k}{k!}$$

and set

$$\psi\{f\} = \sum_{n\in\mathbb{Z}} u(nf)\widehat{\psi}(n), \tag{2.4}$$

choosing ψ so that $\psi(0) = \sum_{n\in\mathbb{Z}} \widehat{\psi}(n) = 0$.

Let (G,ω) satisfy (LpAlg) and (BDna). We suppose ω submultiplicative, so there exists a constant C such that $\omega(x) \leqslant e^{C|x|}$ for all x. Similarly to [54], we show in [6] that for every continuous function f on G with compact support such that $f = f^*$ there exist positive constants A_1, A_2 (depending on f and ω) such that

$$|u(nf)|_{p,\omega} \leq A_1(1+|n|)^2(1+n^2)^{\frac{Q}{2}} s(|n|)^{2\ln(\ln|n|)} e^{A_2\left(\frac{|n|}{(\ln|n|)^C}\right)},$$

where Q denotes the power appearing in the polynomial growth condition of the group G, i. e. $|U^n| \leq Kn^Q$, for all $n \in \mathbb{N}$, for some positive constant K. We then have the following result:

Theorem 2.2.2. *Let (G,ω) satisfy (LpAlg) and (BDna). Let $f = f^*$ be a continuous function with compact support. Then, given a, b, ε such that $0 < a < a+\varepsilon < b-\varepsilon < b < 2\pi$, there exists a function $\psi : \mathbb{R} \to \mathbb{R}$, continuous, periodic of period 2π such that $\mathrm{supp}\psi \cap [0, 2\pi] \subset [a, b]$, $\psi \equiv 1$ on $[a+\varepsilon, b-\varepsilon]$ and*

$$\sum_{n\in\mathbb{Z}} \|u(nf)\|_{p,\omega} |\widehat{\psi}(n)| < +\infty.$$

Hence the equality (2.4) defines a function $\psi\{f\} \in L^p(G,\omega) \cap L^1(G,\omega)$ and the properties of functional calculus are satisfied, i. e.

$$\chi(\psi\{f\}) = \psi(\chi(f))$$

for every character χ of the abelian Banach $*$-subalgebra of $L^p(G,\omega)$ generated by f,

$$\pi(\psi\{f\}) = \psi(\pi(f))$$

for every topologically irreducible $*$-representation π of $L^p(G,\omega)$, and

$$(\varphi\psi)\{f\} = \varphi\{f\} * \psi\{f\},$$

if the functions φ and $\varphi\psi$ still have the correct properties to allow functional calculus.

This theorem applies in particular to the weights polynomial in the sense of Pytlik, including the weights $\omega(x) = (1+|x|)^D$ for $D > 0$ large enough, and to the non-polynomial weights $\omega(x) := e^{C|x|^\gamma}$, $0 < \gamma < 1$, on groups of polynomial growth.

Functional calculus is the main tool used in [6] to prove not only regularity, but also other harmonic analysis properties.

2.2.4 Wiener property

For every proper closed ideal I of $L^1(\mathbb{R})$, the Fourier transforms of functions in I have a common zero, i. e. there exists $a \in \mathbb{R}$ such that

$$I \subset \{f \in L^1(\mathbb{R}) \mid \widehat{f}(a) = \int_{-\infty}^{+\infty} f(x)e^{-iax}dx = 0\}.$$

This result implies Wiener's Tauberian theorem [136].

This remains valid for all locally compact abelian groups, the complex exponentials being replaced by the (unitary) characters of the group and the integral being computed with respect to the Haar measure of the group. See [69] and [121].

In general, one has to distinguish two properties:

Definition 2.2.3. Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is said to be *weakly Wiener* if every proper closed two-sided ideal of $\mathcal{A}$ is contained in the kernel of an algebraically irreducible representation.

Let $\mathcal{A}$ be a Banach $*$-algebra. $\mathcal{A}$ is said to be *Wiener* if every proper closed two-sided ideal in $\mathcal{A}$ is contained in the kernel of a topologically irreducible $*$-representation of $\mathcal{A}$ on a Hilbert space.

In these terms, the algebra $L^1(\mathbb{R})$ is Wiener. More generally, if $\mathcal{A}$ is symmetric and weakly Wiener, then it is also Wiener, so both properties coincide for $L^1(G)$ on an abelian locally compact group G.

For the weighted algebras, $L^1(\mathbb{R}, \omega)$ is Wiener if ω satisfies the Beurling's condition

$$\int_{\mathbb{R}} \frac{\ln \omega(x)}{1 + x^2} dx < \infty.$$

If this integral diverges, then for a large class of weights, $L^1(\mathbb{R}, \omega)$ does not have the Wiener property, as proved by Vretblad [130].

It is shown in [6] that if (G, ω) satisfy (LpAlg) and (BDna), then there exist an approximate identity $(f_s)_s$ of $L^p(G, \omega)$ and a 2π-periodic function φ such that $\varphi\{f_s\}$ is defined for all s. This implies that for every proper closed two-sided ideal I of $L^p(G, \omega)$, there exists s such that $\varphi\{f_s\} \notin I$. Using this fact, we prove

Theorem 2.2.4. *Let (G, ω) satisfy (LpAlg) and (BDna). Then the algebra $L^p(G, \omega)$ has the weak Wiener property.*

If in addition $L^p(G, \omega)$ is symmetric, then it has also the Wiener property. Recall that we prove symmetry for weights polynomial in the sense of Pytlik on non-abelian groups, or for weights satisfying (BDna) on abelian groups. In the symmetric case, we prove also the existence of minimal ideals of a given hull in $L^p(G, \omega)$.

2.2.5 A commutative, symmetric algebra having infinite-dimensional irreducible representations

Let T denote any quasi-nilpotent operator on a Banach space V without non-trivial invariant subspaces. The existence of such operators was proved by Read [111], who constructed an operator on l^1 with the stated properties. The quasi-nipotency means that

$$\|T\| \leqslant 1, \qquad \|T^n\|^{1/n} \to 0, \quad n \to \infty. \tag{2.5}$$

We use such an operator T to construct an example of a weighted algebra $L^p(\mathbb{R},\omega)$ for any $p > 1$ which is commutative, symmetric and has infinite-dimensional topologically irreducible representations. Up to our knowledge, the first application to representation theory, using an operator of the previous type, is due A. Atzmon [18].

We introduce a family of weights on $\mathbb{R}$ which will depend on $p \geqslant 1$. For $p = 1$, we take as a weight

$$\omega(x) = \max\{\, \|e^{xT}\|,\, \|e^{-xT}\|\}. \tag{2.6}$$

Obviously, ω is submultiplicative. Thus, $L^1(\mathbb{R},\omega)$ is an algebra. For $p > 1$, we put $\omega_1(x) = \omega(x)(1+|x|)^2$; by [6], this is an L^p-algebra weight (possibly after multiplication by a constant). The algebra $L^1(\mathbb{R},\omega)$ and every algebra $L^p(\mathbb{R},\omega_1)$ with $p > 1$ are symmetric. One can show also that $\omega(x) > C\exp(x/\ln x)$, and so the algebras which we construct are not regular.

Let $\mathcal{A}$ stand for $L^1(\mathbb{R},\omega)$ or for $L^p(\mathbb{R},\omega_1)$ if $p > 1$. Now we can put

$$U(f) := \int_{\mathbb{R}} \exp(xT) f(x) dx$$

for any $f \in \mathcal{A}$. An easy calculation shows that this integral converges absolutely. Clearly U is a homomorphism.

It remains now to show that (U, V) is a topologically irreducible representation of $\mathcal{A}$. Let I_ε be the indicator function of $[0,\varepsilon]$ and let $\xi_\varepsilon = \varepsilon^{-1} I_\varepsilon$. We show in [6] that $T = \lim_{\varepsilon\to 0} 2\varepsilon^{-1}\big(U(\xi_\varepsilon) - \mathbb{I}\big)$. If now Z is an invariant subspace for U, then its closure $\bar{Z}$ is invariant for T, so $\bar{Z}$ is trivial.

This example is interesting for several reasons. The algebras $L^1(\mathbb{R},\omega)$ and $L^p(\mathbb{R},\omega_1)$, $p > 1$, are abelian, which implies that all the topologically irreducible $*$-representations are one-dimensional, i. e. are characters, by Schur's lemma [99]. Because of the symmetry, all the characters are unitary. Nevertheless these algebras admit infinite-dimensional topologically irreducible representations. This suggests that even for abelian, symmetric Banach $*$-algebras, the description of all topologically irreducible representations on Banach spaces is a highly non-trivial, open problem.

2.3 Density of translates in weighted L^p spaces on locally compact groups

In the article [1] with Evgenii Abakumov, we worked on the following question. Consider the weighted space $L^p(G,\omega) = \{f : \int_G |f\omega|^p < \infty\}$, not supposed to be a convolution algebra, on a locally compact group G. Can *all* functions in $L^p(G,\omega)$ be approximated arbitrarily well by the left translates of some *single* function $f \in L^p(G,\omega)$? The same question can be asked if we allow to translate f only by elements of a fixed subset $S \subset G$.

To be more precise, let S be a subset of G, and suppose that, for any $s \in S$, the (left) translation operator

$$(T_s f)(t) = f(s^{-1}t),\ t \in G$$

is continuous from $L^p(G,\omega)$ into itself. A function $f \in L^p(G,\omega)$ is called *S-dense* if its S-orbit $Orb_S(f) = \{T_s f : s \in S\}$ is dense in $L^p(G,\omega)$. So, we are interested in the existence of S-dense vectors in the space $L^p(G,\omega)$.

This is a particular case of a more general situation, known as the universality phenomenon, see a survey of Grosse-Erdmann [73]. One of the first examples in this area was given in 1929 by Birkhoff (see [73]): there exists an entire function f whose translates are dense in the space $\mathcal{H}(\mathbb{C})$ of entire functions.

In the case when S is the semigroup generated by one element $s \in G$, S-density is equivalent to the *hypercyclicity* of the operator T_s, which means that there exists a single vector x such that the sequence $(T_s^n x)_{n\in\mathbb{N}}$ is everywhere dense. In the well-known example of Read [111] mentionned in Subsection 2.2.5, the operator without invariant subspaces is hypercyclic.

If S is a sub-semigroup or a subgroup of G, S-density means in other terms that the action of S by translations on the space $L^p(G,\omega)$ admits a hypercyclic vector. On a non-weighted L^p-space, the translation operators are isometric and therefore cannot have hypercyclic vectors. In the weighted case hypercyclicity can occur, depending on the weight; moreover, the conditions on weight are as a rule explicit and not difficult to calculate. This problem was considered by many authors. The characterisation in the discrete case $G = \mathbb{Z}$, $S = \mathbb{Z}_+$ was obtained by Salas in 1995 [119]. Desch, Schappacher, and Webb [51] characterized the hypercyclicity of the C_0-semigroup of translations by $S = \mathbb{R}_+$, with $G = \mathbb{R}$. The dynamics of the translation C_0–semigroup indexed by a sector of the complex plane was considered by Conejero and Peris [44]. Chen [38] characterized the hypercyclicity of a single translation operator (then S is the semigroup generated by an element of G) for locally compact groups. Our theorems generalize the results [51], [119], [38] mentioned above.

Let us consider in more detail the case $G = \mathbb{Z}$, assuming that the space $\ell^p(\mathbb{Z},\omega)$ is stable under left and right translations. As proved by Salas [119], the left translation operator T on $\ell^p(\mathbb{Z},\omega)$ is hypercyclic if and only if

$$\inf_{n\in\mathbb{Z}} \max\big(\omega(n),\omega(-n)\big) = 0. \tag{2.7}$$

It is not hard to see the necessity of this condition. If a vector $x \in \ell^p(\mathbb{Z},\omega)$ is $\mathbb{Z}$-dense, then for any $\varepsilon > 0$ there is a translation of x which approximates δ_0, the characteristic function of 0. We have for some n

$$\|T^n x - \delta_0\|_{p,\omega} < \varepsilon, \tag{2.8}$$

and decreasing ε, we can avoid having $n = 0$. We can always scale the weight so that $\omega(0) = 1$. Then (2.8) implies in particular that $|x_n - 1| < \varepsilon$ and $\omega(k)|x_{n+k}| < \varepsilon$, $k \neq n$.

Taking ε' smaller than $\|T^n x - \delta_0\|_{p,\omega}$, we can find another integer m such that

$$\|T^m x - \delta_0\|_{p,\omega} < \varepsilon',$$

and by the choice of ε' we have $m \neq n$. This implies that $|x_m - 1| < \varepsilon'$ and $\omega(k)|x_{m+l}| < \varepsilon$, $l \neq m$.

Setting now $k = m - n$ and $l = n - m$, we get $\omega(m - n) < \varepsilon/(1 - \varepsilon')$ and $\omega(n - m) < \varepsilon'/(1 - \varepsilon)$, what implies immediately (2.7).

Even if the non-discrete, non-abelian case is more complicated, the reasoning above remains a good guidance for intuition.

2.3.1 Density criterion

We do not suppose that the set S has any algebraic structure: it can be an arbitrary subset of G.

Of course, to speak of translations we need to know at least that the operator T_s is bounded for every $s \in S$. In the weighted space $L^p(G, \omega)$ this means that

$$\sup \frac{\omega(st)}{\omega(t)} < \infty$$

for every $s \in S$. Such a weight we call S-admissible. In the abelian case, it is more convenient to use the additive notation, so that this condition is written as

$$\sup \frac{\omega(s+t)}{\omega(t)} < \infty. \tag{2.9}$$

On the real line for example, the weight $\omega(t) = 1 + t^2$ is $\mathbb{R}$-admissible and $\omega(t) = e^{t^2}$ is not.

Let $\mathbf{\Sigma}$ denote the class of non-compact second countable locally compact groups. We show that a locally compact group G is in the class $\mathbf{\Sigma}$ if and only if for some weight ω on G and for some $S \subset G$ there exist S-dense functions.

In the most general case, the criterion has the following form.

Theorem 2.3.1. *Let $G \in \mathbf{\Sigma}$, and let $S \subset G$. Let ω be an S-admissible weight on G. There is an S-dense vector in $L^p(G, \omega)$ if and only if for every increasing sequence $(F_n)_{n \geqslant 1}$ of compact subsets of G and any given $\delta_n > 0$, there exists a sequence $(s_n)_{n \geqslant 1} \subset S$ and compact sets $K_n \subset F_n$ such that the sets $s_n^{-1} F_n$ are pairwise disjoint, $|F_n \backslash K_n| < \delta_n$ and, setting $s_0 = e$, $K_0 = \emptyset$,*

$$\sum_{n,k \geqslant 0 : n \neq k} \|\omega\|^p_{p, s_n s_k^{-1} K_k} < \infty. \tag{2.10}$$

If G is discrete, then necessarily we have $K_n = F_n$. One can note also that it is enough to check the conditions of Theorem 2.3.1 for an increasing sequence (F_n) of compact sets such that $G = \cup_n F_n$.

If we suppose $G = \mathbb{Z}$, $F_n = \{0\}$, $S = \mathbb{N}$, then we arrive at the series

$$\sum_{n,k \geqslant 0 : n \neq k} |\omega(s_n - s_k)| < \infty,$$

and it is easy to see that this condition is equivalent to (2.7). It is moreover demonstrated by examples that it is not the series or the p-norm that makes the non-abelian situation more complicated, but the presence of $s_n s_k^{-1}$ which might be not in S, whereas $s_n, s_k \in S$.

If we suppose that S generates a commutative subgroup of G, the criterion is straightforward:

Theorem 2.3.2. *Let $G \in \mathbf{\Sigma}$, and let $S \subset G$ generate an abelian subgroup in G. Let ω be a weight on G. There is an S-dense vector in $L^p(G, \omega)$ if and only if for every compact set $F \subset G$ and any given $\delta > 0$, there exist $s \in S$ and a compact $E \subset F$ such that $\mu(F \setminus E) < \delta$ and*

$$\operatorname*{ess\,sup}_{sE \cup s^{-1}E} \omega < \delta. \tag{2.11}$$

In the non-abelian case the assumptions of Theorem 2.3.2 are sufficient, but not necessary for S-density, see Example 2.3.6 below.

Even in the case $G = \mathbb{Z}$, one does not arrive at the formula (2.7) if we suppose only the translations by $S = \mathbb{N}$ being bounded (and not both left and right ones); Salas [119] has a formula similar to (2.11) for this case. This difference can be illustrated on the following example. Let us set $\lceil x \rceil$ to be the least integer $> x$ and define the weight $\omega(t) = \lceil t \rceil - t$, so that $\omega(n) = 1$ for all $n \in \mathbb{Z}$. Then (2.9) is satisfied for all $s > 0$, but not for $s < 0$, thus this weight is R_+-, but not $\mathbb{R}$-admissible.

The things become really simple if we suppose that $L^p(G,\omega)$ is stable under any translations, left or right. In this case we have

Corollary 2.3.3. *Let ω be a continuous G-admissible weight on a group $G \in \mathbf{\Sigma}$. Suppose that either G is abelian or the right translations R_s are also bounded on $L^p(G,\omega)$ for all $s \in G$. Then, for any given $S \subset G$, there are S-dense vectors in $L^p(G,\omega)$ for all $1 \leqslant p < \infty$ if and only if*

$$\inf_{s\in S} \max\big(\omega(s), \omega(s^{-1})\big) = 0. \tag{2.12}$$

2.3.2 Examples and counter-examples

Example 2.3.4. We show that every group $G \in \mathbf{\Sigma}$ admits a weight with G-dense vectors, and more exactly, that for every subset $S \subset G$ with non-compact closure there is a weight ω on G such that there is an S-dense vector in $L^p(G,\omega)$.

This is done by choosing $(s_n) \subset S$ far enough and setting

$$\omega(s_n) = 2^{-n}$$

for all n. "Far enough" means that with a generating compact neighbourhood $U \subset G$, the sets $U^{n+1}s_nU^{n+1}$ and $U^{n+1}s_n^{-1}U^{n+1}$ are pairwise disjoint for all n. This allows us to set, denoting $V_{n,k} = U_k s_n U_k \cup U_k s_n^{-1} U_k$ for $1 \leqslant k \leqslant n$,

$$\omega(s) = \begin{cases} 1, & s \notin \bigcup_n V_{n,n}, \\ 2^{-n+k}, & s \in V_{n,k} \setminus V_{n,k-1} \quad (k \leqslant n). \end{cases}$$

We have still $\omega(s_n) = \omega(s_n^{-1}) = 2^{-n}$. By construction, the left and right translations are all bounded (with norm at most 2^n for $s \in U^n$), so that we can apply Corollary 2.3.3. Clearly the condition (2.12) holds, and we conclude that $L^p(G,\omega)$ contains an S-dense vector.

As it was said before, it is important in Corollary 2.3.3 that not only all left translations are bounded, but also all right ones. This is illustrated by the following example, in which we suppose the boundedness of only left translations.

Example 2.3.5. Let $\mathbb{F}_2$ be the free discrete group of two generators a and b. There exists an $\mathbb{F}_2$-admissible weight on $\mathbb{F}_2$ which satisfies the condition (2.12) but does not admit $\mathbb{F}_2$-dense vectors.

Set $U = \{e, a, b, a^{-1}, b^{-1}\}$, then $G = \cup_n U^n$. Set $n_k = 2^{2^k}$, $k = 1, 2, \ldots$. Define the weight ω as follows: $\omega(a^{\pm n_k}) = 2^{-k}$, $\omega(t) = 2^{j-k}$ if $t \in (U^j \setminus U^{j-1})\, a^{\pm n_k}$, $1 \leqslant j \leqslant k$, and 1 elsewhere. In particular, $\omega|_U = 1$.

Clearly, the condition (2.12) holds, but it is shown directly that there is no dense vector in $\ell^p(\mathbb{F}_2,\omega)$.

Example 2.3.6. There exists a sub-semigroup $S \subset \mathbb{F}_2$ and an S-admissible weight on $\mathbb{F}_2$ which admits S-dense vectors but does not satisfy the sufficient condition (2.11).

For $k \in \mathbb{N}$, set $n_k = 2^{2^k}$ and $s_k = a^{n_k}b$. Let S be the semigroup generated by $\{s_k : k \in \mathbb{N}\}$. Set $U = \{e, a, b, a^{-1}, b^{-1}\}$ and $V_{l,k} = s_l s_k^{-1} U^k$.

One can show that the sets $SV_{l,k}$, $k \neq l$, are pairwise disjoint. This allows to set

$$\omega(t) = \begin{cases} 8^{-l-k}, & t \in SV_{l,k}; \\ 1, & \text{otherwise} \end{cases}$$

In particular, $\omega|_S \equiv 1$, so that the condition (2.11) does not hold. But a direct verification shows that the assumptions of Theorem 2.3.1 hold.

Another example shows that

Example 2.3.7. In general, one cannot have $G_n = F_n$ in Theorem 2.3.1, so that

$$\sum_{n,k:n\neq k} \|\omega\|^p_{p,s_n s_k^{-1} F_k} < \infty. \tag{2.13}$$

In [1], it is shown that there exists a weight ω on $G = \mathbb{R}^2$ with $S = \mathbb{R} \times \{0\}$ which is S-admissible and such that the condition (2.11) is satisfied, i.e. $L^p(\mathbb{R}^2, \omega)$ contains an S-dense vector, but the strengthened condition (2.13) does not hold. The constructed weight is piecewise constant on every square $[n, n+1) \times [m, m+1)$ in such a way that for certain $y \in \mathbb{R}$, the sequence $\omega(y, n) \to 0$, whereas for other y we have $\omega(y, n) \to +\infty$.

Chapter 3

Noncommutative duality theorems

During my postdoctoral fellowship in Luxembourg, I started working on problems in noncommutative duality of group algebras. In a series of three papers [4, 3, 8] I constructed duality maps on enlarging categories of operator algebras, which generalize Pontryagin duality of abelian groups and use topological and algebraic structure of the algebras involved but not invariant measures or similar constructions.

Recall that in the context of Pontryagin duality, we start from the notion of a locally compact abelian group, prove the existence of the Haar measure and thus are able to introduce the Fourier transform on $L^1(G)$, and these tools allow us to deduce the duality theorem [116]. The theory of Kac algebras, developed later to topological quantum groups, was initially motivated by the search of noncommutative Pontryagin duality: a category which would have a duality map such that $A \simeq \widehat{\widehat{A}}$ for every object A, and which would extend the duality of locally compact abelian groups. The idea is to pass from groups to their function algebras, such at the algebra $C_0(G)$ of continuous functions vanishing at infinity or $L^\infty(G)$ of measurable bounded functions. The subsequent work of many authors [58] has shown that this approach results in a category with duality covering all locally compact groups, not necessarily abelian.

It makes sense to discuss this theory in more detail. A Kac algebra is a von Neumann algebra M equipped with a comultiplication Δ which is a normal unital *-homomorphism from M to the von Neumann algebraic tensor product $M\bar{\otimes}M$, a coinvolution $S : M \to M$, called also the antipode, which is a *-antihomomorphism, and a faithful semifinite normal weight φ on M, termed the Haar weight, which is left invariant, that is

$$(\mathrm{id} \otimes \varphi)\Delta(x) = \varphi(x)\,1, \quad x \in M^+.$$

This additional structure appears for the following reasons. Alone, the group functional algebras do not allow to identify the groups; it is not rare to have isomorphic group algebras for non-isomorphic groups, such as say $\ell^\infty(\mathbb{Z})$ and $\ell^\infty(\mathbb{F}_2)$. Thus, to encode the group multiplication, we introduce a comultiplication Δ which in the classical case takes form $\Delta : L^\infty(G) \to L^\infty(G \times G)$, $\Delta(f)(s,t) = f(st)$. Next, the coinvolution corresponds to the group inverse: in the classical case, $(Sf)(t) = f(t^{-1})$. Finally, the Haar weight has the meaning of the Haar integral and on $L^\infty(G)$, it is $\varphi(f) = \int_G f$, what might of course be infinite for some f.

A comparison to the classical setting might suggest that making the Haar weight a part of the axioms is redundant and that its existence should follow as a theorem. However, this question remains open, and despite much effort no other axiomatics has been found which would contain significantly less information than described above.

Considering the up-to-day theory of locally compact quantum groups [88] changes radically the class of objects: it is able to describe quantum groups, such as notably the first example of the group $SU_q(2)$ constructed by Woronowicz [139]. It appears that on the structural level, the "essentially quantum" case is expressed by the non-boundedness of the antipode S. As for the Haar measure question, it keeps its place; the axioms require even existence of two invariant weights, left and right, but this is equivalent to having an antipode and one of these weights. Another approach, formally wider, is that of multiplicative unitaries [98]. Instead, it requires much spatial information of the algebra involved, what is coarsely equivalent to be given the regular representation of the (quantum) group. It is not known whether there exist multiplicative unitaries which are not generated by locally compact quantum groups.

In the cited papers [4, 3, 8] I aimed to look at the problem from another side: to define a category $\mathfrak{C}$ of topological algebras equipped with comultiplication and antipode, and to construct a duality map on it which would extend the classical Pontryagin duality. Having no Haar measure at hand, inevitably we have to define the dual algebra and prove duality theorems by only algebraical and topological means.

The price to pay is that for the class $\mathfrak{CRef}$ of reflexive objects, isomorphic to their second dual, we have no explicit description, so that this way does not produce another axiomatics of a quantum group. This may be compared to the duality of Banach spaces: every space has a linear dual, but only reflexive spaces (defined as such) are canonically isomorphic to their second dual.

The idea of the duality map is common to all three papers. Take a bialgebra A in question, pass to its linear dual space A^*, show that this is also an algebra, and set the dual algebra $\widehat{A}$ to be the C^*- or von Neumann envelope of A^*. This brief program hides of course various topological questions to specify and is concretized differently in each of the three cases.

This passage to the enveloping algebra of the dual space appeared for the first time in the PhD thesis of Kirchberg [83]: for a von Neumann bialgebra M with a coinvolution, he considered an algebra W^*M universal with respect to unitary corepresentations (this notion is discussed below in Subsection 3.2.2), and in some cases W^*M is equal to the von Neumann enveloping algebra $W^*(M_*)$ if the predual space of M. Later Kustermans [87] took over this idea and defined the universal C^*-algebra of a locally compact quantum group. And recently, Sołtan and Woronowicz [127] considered universal algebras in the context of multiplicative unitaries. The interest of these constructions is in the possibility to work with algebra representations rather than corepresentations. The minus of the two latter is, however, that they cannot be applied twice to a given algebra. The functor of Kirchberg is close to my functor described in Section 3.2 but is less explicit and differs from it notably in the description of (co-)commutative dual algebras: they do not necessarily correspond to classical groups (to be compared to the results cited at the end of Subsection 3.2.4).

To have an isomorphism $A \simeq \widehat{\widehat{A}}$ with $\widehat{A} = Env(A^*)$, we need to reduce either the dual A^* (by considering a stronger topology on A), or the envelope. In the subsequent sections, I describe three solutions of this problem in three categories.

The first category (Section 3.1) is rather limited and for classical groups, includes the algebras of continuous functions $C(G)$ for all Moore groups G. A locally compact group is a Moore group if all its irreducible representations are finite-dimensional; this class includes all Abelian and compact groups but not all discrete groups. The construction has in return the advantage that its objects are Hopf algebras in strictly algebraic sense, with respect to an appropriate topological tensor product.

The second category (Section 3.2) includes already the algebras $C_0(G)$ and $C^*(G)$ (dual to each other) for any locally compact group G, but still supposes that the antipode of the algebra is bounded. It has two versions, C^*-algebraic and von Neumann.

And finally, in the last paper in this series (Section 3.3) admits the quantum case of an unbounded antipode, and the third category contains all locally compact quantum groups. It allows also for many known quantum semigroups which are not groups; of course, these are not reflexive. Having a duality map for them gives a new tool for their future study.

3.1 A duality for Moore groups

The construction of [4] is done in the class of pro-C^*-algebras which are topological, but not necessarily Banach algebras. By definition, a pro-C^*-algebra A is a topological $*$-algebra which is a locally convex topological space, with a topology defined by a family of seminorms $\mathcal{P}(A)$ with the property that for every $p \in \mathcal{P}(A)$,

$$p(x^*x) = p(x)^2, \quad p(xy) \leqslant p(x)p(y), \qquad x, y \in A.$$

This class can be alternatively defined as the class of inverse limits of C^*-algebras [105]. For a locally compact space G, the algebra $C(G)$ of all continuous functions on G is a pro-C^*-algebra; it is equal to the inverse limit of $C(K)$ over all compact subsets $K \subset G$. The seminorms in $\mathcal{P}(C(G))$ are indexed by compact subsets $K \subset G$ and are equal to $p_K(f) = \sup_{t \in K} |f(t)|$.

3.1.1 Hopf algebra structures and tensor products

In addition to being pro-C^*-algebras, our objects are Hopf algebras in the following sense. Recall that purely algebraically, a Hopf algebra is a vector space A (for us all the vector spaces will be over $\mathbb{C}$) with morphisms $\mathfrak{m} : A \otimes A \to A$ (multiplication), $\imath : \mathbb{C} \to A$ (unit), $\Delta : A \to A \otimes A$ (comultiplication), $\varepsilon : A \to \mathbb{C}$ (counit) and $S : A \to A$ (antipode) such that

$$\mathfrak{m}(\mathfrak{m} \otimes \mathrm{id}) = \mathfrak{m}(\mathrm{id} \otimes \mathfrak{m}), \qquad \mathfrak{m}(\mathrm{id} \otimes \imath) = \mathrm{id}, \;\; \mathfrak{m}(\imath \otimes \mathrm{id}) = \mathrm{id};$$

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \qquad (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}, \;\; (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id};$$

Δ and ε are algebra homomorphisms: $\Delta\mathfrak{m} = \mathfrak{m}_{A\otimes A}(\Delta \otimes \Delta)$, $\varepsilon\,\mathfrak{m} = \mathfrak{m}_{\mathbb{C}\otimes\mathbb{C}}(\varepsilon \otimes \varepsilon)$;

$$\mathfrak{m}(S \otimes \mathrm{id})\Delta = \mathfrak{m}(\mathrm{id} \otimes S)\Delta = \varepsilon\imath.$$

If in addition A is equipped with an involution such that Δ and ε are *-homomorphisms, A is a *Hopf *-algebra*.

In analysis, it would be natural to replace $\otimes$ by a suitable topological tensor product. This would not be possible for every group algebra, for example not for the most common group algebras $C_0(G)$ or $L^\infty(G)$ for a non-compact group G. But as for $C(G)$, it is a Hopf algebra in the above sense with respect to the injective tensor product of locally convex topological vector spaces.

In the duality constructions of [4] I was using the following fact which concerns the linear dual spaces — to make disctinction, say "conjugate spaces" instead. Let X be a locally convex topological vector space, and let $X^\star$ denote its conjugate space. Consider $X^\star$ systematically with the topology of uniform convergence on totally bounded sets. It turns out that then almost all classical spaces become reflexive: $X \simeq (X^\star)^\star$; in particular, every Banach space, moreover every Fréchet space (complete metrizable locally convex space), and the space $C(G)$ for a locally compact group G (see [124], [31], [12], [11]).

Now let $\mathfrak{Ste}$ denote the category of locally convex spaces reflexive in the sense above. As shown in [12], $\mathfrak{Ste}$ is equipped with two tensor products: $\odot$ and $\circledast$ (and is a symmetric monoidal tensor category with both of them). These tensor products are related by the identity $(X \circledast Y)^\star = X^\star \odot Y^\star$. For Fréchet spaces with approximation property, $\odot$ and $\circledast$ coincide respectively with the injective and projective tensor products of topological vector spaces. Further, we use these tensor products to be sure that we stay within the category $\mathfrak{Ste}$.

With this terminology, $C(G)$ is a $\odot$-Hopf *-algebra for any locally compact group G. Its conjugate space is the space $M_c(G)$ of all compactly supported measures, and as said above, we consider it in the topology of uniform convergence on totally bounded subsets of $C(G)$. This is a $\circledast$-Hopf *-algebra, and moreover, in general, if A is a $\odot$-Hopf *-algebra, then $A^\star$ is a $\circledast$-Hopf *-algebra.

3.1.2 The dual algebra

According to the general scheme described in the introduction to this chapter, the dual algebra of a $\odot$-Hopf *-algebra A is defined as the C^*-enveloping algebra of $A^\star$. Let us specify the construction of this envelope.

Let A be a unital algebra with involution, endowed with some locally convex topology. Denote by $\mathcal{P}_C(A)$ the set of all continuous C^*-seminorms on A. For every $p \in \mathcal{P}_C(A)$ the kernel $\ker p$ is a *-ideal in A, so $A/\ker p$ is a normed algebra with the quotient norm $\bar{p}$. Its completion $C^*_p(A)$ with respect to $\bar{p}$ is a (Banach) C^*-algebra. The algebras $C^*_p(A)$ with pointwise ordering on $\mathcal{P}_C(A)$ form an inverse spectrum of C^*-algebras, and we set the *C^*-envelope* $A^\diamondsuit$ of A to be the inverse limit of the algebras $C^*_p(A)$ over $p \in \mathcal{P}_C(A)$.

The C^*-envelope can be alternatively defined as the completion of A/E with respect to all continuous C^*-seminorms, where E is the common kernel of $p \in \mathcal{P}_C(A)$. Any algebra A is continuously, but not always injectively mapped into its envelope $A^\diamondsuit$. It may happen that there are no other C^*-seminorms except zero; then $A^\diamondsuit = \{0\}$.

Definition 3.1.1. Let A be a $\odot$-Hopf *-algebra. The dual algebra $\widehat{A}$ is defined as $(A^\star)^\diamondsuit$.

If $A = C(G)$, then $\widehat{A} = M_c(G)^\diamondsuit$ will be denoted by $\widehat{C}(G)$. For every C^*-seminorm $p \in \mathcal{P}_C(M_c(G))$ the quotient map $i_p : M_c(G) \to C^*_p(M_c(G))$ can be composed with an imbedding into the algebra $\mathcal{B}(H_p)$ of bounded operators on a Hilbert space H_p. Let us denote by $\psi_p : M_c(G) \to \mathcal{B}(H_p)$ this homomorphism. It is easy to see that the map $\pi_p : t \mapsto \psi_p(\delta_t)$, $G \to B(H_p)$ is a unitary representation of G; moreover, since G is mapped continuously into $M_c(G)$, π_p is norm continuous.

This is the point which limits the class of groups with reflexive algebras in this sense: if norm continuous representations do not separate the points of G, we cannot hope to recover $C(G)$ as the dual of $\widehat{C}(G)$.

One can describe $\widehat{C}(G)$ in the following particular cases.

- Let G be an Abelian locally compact group, and let $\widehat{G}$ be its dual group. Then $\widehat{C}(G)$ is the algebra $C(\widehat{G})$ with the topology of uniform convergence on compact sets.

- Let G be a compact group and let $\widehat{G}$ be the set of equivalence classes of its irreducible representations. For $\pi \in \widehat{G}$, let $C^*_\pi(G)$ be the C^*-algebra generated by

$\pi(G)$. Then $\widehat{C}(G) = \prod_{\pi \in \widehat{G}} C^*_\pi(G)$.

- Let G be a discrete group, then $\widehat{C}(G)$ is equal to the classical group C^*-algebra $C^*(G)$. Moreover, a locally compact group G is discrete if and only if $\widehat{C}(G)$ is a Banach algebra.

Remark 3.1.2. The paper [4], as pointed out in the Errata to [13] (contained in the 11th ArXiv version and not in the journal), contained an error in the statement (i) of Lemma 4.6: instead of being extended to a seminorm on $\widehat{C}(G)$, a seminorm on $\widehat{C}(N)$ is only estimated from above by such a seminorm. This influences the description of $\widehat{C}(G)$ in general. However, the description can be achieved with another proof, and the main results of [4] are valid. An article [14] by Akbarov provides a complete proof of the results of [4]. I note that it appeared a few days before the defense of this thesis and this is when I learned of the initial error.

3.1.3 Hopf algebra structure on the dual algebra

If A is a $\odot$-Hopf *-algebra, then $\widehat{A}$ might not be even a $\odot$-algebra. In fact, a sufficient condition is as follows.

For a separable C^*-algebra A, being a $\odot$-algebra is equivalent to being of bounded degree [17], so that there exists a natural number n such that all irreducible representations of A are finite-dimensional and their dimensions do not exceed n. If a pro-C^*-algebra $A = \varprojlim A_p$ is metrizable and every A_p is strict, then A is a $\odot$-algebra.

Now, a locally compact group is called a Moore group if all its irreducible representations are finite-dimensional. This class includes all Abelian and compact groups but not all discrete groups. Using structure theory an approximation by Lie groups, we show that if G is a Moore group, then $\widehat{C}(G)$ is a $\odot$-algebra. For a discrete group G which is not a Moore group, $\widehat{C}(G)$ is not a $\odot$-algebra, so this class of groups is sharp enough.

Having established this fact, we can show easily enough that there exist also other maps turning $\widehat{A}$ into a $\odot$-Hopf *-algebra. On $A^\star$, a natural Hopf structure arises by dualizing the structure of A: $\Delta_{A^\star} = \mathfrak{m}_A^*$, $\varepsilon_{A^\star} = i_A^*$ etc.; next, these maps are extended by continuity to $\widehat{A}$.

We can note that for a discrete group G which is not a Moore group, $\widehat{C}(G)$ is not a $\underset{\max}{\otimes}$-algebra with respect to the maximal C^*-tensor product, so we could not widen the class of groups if we considered $\underset{\max}{\otimes}$ instead of $\odot$.

3.1.4 Duality on the category $\mathfrak{Moore}$

Let $\mathcal{K}(G)$ denote the space $\widehat{C}(G)^\star$. The continuous mapping of G into $\widehat{C}(G)$ allows to consider $\mathcal{K}(G)$ as a (commutative) algebra of functions on G. If G is a Moore group, then this space separates the points of G, and the space of its maximal ideals is homeomorphic to G (simiarly to the case of the Fourier algebra $A(G)$).

By describing all continuous C^*-seminorms on $\mathcal{K}(G)$, we prove that its C^*-envelope is $\mathcal{K}(G)^\diamondsuit = C(G)$. Thus, for a Moore group G we have the following diagram:

$$\begin{array}{ccc} C(G) & \xmapsto{\ \star\ } & M_c(G) \\ {\scriptstyle C^*-\mathrm{env}}\uparrow & & \downarrow{\scriptstyle C^*-\mathrm{env}} \\ \mathcal{K}(G) & \xleftarrow[\ \ ]{\ \star\ } & \widehat{C}(G) \end{array} \tag{3.1}$$

and in particular, the equality $\widehat{\widehat{C}(G)} \simeq C(G)$.

Let $\mathfrak{Moore}$ denote the subcategory of $\odot$-Hopf *-algebras A such that the dual algebra $\widehat{A} = A^{\star\Diamond} \in \mathfrak{Moore}$ is well defined and $\widehat{\widehat{A}} = A$. Morphisms in $\mathfrak{Moore}$ are morphisms of $\odot$-Hopf *-algebras.

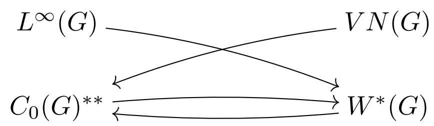
Theorem 3.1.3. *$A \mapsto \widehat{A}$ is a contravariant functor on $\mathfrak{Moore}$. Every algebra $C(G)$ and $\widehat{C}(G)$, where G is a Moore group, is contained in $\mathfrak{Moore}$.*

3.2 Duality of locally compact groups that does not involve the Haar measure

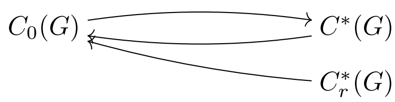
This construction [3] is made in the class of von Neumann algebras and includes group algebras for all locally compact groups. It has also a C^*-algebraic version. The development of this paper [8] is more general and applies in particular to all locally compact quantum groups, so I will leave a general discussion for the next section and will mainly treat the group case in the present one.

Recall that for every C^*-algebra A, there exists its enveloping von Neumann algebra which can be identified with the second dual A^{**}, and has the property that representations of A extend uniquely to normal representations of A^{**}. In this way, to a locally compact group G one can associate, along with the most commonly used $L^\infty(G)$ and $VN(G)$, two other von Neumann algebras: $C_0(G)^{**}$ and the Ernest algebra $W^*(G) = C^*(G)^{**}$.

The functor constructed in [3] acts as follows:



And in the C^*-version,



Thus, the pairs of group algebras in duality are $C_0(G)^{**} \leftrightarrow W^*(G)$ and $C_0(G) \leftrightarrow C^*(G)$. Having no acces to the Haar measure, we make use of *all* representations of the group, so we do not return to the reduced C^*-algebra or to $VN(G)$ which are based on the regular representation.

3.2.1 Coinvolutive Hopf-von Neumann algebras

We work with algebras with the following structure [58, §1.2].

Definition 3.2.1. A *von Neumann bialgebra* is a von Neumann algebra M equipped with a comultiplication $\Delta : M \to M\bar{\otimes}M$ which is an injective normal unital *-homomorphism such that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. If in addition there exists a *-antihomomorphism $S : M \to M$ (coinvolution) such that $S^2 = \mathrm{id}$ and $(S \otimes S)\Delta = \theta\Delta S$, where θ is the flip map, then the triple (M, Δ, S) is called a *coinvolutive Hopf-von Neumann algebra.*

The algebras $L^\infty(G)$, $VN(G)$, $C_0(G)^{**}$ and $W^*(G)$ of a locally compact group G carry this structure.

The predual space M_* of a coinvolutive Hopf-von Neumann algebra M is a Banach algebra with the multiplication Δ_*; it is equipped with the involution defined by $\mu^*(a) = \overline{\mu(S(a^*))}$, $\mu \in M_*$, $a \in M$. If we consider the dual operator space structure on M_* (see [55]), then M_* becomes a completely contractive Banach *-algebra, but in general not an operator algebra.

3.2.2 Unitary (co)representations

In the theory of group representations, the central place belongs to unitary representations. In the setting of bialgebras of functions on groups, such as $C_0(G)$ and $L^\infty(G)$, this corresponds to unitary *corepresentations*.

In algebra, a corepresentation of a Hopf alfebra H on a vector space V is defined as a morphism $\rho : V \to V \otimes H$ which is compatible with the comultiplication: $(\mathrm{id} \otimes \Delta)\rho = (\rho \otimes \mathrm{id})\rho$. If we pass to the dual map $\pi = \rho^* : V^* \otimes H^* \to V^*$ then this condition implies that π is a representation of H^*, with respect to the multiplication $\mathfrak{m}_{H^*} = \Delta_H^*$.

In the setting of von Neumann bialgebras, it is even more natural to speak of representations of M_* rather than of corepresentations of M. If a representation $\pi : M_* \to B(H)$ is completely bounded (in the dual operator space structure as mentioned above), then it has a *generator* $U \in B(H)\bar{\otimes}M$ such that

$$U(\mu, \omega) = \omega(\pi(\mu))$$

for every $\mu \in M_*$, $\omega \in B(H)_*$. Recall that the von Neumann algebraic tensor product $M\bar{\otimes}N$ of von Neumann algebras M, N is canonically isomorphic to the dual of the projective operator space tensor product $M_* \otimes N_*$ [55, 7.2.4].

Now, a representation $\pi : M_* \to B(H)$ is *unitary* if it has a generator which is a unitary in $B(H) \otimes M$. This implies that π is automatically a *-representation of M_*, with the involution defined above.

This definition has its coordinate form. Let (f_α) be an orthonormal basis of H, then for every α, β the formula $\mu \mapsto \langle \pi(\mu) f_\beta, f_\alpha \rangle$, $\mu \in M_*$ defines a bounded linear functional on M_*, and thus an element of M. Denote it by $\pi_{\alpha\beta}$. Then, π is unitary if and only if

$$\sum_\gamma \pi^*_{\gamma\alpha} \cdot \pi_{\gamma\beta} = \sum_\gamma \pi_{\alpha\gamma} \cdot \pi^*_{\beta\gamma} = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} \tag{3.2}$$

for every α, β, the series converging absolutely in the M_*-weak topology of M. One can show that this definition does not depend on the choice of the basis.

3.2.3 Unitary representations of group algebras

If M is $L^\infty(G)$ or $VN(G)$, so that M_* is $L^1(G)$ or the Fourier algebra $A(G)$ respectively, then every non-degenerate representation of M_* is unitary. This is a result of Kirchberg

[83], and the same fact is valid for any locally compact quantum group (in the von Neumann algebraic setting).

In other cases it is sufficient to describe irreducible unitary representations. If $M = W^*(G)$, then $M_* = B(G)$ is the Fourier-Stieltjes algebra of G, so it is a commutative algebra of functions on G. A character π of $B(G)$ can be viewed as an element of $W^*(G)$, and it is unitary in the sense above exactly when it is a unitary in $W^*(G)$. By a theorem of Walter [132], this implies that π is the evaluation at a point of G.

If $M = C_0(G)^{**}$, then $M_* = M(G)$ is the measure algebra of G. One can show that an irreducible representation π is unitary if and only if it is generated by a *continuous* unitary representation $\tilde{\pi}$ of G by the classical integral formula:

$$\pi(\mu) = \int_G \tilde{\pi}(t) d\mu(t), \tag{3.3}$$

for every $\mu \in M(G)$.

This result follows from Kirchberg's [83], but it can be also proved directly by considering the series (3.2) of coordinate functions of π [3].

3.2.4 The dual algebra

The general idea is to define the dual algebra $\widehat{M}$ as a von Neumann algebra whose normal representations would correspond to unitary representations of M_*. The problem is that defined in this way, it might not exist; this is why in known articles [87, 127] the universal dual is defined only in the case of a locally compact quantum group. In [3], the dual algebra $\widehat{M}$ is defined as the von Neumann enveloping algebra of a certain *absolutely continuous ideal* $M_*^0 \subset M_*$. In the case of the measure algebra $M_* = M(G)$, the ideal M_*^0 has exactly the same representations as $L^1(G)$, what explains its name. In certain degenerate cases we arrive at $\widehat{M} = \{0\}$.

This construction is given in more generality in Section 3.3, and below we give only main examples.

If G is a locally compact group, then the von Neumann algebraic construction leads to the following commutative diagram:

$$\begin{array}{ccc} C_0(G)^{**} & \xrightarrow{\ predual\ } & M(G) \\ {\scriptstyle VN(M_*^0)} \uparrow & & \downarrow {\scriptstyle VN(M_*^0)} \\ B(G) & \xleftarrow{\ predual\ } & W^*(G) \end{array} \tag{3.4}$$

so that $M = C_0(G)^{**}$ implies $\widehat{M} = W^*(G)$, and conversely, if $M = W^*(G)$ then $\widehat{M} = C_0(G)^{**}$. If $M = L^\infty(G)$ or $M = VN(G)$, we get as well $W^*(G)$ or $C_0(G)^{**}$ respectively as the dual algebra.

In the C^*-algebraic version, which is also possible in our setting, another commutative diagram holds:

$$\begin{array}{ccc} C_0(G) & \xrightarrow{\ *\ } & M(G) \\ {\scriptstyle C^*(M_*^0)} \uparrow & & \downarrow {\scriptstyle C^*(M_*^0)} \\ B(G) & \xleftarrow{\ *\ } & C^*(G) \end{array} \tag{3.5}$$

so that $\widehat{C_0(G)} = C^*(G)$ and $\widehat{C^*(G)} = \widehat{C_r^*(G)} = C_0(G)$.

Moreover, if M is commutative and $\widehat{M} \neq \{0\}$, then there exists a locally compact group G such that $\widehat{M} \simeq W^*(G)$. If M is cocommutative (M_* is commutative) and $\widehat{\widehat{M}} \neq \{0\}$, then $\widehat{M} \simeq C_0(G)^{**}$ for some locally compact group G.

In general, the dual algebras $\widehat{M}$ are *quantum groups* in a certain sense which is yet to specify. In any case, for all known examples $\widehat{M}$ is isomorphic to the dual of some locally compact quantum group.

3.3 Duals of quantum semigroups with involution

This section continues the previous one. This time [8] we define a duality map on the category $\mathcal{QSI}$ of *quantum semigroups with involution* defined as follows.

An object in $\mathcal{QSI}$ is a von Neumann bialgebra M with comultiplication Δ and a densely defined *proper coinvolution* $S : D(S) \subset M \to M$, which satisfies the following conditions:

1. $D(S)$ is σ-weakly dense in M;
2. $D(S)$ is closed under multiplication and $S : D(S) \to M$ is an anti-homomorphism;
3. $(*S)(D(S)) \subset D(S)$ and $(*S)^2 = \mathbb{I}_{D(S)}$;
4. if $\mu, \nu \in M_*$ are such that $\mu \circ S$ and $\nu \circ S$ extend to normal functionals on M, then for all $x \in D(S)$ holds $\Delta(x)(\nu \circ S \otimes \mu \circ S) = \Delta S(x)(\mu \otimes \nu)$.

M is called in this case a *quantum semigroup with involution.*

Every locally compact quantum group is a quantum semigroup with involution. Another example is given below.

Example 3.3.1. Let P be a compact semitopological semigroup with involution. Recall that an involution on a semigroup is a map $* : P \to P$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in P$. Let $SC(P \times P)$ be the space of separately continuous functions on $P \times P$, then we get a natural map $\check{\Delta} : C(P) \to SC(P \times P)$, $\check{\Delta}(f)(s,t) = f(st)$. Set $M = C(P)^{**}$, with the usual structure of a von Neumann algebra. Since $SC(P \times P)$ is canonically imbedded into $M \otimes M$, $\check{\Delta}$ can be viewed as a map from $C(P)$ to $M \otimes M$, and it is known [47] that it is a unital *-homomorphism and as such can be extended by normality to M, so that the extension Δ satisfies $(\Delta \otimes \mathrm{id})\check{\Delta} = (\mathrm{id} \otimes \Delta)\check{\Delta}$. Altogether, this implies that Δ is a comultiplication on M. Set $D(S) = C(P)$ and $(Sf)(t) = f(t^*)$ for $f \in C(P)$, $t \in P$. It is easily seen that S satisfied conditions (1)–(4), so that $M = C(P)^{**}$ is a quantum semigroup with involution.

As one sees from the definition, S might be unbounded. This means that on M_*, the natural involution might be not everywhere defined, contrary to the case of Hopf–von Neumann algebras.

Definition 3.3.2. For every $\mu \in M_*$, define $\bar{\mu} \in M_*$ by $\bar{\mu}(a) = \overline{\mu(a^*)}$, $a \in M$. Let M_{**} be the subspace of all $\mu \in M_*$ such that $\bar{\mu} \circ S$ extends to a bounded normal functional on M. We will denote by μ^* this extension, so that $\mu^*(x) = \bar{\mu}(Sx)$ for $x \in D(S)$.

The subset M_{**} is a subalgebra of M_*, and is a Banach *-algebra with the norm $\|\mu\|_* = \max\big(\|\mu\|, \|\mu^*\|\big)$. In fact, M_{**} might be even not dense in M_*. However, it is possible to pass to a quotient algebra $M_r = M/(M_{**})^\perp$ which is, with the quotient structure, again a quantum semigroup with involution, and this time $(M_r)_{**}$ is dense in $(M_r)_*$ and the coinvolution is a closed operator on M_r.

3.3.1 Absolutely continuous ideal

In the general case, the dual algebra $\widehat{M}$ of M is defined as the dual of the quotient algebra M_r defined above, $\widehat{M} = \widehat{M_r}$. In the sequel we can thus suppose that the subalgebra with involution M_{**} is dense in M_*, as it is in the case of $(M_r)_*$.

We are seeking for an algebra whose *-representations would be the same as unitary representations of M_*. For this, we define first the *absolutely continuous ideal* $M_{**}^\times \subset M_*$ as the common kernel of all irreducible non-unitary *-representations.

By definition, every representation of M_* which is irreducible on $M_{**}^\times$ must be unitary. But this can still produce too irregular cases, so we exclude them as follows. Let I^0 be the weakly closed ideal in M generated by $(M_{**}^\times)^\perp$, that is by the annihilator of $M_{**}^\times$. Set $M_{**}^0 = M_{**}^\times$ if $I^0 \neq M$ and $M_{**}^0 = \{0\}$ otherwise.

Defined in this way, the ideal M_{**}^0 – unless it is null – has the property that a representation of M_* is unitary if and only if it is non-degenerate on M_{**}^0. This is proved by a careful disintegration into irreducible representations, and integrating back their generators to get a unitary generator for the original representation.

In the case $M_* = L^1(\mathbb{G})$ of a locally compact quantum group, $M_{**}^0 = L^1(\mathbb{G})$, as every non-degenerate representation of this algebra is unitary. In the case of the measure algebra $M_* = M(G)$ of a locally compact group, $M_{**}^0 \supset L^1(G)$ is an ideal studied by J. Taylor [128] under the notation $L^{1/2}(G)$. He has proved that $L^{1/2}(G) \neq L_1(G)$ unless G is discrete. However, $L^{1/2}(G)$ and $L_1(G)$ have the same $*$-representations and thus the same enveloping C^*-algebras.

3.3.2 The dual algebra

We set $\widehat{M} = C^*(M_{**}^0)^{**}$. By construction (except for degenerate cases when we set $\widehat{M} = \{0\}$), normal representations of this algebra corrrespond bijectively to the unitary representations of M_*.

Every dual algebra carries again a structure of a quantum semigroup with involution: on M_*, a Hopf-like structure is obtained by dualizing the maps of M, and then we show that they extend by continuity to $\widehat{M}$. In particular, the domain of its antipode $\widehat{S}$ is the image of M_* under the canonical map into $\widehat{M}$.

Moreover, $\widehat{M}$ is a Hopf algebra (and not just a bialgebra) in the following sense. For every $a \in D(\widehat{S})$,

$$\mathfrak{m}(\mathbb{I} \otimes S)\Delta(a) = \mathfrak{m}(S \otimes \mathbb{I})\Delta(a) = \varepsilon(a)1_{\widehat{M}}, \tag{3.6}$$

with weak-star convergence of the arising series. This corresponds to the axiom of the antipode in the algebraic definition of a Hopf algebra. This suggests that every dual $\widehat{M}$ is a quantum *group* and not just a semigroup.

Let $L^\infty(\mathbb{G})$ be a von Neumann algebraic locally compact quantum group, and let $C_0^u(\widehat{\mathbb{G}})$ denote its universal dual in the sense of Kustermans [87]. We get that $\widehat{L^\infty(\mathbb{G})} = C_0^u(\widehat{\mathbb{G}})^{**} = \widehat{C_0^u(\mathbb{G})^{**}}$, and $\widehat{L^\infty(\widehat{\mathbb{G}})} = C_0^u(\mathbb{G})^{**} = \widehat{C_0^u(\widehat{\mathbb{G}})^{**}}$.

If $M = \widehat{N} \neq \{0\}$ is a dual of another algebra and is either commutative or cocommutative, then it is isomorphic to $C_0(G)^{**}$ or $W^*(G)$ respectively for a classical locally compact group G. Moreover, in all examples known the dual algebra $\widehat{M}$ coincides with a dual of a locally compact quantum group, so in particular $\widehat{M}$ is isomorphic to the third dual of M. This allows to conjecture that $\widehat{M} \simeq \widehat{\widehat{\widehat{M}}}$ in general.

3.3.3 Examples

Below we list several examples of quantum semigroups which are not groups, and calculate their duals. This ends up every time with one of two possibilities: either we get a quantum group (the universal version of a locally compact quantum group), or a zero algebra; the latter case means that there are "too many" non-unitary representations. The duality construction thus "cuts off" the non-invertible part of the given semigroup, and takes into account the "invertible part", if any.

Example 3.3.3. X. Li [94] defines the reduced C^*-algebra $C^*_r(P)$ of a discrete left cancellative semigroup P, using its regular representation on $\ell^2(P)$. The algebra $C^*_r(P)$ is generated by the translation operators T_p, $p \in P$, and their adjoints $T_p^* \in B(\ell^2(P))$. As a linear space, $C^*_r(P)$ is generated by $E_X L_g$, where X is an ideal in P, E_X is the operator of multiplication by the characteristic function of X, and $g = p_1^{\pm 1} \dots p_n^{\pm 1}$ with $p_j \in P$.

Set $M = C^*_r(P)^{**}$. Li does not define a coinvolution on his algebra. In order that it fits into our assumptions, set $S(T_p) = T_p^*$, and accordingly $S(T_p^*) = T_p$, $p \in P$, and extend it as a linear anti-homomorphism onto the algebra generated by these elements. In fact, this makes S isometric (and bounded) on $C^*_r(P)$.

By definition $\Delta(T_p) = T_p \otimes T_p$, so every T_p is a character of M_*. With the coinvolution above, it is involutive. It is unitary as an element of M if and only if p is invertible in P.

By construction, $\Delta(E_X) = E_X \otimes E_X$ for an ideal X in P, so that it is a character as well, which is clearly unitary only if $X = \emptyset$ or $X = P$. It follows that $M^\times_{**}$ is the linear dual space of $\mathrm{lin}\{L_g : g \in P \cap P^{-1}\}$. As $H = P \cap P^{-1}$ is a group, $M^\times_{**}$ is isomorphic to $\ell^1(H)^*$. It is readily seen that in fact, $M^0_{**} = M^\times_{**}$. We conclude that $\widehat{M} = C^*(H)^{**}$.

Example 3.3.4. Let B be the quantum semigroup $C(\widetilde{S}_N^+)$ defined by Banica and Skalski [20]. Recall that it is defined starting with a "submagic" $N \times N$ matrix $u = (u_{ij})$ with entries in a unital C^*-algebra A. Being "submagic" means that $u_{ij} = u_{ij}^* = u_{ij}^2$ for every i, j, and $u_{ij} u_{ik} = u_{ji} u_{ki} = 0$ for every i if $j \neq k$. By definition, B is the universal unital C^*-algebra with the relations above. The authors show that B admits a comultiplication (a unital coassociative *-homomorphism) defined by the formula $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, and a "sub-coinvolution" (everywhere defined *-antihomomorphism) defined by $S(u_{ij}) = u_{ji}$.

Let M be the enveloping von Neumann algebra of B, then Δ and S extend obviously to normal maps on M. It is immediate to verify that S is a proper coinvolution on M, so that M is a quantum semigroup with involution. Since S is bounded, $M_{**} = M_*$. The elements (u_{ij}) are coefficients of a *-representation of M_*. One shows easily that it is irreducible.

However, u is clearly non-unitary, thus every u_{ij} belongs to the annihilator of $M^\times_{**}$. Since (u_{ij}) generate B, by definition $M^0_{**} = \{0\}$ and $\widehat{M} = \{0\}$.

Example 3.3.5. Weakly almost periodic compactifications

Let G be a locally compact group, and let P be its weakly periodic compactification. It is known that P is a compact semitopological semigroup, and we have seen above that $M = C(P)^{**}$ is a quantum semigroup with involution.

Denote by bG the Bohr compactification of G. There exists a homeomorphic imbedding $\tau : bG \to P$, such that $\tau(bG)$ is equal to eP for a central idempotent e [117, Theorem III.1.9]. Let $\tilde{\tau} : C(P) \to C(bG)$ denote the operator of composition with τ.

Let $M = C(P)^{**}$, with $M_* = M(P)$ the measure algebra of P. If $\pi : M(P) \to B(H)$ is a unitary representation, then $\pi \circ \tilde{\tau}^*$ is a unitary representation of $M(bG)$.

Conversely, every unitary representation of $M(bG)$ comes from an irreducible continuous representation of bG, and if σ is such a representation then $\pi(\mu) = \int_P \sigma(\tau^{-1}(et))d\mu(t)$ defines a representation of $M(P)$, and one easily verifies that π is unitary.

This allows to show that $C^*(M^0_{**})$ is isomorphic to $C^*(M(bG)^0)$. As we know, this equals to $C^*(bG)$, so finally $\widehat{M} = C^*(bG)^{**}$.

Example 3.3.6. Let $M = L^\infty(\mathbb{R}^2)$. Change the coinvolution to be $S(\varphi)(s,t) = \underline{\varphi(s,-t)}$. Then on $M_* = L_1(\mathbb{R}^2)$ we get the usual convolution and the involution $f^\circ(s,t) = \overline{f(s,-t)}$. The *involutive* characters are described as $\chi_\alpha(f) = \int e^{i\alpha t} f(s,t)dsdt$, $\alpha \in \mathbb{R}$ and are all unitary, thus we have $C^*(M^0_*) = C_0(\mathbb{R})$ and $\widehat{M} = C_0(\mathbb{R})^{**}$.

Chapter 4

Homomorphisms of groups and group algebras

This chapter reviews three articles: on automatic continuity of group homomorphisms [5], on almost isometric homomorphisms of Fourier and Fourier-Stieltjes algebras [2] with Jean Roydor, and on a new class of quantum semigroups associated to locally compact semigroups [9] with Marat Aukhadiev.

4.1 On continuity of measurable group representations and homomorphisms

In the theory of representations of locally compact groups, by far the most important class is of continuous unitary representations. To remind the definitions, a unitary representation is a homomorphism $U : G \to \mathcal{U}(H)$ into the group of unitary operators on a Hilbert space H; U is called continuous if for every $x, y \in H$ the coefficient $f(t) = \langle U(t)x, y\rangle$ is a continuous function on G. This is equivalent the requirement that the function $F(t) = \|U(t)x\|$ is continuous for every $x \in H$.

In certain cases it happens that every representation is automatically continuous, as, notably, every finite dimensional unitary representation of a connected semisimple Lie group [131]. But in general it is easy to construct discontinuous representations, as shows the following example:

Example 4.1.1. Let $\mathbb{R}_d$ be the real line considered with the discrete topology. On the space $\ell^2(\mathbb{R}_d)$ of square-summable functions on $\mathbb{R}_d$, the group $\mathbb{R}$ acts by translations. Denote this representation by λ_d.

Every $x, y \in \ell^2(\mathbb{R}_d)$ have countable support, so the support of every coefficient f_{xy} is also countable:

$$f_{xy}(t) = \langle \lambda_d(t)x, y\rangle = \sum_{s\in\mathbb{R}} x(s-t)\bar{y}(s),$$

and it vanishes for t outside the set $\operatorname{supp} y - \operatorname{supp} x$ which is countable. As every function f_{xx}, $x \neq 0$, is nonzero, it cannot be continuous.

In [5], I was interested in finding sufficient conditions, as weak as possible, which imply automatic continuity of a general unitary representation. Recall in this respect the Cauchy functional equation (see a historical survey in [77]):

$$f(x+y) = f(x) + f(y). \tag{4.1}$$

It was shown by Cauchy in 1821 that every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying (4.1) is of the form $f(x) = ax$, $a \in \mathbb{R}$. But later, by subsequent generalizations, it has been shown that it is sufficient to suppose f measurable to arrive at the same conclusion, and thus to guarantee continuity.

Say that a representation U is *weakly measurable* if f_{xy} is measurable for every $x, y \in H$. It is known that every weakly measurable unitary representation must be continuous if it acts on a separable Hilbert space [64, Theorem V.7.3].

However, in general this does not imply continuity: in the example above, every f_{xy} is measurable because it is countably supported. Of course, the space $\ell^2(\mathbb{R}_d)$ is non-separable.

In [5], I prove that separability restriction can be removed if we use a slightly stronger notion of measurability. Let $\mathcal{L}(H)$ be the space of bounded linear operators on the Hilbert space H, endowed with the weak operator topology (it is generated by the functions $\varphi_{xy} : \mathcal{L}(H) \to \mathbb{C}$ for all $x, y \in H$, where $\varphi_{xy}(A) = \langle Ax, y \rangle$, $A \in \mathcal{L}(H)$). Say that U is *weakly operator measurable* if $U^{-1}(V)$ is measurable for every open set $V \subset \mathcal{L}(H)$. Now we can formulate the main result of this paper (Theorem 4.1.3 below): every weak operator measurable unitary representation of a locally compact group is continuous.

To the Example 4.1.1 above, Theorem 4.1.3 is of course not applicable. It is not difficult to verify that λ_d is not weakly operator measurable. Let $e_t \in \ell^2(\mathbb{R}_d)$ be the characteristic function of $t \in \mathbb{R}$. The set

$$V_t = \{A \in \mathcal{L}(\ell^2(\mathbb{R}_d)) : |\langle Ae_0, e_t \rangle| > 0\}$$

is open in $\mathcal{L}(\ell^2(\mathbb{R}_d))$. Pick a non-measurable set $E \subset \mathbb{R}$ and set $V = \cup_{t \in E} V_t$. This is also an open set in $\mathcal{L}(\ell^2(\mathbb{R}_d))$. Its inverse image is

$$\lambda_d^{-1}(V) = \bigcup_{t \in E} \{s : |\langle \lambda_d(s) e_0, e_t \rangle| > 0\} = \bigcup_{t \in E} \{s : |\langle e_s, e_t \rangle| > 0\} = E$$

and is non-measurable.

The second part of the paper deals with automatic continuity of more general group homomorphisms. Most actively this question is studied for homomorphisms between Polish groups, see a review of C. Rosendal [114]. The notion of Haar measurability of $f : G \to H$ is here replaced by universal measurability: the inverse image of every open set is measurable with respect to every Radon measure on G. It is known that every universally measurable homomorphism from a locally compact or abelian Polish group into a Polish group, or from a Polish group to a metric group is continuous. There are also generalizations to other subclasses of Polish groups by S. Solecki and Rosendal. We omit results on other types of measurability (in the sense of Souslin, Christensen etc.)

If G is not supposed to be Polish, the results are fewer. The most general statement is probably the theorem of A. Kleppner [84]: every measurable homomorphism between two locally compact groups is continuous. It has been generalized to some special classes of groups by J. Brzdęk [34]. If one makes no assumptions on the image group, it seems inevitable to impose additional set-theoretic axioms instead. The only result known to me in this direction belongs to J. P. R. Christensen [39]: under Luzin's hypothesis, every Baire, in particular, every Borel measurable homomorphism from a Polish group to any topological group is continuous. My Theorem 4.1.4 is proved under Martin's axiom (MA) and states that every measurable homomorphism from a locally compact group to any topological group is continuous.

4.1.1 Continuity of unitary representations

It is known [64, IV.2.16 and V.7.2] that every unitary representation of a locally compact group may be decomposed into a direct sum $U = U_1 \oplus U_2$, where U_1 is continuous and every coefficient of U_2 is (locally) almost everywhere zero. We will say that U_2 is *singular*. If U acts on a separable space then $U_2 = 0$ [64, Theorem V.7.3].

The proof of Theorem 4.1.3 is based on a generalization of the so called Four Poles Theorem: if $\mathcal{A}$ is a point-finite family of null sets with non-null union in a Polish space, then there is a subfamily in $\mathcal{A}$ with a nonmeasurable union. This was proved initially by L. Bukovsky [36] and then much simpler by J. Brzuchowski, J. Cichoń, E. Grzegorek and C. Ryll-Nardzewski [35]. In the Lemma 4.1.2 below, I prove the same result for subsets of any locally compact group, with a restriction that cardinality of $\mathcal{A}$ is not more than continuum.

Lemma 4.1.2. *Let $\mathcal{A} = \{A_s : s \in S\}$ be a point finite family of null subsets of a σ-compact locally compact group G. If $|\mathcal{A}| \leqslant \mathfrak{c}$ and $\cup\mathcal{A}$ is non-null, then there is $\mathcal{B} \subset \mathcal{A}$ such that $\cup\mathcal{B}$ is nonmeasurable.*

Every σ-compact Lie group is Polish and the Four Poles Theorem applies to it. The proof of the lemma goes by a careful reduction to the case of pro-Lie groups (inverse limits of Lie groups) and then finding a Polish quotient group such that the projection of $\cup\mathcal{B}$ would be non-measurable.

Theorem 4.1.3. *Let G be a locally compact group. Then every weakly operator measurable unitary representation of G is continuous.*

Idea of the proof. As it is said above, U is decomposed into a continuous and singular component, so it is sufficient to prove that its singular component vanishes; moreover, we can assume that U is singular itself. Then our aim is to prove that it is in fact zero.

Suppose on the contrary that there is $x \in H$ of norm one. Put $f(t) = \langle U(t)x, x\rangle$ and consider

$$S = \{t \in G : f(t) \neq 0\}.$$

We must have $f(e) = 1$, so $e \in S$.

The main idea is best seen in the case when G is σ-compact and Polish. We choose by transfinite induction a family $\{t_\alpha : \alpha < \mathfrak{n}\}$ in G, indexed by an ordinal number $\mathfrak{n} \leqslant \mathfrak{c}$ so that $\bigcup_\alpha t_\alpha S$ is non-null and $t_\beta^{-1} t_\alpha \notin S$ for all $\beta < \alpha$. This implies that

$$f(t_\beta^{-1} t_\alpha) = \langle U(t_\alpha)x, U(t_\beta)x\rangle = 0,$$

so that the vectors $\{U(t_\alpha)x : \alpha < \mathfrak{n}\}$ are pairwise orthogonal.

Next, S is the countable union of the sets

$$S_n = \{t \in G : |f(t)| > 1/n\},$$

and (recall that $\bigcup_\alpha t_\alpha S$ is non-null) one can see that for some $N \in \mathbb{N}$ the family

$$\mathcal{A} = \{t_\alpha S_N : \alpha < \mathfrak{n}\}$$

has a non-null union. Now, the orthogonality implies that $\mathcal{A}$ is point-finite:

$$\begin{aligned} 1 = \|x\|^2 = \|U(t)x\|^2 &\geqslant \sum_{\alpha:\, t \in t_\alpha S_N} |\langle U(t)x, U(t_\alpha)x\rangle|^2 \\ &> N^{-2} \cdot \#\{\alpha : t \in t_\alpha S_N\}, \end{aligned}$$

and we can apply the Lemma to extract a subset $\mathcal{B} \subset \mathcal{A}$ with a nonmeasurable union. However, S_N and its every translation is the inverse image of an open set in $\mathcal{L}(H)$, so their union must be measurable. This contradiciton proves the theorem.

4.1.2 Continuity of group homomorphisms

The second main theorem of [5] is proved under the Martin's axiom (MA), an axiom additional to ZFC and independent of it. Leaving a discussion of the axiom apart for a while, let us cite the statement of the theorem:

Theorem 4.1.4 (MA)**.** *Every measurable homomorphism π from a locally compact group G to any topological group H is continuous.*

Here a homomorphism is said to be measurable if the inverse image of every open set is measurable. The local compactness is used in the proof as follows. If $U \subset H$ is an open neighbourhood of identity, then $\pi^{-1}(U) = A$ is measurable. If in addition it has positive measure, then $A^{-1}A$ contains a neighbourhood of identity V (this is a well known fact in locally compact groups), so that $\pi(V) \subset U^{-1}U$. Taking U small enough, we can thus prove the continuity of π in the identity, and this implies its continuity everywhere.

This discussion shows that for a discontinuous homomorphism π, the sets $A = \pi^{-1}(U)$ must be null (for U sufficiently small). But at the same time, they are measurable together with all their translates, and more generally, the set $SA = \{sa : s \in S, a \in A\}$ is measurable whatever is $S \subset G$ (since this is still the inverse image of an open set in H). Call the a set A *extra-measurable* if it has the property above.

This leads us to the study of extra-measurable sets and explains the appearance of the Martin's axiom.

Let $\mathrm{add}(\mathcal{N})$ be the minimal cardinality of a family $\mathcal{J}$ of null sets on the real line $\mathbb{R}$ such that $\cup\mathcal{J}$ is non null. This is called the additivity of the ideal $\mathcal{N}$ of Lebesgue null sets in $\mathbb{R}$. It is known that additivity of the ideal of Haar null sets is the same for every non-discrete locally compact Polish group [63, 522Va]. It is consistent with ZFC that $\mathrm{add}(\mathcal{N}) < \mathfrak{c}$, but it follows from Martin's axiom (MA) that $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$ (see [62]). This is, in fact, the assumption that we use in our proof. It is known that Martin's axiom follows from the Continuum hypothesis, but is consistent also with its negation. For further discussion of Martin's axiom, we refer to the Fremlin's monograph [62].

Under Martin's axiom, there are no extra-measurable sets of zero measure:

Theorem 4.1.5 (MA)**.** *Let G be a locally compact group, and let $A \subset G$ be a nonempty [locally] null set; then there is a set $S \subset G$ such that SA is nonmeasurable.*

In the proof, after reducing it to the case of G Polish non-discrete, we choose the set S by a transfinite induction so that SA and $G \setminus SA$ both intersect every perfect set of positive measure. This implies that they must both have full measure, what is impossible — or to be nonmeasurable, what proves the theorem. This choice of S is possible at every step thank to the Martin's axiom, since the intermediate unions of null sets are of cardinality less than continuum and this cannot cover G.

It is unknown whether Theorem 4.1.4 is true in ZFC without any additional axioms. Already in the basic case of the real line the question is open, but for commutative groups one can make the question more precise:

Proposition 4.1.6. *Let G be a commutative locally compact group. The following are equivalent:*
(i) There is a homomorphism $\varphi : G \to H$ to a topological group H which is measurable

but discontinuous;
(ii) There is a sequence of [locally] null extra-measurable sets A_n *such that for every* n*:* $A_n^{-1} = A_n$*;* $A_{n+1}^2 \subset A_n$*.*

The properties (i) and (ii) guarantee that if we take (A_n) as a base of neighbourhoods of identity in G, this turns G into a topological group [30, IV, §2], and one can show that the identical map on G is measurable and discontinuous.

Existence of sets as in Proposition 4.1.6(ii) is an open question even on the real line. Known results on automatic continuity mostly concern Polish groups; here they do not give a ready answer, since the group H obtained in the proof may be not complete (i.e. not Polish).

In conclusion, let us review some close results. Say that a set S is *small* if the union of every family of translates of S of cardinality less than continuum is null. We use Martin's axiom to guarantee that every null set is small. Without MA, this depends on the set S. Gruenhage [46] has proved that the ternary Cantor set is small, and Darji and Keleti — that every subset of $\mathbb{R}$ of packing dimension less than 1 is small. From the other side, Elekes and Tóth [57] and Abért [10] proved the following: it is consistent with ZFC that in every locally compact group there is a non-small compact set of measure zero. It is however unknown whether for a non-small set the statement of Theorem 4.1.5 is false.

Finally, we say a few words on results in ZFC concerning nonmeasurable products of sets. One should better say "sums of sets" because there is a tradition to do everything in the commutative case. This restriction is reasonable since the principal difficulties lie already in the case of the real line. The advances most close to our topic are: for every null set S on the real line such that $S + S$ has positive outer measure there is a set $A \subset S$ such that $A + A$ is nonmeasurable (Ciesielski, Fejzic and Freiling [41]). Cichoń, Morayne, Rałowski, Ryll-Nardzewski, and Żeberski [40] proved that there is a subset A of the Cantor set C such that $A + C$ is nonmeasurable, and under additional axioms they prove the same statement for every closed null set P such that $P + P$ has positive measure.

4.2 Homomorphisms with small bound between Fourier algebras

To a locally compact group, one can associate several different algebras. The general question is: does this algebra remember the group? Or more precisely, which relation does one need between two algebras to be able to identify the groups? For instance, let G and H be two locally compact groups; it is a classical result of J. Wendel [133] that G and H are isomorphic as topological groups if and only if there exists a contractive algebra isomorphism between the group algebras $L_1(G)$ and $L_1(H)$ (equipped with the convolution product). This improves his earlier paper [134] concerning isometries, note also that B.E. Johnson [76] and R. Rigelhof [113] proved analogous results for measure algebras. It is classical in Banach space theory to look for stability of isometric results (see e.g. [16], [37] and [23]). In this vein, N. Kalton and G. Wood improved Wendel's result in weakening the relation between the group algebras. The algebraic hypothesis in Wendel's result can not be dropped, because surjective linear isometries between L_1-spaces only determine the underlying measure spaces. But one can relax the restriction on the norm of the algebra isomorphism: in [80], N. Kalton and G. Wood showed that G and H are isomorphic if and only if there exists an algebra isomorphism T of $L_1(G)$ onto $L_1(H)$ with $\|T\| < \gamma \approx 1.246$. In the case of two locally compact abelian groups, they show that the bound can be improved to equal $\sqrt{2}$ and is actually optimal, what is shown in [80] on the example of $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In the paper [2] written with Jean Roydor, we considered similar questions for the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ associated to a locally compact group G (defined by P. Eymard in [59]). These algebras are isometric respectively to the predual of the group von Neumann algebra $VN(G)$ and to the predual of the universal von Neumann algebra $W^*(G)$ of the group (in particular, they are noncommutative L_1-spaces). In the case of an abelian group G, the algebra $A(G)$, respectively $B(G)$, is identified (via the Fourier transform) with the group algebra $L_1(\widehat{G})$, respectively with the measure algebra $M(\widehat{G})$ (where $\widehat{G}$ denotes the dual group of G).

It is a well-known result of M. Walter [132] that the Fourier algebras $A(G)$ and $A(H)$ are isometrically isomorphic as Banach algebras if and only if G and H are isomorphic as topological groups. Actually contractivity of the algebra isomorphism is sufficient (see Corollary 5.4 [104]), which is analogous to Wendel's results. Walter proved a similar theorem for Fourier-Stieltjes algebras, which is thus the analog of Johnson's result mentioned above. Inspired by the improvement of Kalton and Wood described above, one can wonder whether the isometric or contractive assumption on the algebra isomorphism in Walter's result is really needed to recover the groups structure? To our knowledge, this question has never been studied before. All the known results (see [132], [104] and [75]) on homomorphisms of Fourier algebras require posivity, contractivity or complete contractivity of the homomorphism, except Theorem 3.7 [75] which needs in return the amenability of the group and complete boundedness of the algebra homomorphism.

We prove two theorems 4.2.1 and 4.2.4, one for completely bounded case and second for the almost isometric case. Our proofs are by duality, they are entirely different from Kalton and Wood's ones (as we want to deal with non-abelian groups). For the proof of our Theorem 4.2.1, we use two 2×2 matrix tricks and for Theorem 4.2.4 we use ultraproducts of C^*-algebras, what allows us to obtain a universal but non-explicit bound. It is important to notice that all the known structural results (see [132], [104]) rely on the linear independence of the translation operators λ_g's in $VN(G)$. Our novelty is to deal with homomorphisms with norm greater than one, we need more precision and we base our proofs on the fact the λ_g's form a uniformly discrete subset of $VN(G)$.

4.2.1 The almost completely contractive case

In Theorem 4.2.1, we consider Fourier algebras and Fourier-Stieltjes algebras as completely contractive algebras, i.e. endowed with their canonical operator space structure, E. Effros and Z.-J. Ruan were the first to consider these operator space structures (see [55] for more details). This means in particular that instead of the class of all bounded operators, we consider a smaller subclass of completely bounded operators with the completely bounded norm $\|\cdot\|_{cb}$ (see [28], [106] or [103] for general theory of operator spaces). Usually, analogues between group algebras and Fourier algebras work better in the category of operator spaces (see e.g. the nice result of [115]). Therefore, we first treat the almost completely contractive case (note that $\|T\| \leqslant \|\mathrm{id}_{\mathbb{M}_2} \otimes T\| \leqslant \|T\|_{cb}$):

Theorem 4.2.1. *Let G and H be locally compact groups.*

1. *Let $T : A(G) \to B(H)$ be a nonzero algebra homomorphism. Suppose that $\|\mathrm{id}_{\mathbb{M}_2} \otimes T\| < \sqrt{5}/2$, then there exist an open subgroup Ω of H, $t_0 \in G$, $h_0 \in H$ and a*

continuous group morphism $\tau : \Omega \to G$ such that

$$T(f)(h) = \begin{cases} f(t_0\tau(h_0h)) & \text{if } h \in h_0^{-1}\Omega \\ 0 & \text{if not,} \end{cases}$$

for any $f \in A(G)$, $h \in H$. Hence T is actually completely contractive, i.e. $\|T\|_{cb} \leqslant 1$.

2. *Let $T : A(G) \to A(H)$ be a surjective algebra isomorphism between the Fourier algebras. If $\|\mathrm{id}_{\mathbb{M}_2} \otimes T\| < \sqrt{3/2}$, then there exist $t_0 \in G$ and a topological isomorphism $\tau : H \to G$ such that*

$$T(f)(h) = f(t_0\tau(h)),$$

for any $f \in A(G)$, $h \in H$. Hence T is actually completely isometric.

3. *Let $T : B(G) \to B(H)$ be a surjective algebra isomorphism between Fourier-Stieltjes algebras. If $\|\mathrm{id}_{\mathbb{M}_2} \otimes T\| < \sqrt{5}/2$, the same conclusion as in 2. holds (for any $f \in B(G)$).*

The important point here is that our result enables us to observe the curious phenomenon of a "norm gap": for any nonzero algebra homomorphism $T : A(G) \to B(H)$, we have either $\|T\|_{cb} = 1$ or $\|T\|_{cb} \geq \sqrt{5}/2$. Idem for cases (2) and (3). This can be compared with the result of N. Kalton and G. Wood [80]: for G, H abelian groups, if $T : L_1(G) \to L_1(H)$ is a surjective algebra isomorphism then either T is isometric or $\|T\| \geqslant (1 + \sqrt{3})/2$. 4.2.2 We use the following matrix lemma. Here K denotes a Hilbert space, and as usual in operator space theory, we identify isometrically $\mathbb{M}_2(\mathbb{B}(K)) = \mathbb{B}(K \oplus^2 K)$ (where $\oplus^2$ is the Hilbert space direct sum).

Lemma 4.2.2. *Let u, v be two unitaries in $\mathbb{B}(K)$. If $x \in \mathbb{B}(K)$ and $c \geq 1$ are such that*

$$\left\| \begin{bmatrix} u & 1 \\ -1 & x \end{bmatrix} \right\| \leq c\sqrt{2},$$

then $\|x - u^*\| \leq 2\sqrt{c^2 - 1}$. If $x \in \mathbb{B}(K)$ and $c \geq 1$ are such that

$$\left\| \begin{bmatrix} u & x \\ -1 & v \end{bmatrix} \right\| \leq c\sqrt{2},$$

then $\|x - uv\| \leq 2\sqrt{c^2 - 1}$.

Remark 4.2.3. The fact that tensorization by 2×2 matrices is sufficient to obtain a completely contractive or completely isometric conclusion can be compared with the main result of [78], where the authors prove that a 2-isometry between noncommutative L_p-spaces is necessarily a complete isometry.

The proof of Theorem 4.2.1 is most straightforward in the case (2) of two Fourier algebras, and we can cite it entirely. The two other cases differ by technical considerations linked to the more complicated description of the spectrum of the Fourier-Stieltjes algebras.

Proof of 2. of Theorem 4.2.1. Let $T^* : VN(H) \to VN(G)$ be the dual map of T. Due to the fact that T is an algebra isomorphism, its restriction to the spectrum of $A(H)$ (which can be identified with H) maps it into the spectrum of $A(G)$, which can be identified with G; moreover, this is a homeomorphism between H and G (not necessarily a group morphism). Denote this map by t. Replacing if necessary T by $f \mapsto T(t(e_H) \cdot f)$, $f \in A(G)$, we can assume that $t(e_H) = e_G$.

Form here we just need to prove that t is a group morphism. Let λ_t denote the operator of left translation by t. For $h_1, h_2 \in H$,

$$\left\| \begin{bmatrix} \lambda_{t(h_1)} & \lambda_{t(h_1 h_2))} \\ -1 & \lambda_{t(h_2)} \end{bmatrix} \right\| = \left\| \begin{bmatrix} T^*(\lambda_{h_1}) & T^*(\lambda_{h_1 h_2}) \\ T^*(-1) & T^*(\lambda_{h_2}) \end{bmatrix} \right\|$$
$$\leq \|\mathrm{id}_{\mathbb{M}_2} \otimes T^*\| \left\| \begin{bmatrix} \lambda_{h_1} & \lambda_{h_1 h_2} \\ -1 & \lambda_{h_2} \end{bmatrix} \right\| = \sqrt{2}\|\mathrm{id}_{\mathbb{M}_2} \otimes T^*\|$$

(the last equality is an easy calculation). By Lemma 4.2.2,

$$\left\|\lambda_{t(h_1 h_2)} - \lambda_{t(h_1)t(h_2)}\right\| \leq 2\sqrt{\|\mathrm{id}_{\mathbb{M}_2} \otimes T^*\|^2 - 1} < \sqrt{2}.$$

As two disctinct translation operators on $L^2(G)$ are on the distance at least $\sqrt{2}$, we conclude that $t(h_1 h_2) = t(h_1)t(h_2)$ and this completes the proof. □

4.2.2 The nearly isometric case

In Theorem 4.2.44, we consider Fourier algebras and Fourier-Stieltjes algebras as Banach algebras (and the usual operator norm of T). We are still able to prove that Walter's result is stable but we need a hypothesis on the norm distortion now:

Theorem 4.2.4. *There exists a universal constant $\varepsilon_0 > 0$ such that for any locally compact groups G and H,*

1. *if $T : A(G) \to A(H)$ is a surjective algebra isomorphism between the Fourier algebras and $\|T\|\|T^{-1}\| < 1+\varepsilon_0$, then there exist $t_0 \in G$ and a topological isomorphism or anti-isomorphism $\tau : H \to G$ such that for any $f \in A(G)$, $h \in H$,*

$$T(f)(h) = f(t_0 \tau(h)),$$

   hence T is actually isometric.

2. *The same result holds for surjective algebra isomorphisms between Fourier-Stieltjes algebras.*

Recall that the Jordan product of two elements a, a' in a C^*-algebra A is defined by:

$$a \circ a' = \frac{aa' + a'a}{2}.$$

It is a well-known result of R. Kadison [79] that a unital surjective isometry between C^*-algebras preserves the Jordan product. The next lemma, using ultraproducts in the category of C^*-algebras, asserts that a unital nearly isometric bijection between C^*-algebras almost preserves the Jordan product.

For C^*-algebras A, B and a linear map $T : A \to B$, we define the bilinear map $T^J : A^2 \to B$ by

$$T^J(a, a') = T(aa') + T(a'a) - T(a)T(a') - T(a')T(a),$$

for $a, a' \in A$.

Lemma 4.2.5. *For any $\eta > 0$, there exists $\rho > 0$ such that for any unital C^*-algebras A, B, for any unital isomorphism $T : A \to B$, $\|T\| \leq 1 + \rho$ and $\|T^{-1}\| \leq 1 + \rho$ imply $\|T^J\| < \eta$.*

This lemma is proved using ultrafilters, so we cannot expect any explicit value of the constant ρ.

The proof of Theorem 4.2.4, part (1) starts similarly to the proof of Theorem 4.2.1: we get a homeomorphism $t : H \to G$ and can assume that $t(e_H) = e_G$. To prove that t is multiplicative, we set ε_0 small enough to apply Lemma 4.2.5 to the dual map $T^* : VN(H) \to VN(G)$, and so get $\|T^{*J}\| < 1$. Now, for $h_1, h_2 \in H$

$$\|T^{*J}(\lambda_{h_1}, \lambda_{h_2})\| = \|\lambda_{t(h_1h_2)} + \lambda_{t(h_2h_1)} - \lambda_{t(h_1)t(h_2)} - \lambda_{t(h_2)t(h_1)}\| < \eta.$$

By direct norm estimates in $VN(G)$, it follows that

$$t(h_1h_2) = t(h_1)t(h_2) \quad \text{and} \quad t(h_2h_1) = t(h_2)t(h_1)$$

or

$$t(h_1h_2) = t(h_2)t(h_1) \quad \text{and} \quad t(h_2h_1) = t(h_1)t(h_2).$$

Consequently, in both cases, we have that

$$T^*(\lambda_{h_1} \circ \lambda_{h_2}) = T^*(\lambda_{h_2}) \circ T^*(\lambda_{h_1}).$$

By w^*-density, it follows that $T^* : VN(H) \to VN(G)$ preserves the Jordan product. Further, we can follow the proof of [132] Theorem 2 to obtain that the restriction of S^* to H is topological group isomorphism or anti-isomorphism from H onto G.

Remark 4.2.6. Our proof of Theorem 4.2.4 does not yield an explicit value of ε_0, but we can estimate it from above. By Example 1 of [80], there is an isomorphism $T : A(\mathbb{Z}_4) \to A(\mathbb{Z}_2 \times \mathbb{Z}_2)$ of norm $\sqrt{2}$ and one can calculate that the distortion $\|T\|\ \|T^{-1}\| = 2$, hence $\varepsilon_0 \leqslant 1$.

Remark 4.2.7. To our knowledge, Kalton and Wood's algebra isomorphism mentioned in the previous remark is the only known computation of the distortion of an algebra isomorphism between Fourier algebras. As it involves only abelian groups, we find it interesting to give another example involving a non-abelian group. We claim that there exists an algebra isomorphism between $A(\mathbb{Z}_6)$ and $A(S_3)$ of distortion 2 (where S_3 denotes the symmetric group on three elements).
A function $f \in A(\mathbb{Z}_6)$ is represented by its values: $(f_0, \ldots, f_5)$. The group $\mathbb{Z}_6$ has 6 characters: $\chi_j(k) = e^{i\pi jk/3}$, $j = 0, \ldots, 5$. Hence for $j = 0, 1, \ldots, 5$, its Fourier coefficients are $\widehat{f}_j = \sum_{k=0}^{5} f_k\, e^{i\pi kj/3}$ and

$$\|f\|_{A(\mathbb{Z}_6)} = \sum_{j=0}^{5} |\widehat{f}_j|.$$

The group S_3 is generated by the transposition $s = (12)$ and the cycle $r = (123)$. We have thus $S_3 = \{\mathrm{id}, s, r, sr, r^2, sr^2\}$. Define $\varphi : S_3 \to \mathbb{Z}_6$ by $\varphi(\mathrm{id}) = 0$, $\varphi(s) = 1$, $\varphi(r) = 2$, $\varphi(sr) = 3$, $\varphi(r^2) = 4$, $\varphi(sr^2) = 5$. Now define a surjective algebra isomorphism $\Phi : A(\mathbb{Z}_6) \to A(S_3)$ by $\Phi(f)(h) = f(\varphi(h))$, $f \in A(\mathbb{Z}_6)$, $h \in S_3$. Explicit calculations show that $\|\Phi\| = \|\Phi^{-1}\| = \sqrt{2}$ and so the distortion is equal to 2.

Remark 4.2.8. Theorem 4.2.4 was strengthened later by Éric Ricard and Jean Roydor [112]: they get an explicit value of ε_0 proving that if $T : A(G) \to A(H)$ is an isomorphism of norm $\|T\| < 1.0005$, then T is in fact an isometry.

4.3 Quantum semigroups generated by locally compact semigroups

The notion of a quantum semigroup, as a C^*- or von Neumann algebra with a comultiplication, appeared well before the term and before the notion of a locally compact quantum group. But it is especially these last years that substantial examples of quantum semigroups are considered; we would like to mention families of maps on finite quantum spaces [125], quantum semigroups of quantum partial permutations [20], quantum weakly almost periodic functionals [48], quantum Bohr compactifications [120, 126].

In the article written with Marat Aukhadiev [9] we construct a rather "classical" family of compact quantum semigroups, which are associated to sub-semigroups of locally compact groups. The interest of our objects is in fact that they provide natural examples of C^*-bialgebras which are co-commutative and are not however duals of functions algebras. Recall that the classical examples of quantum groups belong to one of the two following types: they are either function algebras, such as the algebra $C_0(G)$ of continuous functions vanishing at infinity on a locally compact group G, or their duals, such as the reduced group C^*-algebras $C^*_r(G)$. In the semigroup situation one can go beyond this dichotomy.

If S is a discrete semigroup, then the algebra $C^*_\delta(S)$ which we consider coincides with the reduced semigroup C^*-algebra $C^*_r(S)$ which has been known since long ago [42, 43, 21, 137]. If $S = G$ is a locally compact group, then $C^*_\delta(S) = C^*_\delta(G)$ is the C^*-algebra generated by all left translation operators in $B(L^2(G))$ [85, 22]. If G is moreover abelian, then $C^*_\delta(G)$ equals to the algebra $C(\widehat{G_d})$ of continuous functions on the dual of the discrete group G_d [85].

The new case considered in this paper concerns non-discrete nontrivial subsemigroups of locally compact groups, including the case of $\mathbb{R}_+$, and our objective is to show that their algebras admit a natural coalgebra structure. Let G be a second countable locally compact group, and let S be its sub-semigroup such that $S^{-1}S = G$. Set $H_S = \{f \in L^2(G) : \operatorname{supp} f \subset S\}$, a space isomorphic to $L^2(S)$. Let $L\colon G \to B(L^2(G))$ be the left regular representation of G, i.e. for any $a, b \in G,\ f \in L^2(G)$

$$(L_a f)(b) = f(a^{-1}b). \tag{4.2}$$

The subspace H_S is invariant under L_a, $a \in S$. We denote this restriction by $T_a \in B(H_S)$ and define $C^*_\delta(S)$ as the C^*-algebra generated in $B(H_S)$ by the operators T_a over all $a \in S$.

Defined in this way, we have

$$(T_a^* f)(b) = I_S(b) f(ab), \tag{4.3}$$

so that $T_a^* T_a = I$ and $T_a T_a^*$ is the operator E_{aS} of multiplication by I_{aS} (the characteristic function of aS). Thus, every T_a is an isometry, but is not unitary unless $aS = S$.

The strong closure of $C^*_\delta(S)$ in $B(H_S)$ is denoted $VN(S)$ and is said to be the semigroup von Neumann algebra. In the case $S = G$, this is the classical group von Neumann algebra, and in the case when the interiour of S is dense in it, this equals to the von Neumann algebra generated by the reduced C^*-algebra $C^*_r(S)$ introduced by Muhly and Renault [100].

4.3.1 Semigroup ideals

A subset $X \subset S$ is a right ideal if $Xa \subset X$ for all $a \in S$. For any $a \in S$, the set aS is a right ideal. As we have $T_a T_a^* = E_{aS}$, it is not surprising that the properties of our algebra are linked to the structure of ideals of S.

More generally, for any subset $X \subset S$ and any $a \in S$, define its *translation in S* as

$$a^{-1}X = \{t \in S \colon at \in X\}. \tag{4.4}$$

We always suppose that S contains the identity e of G. Any finite word w in the alphabet $S \subset S^{-1}$ can be written as $w = p_1^{-1}q_1p_2^{-1}q_2 \ldots p_n^{-1}q_n$ with $p_j, q_j \in S$, maybe with $p_1 = e$ or $q_n = e$. Define by induction

$$wS = p_1^{-1}(q_1(\ldots p_n^{-1}(q_nS)\ldots)).$$

This is a right ideal, and we call the ideals of this type *constructible*. We set

$$\mathcal{J} = \{wS|\ w \in F\} \cup \{\emptyset\}.$$

It is easy to see that the operator of multiplication E_X by the characteristic function of X is in the algebra $C^*_\delta(S)$, for every $X \in \mathcal{J}$.

For example, if $G = \mathbb{R}$ and $S = \mathbb{R}_+$, then

$$\mathcal{J} = \{[t, +\infty) : t \in \mathbb{R}_+\} \cup \{\emptyset\}.$$

We define the comultiplication on $C^*_\delta(S)$ in such a way that

$$\Delta(T_a) = T_a \otimes T_a, \tag{4.5}$$

so that it reflects the semigroup structure of S. This implies that we should have

$$\Delta(E_X) = E_X \otimes E_X \tag{4.6}$$

for every $X \in \mathcal{J}$. It was shown by Li [94] that in the discrete case, to guarantee that Δ is well defined we need the following restriction on the constructible ideals which he calls independence: if $X = \cup_{j=1}^n X_j$ for $X, X_1, \ldots, X_n \in \mathcal{J}$, $n \in \mathbb{N}$ then $X = X_j$ should hold for some $1 \le j \le n$.

In the non-discrete case, this definition should be adjusted as follows. Say that two measurable subsets X, Y of G are equivalent, $X \sim Y$, if their symmetric difference is of zero measure. We have $X \sim Y$ iff $E_X = E_Y$ as operators on $L^2(G)$. Now, we say that the constructible right ideals of S are *topologically independent* if

$$X \sim \cup_{j=1}^n X_j \text{ for } X, X_1, \ldots, X_n \in \mathcal{J} \text{ implies}$$
$$X \sim X_j \text{ for some } 1 \le j \le n.$$

Further on we will assume that the constructible right ideals of S are topologically independent. This is the case, for example, in $\mathbb{R}^n_+$, $n \in \mathbb{N}$.

The following is a typical example of a semigroup whose ideals are not topologically independent.

Example 4.3.1. Consider $S = \{0\} \cup [1; 1.5] \cup [2; \infty)$ as a subsemigroup of the group $\mathbb{R}$ with respect to usual addition and the usual topology. Compute the following ideals:

$$1 + S = \{1\} \cup [2; 2.5] \cup [3; \infty),$$

$$1.5 + S = \{1.5\} \cup [2.5; 3] \cup [3.5; \infty),$$
$$-1.5 + (1 + S) = \{1\} \cup \{1.5\} \cup [2; \infty).$$

We see that $-1.5 + (1 + S) = (1 + S) \cup (1.5 + S)$, and the same is true for the equivalence classes of these ideals. Hence, the ideals of S are not independent and not topologically independent.

4.3.2 The quantum semigroup associated to S

We define the comultiplication in several steps, via the intermediate algebras introduced below. Let $\mathcal{J}'$ be the set of equivalence classes of ideals in $\mathcal{J}$ (modulo the equivalence relation defined above). Let $C^*(S)$ be the universal C^*-algebra generated by isometries $\{v_p : p \in S\}$ and projections $\{e_X : X \in \mathcal{J}'\}$, with relations imposed by the semigroup structure: $v_{pq} = v_p v_q$, $v_p e_X v_p^* = e_{pX}$, $e_{X\cap Y} = e_X e_Y$. The identities (4.5) and (4.6) extend to a *-homomorphism of $C^*(S)$ by universality.

Next, in $C^*(S)$ we consider the commutative subalgebra D generated by the projections $\{e_X : X \in \mathcal{J}'\}$. It is shown that an inverse limit $\varprojlim_{p\in S} D_p$ of copies of $D = D_p$ indexed by $p \in S$ is isomorphic to the C^*-subalgebra in $B(L^2(G))$

$$D(S) = C^*\big(\{E_{q^{-1}X} : q \in S, X \in \mathcal{J}\}\big)$$

(where the translations are considered in G and not reduced to S as in the definition of semigroup ideals). We show that the comultiplication agrees with the inverse limit structure, so it generates a limit map on $D(S)$ which is a comultiplication on it.

If $G = \mathbb{R}$ and $S = \mathbb{R}_+$, then the algebra $D(\mathbb{R}_+)$ can be described as the space of functions supported in $\mathbb{R}_+$ and such that $\lim_{t\to t_0-0} f(t)$ exists and $f(t_0) = \lim_{t\to t_0+0} f(t)$ for every $t_0 \in [0, +\infty)$. This is the uniform closure of the algebra of piecewise continuous functions, and is sometimes called by the same name.

Up to now, the construction follows the lines of [94]. The difference appears when we extend the comultiplication to the non-commutative part of our algebra, and for this we need to extend the comultiplication to the strong closure $D(S)''$ of $D(S)$ in $B(L^2(G))$. This is done by considering in general von Neumann algebras generated by a semilattice $\mathcal{E}$ of linearly independent commuting projections; in our case $\mathcal{E} = \{E_{q^{-1}X} : q \in S, X \in \mathcal{J}'\}$. The strong closure of $C^*(\mathcal{E})$ is isomorphic to $L^\infty(\widehat{\mathcal{E}})$ with a suitable measure, where the compact zero-dimensional space $\widehat{\mathcal{E}}$ is the dual semilattice of $\mathcal{E}$. This allows to define a comultiplication by the usual formula $\Delta(f)(x, y) = f(x\wedge y)$, in which we consider $x, y \in \widehat{\mathcal{E}}$ and $f \in L^\infty(\widehat{\mathcal{E}})$.

The next step is to consider the von Neumann algebraic crossed product $D(S) \rtimes_\alpha G$ under the action $\alpha_g(E_{q^{-1}X}) = E_{gq^{-1}X}$. It is isomorphic to the von Neumann subalgebra $\mathcal{M}$ in $B(L^2(G))$ generated by $D(S)$ and all left translations by elements of G. The action α commutes with the comultiplication, what allows to define a comultiplication on the crossed product. And finally, one shows that $VN(S)$ is isomorphic to $E_S\mathcal{M}E_S$, and that the comultiplication restricts to it from $\mathcal{M}$ and has a well defined restriction onto $C^*_\delta(S) \subset VN(S)$.

In the abelian case, one has a more explicit description of $C^*_\delta(S)$. We will give it in the case $S = \mathbb{R}_+$.

First,

$$C^*_\delta(\mathbb{R}_+) \simeq \overline{\text{lin}}\{E_t L_g : t \in \mathbb{R}_+, g \in \mathbb{R}, t \geq g\},$$

where L_g is the operator of translation by g on $L^2(\mathbb{R})$.

Moreover, there exists a short exact sequence

$$0 \to K \to C^*_\delta(\mathbb{R}_+) \to C^*_\delta(\mathbb{R}) \to 0,$$

where K is the commutator ideal in $C^*_\delta(\mathbb{R}_+)$ generated by all operators $AB - BA$, $A, B \in C^*_\delta(S)$. Recall that $C^*_\delta(\mathbb{R})$ is the space $C(\mathbb{R}_b)$ of continuous functions on the Bohr compactification of the real line. The ideal K can be also described as

$$K = \overline{\text{lin}}\{E_{[a;b)} L_g : a, b \in \mathbb{R}_+, g \in \mathbb{R}, b \geq a \geq g\}.$$

Conclusion and perspectives

Currently I am working on two projects linked to the topics included into this thesis. During the year 2016/17, Safoura Zadeh (currently at IMPAN, Warsaw) was a postdoc in Besançon, and we worked with her on homomorphisms of weighted L^p-algebras. The questions are in the style of Section 4.2: given an isomorphism $T : L^p(G, \omega) \to L^p(H, \upsilon)$, subject to certain conditions, determine whether the groups G and H are isomorphic and T is given by a composition with a group isomorphism. Our paper is currently in preparation.

Another project is starting with Biswarup Das (Wrocław). In view of Example 3.3.5, we would like to apply the duality construction of [8] to quantum semigroup compactifications constructed by M. Daws [48] and later studied by Das and C. Mrozinski [47]. More generally, we would like to know what information one can get of a quantum semigroup by examining its dual.

There remain other questions to be explored in connection to the results collected in this thesis. I plan in particular to return to the description of almost isometries, this time of quantum group algebras; to work in the developing domain of Fourier analysis on quantum groups. In general, I will continue research in both classical and quantum branches of harmonic analysis.

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