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Existence et propriétés qualitatives des solutions de quelques problèmes elliptiques non-linéaires

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Existence et propriétés qualitatives des solutions de quelques problèmes elliptiques non-linéaires

Mémoire présenté pour obtenir l'habilitation à diriger des recherches

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Les travaux présentés dans ce mémoire portent sur l'existence et l'étude qualitative des solutions des équations aux dérivées partielles non linéaires, et tout particulièrement des solutions ayant une structure spatiale et temporelle bien définie, appelées suivant les cas ondes stationnaires, ondes progressives, ondes solitaires, ou de manière générale *ondes non linéaires*. Ces structures sont bien observées expérimentalement et numériquement, et très souvent jouent un rôle majeur dans la dynamique des systèmes correspondants.

Les systèmes considérés sont des modèles concrets issus de la mécanique des fluides, de la superfluidité, de la superconductivité ou de la physique des transitions de phase. A titre d'exemples, on peut citer les très nombreux modèles représentant différentes approximations de la propagation des ondes à la surface libre d'un fluide (équations de Benjamin-Ono, de Kadomtsev-Petviashvili, ou de Benney-Luke). D'autre part, il convient de mentionner les différentes variantes de l'équation de Schrödinger non-linéaire (comme l'équation de Gross-Pitaevskii, l'équation de Hartree ou l'équation de Schrödinger avec nonlinéarité de type "$\psi^3 - \psi^5$") qui interviennent dans l'étude des condensats de Bose-Einstein, la supraconductivité et la superfluidité.

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1 Travaux de recherche pendant la thèse

1.1 Régularité et décroissance

Dans un premier temps, je me suis intéressé aux propriétés qualitatives des ondes non linéaires pour quelques équations issues de la mécanique des fluides, plus précisément à leur régularité et à leur taux de décroissance à l'infini. On a utilisé la théorie classique des multiplicateurs de Fourier pour obtenir la régularité dans les espaces de Sobolev $W^{k,p}$, respectivement la théorie de Paley-Wiener pour démontrer l'analyticité des solutions.

Les propriétés de décroissance ont été prouvées en utilisant une technique générale qui consiste à transformer une équation aux dérivées partielles en une équation de convolution. Pour donner un exemple, considérons une équation de la forme $P(D)(u) = F(u)$ dans $\mathbf{R}^N$. En utilisant la transformation de Fourier cette équation est équivalente à $P(i\xi)\widehat{u} = \mathcal{F}(F(u))$. Si l'opérateur $P(D)$ est elliptique, on peut écrire $\widehat{u} = \frac{1}{P(i\xi)}\mathcal{F}(F(u))$ ou encore $u = k * F(u)$, où $k = \mathcal{F}^{-1}(\frac{1}{P(i\xi)})$. Dans beaucoup d'exemples concrets, le noyau k est une fonction qui décroît assez rapidement lorsque $|x|$ tend vers l'infini. Si $|F(u)| \leq C|u|^r$ pour u proche de zéro, où $r > 1$, et si l'on dispose d'une estimation sur la vitesse de convergence de u vers 0 à l'infini, en utilisant l'équation de convolution on peut améliorer successivement cette estimation. Dans la plupart des applications on obtient que u tend vers zéro (au moins) aussi rapidement que le noyau k.

Cette technique avait été utilisée dans [LiBo96] et [BoLi97] pour des problèmes unidimensionnels et dans [dBS97] pour les ondes solitaires de l'équation de Kadomtsev-Petviashvili en dimension 2 et 3. Elle a été utilisée par la suite par P. Gravejat pour les ondes progressives de l'équation de Gross-Pitaevskii.

On a montré dans [1] que les ondes solitaires de l'équation de Benney-Luke (dont l'existence avait été prouvée en 1998 par R. L. Pego et J. L. Quintero) sont des fonctions analytiques et on a trouvé leur taux algébrique optimal de décroissance à l'infini. Dans [2] on a montré l'existence, l'analyticité et on a trouvé le taux optimal de décroissance des ondes solitaires d'une généralisation bidimensionnelle de l'équation de Benjamin-Ono.

1.2 Existence des ondes non-linéaires

A. Une équation de Schrödinger non-linéaire avec potentiel en dimension 1. Dans [4] on a étudié l'équation

$$(1.1) \qquad iA_t - ivA_x = -A_{xx} - A + |A|^2 A + U(x)A, \qquad x \in \mathbf{R}, \quad t \in \mathbf{R}_+,$$

qui décrit l'écoulement derrière un obstacle fixe d'un fluide injecté avec une vitesse constante v à l'infini. Le potentiel U est une mesure positive qui modélise l'obstacle. L'équation (1.1) a été étudiée à l'aide des développements asymptotiques formels et des simulations numériques par V. Hakim pour quelques types particuliers de potentiel.

On a cherché des solutions stationnaires (i.e. indépendantes de t) dont le module tend vers ± 1 à l'infini. On a réussi à montrer que, si le potentiel U n'est pas trop grand, deux telles solutions existent : l'une est obtenue comme un minimiseur de l'énergie associée à (1.1), l'autre est un point selle de l'énergie. L'existence d'un minimiseur est classique. La preuve de l'existence d'une deuxième solution est beaucoup plus délicate et repose sur une variante du Lemme du Col due à Ghoussoub et Preiss. La difficulté majeure est d'obtenir des informations assez précises sur les suites de Palais-Smale afin de déduire leur convergence et de montrer que leur limite est différente de la solution obtenue par minimisation.

B. Existence des bulles instationnaires. L'objectif de l'article [3] a été d'étudier l'existence des ondes progressives (appelées également "bulles instationnaires") de petite vitesse pour l'équation de Schrödinger non-linéaire

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + F(|\psi|^2)\psi = 0 \quad \text{dans } \mathbf{R}^N, \tag{1.2}$$

où ψ est une fonction complexe qui satisfait la "condition aux limites" $|\psi| \longrightarrow r_0 > 0$ quand $|x| \longrightarrow \infty$ et la nonlinéarité est de type "$\psi^3 - \psi^5$". Les bulles sont des solutions de la forme $\psi(x,t) = \phi(x_1 - ct, x_2, \ldots, x_N)$. L'existence de telles solutions en dimension un d'espace a été prouvée dans [BaMa88].

On a utilisé une approche variationnelle : les bulles sont des points critiques d'une fonctionnelle $E_c(u) = E(u) + cQ(u)$, où E est "l'énergie" associée à (1.2) et Q est le moment. Une technique classique pour montrer l'existence des points critiques consiste à mettre en évidence un changement de topologie entre deux ensembles de niveau de la fonctionnelle, et ensuite à prouver une propriété de compacité des suites de Palais-Smale. Cependant, les ensembles de niveau de E_c ont une structure bien compliquée et il semble très difficile de montrer un changement au niveau global dans leur topologie. D'autre part, il est également difficile de montrer la compacité des suites de Palais-Smale. Pour surmonter ces difficultés, nous avons prouvé une variante locale du Lemme du Col. Ce résultat abstrait permet de trouver des suites de Palais-Smale bornées lorsqu'on dispose d'une information concernant un changement dans la structure des ensembles de niveau uniquement localement, au voisinage d'un point. D'autre part, nous disposons d'un *état fondamental* u_0 de E qui possède des propriétés tout à fait remarquables : il est un point critique de E et il est un minimum local strict de E sur un sous-espace fonctionnel de codimension 1. D'une manière heuristique, il existe une "cuvette" autour de u_0 sur un sous-espace de codimension 1. En dimension au moins égale à 4, la hauteur de la "cuvette" est strictement positive et cette structure subsiste lorsqu'on rajoute à E une "perturbation" cQ avec c suffisamment petit. Cette observation ainsi que le résultat abstrait mentionné nous ont permis de montrer l'existence des bulles instationnaires de petite vitesse.

2 Symétrie des solutions des équations aux dérivées partielles

Le fait de savoir que les solutions d'une EDP présentent des symétries est très important à la fois pour leur étude théorique que pour leur approximation numérique. Les symétries peuvent aussi s'avérer très utiles pour l'étude de la stabilité des ondes solitaires ou des ondes stationnaires de certaines EDP d'évolution. En général, la symétrie constitue la première étape dans la preuve de l'unicité des solutions de certaines EDP elliptiques.

Jusqu'à présent il existe dans la littérature trois méthodes générales pour montrer de telles symétries. La première a été développée par Gidas, Ni et Nirenberg à la fin des années '70 et est basée sur les "moving planes" et le principe de maximum. Elle est applicable aux solutions positives dans des problèmes qui font intervenir le Laplacien. Une autre méthode repose sur l'utilisation de la symétrisation de Schwarz d'une fonction. Elle permet de montrer qu'il existe des solutions symétriques pour un probleme de minimisation. (Notons, par ailleurs, que dans beaucoup de situations les ondes non-linéaires sont obtenues en minimisant une certaine fonctionnelle, avec ou sans contrainte). En général, cette méthode n'implique pas directement que *toutes* les solutions sont symétriques. L'utilisation de ces deux méthodes dans le cas des systèmes est parfois possible, mais elle reste assez limitée car, d'une part, le signe de chacune des composantes de la solution doit être constant, d'autre part on a besoin de conditions assez fortes (et souvent irréalistes) sur les nonlinéarités. Afin d'éviter ces inconvenients, O. Lopes a proposé en 1996 dans [Lop1, Lop2] une méthode étonnament simple et efficace pour montrer la symétrie des minimiseurs. Nous présentons ci-dessous le résultat de [Lop1].

Théorème 2.1 *([Lop1]) Soit $u \in H^1(\mathbf{R}^N, \mathbf{R}^m)$ un minimiseur de la fonctionnelle*

$$V(u) := \int_{\mathbf{R}^N} |\nabla u|^2 dx + \int_{\mathbf{R}^N} F(u)\, dx$$

sous la contrainte $I(u) := \int_{\mathbf{R}^N} G(u)\, dx = \lambda \neq 0$. On suppose que les fonctions F et G sont C^1, qu'il existe $p \leq 2^*$ tel que $|F(u)| \leq C|u|^{2^*}$ et $|G(u)| \leq C|u|^{2^*}$ pour $|u| \geq 1$ et que $G'(u) \not\equiv 0$ si $u \not\equiv 0$.

Alors la fonction u est à symétrie radiale (après une translation dans $\mathbf{R}^N$).

Preuve. On va d'abord montrer que la fonction u présente une symétrie par rapport à la variable x_1. Après une translation dans la direction de x_1 on peut supposer que

$$\int_{\{x_1<0\}} G(u)dx = \int_{\{x_1>0\}} G(u)dx = \lambda/2.$$

On définit

$$v_1(x) = \begin{cases} u(x_1, x') & \text{si } x_1 \leq 0 \\ u(-x_1, x') & \text{si } x_1 > 0 \end{cases} \quad \text{et} \quad v_2(x) = \begin{cases} u(-x_1, x') & \text{si } x_1 \leq 0 \\ u(x_1, x') & \text{si } x_1 > 0. \end{cases} \tag{2.1}$$

Il est facile de voir que $v_1, v_2 \in H^1(\mathbf{R}^N, \mathbf{R}^m)$ et on a $I(u) = I(v_1) = I(v_2) = \lambda$. Come u est un minimiseur, ceci implique $V(v_1) \geq V(u)$ et $V(v_2) \geq V(u)$.

D'autre part on a $V(v_1) + V(v_2) = 2V(u)$.

On en déduit que nécessairement $V(v_1) = V(v_2) = V(u)$ et v_1 et v_2 sont aussi des minimiseurs. Par conséquent, il existe des multiplicateurs de Lagrange α et β tels que

$$\begin{gathered} -\Delta u + 2F'(u) + \alpha G'(u) = 0 \quad \text{et} \\ -\Delta v_1 + 2F'(v_1) + \beta G'(v_1) = 0 \quad \text{dans } \mathbf{R}^N. \end{gathered} \tag{2.2}$$

En utilisant (2.2) et la théorie de la régularité elliptique, on déduit que les fonctions u et v_1 sont bornées et régulières.

On ne peut pas avoir $v_1 = 0$ car $I(v_1) = \lambda \neq 0$. Par conséquent, il existe x^* tel que $x_1^* < 0$ et $G'(v_1(x^*)) \neq 0$. Comme $u = v_1$ dans $\{x_1 < 0\}$, de (2.2) on déduit que $\alpha = \beta$. Il est alors facile de voir que la fonction $w := u - v_1$ satisfait une équation de la forme

$$-\Delta w + A(x)w = 0 \quad \text{dans } \mathbf{R}^N, \text{ où } A \in L^\infty(\mathbf{R}^N, \mathbf{R}^m \times \mathbf{R}^m). \tag{2.3}$$

On utilise ensuite le

Théorème 2.2 *(Théorème de Prolongement Unique) Supposons que $\Phi \in H^1_{loc}(\mathbf{R}^N, \mathbf{R}^m)$ satisfait*

$$-\Delta\Phi + A(x)\Phi = 0 \quad \textit{dans } \mathbf{R}^N, \quad \textit{où } A \in L^\infty(\mathbf{R}^N, \mathbf{R}^m \times \mathbf{R}^m).$$

Si $\Phi \equiv 0$ dans un ouvert $\Omega \subset \mathbf{R}^N$, alors $\Phi \equiv 0$ dans $\mathbf{R}^N$.

Comme $w \equiv 0$ dans le demi-espace $\{x_1 < 0\}$, on déduit du théorème de prolongement unique et de (2.3) que $w = 0$ dans $\mathbf{R}^N$, c'est-à-dire $u = v_1$. Donc u est symétrique par rapport à x_1.

De la même façon, après translation u est symétrique par rapport à chacune des variables $x_2, \ldots, x_N$. En particulier, $u(x) = u(-x)$. Par conséquent, tout hyperplan Π qui contient l'origine O coupe la contrainte en deux quantités égales. Le même argument que ci-dessus implique alors que u est symétrique par rapport à tout hyperplan contenant O, donc u est à symétrie radiale. □

2.1 Un résultat général de symétrie

Dans un travail récent [7], nous avons étudié la symétrie des solutions d'un problème $(\mathcal{P})$ qui consiste à minimiser une fonctionnelle

$$E(u) = \int_\Omega F(|x|, u(x), |\nabla u(x)|)\, dx$$

avec un nombre fini de contraintes

$$Q_j(u) = \int_\Omega G_j(|x|, u(x), |\nabla u(x)|)\, dx = \lambda_j, \qquad j = 1, \ldots, k,$$

où $\Omega \subset \mathbf{R}^N$ est un ensemble ouvert invariant par rotations. On suppose que :

A1. On travaille dans un espace $\mathcal{X}$ de fonctions ayant la propriété que pour tout $u \in \mathcal{X}$ et pour tout hyperplan Π de $\mathbf{R}^N$ contenant le centre de Ω, les deux fonctions u_{Π^+} et u_{Π^-} obtenues de u par symétrie miroir par rapport à Π (comme dans (2.1)) appartiennent encore à $\mathcal{X}$.

A2. Le problème $(\mathcal{P})$ admet des solutions dans $\mathcal{X}$ et toute solution est de classe C^1 sur Ω.

Notons que ces hypothèses sont très générales, donc les résultats obtenus s'appliquent à un grand nombre de situations concrètes. Par exemple, la condition (A1) est vérifiée par tous les espaces de Sobolev $W^{1,p}(\Omega)$. Sous des hypothèses de régularité et de croissance raisonnables sur $F, G_1, \ldots, G_k$, les fonctionnelles $E, Q_1, \ldots, Q_k$ sont différentiables sur $\mathcal{X}$ et les minimiseurs de $(\mathcal{P})$ satisfont des équations d'Euler-Lagrange. Très souvent, ces équations sont des systèmes elliptiques quasi-linéaires. La théorie de la régularité des solutions de tels systèmes a connu un développement spectaculaire les 50 dernières années. Dans des situations très générales, elle permet de montrer que les solutions des équations d'Euler-Lagrange sont (au moins) C^1, donc (A2) est satisfaite.

Le résultat obtenu est le suivant :

Théorème 2.3 *Supposons que (A1), (A2) sont satisfaites et $0 \le k \le N-2$. Soit $u \in \mathcal{X}$ un minimiseur de $(\mathcal{P})$. Alors il existe un sous-espace vectoriel V de $\mathbf{R}^N$ de dimension k tel que u est à symétrie radiale par rapport à V (c'est-à-dire $u(x)$ dépend uniquement de la projection orthogonale de x sur V et de la distance de x à V).*

Dans le cas où $\Omega = \mathbf{R}^N$, les fonctionnelles $E, Q_1, \ldots Q_k$ sont invariantes par translations et (A1) a lieu pour tout hyperplan affine Π de $\mathbf{R}^N$ (et non seulement pour les hyperplans contenant l'origine), on a montré :

Théorème 2.4 *Si $u \in \mathcal{X}$ est un minimiseur de $(\mathcal{P})$ et $1 \le k \le N-1$, alors il existe un sous-espace affine V de $\mathbf{R}^N$ de dimension $k-1$ tel que u est à symétrie radiale par rapport à V.*

En particulier, dans le cas d'une seule contrainte, tous les minimiseurs sont à symétrie radiale par rapport à un point. Notons que tous ces résultats sont valables pour des minimiseurs à valeurs vectorielles et on ne demande aucune hypothèse sur les signes des composantes des minimiseurs. D'autre part, les exemples présentés dans [7] montrent que les résultats ci-dessus sont optimaux même pour des minimiseurs à valeurs scalaires.

Afin de donner une idée des preuves des résultats énoncés plus haut, nous allons présenter la démonstration du Théorème 2.4 dans le cas particulier où $N = 2$ et $k = 1$. Le problème $(\mathcal{P})$ devient

$$(\mathcal{P}') \qquad \begin{array}{l} \text{Minimiser } E(u) = \displaystyle\int_{\mathbf{R}^2} F(u(x), |\nabla u(x)|)\, dx \quad \text{sous la contrainte} \\ Q(u) = \displaystyle\int_{\mathbf{R}^2} G(u(x), |\nabla u(x)|)\, dx = \lambda \ne 0. \end{array}$$

Lemme 2.5 *Soit u un minimiseur de $(\mathcal{P}')$. On suppose que toute droite Π passant par O a la propriété :*

$$\int_{\Pi^+} G(u(x), |\nabla u(x)|)\, dx = \int_{\Pi^-} G(u(x), |\nabla u(x)|)\, dx = \frac{\lambda}{2}. \tag{2.4}$$

Alors u est à symétrie radiale par rapport à O.

Preuve du Lemme 2.5. Soit Π une droite quelconque contenant O. On choisit un système de coordonnées tel que $\Pi = Oy$. On définit

$$v_1(x,y) = \begin{cases} u(x,y) \text{ si } x \le 0 \\ u(-x,y) \text{ si } x > 0, \end{cases} \qquad v_2(x,y) = \begin{cases} u(-x,y) \text{ si } x < 0 \\ u(x,y) \text{ si } x \ge 0. \end{cases}$$

Alors $v_1, v_2 \in \mathcal{X}$ et on a $Q(v_1) = Q(v_2) = \lambda$ ce qui implique $E(v_1) \ge E(u)$ et $E(v_2) \ge E(u)$. D'autre part, on a $E(v_1) + E(v_2) = 2E(u)$. Donc $v_1,\ v_2$ sont aussi des minimiseurs et en utilisant (A2) on déduit que $v_1,\ v_2 \in C^1(\mathbf{R}^2)$.

La symétrie de v_1 et de v_2 par rapport à x_1 implique $\dfrac{\partial v_1}{\partial x}(0,y) = \dfrac{\partial v_2}{\partial x}(0,y) = 0$ pour tout y. Comme $u = v_1$ pour $x < 0$, on a

$$\frac{\partial u}{\partial x}(0,y) = \lim_{s \uparrow 0} \frac{\partial u}{\partial x}(s,y) = \lim_{s \uparrow 0} \frac{\partial v_1}{\partial x}(s,y) = \frac{\partial v_1}{\partial x}(0,y) = 0.$$

Ainsi on a montré que pour toute droite Π contenant O,

$$\frac{\partial u}{\partial n} = 0 \quad \text{sur } \Pi, \text{ où } n \text{ est la normale à } \Pi. \tag{2.5}$$

En coordonnées polaires $x = r\cos\theta$, $y = r\sin\theta$, ceci implique $\dfrac{\partial u}{\partial \theta} = 0$ sur $\mathbf{R}^2 \setminus \{O\}$ et on en déduit que u ne dépend pas de θ, c'est-à-dire u est une fonction radiale. □

Démonstration du Théorème 2.4 pour $N = 2$ et $k = 1$.

Soit u un minimiseur de $(\mathcal{P}')$. Après translation, on peut supposer que

$$\int_{\{x<0\}} G(u, |\nabla u|)\, dx\, dy = \int_{\{x>0\}} G(u, |\nabla u|)\, dx\, dy = \frac{\lambda}{2}.$$

Soient u_1 et u_2 les deux fonctions obtenues de u par symétrie miroir par rapport à Oy. Alors u_1 et u_2 sont aussi des minimiseurs et, de plus, sont paires en x.

Après translation en y, on peut supposer que

$$\int_{\{y<0\}} G(u_1, |\nabla u_1|)\, dx\, dy = \int_{\{y>0\}} G(u_1, |\nabla u_1|)\, dx\, dy = \frac{\lambda}{2}.$$

Soient $u_{1,1}$ et $u_{1,2}$ les deux fonctions obtenues de u_1 par symétrie miroir par rapport à Ox. Il est évident que $u_{1,1}$ et $u_{1,2}$ sont aussi des minimiseurs et sont paires en x et en y. Par le Lemme 2.5 on déduit que $u_{1,1}$ et $u_{1,2}$ sont des fonctions radiales par rapport à O. Comme $u_{1,1}(x, 0) = u_1(x, 0) = u_{1,2}(x, 0)$ pour tout x, on a nécessairement $u_{1,1} = u_{1,2} = u_1$ sur $\mathbf{R}^2$, donc u_1 est une fonction radiale.

De la même façon, il existe $k \in \mathbf{R}$ tel que

$$\int_{\{y<k\}} G(u_2, |\nabla u_2|)\, dx\, dy = \int_{\{y>k\}} G(u_2, |\nabla u_2|)\, dx\, dy = \frac{\lambda}{2}.$$

Comme ci-dessus on déduit que u_2 est radiale par rapport à $(0, k)$.

Nous allons montrer que $k = 0$. Supposons, par l'absurde, que $k \neq 0$. Alors la fonction d'une variable $y \longmapsto u(0, y) = u_1(0, y) = u_2(0, y)$ est une fonction symétrique par rapport à 0 et par rapport à k, donc c'est une fonction $2|k|-$périodique. Donc $G(u_1, |\nabla u_1|)$ est une fonction radiale dont le profil est $2|k|-$périodique. On en déduit que si l'intégrale $\int_{\mathbf{R}^2} G(u_1, |\nabla u_1|)\, dx\, dy$ converge, sa valeur est nécessairement 0, ce qui implique $\lambda = 0$, absurde.

Par conséquent, on a $k = 0$ et alors u_1, u_2 sont deux fonctions radiales par rapport à O. Comme $u_1(0, \cdot) = u(0, \cdot) = u_2(0, \cdot)$ on en déduit que $u_1 = u_2 = u$, c'est-à-dire u est radiale. □

Pour démontrer les Théorèmes 2.3 et 2.4 dans le cas général, on utilise le Théorème de Borsuk-Ulam pour trouver des hyperplans qui "coupent les contraintes en deux", un résultat analogue au Lemme 2.5, et un peu de géométrie élémentaire et de combinatoire pour "recoller les morceaux."

2.2 Symétrie et monotonie des solutions d'énergie minimale

Dans [9], nous étudions le comportement des solutions d'énergie minimale du système

$$-\text{div}(|\nabla u_i|^{p-2} \nabla u_i) = g_i(u), \qquad i = 1, \ldots, m, \tag{2.6}$$

où $u=(u_1,\dots,u_m):\mathbf{R}^N\longrightarrow\mathbf{R}^m$, $1<p<\infty$, $|(y_1,\dots,y_N)|^p=\left(\sum_{j=1}^N y_j^2\right)^{\frac{p}{2}}$, $g_i(0)=0$ et il existe $G\in C^1(\mathbf{R}^m\setminus\{0\},\mathbf{R})$ telle que $g_i(u)=\frac{\partial G}{\partial u_i}(u)$.

Ce système, notamment dans le cas $p=2$, intervient dans un nombre important de problèmes issus de la physique. La fonctionnelle d'énergie associée est

$$S(u)=\frac{1}{p}\int_{\mathbf{R}^N}\sum_{i=1}^m|\nabla u_i|^p\,dx-\int_{\mathbf{R}^N}G(u)\,dx.$$

Il est facile de voir que les solutions de (2.6) sont précisément les points critiques de S. On appelle *solution d'énergie minimale* une solution qui minimise S dans l'ensemble de toutes les solutions.

L'existence des solutions d'énergie minimale a été prouvée dans une série de travaux classiques (le lecteur pourra consulter les articles [BeLi83], [BeGK83], [BrLieb84] et les références qu'ils contiennent). Cependant, dans le cas des conditions générales sur la nonlinéarité g, la symétrie de telles solutions et la monotonie de leur profil dans le cas scalaire ($m=1$) ont été des problèmes longtemps non résolus.

Nous avons trouvé une caractérisation variationnelle équivalente des solutions d'énergie minimale de (2.6). On introduit les fonctionnelles $J(u)=\frac{1}{p}\int_{\mathbf{R}^N}\sum_{i=1}^m|\nabla u_i|^p\,dx$ et $V(u)=\int_{\mathbf{R}^N}G(u)\,dx$. On a :

Proposition 2.6 *On suppose que $1<p\le N$. Soit u une solution d'énergie minimale de (2.6). Alors u est une solution du problème de minimisation*

(2.7) *minimiser $J(v)$ sous la contrainte $v\neq 0$ et $V(v)=V(u)$.*

En utilisant la Proposition 2.6 et le Théorème 2.4, on déduit :

Corollaire 2.7 *Si u est une solution d'énergie minimale de (2.6), alors u est à symétrie radiale (modulo une translation dans $\mathbf{R}^N$).*

Dans le cas scalaire ($m=1$) nous avons montré le résultat de monotonie suivant :

Proposition 2.8 *Soit $m=1$ et soit u une solution d'énergie minimale de (2.6). Alors :*
i) La fonction u a un signe constant sur $\mathbf{R}^N$.
ii) Le profil radial de u est une fonction monotone sur $[0,\infty)$.

La preuve de (i) est assez simple (et présente un lien évident avec le fait que le principe de concentration-compacité peut être utilisé pour montrer la compacité des suites minimisantes du problème (2.7)) : si u change de signe, on peut considérer séparement les fonctions u_+ et u_-. Alors $J(u_+)+J(u_-)=J(u)$ et $V(u_+)+V(u_-)=V(u)$, ce qui implique que la dichotomie se produit pour les suites minimisantes de (2.7).

La preuve de (ii) repose sur un résultat de [BroZi88] qui affirme que pour une fonction positive $v\in\mathcal{D}^{1,p}(\mathbf{R}^N)$ on a toujours $J(v)\ge J(v^*)$ (où v^* est le réarrangement de Schwarz de v) et on peut avoir $J(v)=J(v^*)$ uniquement si les ensembles de niveau $v^{-1}(t)$ sont des sphères pour presque tout $t>0$. Or, si u est une solution d'énergie minimale, on sait déjà que u est positive et radiale, $u(x)=\tilde{u}(|x|)$, où $\tilde{u}(r)\longrightarrow 0$ quand $r\longrightarrow\infty$. Si $\tilde{u}$ n'est pas décroissante, il existe $0<a<b<c$ tels que $\tilde{u}(a)<\tilde{u}(b)$ et $\tilde{u}(b)>\tilde{u}(c)$. Soit $m_1=\min(\tilde{u}(a),\tilde{u}(c))$ et $m_2=\tilde{u}(b)$. Alors pour tout $t\in[m_1,m_2]$, $u^{-1}(t)$ contient au moins deux sphères concentriques, donc ce n'est pas une sphère. Par le théorème de [BroZi88] on déduit que $J(u^*)<J(u)$. Comme $V(u^*)=V(u)$, on obtient une contradiction avec le fait que u est un minimiseur de (2.7).

2.3 Symétrie dans des problèmes non-locaux

La symétrie des minimiseurs dans des problèmes qui font intervenir des opérateurs non-locaux a été étudiée dans [6]. Nous décrivons ci-dessous quelques exemples.

A. Problèmes qui font intervenir les puissances fractionnaires du Laplacien. On considère l'équation de Benjamin-Ono généralisée

$$A_t + \alpha A A_x - \beta(-\Delta)^{\frac{1}{2}} A_x = 0 \quad \text{dans } \mathbf{R}^2, \quad \alpha,\ \beta > 0.$$

Les ondes progressives de cette équation sont des solutions de la forme $A(x,\, y,\, t) = u(x - ct,\, y)$. Après changement d'échelle, on trouve que le profil u satisfait

$$u + (-\Delta)^{\frac{1}{2}} u = u^2 \quad \text{dans } \mathbf{R}^2.$$

L'existence des ondes progressives a été démontrée dans [2] en minimisant

$$V(u) := \frac{1}{2}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u|^2 dx + \int_{\mathbf{R}^2} u^2 dx$$

sous la contrainte $I(u) = \displaystyle\int_{\mathbf{R}^2} u^3 dx = \textit{constant}$. Plus généralement, dans [Lop3] on a montré l'existence des minimiseurs des fonctionnelles de type

$$V(u) = \int_{\mathbf{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi + \int_{\mathbf{R}^N} F(u) dx \tag{2.8}$$

sous une contrainte $I(u) = \displaystyle\int_{\mathbf{R}^N} G(u) dx = \lambda \neq 0$. Il est évident que ce problème de minimisation ressemble à celui considéré dans le Théorème 2.1. La question qui se pose naturellement est de savoir si les minimiseurs de (2.8) présentent aussi une symétrie. Le résultat suivant permet d'apporter une réponse affirmative à cette question.

Lemme 2.9 *Soit $s \in (-\frac{1}{2}, \frac{3}{2})$ et soit $u \in \dot{H}^s(\mathbf{R}^N)$. On définit v_1, v_2 comme dans (2.1). Si V est donné par (2.8), on a*

$$V(v_1) + V(v_2) - 2V(u) = -\frac{16\sin(s\pi)}{\pi^2} N_s^2(f), \qquad \forall\, s \in (-\frac{1}{2}, \frac{3}{2}), \tag{2.9}$$

où $f(x) = \frac{1}{2}(u(x_1, x') - u(-x_1, x'))$ et

$$N_s^2(f) = \int_{\mathbf{R}^{N-1}} \int_{|\xi'|}^{\infty} \left(t^2 - |\xi'|^2\right)^s \left| \int_0^{\infty} \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2}\, d\xi_1 \right|^2 dt\, d\xi'. \tag{2.10}$$

De plus, N_s est une norme sur $\dot{H}^s_{1,odd}(\mathbf{R}^N) = \{u \in \dot{H}^s(\mathbf{R}^N) \mid u \text{ est antisymétrique en } x_1\}$ et cette norme est continue par rapport à la norme usuelle de $\dot{H}^s$.

Supposons que $s \in (0, 1)$ et que u est un minimiseur de (2.8) sous la contrainte $I(u) = \lambda \neq 0$. Après une translation dans la direction de x_1, on peut supposer que $\displaystyle\int_{\{x_1<0\}} G(u)dx = \int_{\{x_1>0\}} G(u)dx = \lambda/2$. On définit v_1 et v_2 comme dans (2.1). Il est clair que $I(v_1) = I(v_2) = \lambda$ et (2.9) implique que $N_s(f) = 0$ (car sinon on aurait $V(v_1) + V(v_2) - 2V(u) < 0$, donc $V(v_1) < V(u)$ ou $V(v_2) < V(u)$, en contradiction avec le fait que u est un minimiseur). Comme $N_s(f)$ est une norme, on en déduit que $f = 0$, donc u

est symétrique par rapport à x_1. De la même façon, u est symétrique par rapport à toute direction (modulo translation) et finalement on obtient que u est une fonction radiale.

Notons que dans le cas $s \in (1, \frac{3}{2})$, (2.9) implique $V(v_1) + V(v_2) - 2V(u) \geq 0$ (avec inégalité stricte si u n'est pas symétrique) et la méthode n'est plus applicable. La symétrie des minimiseurs dans ce cas reste un problème ouvert.

Preuve du Lemme 2.9 dans le cas N=1.

On va démontrer que (2.9) a lieu pour $V(u) = ||u||^2_{\dot{H}^s} = \int_{\mathbf{R}^N} |\xi|^{2s} |\widehat{u}|^2 \, d\xi$ quelque soit $s \in (-\frac{1}{2}, \frac{3}{2})$. On note $f(x) = \frac{1}{2}(u(x) - u(-x))$.

Étape 1 : $u \in C_c^\infty(\mathbf{R})$. On note $g(x) = \frac{1}{2}(u(x) + u(-x))$, $f_*(x) = \begin{cases} -f(x), & x \leq 0, \\ f(x), & x > 0, \end{cases}$ de sorte que g, f_* sont paires, f est impaire, $u = g + f$, $u_1 = g - f_*$, $u_2 = g + f_*$. On a

$$\widehat{f}(\xi) = \int_0^\infty (e^{-ix\xi} - e^{ix\xi}) f(x) \, dx = 2i \int_0^\infty \sin(x\xi) f(x) \, dx,$$

$$\widehat{f_*}(\xi) = \int_0^\infty (e^{-ix\xi} + e^{ix\xi}) f(x) \, dx = 2 \int_0^\infty \cos(x\xi) f(x) \, dx.$$

On obtient ensuite

$$\begin{aligned}
\|u_1\|^2_{\dot{H}^s} + \|u_2\|^2_{\dot{H}^s} - 2\|u\|^2_{\dot{H}^s} &= \int_{\mathbf{R}} |\xi|^{2s} \left(|\widehat{g} - \widehat{f_*}|^2 + |\widehat{g} + \widehat{f_*}|^2 - 2|\widehat{g} + \widehat{f}|^2 \right) d\xi \\
&= 2 \int_{\mathbf{R}} |\xi|^{2s} \left(|\widehat{f_*}|^2 - |\widehat{f}|^2 \right) d\xi \\
&= 8 \int_{\mathbf{R}} |\xi|^{2s} \left(\left| \int_0^\infty \cos(x\xi) f(x) \, dx \right|^2 - \left| \int_0^\infty \sin(x\xi) f(x) \, dx \right|^2 \right) d\xi \\
&= 8 \int_{\mathbf{R}} |\xi|^{2s} \Big(\int_0^\infty \int_0^\infty \cos(x\xi) f(x) \overline{\cos(y\xi) f(y)} \, dxdy \\
&\qquad - \int_0^\infty \int_0^\infty \sin(x\xi) f(x) \overline{\sin(y\xi) f(y)} \, dxdy \Big) \qquad \text{(Fubini)} \\
&= 8 \int_{\mathbf{R}} |\xi|^{2s} \int_0^\infty \int_0^\infty \cos((x+y)\xi) f(x) \overline{f(y)} \, dxdy \, d\xi.
\end{aligned} \tag{2.11}$$

On définit $h(z) = \int_0^\infty \int_0^\infty e^{i(x+y)z} f(x) \overline{f(y)} \, dxdy$ et on prouve que :

- La fonction h est holomorphe sur $\mathbf{C}$ et bornée sur $\{z \in \mathbf{C} \mid \mathrm{Im}(z) \geq 0\}$.
- Si $z \in \mathbf{R}$, alors $h(-z) = \overline{h(z)}$ et $\mathrm{Re}(h(z)) = \int_0^\infty \int_0^\infty \cos((x+y)\xi) f(x) \overline{f(y)} \, dxdy$.
- On a $|h(z)| \leq \frac{C}{|z|^4}$ si $\mathrm{Im}(z) \geq 0$ car $\int_0^\infty e^{ixz} f(x) \, dx = -\frac{1}{z^2} \left(f'(0) + \int_0^\infty e^{ixz} f''(x) \, dx \right)$.
- Pour $t \in \mathbf{R}_+$ on a

$$\begin{aligned}
h(it) &= \left| \int_0^\infty e^{-tx} f(x) \, dx \right|^2 = \left| \langle f, e^{-t\cdot} \mathbf{1}_{[0,\infty)}(\cdot) \rangle_{L^2} \right|^2 \\
&= \frac{1}{(2\pi)^2} \left| \langle \widehat{f}, \mathcal{F} \left(e^{-t\cdot} \mathbf{1}_{[0,\infty)}(\cdot) \right) \rangle_{L^2} \right|^2 \qquad \text{(Plancherel)} \\
&= \frac{1}{(2\pi)^2} \left| \int_{-\infty}^\infty \widehat{f}(\xi) \frac{1}{t - i\xi} \, d\xi \right|^2 = \frac{1}{\pi^2} \left| \int_0^\infty \widehat{f}(\xi) \frac{\xi}{t^2 + \xi^2} \, d\xi \right|^2.
\end{aligned} \tag{2.12}$$

On définit $m(z) = (z^2)^s = e^{s\log(z^2)} = e^{2s\ln|z| + is\arg(z^2)}$. La fonction m est holomorphe sur $\mathbf{C} \setminus \{it \mid t \in \mathbf{R}\}$. En intégrant la fonction holomorphe $z \longmapsto h(z)m(z)$ sur un chemin bien choisi, on obtient que pour tout $s \in (-\frac{1}{2}, \frac{3}{2})$ on a

$$\int_\varepsilon^\infty m(z)h(z)\,dz = i\int_0^\infty m(\varepsilon + it)h(\varepsilon + it)\,dt.$$

On passe à la limite lorsqeue $\varepsilon \longrightarrow 0$ et on trouve

$$\int_0^\infty m(z)h(z)\,dz = i\int_0^\infty t^{2s}e^{is\pi}h(it)\,dt.$$

On a aussi

$$\int_{-\infty}^0 |z|^{2s}h(z)\,dz = \int_0^\infty |z|^{2s}\overline{h(z)}\,dz = -i\int_0^\infty t^{2s}e^{-is\pi}h(it)\,dt.$$

Par conséquent,

$$\int_{-\infty}^\infty |z|^{2s}h(z)\,dz = -2\sin(s\pi)\int_0^\infty t^{2s}h(it)\,dt.$$

D'où finalement, en utilisant (2.11) et (2.12),

$$\begin{aligned} \|u_1\|_{\dot H^s}^2 + \|u_2\|_{\dot H^s}^2 - 2\|u\|_{\dot H^s}^2 &= 8\int_{\mathbf{R}} |\xi|^{2s}\mathrm{Re}(h(\xi))\,d\xi \\ &= -16\sin(s\pi)\int_0^\infty t^{2s}h(it)\,dt \\ &= -\frac{16\sin(s\pi)}{\pi^2}\int_0^\infty t^{2s}\left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2 dt. \\ &= -\frac{16\sin(s\pi)}{\pi^2}N_s^2(f). \end{aligned} \tag{2.13}$$

Étape 2 : argument de densité. On a montré l'identité (2.13) pour tout $s \in (-\frac{1}{2}, \frac{3}{2})$ et pour tout $u \in C_c^\infty(\mathbf{R})$. Pour étendre cette identité à $\dot H^s(\mathbf{R}^N)$, il suffit de prouver que $N_s(f) \leq C\|f\|_{\dot H^s}$ avec C indépendant de f. On a :

$$\begin{aligned} N_s^2(f) &= \int_0^\infty t^{2s}\Big|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\Big|^2 dt \\ &= \int_0^\infty\int_0^\infty\int_0^\infty t^{2s}\frac{\xi}{t^2+\xi^2}\frac{\eta}{t^2+\eta^2}\widehat{f}(\xi)\overline{\widehat{f}(\eta)}\,d\xi\,d\eta\,dt \qquad \text{(Fubini)} \\ &= \int_0^\infty\int_0^\infty I_s(\xi,\eta)\widehat{f}(\xi)\overline{\widehat{f}(\eta)}\,d\xi\,d\eta \qquad \text{où } I_s(\xi,\eta) = \int_0^\infty t^{2s}\frac{\xi}{t^2+\xi^2}\frac{\eta}{t^2+\eta^2}\,dt \\ &= \int_0^\infty\int_0^\infty |\xi|^{-s}|\eta|^{-s}I_s(\xi,\eta)|\xi|^s\widehat{f}(\xi)|\eta|^s\overline{\widehat{f}(\eta)}\,d\xi\,d\eta \\ &= \int_0^\infty\int_0^\infty K_s(\xi,\eta)|\xi|^s\widehat{f}(\xi)|\eta|^s\overline{\widehat{f}(\eta)}\,d\xi\,d\eta, \end{aligned}$$

où $K_s(\xi,\eta)) = |\xi|^{-s}|\eta|^{-s}I_s(\xi,\eta)$. Il suffit de montrer que

$$\left|\int_0^\infty\int_0^\infty K_s(\xi,\eta)\varphi(\xi)\overline{\psi(\eta)}\,d\xi\,d\eta\right| \le C\|\varphi\|_{L^2}\|\psi\|_{L^2}, \qquad \forall\varphi,\psi\in L^2(0,\infty).$$

Par calcul, on trouve $I_s(\xi,\eta) = \frac{\pi}{2\cos(s\pi)}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}$ si $s\neq\frac{1}{2}$, respectivement $I_s(\xi,\eta) = \frac{\ln\eta-\ln\xi}{\eta^2-\xi^2}$ si $s=\frac{1}{2}$.

Notons que ni les résultats généraux sur les opérateurs de Hilbert-Schmidt, ni l'inégalité de Hardy-Littlewood-Sobolev n'impliquent que le noyau K définit un opérateur borné sur $L^2(\mathbf{R}^N)$. Pour montrer ceci, on effectue les calculs en coordonnées polaires $\xi = r\cos\theta$, $\eta = r\sin\theta$ et on trouve une fonction $L_s(\theta)$ telle que $K_s(\xi,\eta) = \frac{1}{r}L_s(\theta)$.

Pour $\varphi,\psi\in L^2(0,\infty)$ on a :

$$\int_0^\infty\int_0^\infty \left|K_s(\xi,\eta)\varphi(\xi)\overline{\psi(\eta)}\right|d\xi\,d\eta = \int_0^{\frac{\pi}{2}}\int_0^\infty\left|\varphi(r\cos\theta)\overline{\psi(r\sin\theta)}\right|dr|L_s(\theta)|\,d\theta$$

$$\le \int_0^{\frac{\pi}{2}}\|\varphi(\cdot\cos\theta)\|_{L^2}\|\psi(\cdot\sin\theta)\|_{L^2}|L_s(\theta)|\,d\theta \qquad \text{(Cauchy-Schwarz)}$$

$$= \|\varphi\|_{L^2}\|\psi\|_{L^2}\int_0^{\frac{\pi}{2}}\frac{|L_s(\theta)|}{\sqrt{\sin\theta\cos\theta}}\,d\theta.$$

Pour $s\in(-\frac{1}{2},\frac{3}{2})$ on montre par calcul direct que $\displaystyle\int_0^{\frac{\pi}{2}}\frac{|L_s(\theta)|}{\sqrt{\sin\theta\cos\theta}}\,d\theta<\infty$. Par conséquent, on peut étendre (2.9) par densité à $\dot{H}^s(\mathbf{R})$. □

B. Le problème de Choquard généralisé.

Ce problème consiste à minimiser

$$E(u) := \frac{1}{2}\int_{\mathbf{R}^3}|\nabla u|^2dx - \int_{\mathbf{R}^3}\int_{\mathbf{R}^3}u^2(x)\frac{1}{|x-y|}u^2(y)\,dx\,dy$$

sous la contrainte $Q(u) := \displaystyle\int_{\mathbf{R}^3}u^2dx = \lambda$.

Il a été prouvé dans [Lieb77] qu'il existe un minimiseur $u\in H^1(\mathbf{R}^3)$. De plus, ce minimiseur est radial et unique (modulo translations). La démonstration repose sur des inégalités strictes pour les réarrangements sphériques.

Nous avons considéré le problème $(\mathcal{CG})$ qui consiste à minimiser

$$E(u) := \frac{1}{2}\int_{\mathbf{R}^N}|\nabla u|^2dx - \int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\frac{F(u(x))F(u(y))}{|x-y|^{N-2}}\,dx\,dy + \int_{\mathbf{R}^N}H(u)dx$$

sous la contrainte $Q(u) := \displaystyle\int_{\mathbf{R}^N}G(u)dx = \lambda$. L'intérêt de ce problème vient du fait que les minimiseurs sont des ondes stationnaires pour l'équation de Hartree

$$iu_t + \Delta u + 2\left(\int_{\mathbf{R}^N}\frac{F(u(y))}{|x-y|^{N-2}}\,dy\right)F'(u(x)) - H'(u(x)) = 0.$$

Théorème 2.10 *On suppose que* $N\ge 3$ *et*

• F, G, H *sont* C^2, $F(0)=G(0)=H(0)=0$, $F'(0)=G'(0)=H'(0)=0$ *et* F, G, H *ont un comportement sous-critique à l'infini,*

• $G'\not\equiv 0$ *sur un voisinage de 0.*

Alors tout minimiseur $u\in H^1(\mathbf{R}^N)$ *du problème* $(\mathcal{CG})$ *est à symétrie radiale.*

Preuve. On définit $I(\varphi) = \frac{1}{|\cdot|^{N-2}} * \varphi$. Il est bien connu que $\widehat{I(\varphi)} = \frac{c_N}{|\xi|^2}\widehat{\varphi}(\xi)$ et $-\Delta(I(\varphi)) = c_N \cdot \varphi$. Le terme nonlocal peut alors être écrit sous la forme

$$\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} F(u(x))\frac{1}{|x-y|^{N-2}}F(u(y))\,dx\,dy$$

$$= \langle I(F(u))\ ,\ F(u)\rangle = \tfrac{1}{(2\pi)^N}\langle \widehat{I(F(u))}\ ,\ \widehat{F(u)}\rangle \quad \text{(Plancherel)}$$

$$= \frac{c_N}{(2\pi)^N}\int_{\mathbf{R}^N}\frac{1}{|\xi|^2}\left|\widehat{F(u)}(\xi)\right|^2 d\xi.$$

Après translation, on peut supposer que $\displaystyle\int_{\{x_1<0\}} G(u)dx = \int_{\{x_1>0\}} G(u)dx = \lambda/2$. On définit v_1, v_2 comme dans (2.1) et on trouve

$$\begin{aligned}&E(v_1)+E(v_2)-2E(u)\\ &= -\frac{4c_N}{\pi(2\pi)^N}\int_{\mathbf{R}^{N-1}}\frac{1}{|\xi'|}\left|\int_0^\infty\left(\widehat{F(u)}(\xi_1,\xi')-\widehat{F(u)}(-\xi_1,\xi')\right)\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'.\end{aligned} \tag{2.14}$$

On obtient $E(v_1)+E(v_2)\le 2E(u)$. Donc v_1 et v_2 sont aussi minimiseurs et l'intégrale du membre de droite de (2.14) est nulle. Le fait que cette intégrale s'annulle équivaut à $\frac{\partial}{\partial x_1}(I(F(u)))(0,x') = 0$, $\forall x' \in \mathbf{R}^{N-1}$. Contrairement à l'exemple précédent, cette information n'implique pas directement la symétrie de u.

L'équation d'Euler-Lagrange pour un minimiseur est

$$-\Delta u - 2I(F(u))\cdot F'(u) + H'(u) + \alpha G'(u) = 0. \tag{2.15}$$

Nous ne connaissons pas de théorème de prolongement unique pour cette équation. On a le résultat de régularité suivant :

Lemme 2.11 *Soit $u \in H^1(\mathbf{R}^N)$ une solution de (2.15). Alors $u \in W^{3,p}(\mathbf{R}^N)$, $\forall p \in [2,\infty)$. En particulier, $u \in C^2(\mathbf{R}^N)$.*

On a ainsi montré que

• Pour tout minimiseur u et tout hyperplan Π qui "coupe la contrainte en deux", u_{Π^+} et u_{Π^-} sont aussi des minimiseurs.

• Tous les minimiseurs sont réguliers.

On peut alors conclure en utilisant la même technique que dans le Théorème 2.4. □

C. Le système de Davey-Stewartson. On considère le système

$$\begin{cases} iu_t + \Delta u = f(|u|^2)u - uv_{x_1} \\ \Delta v = \frac{\partial}{\partial x_1}\left(|u|^2\right)\end{cases} \quad \text{dans } \mathbf{R}\times\mathbf{R}^3.$$

Ce système peut être écrit sous la forme

$$iu_t = -\Delta u + f(|u|^2)u + R_1^2\left(|u|^2\right)u, \tag{2.16}$$

où R_1 est la transformation de Riesz donnée par $\widehat{R_1\varphi} = \frac{i\xi_1}{|\xi|}\widehat{\varphi}$. L'équation (2.16) est Hamiltonienne, deux quantités conservées étant

$$\tilde{E}(u) := \frac{1}{2}\int_{\mathbf{R}^3}|\nabla u|^2 + F_1(|u|^2)dx - \frac{1}{4}\int_{\mathbf{R}^3}\left|R_1(|u|^2)\right|^2 dx \quad \text{et} \quad \tilde{Q}(u) = \int_{\mathbf{R}^3}|u|^2dx.$$

Les minimiseurs de $\tilde{E}$ sous la contrainte $\tilde{Q} = constant$ sont des ondes stationnaires de (2.16). On a le résultat suivant concernant la symétrie de ces minimiseurs :

Théorème 2.12 *Soit $u \in H^1(\mathbf{R}^3)$ un minimiseur de*

$$E(u) := \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + F(u)dx - \frac{1}{4}\int_{\mathbf{R}^3} \left|R_1(|u|^2)\right|^2 dx$$

sous une contrainte $Q(u) = \int_{\mathbf{R}^3} G(u)dx = \lambda$. On suppose que $F, G \in C^1(\mathbf{C})$, $F(0) = G(0) = 0$, $\nabla F(0) = \nabla G(0) = 0$ et F, G ont un comportement sous-critique pour $|u| > 1$.

Alors modulo translation, u est à symétrie radiale dans les variables (x_2, x_3) (i.e. u est à symétrie axiale).

Preuve. On peut supposer que $\int_{\{x_2<0\}} G(u)dx = \int_{\{x_2>0\}} G(u)dx = \lambda/2$. On définit v_1, v_2 par symétrie miroir par rapport à x_2. Alors on a l'identité

$$\begin{aligned} &E(v_1) + E(v_2) - 2E(u) \\ &= \frac{-1}{\pi(2\pi)^3}\int_{\mathbf{R}^2} \frac{\xi_1^2}{\sqrt{\xi_1^2+\xi_3^2}} \left|\int_0^\infty \Big(\widehat{|u|^2}(\xi_1,\xi_2,\xi_3) - \widehat{|u|^2}(\xi_1,-\xi_2,\xi_3)\Big)\frac{\xi_2}{|\xi|^2}d\xi_2\right|^2 d\xi_1\, d\xi_3. \end{aligned} \tag{2.17}$$

Comme u est un minimiseur, on obtient que v_1 et v_2 sont aussi des minimiseurs et l'intégrale du membre de droite de (2.17) s'annulle, ce qui implique $\frac{\partial}{\partial x_2}(I(|u|^2))(x_1, 0, x_3) = 0$ pour tout $(x_1, x_3) \in \mathbf{R}^2$. Comme pour les minimiseurs du problème de Choquard généralisé, seule cette information ne suffit pas pour montrer la symétrie.

L'équation d'Euler-Lagrange satisfaite par les minimiseurs s'écrit

$$-\Delta u + \nabla F(u) + R_1^2(|u|^2)u + \alpha\nabla G(u) = 0. \tag{2.18}$$

On a le résultat de régularité suivant :

Lemme 2.13 *Soit $u \in H^1(\mathbf{R}^3)$ une solution de (2.18). Alors $u \in W^{2,p}(\mathbf{R}^3)$, $\forall p \in [2, \infty)$. En particulier, $u \in C^1(\mathbf{R}^3)$.*

On a ainsi montré que :

• Pour tout minimiseur u et tout hyperplan Π parallèle à Ox_1 et qui coupe la contrainte en deux, u_{Π^+} et u_{Π^-} sont aussi des minimiseurs.

• Tous les minimiseurs sont réguliers.

En utilisant le Théorème 2.4, on en déduit que tout minimiseur est radial par rapport aux variables (x_2, x_3). □

3 Ondes progressives pour des équations de Schrödinger non-linéaires avec des conditions non-nulles à l'infini

Une partie importante de mon activité de recherche a été consacrée à l'étude des équations de type

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + F(|\Phi|^2)\Phi = 0 \qquad \text{dans } \mathbf{R}^N, \tag{3.1}$$

où Φ est une fonction complexe qui satisfait $|\Phi| \longrightarrow r_0 > 0$ quand $|x| \longrightarrow \infty$ et $F(r_0^2) = 0$, $F'(r_0^2) < 0$. Deux cas particuliers importants de (3.1) ont été très étudiés par les physiciens et par les mathématiciens : l'équation de Gross-Pitaevskii (où $F(s) = 1 - s$) et l'équation appelée "cubique-quintique" (où $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, $\alpha_1, \alpha_3, \alpha_5 > 0$ et F admet deux racines réelles positives).

L'équation (3.1) est hamiltonienne. L'énergie correspondante est

$$E(\Phi) = \int_{\mathbf{R}^N} |\nabla\Phi|^2 \, dx + \int_{\mathbf{R}^N} V(|\Phi|^2)\, dx, \qquad \text{où } V(s) = \int_s^{r_0^2} F(\tau)\, d\tau. \tag{3.2}$$

Cette quantité est conservée par la dynamique associée à l'équation (3.1).

Des équations de type (3.1), avec les conditions aux limites non-nulles considérées ci-dessus, apparaîssent dans la modélisation d'un grand nombre de phénomènes en plusieurs domaines de la physique, comme la supraconductivité, la superfluidité dans Hélium II, les transitions de phase et les condensats de Bose-Einstein. Dans une longue série de travaux (v. [GR74], [JR82], [JPR86] et les références de ces articles), J. Grant, C.A. Jones, S.J. Putterman, P.H. Roberts et al. ont étudié formellement et numériquement des équations de type (3.1). Une attention particulière a été accordée à une classe spéciale de solutions de (3.1), les ondes progressives. Une onde progressive de vitesse c est une solution de la forme $\Phi(x,t) = \psi(x_1 - ct, x_2, \ldots, x_N)$. La fonction ψ satisfait alors l'équation

$$-ic\frac{\partial \psi}{\partial x_1} + \Delta\psi + F(|\psi|^2)\psi = 0 \qquad \text{dans } \mathbf{R}^N. \tag{3.3}$$

Des développements asymptotiques formels et des simulations numériques ont conduit à la formulation d'un ensemble de conjectures (parfois appelé *le programme de Roberts*) concernant l'existence, les propriétés structurelles et la stabilité des ondes progressives. La démonstration rigoureuse de ces conjectures conduit à des problèmes mathématiques intéressants et souvent très difficiles. Malgré de nombreux efforts qui ont été faits pendant les vingt dernières années, beaucoup de ces conjectures restent encore non résolues.

Notons que l'équation (1) admet une formulation hydrodynamique : en utilisant la transformation de Madelung $\Phi = \sqrt{\rho}e^{i\theta}$, elle est équivalente à un système en (ρ, θ) qui est semblable au système d'Euler pour un fluide non-visqueux compressible de densité ρ et de vitesse $2\nabla\theta$. Dans ce contexte, on peut calculer la vitesse du son à l'infini : $v_s = r_0\sqrt{-2F'(r_0^2)}$. Il a été conjecturé que des ondes progressives de vitesse c existent si et seulement si $|c| < v_s$. Nous avons tenté de donner une preuve rigoureuse à cette conjecture.

3.1 Non-existence des ondes progressives subsoniques

Dans le cas de l'équation de Gross-Pitaevskii, en utilisant une identité intégrale astucieuse, P. Gravejat a réussi à montrer la non-existence des ondes progressives supersoniques d'énergie finie ([Gr03]). Il a également prouvé la non-existence des ondes soniques en dimension 2. En simplifiant les arguments de P. Gravejat, dans [8] nous avons généralisé son identité intégrale et nous avons montré la non-existence des ondes progressives supersoniques de (3.1) pour une large classe de nonlinéarités (qui inclût les nonlinéarités de type Gross-Pitaevskii et de type "$\psi^3 - \psi^5$"), ainsi que la non-existence, en toute dimension d'espace, des ondes soniques ayant une énergie finie et une phase intégrable.

Nous allons décrire plus en détail les résultats de [8]. Dans cette section on suppose partout que les conditions suivantes sont vérifiées :

C1. La fonction F est continue sur $[0, \infty)$, C^1 au voisinage de r_0^2, $F(r_0^2) = 0$ et $F'(r_0^2) < 0$.

C2. Il existe $C, \alpha > 0$ tels que pour s suffisamment grand on a $F(s) \leq -Cs^\alpha$.

Le premier résultat de [8] concerne la régularité des solutions d'énergie finie de (3.3). Par solution d'énergie finie nous entendons une fonction $\psi \in L^1_{loc}(\mathbf{R}^N)$ qui vérifie (3.3) dans $\mathcal{D}'(\mathbf{R}^N)$ et qui a la propriété que $\nabla\psi \in L^2(\mathbf{R}^N)$ et $V(|\psi|^2) \in L^1(\mathbf{R}^N)$.

Proposition 3.1 *On suppose que les conditions* **C1** *et* **C2** *sont satisfaites. Soit ψ une solution d'énergie finie de (3.3). Alors :*

i) On a $\psi \in L^\infty \cap W^{2,p}_{loc}(\mathbf{R}^N)$ pour tout $p \in [1, \infty)$.

ii) On a $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ pour tout $p \in [2, \infty)$ et il existe $R_* > 0$ tel que sur $\mathbf{R}^N \setminus B(0, R_*)$, ψ admet un relèvement $\psi = \rho e^{i\theta}$ avec $\rho, \theta \in W^{2,p}_{loc}(\mathbf{R}^N)$, $p \in [1, \infty)$.

iii) Si, de plus, $F \in C^k([0, \infty))$, alors $\psi \in W^{k+2,p}_{loc}(\mathbf{R}^N)$ pour tout $p \in [1, \infty)$.

Notons que le schéma classique pour obtenir la régularité des solutions des équations elliptiques (et qui consiste à utiliser l'équation, les estimations elliptiques standard et les injections de Sobolev pour améliorer successivement la régularité de la solution) ne s'applique pas car dans la plupart des applications la nonlinéarité a une croissance critique ou surcritique à l'infini. On a utilisé une méthode développée par A. Farina dans [Fa98, Fa03] pour des systèmes de type Ginzburg-Landau (et basée sur l'inégalité de Kato, voir [Ka72]) pour montrer que les solutions de (3.3) sont bornées. Ensuite la théorie classique de la régularité elliptique permet de montrer les autres assertions de la Proposition 3.1.

Au moins formellement, les solutions de (3.3) sont des points critiques de la fonctionnelle $\tilde{E}_c(\psi) = E(\psi) + \tilde{Q}(\psi)$, où E est donnée par (3.2) et $\tilde{Q}$ est le "moment" par rapport à la direction Ox_1 (une définition plus précise sera donnée plus tard ; pour l'instant, notons que c'est une fonctionnelle dont la différentielle est $\tilde{Q}'(\psi) = 2i\psi_{x1}$). Cette caractérisation variationnelle nous permet de montrer des identités de type Pohozaev :

Proposition 3.2 *Soit ψ une solution d'énergie finie de (3.3). Alors on a*

$$(3.4) \qquad -\int_{\mathbf{R}^N} \left|\frac{\partial\psi}{\partial x_1}\right|^2 dx + \int_{\mathbf{R}^N} \sum_{j=2}^N \left|\frac{\partial\psi}{\partial x_j}\right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2)\, dx = 0 \qquad \textit{et}$$

$$(3.5) \qquad \int_{\mathbf{R}^N} \left|\frac{\partial\psi}{\partial x_1}\right|^2 + \frac{N-3}{N-1}\sum_{k=2}^N \left|\frac{\partial\psi}{\partial x_k}\right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2)\, dx + c\tilde{Q}(\psi) = 0.$$

Les identités de Pohozaev découlent du comportement de $\tilde{E}_c$ par rapport aux dilatations de $\mathbf{R}^N$. Plus précisément, pour $x = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N$ on note $x' = (x_2, \dots, x_N)$ et pour $\lambda,\ \sigma > 0$ on note $\psi_{\lambda,\sigma}(x) = \psi\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right)$. Alors (3.4) et (3.5) expriment le fait que si ψ est point critique de $\tilde{E}_c$, on a $\frac{d}{d\lambda}\Big|_{\lambda=1} \tilde{E}_c(\psi_{\lambda,1}) = 0$, respectivement $\frac{d}{d\sigma}\Big|_{\sigma=1} \tilde{E}_c(\psi_{1,\sigma}) = 0$. Bien sûr, cet argument est purement formel car, en général, $\frac{d}{d\lambda}\Big|_{\lambda=1}(\psi_{\lambda,1}) = -x_1\frac{\partial\psi}{\partial x_1}$ et $\frac{d}{d\sigma}\Big|_{\sigma=1}(\psi_{1,\sigma}) = -\sum_{j=2}^N x_j\frac{\partial\psi}{\partial x_j}$ n'appartiennent pas à l'espace fonctionnel sur lequel $\tilde{E}_c'(\psi)$ est définie.

Pour démontrer (3.4) et (3.5) rigoureusement, on multiplie (3.3) par $\chi\left(\frac{x}{n}\right) x_j\frac{\partial\psi}{\partial x_j}$, où $\chi \in C_c^\infty(\mathbf{R}^N)$ est une fonction qui est égale à 1 dans un voisinage de zéro, on effectue des intégrations par parties, puis on passe à la limite lorsque $n \longrightarrow \infty$. Pour pouvoir intégrer par parties on a besoin de connaître que ψ est une fonction suffisamment régulière. La régularité donnée par la Proposition 3.1 ($\psi \in L^\infty \cap W_{loc}^{2,p}(\mathbf{R}^N)$ et $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ pour $p \in [2,\infty)$) suffit pour obtenir les identités de Pohozaev.

Théorème 3.3 *Supposons que $c^2 > v_s^2$ et ψ est une solution d'énergie finie de (3.3). Alors ψ satisfait l'identité*

$$\int_{\mathbf{R}^N} |\nabla\psi|^2 - F(|\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2)\,dx + c(1-\frac{v_s^2}{c^2})\tilde{Q}(\psi) = 0. \tag{3.6}$$

Preuve. On note $\psi_1 = \mathrm{Re}(\psi)$, $\psi_2 = \mathrm{Im}(\psi)$. L'équation (3.3) équivaut au système

$$c\frac{\partial\psi_2}{\partial x_1} + \Delta\psi_1 + F(|\psi|^2)\psi_1 = 0 \qquad \text{et} \tag{3.7}$$

$$-c\frac{\partial\psi_1}{\partial x_1} + \Delta\psi_2 + F(|\psi|^2)\psi_2 = 0. \tag{3.8}$$

On multiplie (3.7) par ψ_2 et (3.8) par ψ_1, ensuite on soustrait les égalités obtenues. On trouve

$$\frac{c}{2}\frac{\partial}{\partial x_1}(|\psi|^2 - r_0^2) = \mathrm{div}(\psi_1\nabla\psi_2 - \psi_2\nabla\psi_1). \tag{3.9}$$

On multiplie (3.7) par ψ_1 et (3.8) par ψ_2, puis on rajoute les égalités obtenues. On obtient :

$$|\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(|\psi|^2)|\psi|^2 - c(\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1}) = \frac{1}{2}\Delta(|\psi|^2 - r_0^2). \tag{3.10}$$

Soit R_* comme dans la Proposition 3.1 (ii). Alors sur $\mathbf{R}^N \setminus B(0,R_*)$ on a un relèvement $\psi = \rho e^{i\theta}$. Soit $\chi \in C^\infty(\mathbf{R}^N)$ une fonction telle que $\chi = 0$ sur $B(0,2R_*)$ et $\chi = 1$ sur $\mathbf{R}^N \setminus B(0,3R_*)$. On note $G_j = \psi_1\frac{\partial\psi_2}{\partial x_j} - \psi_2\frac{\partial\psi_1}{\partial x_j} - r_0^2\frac{\partial}{\partial x_j}(\chi\theta)$, $\quad j = 1,\dots,N$. On peut montrer que $G_j \in L^1 \cap L^\infty(\mathbf{R}^N)$ et que le moment $\tilde{Q}(\psi)$ par rapport à la direction de x_1 est donné par

$$\tilde{Q}(\psi) = -\int_{\mathbf{R}^N} \psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta)\,dx = -\int_{\mathbf{R}^N} G_1\,dx.$$

De (3.9) et (3.10) on déduit

$$\frac{c}{2}\frac{\partial}{\partial x_1}(|\psi|^2 - r_0^2) = \mathrm{div}(\psi_1\nabla\psi_2 - \psi_2\nabla\psi_1 - r_0^2\nabla(\chi\theta)) + r_0^2\Delta(\chi\theta), \tag{3.11}$$

respectivement

$$\frac{1}{2}\Delta(|\psi|^2 - r_0^2) - \frac{v_s^2}{2}(|\psi|^2 - r_0^2)$$

$$= |\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(x,|\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) \tag{3.12}$$

$$-c\left(\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta)\right) - cr_0^2\frac{\partial}{\partial x_1}(\chi\theta).$$

On note

$$H = |\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(x,|\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) - c(\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta)).$$

On prend la dérivée de (3.11) par rappport à x_1 et on la multiplie par c, ensuite on prend la Laplacien de (3.12). En rajoutant les égalités obtenues on trouve

$$\frac{1}{2}\Big(\Delta^2 - v_s^2\Delta + c^2\frac{\partial^2}{\partial x_1^2}\Big)(|\psi|^2 - r_0^2) = \Delta H + c\frac{\partial}{\partial x_1}(\mathrm{div}(G)). \tag{3.13}$$

En prenant la transformation de Fourier de (3.13) on obtient

$$\frac{1}{2}\left(|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2\right)\mathcal{F}(|\psi|^2 - r_0^2) = -|\xi|^2\widehat{H} - c\sum_{k=1}^{N}\xi_1\xi_k\widehat{G_k}. \tag{3.14}$$

Soit $\Gamma = \{\xi \in \mathbf{R}^N \mid |\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2 = 0\}$. Si $c^2 \le v_s^2$ on a $\Gamma = \{0\}$. Dans le cas où $c^2 > v_s^2$, Γ est une sous-variété de $\mathbf{R}^N$ et en utilisant (3.14) on obtient

$$|\xi|^2\widehat{H}(\xi) + c\sum_{k=1}^{N}\xi_1\xi_k\widehat{G_k}(\xi) = 0 \qquad \text{pour tout } \xi \in \Gamma. \tag{3.15}$$

Il est clair que $\Gamma = \{(\xi_1,\xi') \in \mathbf{R}\times\mathbf{R}^{N-1} \mid |\xi'|^2 = \frac{1}{2}(-v_s^2 - 2\xi_1^2 + \sqrt{v_s^4 + 4c^2\xi_1^2})\}$. Soit $f(t) = \sqrt{\frac{1}{2}\left(-v_s^2 - 2t^2 + \sqrt{v_s^4 + 4c^2t^2}\right)}$. La function f est définie sur $[-\sqrt{c^2 - v_s^2}, \sqrt{c^2 - v_s^2}]$, on a $f(0) = 0$ et $\lim_{t\to 0}\frac{f^2(t)}{t^2} = -1 + \frac{c^2}{v_s^2}$. Pour $j \in \{2,\dots,N\}$ et $t \in (0, \sqrt{c^2 - v_s^2}]$, on note $\xi(t) = (t,0,\dots,0,f(t),0,\dots,0)$ et $\tilde{\xi}(t) = (t,0,\dots,0,-f(t),0,\dots,0)$, où $f(t)$, respectivement $-f(t)$, sont à la $j^{\text{ième}}$ place. Il est évident que $\xi(t), \tilde{\xi}(t) \in \Gamma$. De (3.15) on obtient

$$(t^2 + f^2(t))\widehat{H}(\xi(t)) + ct^2\widehat{G_1}(\xi(t)) + ctf(t)\widehat{G_j}(\xi(t)) = 0, \qquad \text{respectivement} \tag{3.16}$$

$$(t^2 + f^2(t))\widehat{H}(\tilde{\xi}(t)) + ct^2\widehat{G_1}(\tilde{\xi}(t)) - ctf(t)\widehat{G_j}(\tilde{\xi}(t)) = 0. \tag{3.17}$$

On multiplie (3.16) et (3.17) par $\frac{1}{t^2}$, on prend la limite lorsque $t \downarrow 0$ et on trouve

$$\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) + c\sqrt{-1 + \frac{c^2}{v_s^2}}\widehat{G_j}(0) = 0, \text{ respectivement} \tag{3.18}$$

$$\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) - c\sqrt{-1 + \frac{c^2}{v_s^2}}\widehat{G_j}(0) = 0. \tag{3.19}$$

De (3.18) et (3.19) on déduit que $\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) = 0$, et cette égalité est exactement (3.6). □

Théorème 3.4 *On suppose que $N \geq 2$, les conditions (C1) et (C2) sont satisfaites et $c^2 > v_s^2$. De plus, on suppose qu'il existe $\alpha \in [-1 + \frac{N-3}{N-1}(1 - \frac{v_s^2}{c^2}), \frac{v_s^2}{c^2}]$ tel que*

$$sF(s) + \frac{v_s^2}{2}(s - r_0^2) + \Big(1 - \alpha - \frac{v_s^2}{c^2}\Big)V(s) \leq 0 \qquad \textit{pour } s \geq 0.$$

Soit ψ une onde progressive d'énergie finie et de vitesse c de (3.1). Alors ψ est constante.

Preuve. On multiplie (3.5) par $1 - \frac{v_s^2}{c^2}$ et on soustrait l'égalité qui en résulte de (3.6). On obtient

$$\begin{aligned} &\int_{\mathbf{R}^N} \frac{v_s^2}{c^2}\Big|\frac{\partial\psi}{\partial x_1}\Big|^2 + \Big(1 - (1 - \frac{v_s^2}{c^2})\frac{N-3}{N-1}\Big)\sum_{k=2}^{N}\Big|\frac{\partial\psi}{\partial x_k}\Big|^2\,dx \\ &- \int_{\mathbf{R}^N} F(|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + \Big(1 - \frac{v_s^2}{c^2}\Big)V(|\psi|^2)\,dx = 0. \end{aligned} \tag{3.20}$$

Si α vérifie la condition du Théorème 3.4, on multiplie (3.4) par α et on rajoute le résultat à (3.20). On trouve

$$\begin{aligned} &\int_{\mathbf{R}^N} \Big(\frac{v_s^2}{c^2} - \alpha\Big)\Big|\frac{\partial\psi}{\partial x_1}\Big|^2 + \Big(\alpha + 1 - (1 - \frac{v_s^2}{c^2})\frac{N-3}{N-1}\Big)\sum_{k=2}^{N}\Big|\frac{\partial\psi}{\partial x_k}\Big|^2\,dx \\ &= \int_{\mathbf{R}^N} F(|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + (1 - \alpha - \frac{v_s^2}{c^2})V(|\psi|^2)\,dx. \end{aligned} \tag{3.21}$$

On observe alors que le membre de droite de (3.21) est négatif ou nul, alors que les coefficients qui apparaîssent dans le membre de gauche sont positifs (et au moins un est strictement positif). Comme $\nabla\psi \in L^2(\mathbf{R}^N)$, on en déduit que ψ est constante. □

Remarquons que les hypothèses du Théorème 3.4 sont vérifiées aussi bien par $F(s) = 1 - s$ que par $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, où $\alpha_i > 0$ et F admet deux racines positives. La conclusion du Théorème 3.4 est donc valable pour l'équation de Gross-Pitaevskii comme pour l'équation de Schrödinger avec non-linéarité cubique-quintique.

3.2 Existence des ondes progressives pour toute vitesse subsonique

Beaucoup d'efforts ont été consacrés à la démonstration de l'existence des ondes progressives pour (3.1). La plupart des résultats portent sur l'équation de Gross-Pitaevskii. Dans [BS99], l'existence de telles solutions a été prouvée en dimension deux d'espace et pour toute vitesse $c \in]-\varepsilon, \varepsilon[$, où ε est petit. En dimension $N \geq 3$, il a été prouvé dans [BOS04] qu'il existe des ondes progressives pour une suite de vitesses $c_n \longrightarrow 0$. Le même résultat a été obtenu pour toute vitesse $c \in]-\varepsilon, \varepsilon[$ dans [Ch04]. Dans un travail récent [BGS09], l'existence a été obtenue en dimensions 2 et 3 pour une plage plus large de vitesses (qui contient des vitesses proches de v_s en dimension 2). Cependant, même en dimension 2 les résultats de [BGS09] ne couvrent pas toutes les vitesses subsoniques. Pour des nonlinéarités de type "cubique-quintique" il a été prouvé dans [3] qu'il existe des ondes progressives de petite vitesse en dimension $N \geq 4$.

Dans [10], mon objectif a été à la fois de donner une preuve de l'existence des ondes progressives pour toute vitesse subsonique et de trouver une approche qui soit valable pour les différents types de nonlinéarité qui peuvent apparaître dans (3.1).

Notons que, si les conditions (C1) et (C2) dans la section précedente sont vérifiées, la Proposition 3.1 nous donne une estimation uniforme pour la norme L^∞ des solutions d'énergie finie de (3.3) : on sait qu'il existe une constante $M > 0$ (qui dépend uniquement de F) ayant la propriété que toute solution ψ satisfait $|\psi(x)| \leq M$ sur $\mathbf{R}$. On peut alors remplacer la fonction F par une fonction $\tilde{F}$ telle que $F = \tilde{F}$ sur $[0, M_1]$, où $M_1 > M$, $\tilde{F}$ satisfait (C1) et (C2) (eventuellement avec une constante $\beta \in (0, \alpha)$ au lieu de α) et, de plus, $\tilde{F}$ a une croissance sous-critique à l'infini. On peut donc supposer que la condition suivante est satisfaite :

C3. Il existe $p_0 < \frac{2^*}{2} - 2 = \frac{2}{N-2}$ et $C > 0$ tels que $|\tilde{F}(s)| \leq Cs^{p_0}$ pour $s > M_1$.

Si $\tilde{F}$ est comme ci-dessus et si ψ satisfait l'équation (3.3) avec $\tilde{F}$ à la place de F, par la Proposition 3.1 on sait que $|\psi| \leq M$, donc ψ est bien une solution de (3.3).

Le résultat principal de l'article [10] est le suivant :

Théorème 3.5 *Soit $N \geq 3$. On suppose que (C1) et une des conditions (C2) ou (C3) sont vérifiées. Alors pour toute vitesse $c \in (-v_s, v_s)$ il existe des ondes progressives de (3.1) de vitesse c et d'énergie finie.*

Nous allons décrire les idées qui ont conduit à la preuve de ce résultat. Compte tenu des conditions aux limites à l'infini, on a cherché des solutions de la forme $\psi = r_0 - u$, où $u \longrightarrow 0$ quand $|x| \longrightarrow \infty$. Alors u satisfait l'équation

$$icu_{x_1} - \Delta u + F(|r_0 - u|^2)(r_0 - u) = 0 \qquad \text{dans } \mathbf{R}^N. \tag{3.22}$$

Formellement, les solutions de (3.22) sont des points critiques de la fonctionnelle

$$E_c(u) = \int_{\mathbf{R}^N} |\nabla u|^2 \, dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx, \tag{3.23}$$

où Q est le moment par rapport à x_1. On considère également les fonctionnelles

$$A(u) = \int_{\mathbf{R}^N} \sum_{k=2}^{N} \left| \frac{\partial u}{\partial x_k} \right|^2 dx,$$

$$B_c(u) = \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx, \tag{3.24}$$

$$P_c(u) = \frac{N-3}{N-1} A(u) + B_c(u),$$

en sorte que $E_c(u) = A(u) + B_c(u) = \frac{2}{N-1} A(u) + P_c(u)$. D'après la Proposition 3.2, toute solution de (3.22) satisfait l'identité de Pohozaev $P_c(u) = 0$. Par conséquent $B_c(u) = -\frac{N-3}{N-1} A(u) < 0$, ce qui implique $B_c(u) < 0$ si $N \geq 4$, respectivement $B_c(u) = 0$ si $N = 3$. Pour toute fonction v, en utilisant la notation $v_{\lambda,\sigma}(x) = v\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right)$, on trouve

$$\begin{aligned} &E_c(v_{1,\sigma}) = \sigma^{N-3} A(v) + \sigma^{N-1} B_c(v) \qquad \text{et} \\ &\tfrac{d}{d\sigma}\left(E_c(v_{1,\sigma})\right) = (N-3)\sigma^{N-4} A(v) + (N-1)\sigma^{N-2} B_c(v). \end{aligned} \tag{3.25}$$

En dimension $N \geq 4$, si la fonction v qui satisfait $B_c(v) < 0$ il existe un unique $\sigma_v > 0$ tel que $P_c(v_{1,\sigma_v}) = 0$. De plus, (3.25) implique que la fonction $\sigma \longmapsto E_c(v_{1,\sigma_v})$ est croissante sur $(0, \sigma_v]$ et décroissante sur $[\sigma_v, \infty)$. Cette observation suggère qu'il est intéressant de minimiser E_c sous la contrainte $P_c = 0$. C'est exactement la démarche que nous avons suivie pour trouver des points critiques de E_c.

Soit $a = \sqrt{-\frac{1}{2}F'(r_0^2)}$. Alors $v_s = 2ar_0$ et la condition (C1) implique que pour s dans un voisinage de r_0^2 on a

$$(3.26)\quad V(s) = \frac{1}{2}V''(r_0^2)(s - r_0^2)^2 + (s - r_0^2)^2\varepsilon(s - r_0^2) = a^2(s - r_0^2)^2 + (s - r_0^2)^2\varepsilon(s - r_0^2),$$

où $\varepsilon(t) \longrightarrow 0$ lorsque $t \longrightarrow 0$. Par conséquent, pour u proche de zéro, on peut approximer $V(|r_0 - u|^2)$ par $a^2(|r_0 - u|^2 - r_0^2)^2$.

On fixe une fonction $\varphi \in C^\infty([0, \infty), \mathbf{R})$ telle que $\varphi(s) = s$ pour $s \in [0, 2r_0]$, φ est croissante et $\varphi(s) = 3r_0$ pour $s \geq 4r_0$. Pour un domaine $\Omega \subset \mathbf{R}^N$, on considère l'énergie de Ginzburg-Landau

$$E_{GL}^{\Omega}(u) = \int_\Omega |\nabla u|^2 dx + a^2 \int_\Omega \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 \, dx.$$

On note $E_{GL}(u) = E_{GL}^{\mathbf{R}^N}(u)$. Compte tenu de (3.26), l'espace naturel de fonctions sur lequel on doit étudier la fonctionnelle E_c est

$$\mathcal{X} = \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N) \mid E_{GL}(u) < \infty\}.$$

Par l'injection de Sobolev on a $\mathcal{X} \subset L^{2^*}(\mathbf{R}^N)$. Soit $u \in \mathcal{X}$. Lorsque $u(x)$ se trouve dans un voisinage de zéro, on pout majorer $|V(|r_0 - u(x)|^2)|$ grâce à (3.26) ; lorsque $u(x)$ est "loin" de zéro, par (C3) on obtient une majoration $|V(|r_0 - u(x)|^2)| \leq C|u|^{2^*}(x)$. Donc $V(|r_0 - u|^2) \in L^1(\mathbf{R}^N)$ pour tout $u \in \mathcal{X}$.

Nous allons indiquer comment définir le moment Q pour toutes les fonctions de $\mathcal{X}$. Remarquons que pour tout $u \in H^1(\mathbf{R}^N)$ on doit avoir $Q(u) = \int_{\mathbf{R}^N} \langle iu_{x_1}, u\rangle \, dx$, alors que pour toute fonction u qui admet un relèvement $r_0 - u = \rho e^{i\theta}$ on a (au moins formellement) $Q(u) = -\int_{\mathbf{R}^N} \rho^2\theta_{x_1} \, dx = -\int_{\mathbf{R}^N} (\rho^2 - r_0^2)\theta_{x_1} \, dx$, où $\rho^2 - r_0^2,\ \theta_{x_1} \in L^2(\mathbf{R}^N)$.

On observe que pour tout $u \in \mathcal{X}$ on a $\langle iu_{x_1}, u\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$, où $\mathcal{Y} = \{\partial_{x_1}\phi \mid \phi \in \mathcal{D}^{1,2}(\mathbf{R}^N)\}$. En posant $L(v + w) = \int_{\mathbf{R}^N} v \, dx$ pour $v \in L^1(\mathbf{R}^N)$ et $w \in \mathcal{Y}$, on vérifie sans peine que L est définie sans ambiguïté et constitue une forme linéaire sur $L^1(\mathbf{R}^N) + \mathcal{Y}$. Ceci nous permet de définir

$$Q(u) = L(\langle iu_{x_1}, u\rangle) \qquad \text{pour tout } u \in \mathcal{X}.$$

On vérifie ensuite que la fonctionnelle Q a les propriétés convenables pour notre approche variationnelle.

Un outil technique essentiel dans la démonstration du Théorème 3.5 est une procédure de "régularisation" pour les fonctions de $\mathcal{X}$ qui a pour but d'éliminer les défauts topologiques à petite échelle des fonctions. Plus précisément, pour $u \in \mathcal{X}$, $h > 0$ et pour un domaine $\Omega \subset \mathbf{R}^N$ on considère la fonctionnelle

$$G_{h,\Omega}^u(v) = E_{GL}^{\Omega}(v) + \frac{1}{h^2}\int_\Omega \varphi\left(\frac{|v - u|^2}{32r_0}\right) dx.$$

On montre que $G_{h,\Omega}^u$ admet des minimiseurs dans l'ensemble

$$\{v \in \mathcal{X} \mid v = u \text{ sur } \mathbf{R}^N \setminus \Omega,\ v - u \in H_0^1(\Omega)\}.$$

De plus, les minimiseurs v_h de cette fonctionnelle ont des propriétés remarquables. Ainsi,

• $||v_h - u||_{L^2(\mathbf{R}^N)} \longrightarrow 0$ quand $h \longrightarrow 0$,

• pour tout compact $\omega \subset \Omega$ on peut estimer $||\ |v_h - r_0| - r_0\ ||_{L^\infty(\omega)}$ en termes de h et de $E^{\Omega}_{GL}(u)$ et on trouve que $||\ |v_h - r_0| - r_0\ ||_{L^\infty(\omega)}$ est arbitrairement petite si l'énergie $E^{\Omega}_{GL}(u)$ est suffisamment petite.

On peut alors montrer les résultats suivants :

Lemme 3.6 *On suppose que $0 \le c < v_s$. Alors pour tout $\varepsilon \in (0, 1 - \frac{c}{v_s})$ il existe $K > 0$ tel que pour tout $u \in \mathcal{X}$ avec $E_{GL}(u) < K$ on a*

$$E_c(u) > \varepsilon E_{GL}(u).$$

Nous allons tenter de donner une idée de la démonstration.

Soit $\delta > 0$ suffisamment petit, tel que $\delta < \frac{r_0}{2}$ et $\frac{c}{2a(r_0-\delta)} < 1 - \varepsilon$ (un tel δ existe car $v_s = 2ar_0$ et $\varepsilon < 1 - \frac{c}{v_s}$). Considérons d'abord le cas d'une fonction $v \in \mathcal{X}$ qui vérifie $r_0 - \delta < |r_0 - v| \le r_0 + \delta$ sur $\mathbf{R}^N$. Si v est une telle fonction, il existe un relèvement $r_0 - v = \rho e^{i\theta}$ et on a $|\nabla v|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \theta|^2$ et $Q(v) = -\displaystyle\int_{\mathbf{R}^N} (\rho^2 - r_0^2)\theta_{x_1}\,dx$. Par l'inégalité de Cauchy-Schwarz on obtient

$$\frac{c}{1-\varepsilon}|Q(v)| \le 2a(r_0-\delta)|Q(v)| \le 2a(r_0-\delta)||\theta_{x_1}||_{L^2(\mathbf{R}^N)}||\rho^2 - r_0^2||_{L^2(\mathbf{R}^N)}$$

$$\le (r_0-\delta)^2 \int_{\mathbf{R}^N} |\theta_{x_1}|^2\,dx + a^2 \int_{\mathbf{R}^N} \left(\rho^2 - r_0^2\right)^2\,dx$$

$$\le \int_{\mathbf{R}^N} \rho^2|\nabla\theta|^2 + a^2\left(\rho^2 - r_0^2\right)^2\,dx \le E_{GL}(v).$$

Par conséquent, $E_{GL}(v) - c|Q(v)| > \varepsilon E_{GL}(v)$. Si $E_{GL}(v)$ est suffisamment petite, alors $\displaystyle\int_{\mathbf{R}^N} V(|r_0 - v|^2)\,dx$ est "proche" de $a^2 \displaystyle\int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - v|) - r_0^2\right)^2\,dx$ et on en déduit que v satisfait la conclusion du Lemme 3.6.

Dans le cas général : si $u \in \mathcal{X}$ est une fonction quelconque, on choisit $h > 0$ petit et on prend un minimiseur v_h de $G^u_{h,\mathbf{R}^N}$. Si l'énergie $E_{GL}(u)$ est suffisamment petite, on a $||\ |v_h - r_0| - r_0\ ||_{L^\infty(\mathbf{R}^N)} < \delta$, donc v_h vérifie la conclusion du lemme. Si h a été choisi suffisamment petit, v_h est "proche" de u et on peut montrer que u vérifie aussi la conclusion du Lemme 3.6.

En utilisant le Lemme 3.6, il est assez facile de voir que pour tout $k > 0$, la fonctionnelle E_c est bornée sur $\{u \in \mathcal{X} \mid E_{GL}(u) \le k\}$. On définit alors

$$E_{c,min}(k) = \inf\{E_c(u) \mid u \in \mathcal{X},\ E_{GL}(u) = k\}.$$

Lemme 3.7 *On suppose que $0 < c < v_s$. La fonction $E_{c,min}$ a les propriétés suivantes :*
i) Il existe $k_0 > 0$ tel que $E_{c,min}(k) > 0$ pour tout $k \in (0, k_0)$.
ii) On a $\lim_{k\to\infty} E_{c,min}(k) = -\infty$.
iii) Pour tout $k > 0$ on a $E_{c,min}(k) < k$.

La partie (i) découle directement du Lemme 3.6. Notons que pour $c > v_s$, les Lemmes 3.6 et 3.7 (i) ne sont plus valables. Plus précisément, on peut montrer que la fonction $k \longmapsto E_{c,min}(k)$ est strictement décroissante (et négative) sur $(0, \infty)$.

Le Lemme 3.7 nous permet de déduire que

$$S_c := \sup \{E_{c,min}(k) \mid k > 0\} > 0. \tag{3.27}$$

Lemme 3.8 *L'ensemble* $\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0,\ P_c(u) = 0\}$ *est non vide et on a*

$$T_c := \inf \{E_c(u) \mid u \in \mathcal{C}\} \geq S_c > 0.$$

Preuve. Soit $w \in \mathcal{X}$ une fonction telle que $E_c(w) < 0$ (une telle fonction existe par le Lemme 3.7 (ii)). Alors $P_c(w) = E_c(w) - \frac{2}{N-1}A(w) < 0$. On a

$$P_c(w_{\sigma,1}) = \frac{1}{\sigma}\int_{\mathbf{R}^N} \left|\frac{\partial w}{\partial x_1}\right|^2 dx + \frac{N-3}{N-1}\sigma A(w) + cQ(w) + \sigma \int_{\mathbf{R}^N} V(|r_0 - w|^2)\, dx. \tag{3.28}$$

Comme $P_c(w_{1,1}) = P_c(w) < 0$ et $\lim_{\sigma\to 0} P_c(w_{\sigma,1}) = \infty$, il existe $\sigma_0 \in (0,1)$ tel que $P_c(w_{\sigma_0,1}) = 0$, donc $w_{\sigma_0,1} \in \mathcal{C}$.

Pour la seconde partie, supposons d'abord que $N \geq 4$. Soit $v \in \mathcal{C}$. Alors $A(v) > 0$ et $B_c(v) = -\frac{N-3}{N-1}A(v) < 0$. En utilisant (3.25), on obtient que $\sigma \longmapsto E_c(v_{1,\sigma})$ est croissante sur $(0,1]$ et décroissante sur $[1,\infty)$, donc atteint son maximum en $\sigma = 1$. Soit $k > 0$ fixé. On voit facilement qu'il existe un unique $\sigma(k,v) > 0$ tel que $E_{GL}(v_{1,\sigma(k,v)}) = k$. Alors

$$E_{c,min}(k) \leq E_c(v_{1,\sigma(k,v)}) \leq E_c(v_{1,1}) = E_c(v).$$

En prenant le sup pour $k \geq 0$ dans cette inégalité on obtient $S_c \leq E_c(v)$.

Considérons maintenant le cas $N = 3$. Soit $v \in \mathcal{C}$. Alors $E_c(v_{1,\sigma}) = E_c(v) = A(v) =$ *constant* pour $\sigma > 0$. Soit $k > 0$. On distingue deux cas :

• Si $A(v) \geq k$, on a $E_c(v) = A(v) \geq k > E_{c,min}(k)$ par le Lemme 3.7 (iii).

• Si $A(v) < k$, il existe un unique $\sigma(k,v) > 0$ tel que $E_{GL}(v_{1,\sigma(k,v)}) = k$. Alors $E_c(v) = E_c(v_{1,\sigma(k,v)}) \geq E_{c,min}(k)$.

Dans les deux cas on obtient $E_c(v) \geq E_{c,min}(k)$ quelque soient $k > 0$ et $v \in \mathcal{C}$ et le lemme est prouvé. □

Lemme 3.9 *Soit* T_c *comme dans le lemme précedent. Alors :*

i) Pour tout $w \in \mathcal{X}$ *qui satisfait* $P_c(w) < 0$ *on a* $A(w) > \frac{N-1}{2}T_c$.

ii) Soit $(u_n)_{n\geq 1} \subset \mathcal{X}$ *une suite telle que* $(E_{GL}(u_n))_{n\geq 1}$ *est bornée et* $\lim_{n\to\infty} P_c(u_n) = \mu < 0$. *Alors* $\liminf_{n\to\infty} A(u_n) > \frac{N-1}{2}T_c$.

Preuve. Nous démontrons seulement (i). Pour tout $\sigma > 0$, $P_c(w_{\sigma,1})$ est donné par (3.28). Comme dans la preuve du Lemme 3.7, il existe $\sigma_0 \in (0,1)$ tel que $P_c(w_{\sigma_0,1}) = 0$, donc $w_{\sigma_0,1} \in \mathcal{C}$. Par la définition de T_c on a $E_c(w_{\sigma_0,1}) \geq T_c$, et on en déduit que $A(w_{\sigma_0,1}) = \frac{N-1}{2}\left(E_c(w_{\sigma_0,1}) - P_c(w_{\sigma_0,1})\right) \geq \frac{N-1}{2}T_c$. Ceci implique $A(w) \geq \frac{N-1}{2}\frac{1}{\sigma_0}T_c > \frac{N-1}{2}T_c$. □

Pour prouver le Théorème 3.5, on montre que la fonctionnelle E_c admet un minimiseur dans $\mathcal{C}$. Ensuite on prouve que tout minimiseur satisfait (3.3). La démonstration est assez différente dans le cas $N = 3$ par rapport au cas $N \geq 4$. Nous commençons par le cas (plus facile) $N \geq 4$.

Théorème 3.10 *On suppose que* $N \geq 4$. *Soit* $(u_n)_{n\geq 1} \subset \mathcal{X} \setminus \{0\}$ *une suite telle que*

$$P_c(u_n) \longrightarrow 0 \qquad et \qquad E_c(u_n) \longrightarrow T_c \qquad lorsque\ n \longrightarrow \infty. \tag{3.29}$$

Il existe une sous-suite $(u_{n_k})_{k\geq 1}$, *une suite de points* $(x_k)_{k\geq 1} \subset \mathbf{R}^N$ *et une fonction* $u \in \mathcal{C}$ *telles que*

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad et \quad \varphi^2(|r_0 - u_{n_k}(\cdot + x_k)|) - r_0^2 \longrightarrow \varphi^2(|r_0 - u|) - r_0^2 \quad dans\ L^2(\mathbf{R}^N).$$

De plus, on a $E_c(u) = T_c$, *donc* u *est un minimiseur de* E_c *dans* $\mathcal{C}$.

Résumé de la preuve. Comme $A(u_n) = \frac{N-1}{2}\left(E_c(u_n) - P_c(u_n)\right) \longrightarrow \frac{N-1}{2}T_c$, par (3.29) on déduit que $(A(u_n))_{n\geq 1}$ est bornée. Ensuite on montre que $(E_{GL}(u_n))_{n\geq 1}$ est bornée. On utilise la méthode de concentration-compacité de P.-L. Lions [Lio84] pour montrer la convergence d'une sous-suite de $(u_n)_{n\geq 1}$.

En passant à une sous-suite, on peut supposer que $E_{GL}(u_n) \longrightarrow \alpha_0 > 0$ lorsque $n \longrightarrow \infty$. Soit $q_n(t)$ la fonction de concentration de $E_{GL}(u_n)$, c'est-à-dire

$$q_n(t) = \sup_{x\in\mathbf{R}^N} E_{GL}^{B(x,t)}(u_n).$$

Pour chaque n, q_n est une fonction croissante sur $[0,\infty)$ qui tend vers $E_{GL}(u_n)$ lorsque $t \longrightarrow \infty$. Alors il existe une sous-suite (encore notée $(u_n)_{n\geq 1}$) et une fonction croissante $q : [0,\infty) \longrightarrow \mathbf{R}_+$ telles que $q_n(t) \longrightarrow q(t)$ quand $n \longrightarrow \infty$ pour presque tout $t \in [0,\infty)$.

On note $\alpha = \lim_{t\to\infty} q(t)$. Il est évident que $\alpha \in [0,\alpha_0]$. L'objectif est de prouver que l'énergie de u_n "se concentre", c'est-à-dire $\alpha = \alpha_0$.

Le fait que $\alpha > 0$ résulte du lemme suivant.

Lemme 3.11 *Soit $(u_n)_{n\geq 1} \subset \mathcal{X}$ une suite ayant les propriétés suivantes :*

a) $M_1 \leq E_{GL}(u_n) \leq M_2$, où M_1, M_2 sont deux constantes strictement positives, et

b) $\lim_{n\to\infty} P_c(u_n) = 0$.

Alors il existe $k > 0$ tel que $\sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \geq k$ pour tout n suffisamment grand.

La preuve du Lemme 3.11 est délicate. Elle repose sur la procédure de régularisation des fonctions décrite plus haut ainsi que sur le lemme de Lieb. L'idée de base de la démonstration est la suivante :

1. On suppose, par l'absurde, que $\lim_{n\to\infty} \sup_{x\in\mathbf{R}^N} E_{GL}^{B(x,1)}(u_n) = 0$.

On montre alors qu'il existe une suite $h_n \longrightarrow 0$ et pour chaque n il existe un minimiseur v_n de $G^{u_n}_{h_n,\mathbf{R}^N}$ tel que

$$|| \, |v_n - r_0| - r_0||_{L^\infty(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{quand } n \longrightarrow \infty. \tag{3.30}$$

2. Soit $\varepsilon \in (0, 1 - \frac{c}{v_s})$. En utilisant (3.30), on montre comme dans la preuve du Lemme 3.6 que pour tout n sufisamment grand on a

$$\begin{aligned} &\int_{\mathbf{R}^N} \left|\frac{\partial v_n}{\partial x_1}\right|^2 dx + \frac{N-3}{N-1}A(v_n) + \int_{\mathbf{R}^N} V(|r_0 - v_n|^2)\, dx + cQ(u_n) \\ &\geq \varepsilon \left(\int_{\mathbf{R}^N} \left|\frac{\partial v_n}{\partial x_1}\right|^2 dx + \frac{N-3}{N-1}A(v_n) + a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - v_n|) - r_0^2\right)^2 dx \right) \end{aligned} \tag{3.31}$$

3. Comme $h_n \longrightarrow 0$, pour n grand v_n est proche de u_n, donc (3.31) a lieu pour (u_n) à la place de v_n et pour un $\varepsilon_1 \in (0,\varepsilon)$ à la place de ε. On obtient ainsi une contradiction car $P_c(u_n) \longrightarrow 0$ et $E_{GL}(u_n) \geq M_1 > 0$.

L'étape suivante est de montrer qu'on ne peut pas avoir $\alpha \in (0,\alpha_0)$. On procède à nouveau par l'absurde et on supose que $\alpha \in (0,\alpha_0)$. Par un argument général on déduit qu'il existe une suite $R_n \longrightarrow \infty$ et une suite $(x_n)_{n\geq 1} \subset \mathbf{R}^N$ telles que :

$$E_{GL}^{B(x_n,R_n)}(u_n) \longrightarrow \alpha \qquad \text{et} \qquad E_{GL}^{\mathbf{R}^N\setminus B(x_n,2R_n)}(u_n) \longrightarrow \alpha_0 - \alpha. \tag{3.32}$$

Il est évident que (3.32) implique

$$E_{GL}^{B(x_n,2R_n)\setminus B(x_n,R_n)}(u_n) \longrightarrow 0.$$

Comme l'énergie de u_n sur la couronne $B(x_n, 2R_n) \setminus B(x_n, R_n)$ est petite, en utilisant à nouveau la procédure de régularisation on montre que pour chaque n il existe deux fonctions $u_{n,1}$ et $u_{n,2}$ telles que $r_0 - u_{n,1} = e^{i\theta_n}(r_0 - u_n)$ sur $B(x_n, R_n)$ (où θ_n est constante), $\text{supp}(u_{n,1}) \subset B(x_n, 2R_n)$, $u_{n,2} = u_n$ sur $\mathbf{R}^N \setminus B(x_n, 2R_n)$, $u_{n,2}$ est constante sur $B(x_n, R_n)$ et

$$E_{GL}(u_{n,1}) \longrightarrow \alpha \qquad \text{et} \qquad E_{GL}(u_{n,2}) \longrightarrow \alpha_0 - \alpha, \tag{3.33}$$

$$|A(u_n) - A(u_{n,1}) - A(u_{n,2})| \longrightarrow 0, \tag{3.34}$$

$$|P_c(u_n) - P_c(u_{n,1}) - P_c(u_{n,2})| \longrightarrow 0 \qquad \text{lorsque } n \longrightarrow \infty. \tag{3.35}$$

Il est facile de voir que les suites $(P_c(u_{n,i}))_{n\geq 1}$ sont bornées, $i = 1, 2$. En passant à nouveau à une sous-suite, on peut supposer que

$$P_c(u_{n,1}) \longrightarrow p_1 \qquad \text{et} \qquad P_c(u_{n,2}) \longrightarrow p_2 \qquad \text{lorsque } n \longrightarrow \infty,$$

où $p_1,\ p_2 \in \mathbf{R}$. Par (3.35) on a $p_1 + p_2 = 0$ et on distingue deux cas :

a) Un des p_i est négatif, par exemple $p_1 < 0$. Par le Lemme 3.9 (ii) on déduit que $\liminf\limits_{n\to\infty} A(u_{n,1}) > \frac{N-1}{2}T_c$. Alors (3.34) implique $\liminf\limits_{n\to\infty} A(u_n) > \frac{N-1}{2}T_c$ et en utilisant le fait que $P_c(u_n) \longrightarrow 0$ on trouve $\liminf\limits_{n\to\infty} E_c(u_n) > T_c$, ce qui contredit l'hypothèse du Théorème 3.10.

b) On a $p_1 = p_2 = 0$. Dans ce cas on utilise le

Lemme 3.12 *Soit $(u_n)_{n\geq 1} \subset \mathcal{X}$ une suite qui satisfait les propriétés suivantes :*
a) Il existe $C_1, C_2 > 0$ tels que $C_1 \leq E_{GL}(u_n)$ et $A(u_n) \leq C_2$ pour tout $n \geq 1$.
b) $P_c(u_n) \longrightarrow 0$ lorsque $n \longrightarrow \infty$.
Alors on a $\liminf\limits_{n\to\infty} E_c(u_n) \geq T_c$, où T_c est comme dans le Lemme 3.8.

Dans le cas (b), par le Lemme 3.12 on obtient $\liminf\limits_{n\to\infty} E_c(u_{n,i}) \geq T_c$ pour $i = 1,\ 2$. En utilisant (3.34) et (3.35) on trouve $\liminf\limits_{n\to\infty} E_c(u_n) \geq 2T_c$, ce qui est à nouveau une contradiction.

De ce qui précède on déduit que $\lim\limits_{t\to\infty} q(t) = \alpha_0$. Il est alors classique de montrer qu'il existe une suite $(x_n)_{n\geq 1} \subset \mathbf{R}^N$ telle que, en notant $\tilde{u}_n = u_n(\cdot + x_n)$, on a :

$$\begin{array}{c} \text{pour tout } \varepsilon > 0, \text{ il existe } R_\varepsilon > 0 \text{ et } n_\varepsilon \in \mathbf{N}^* \text{ tels que} \\ E_{GL}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\tilde{u}_{n_k}) < \varepsilon \text{ pour tout } n \geq n_\varepsilon. \end{array} \tag{3.36}$$

Comme $(E_{GL}(\tilde{u}_n))_{n\geq 1}$ est bornée, il existe une sous-suite $(\tilde{u}_{n_k})_{k\geq 1}$ telle que

$$\begin{array}{ll} \tilde{u}_{n_k} \rightharpoonup u & \text{faiblement dans } \mathcal{D}^{1,2}(\mathbf{R}^N), \\ \tilde{u}_{n_k} \longrightarrow u & \text{fortement dans } L^p_{loc}(\mathbf{R}^N) \text{ et presque partout sur } \mathbf{R}^N. \end{array} \tag{3.37}$$

On en déduit que $u \in \mathcal{X}$ et $\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \rightharpoonup \varphi^2(|r_0 - u|) - r_0^2$ faiblement dans $L^2(\mathbf{R}^N)$.

On prouve ensuite que

$$\lim_{k\to\infty}\int_{\mathbf{R}^N} V(|r_0-\tilde{u}_{n_k}|^2)\,dx = \int_{\mathbf{R}^N} V(|r_0-u|^2)\,dx \tag{3.38}$$

et

$$\lim_{k\to\infty} Q(\tilde{u}_{n_k}) = Q(u). \tag{3.39}$$

En utilisant (3.37), (3.38), (3.39) et le Lemme 3.9 (i) on montre que la sous-suite $(\tilde{u}_{n_k})_{k\geq 1}$ satisfait la conclusion du Théorème 3.10. □

Proposition 3.13 *On suppose que $N \geq 4$, $0 \leq c < v_s$, (C1) et une des conditions (C2) ou (C3) sont vérifiées. Soit u un minimiseur de E_c dans l'ensemble $\mathcal{C}$. Alors $u \in W^{2,p}_{loc}(\mathbf{R}^N)$, $\nabla u \in W^{1,p}(\mathbf{R}^N)$ pour $p \in [2,\infty)$ et u est une solution de (3.22).*

Preuve. Il est évident que u minimise la fonctionnelle A sous la contrainte $P_c = 0$ et il est facile de voir que u satisfait une équation d'Euler-Lagrange $A'(u) = \alpha P_c'(u)$. On ne peut pas avoir $\alpha > 0$. En effet, supposons par l'absurde que $\alpha > 0$. Soit w tel que $P_c'(u).w > 0$. Alors pour $t < 0$ et t proche de zéro on a $P_c(u+tw) < 0$ et $A(u+tw) < A(u) = \frac{N-1}{2}T_c$, en contradiction avec le Lemme 3.9 (i). De même, on ne peut pas avoir $\alpha = 0$ (car ceci impliquerait $A'(u) = 0$, donc $u = 0$). Par conséquent, on a $\alpha < 0$ et l'équation d'Euler-Lagrange équivaut à

$$-\frac{\partial^2 u}{\partial x_1^2} - \left(\frac{N-3}{N-1} - \frac{1}{\alpha}\right)\sum_{k=2}^{N}\frac{\partial^2 u}{\partial x_k^2} + icu_{x_1} + F(|r_0-u|^2)(r_0-u) = 0. \tag{3.40}$$

Comme dans la Proposition 3.2 on montre alors que u satisfait une identité de Pohozaev analogue à (3.5) qui s'écrit

$$\frac{N-3}{N-1}\left(\frac{N-3}{N-1} - \frac{1}{\alpha}\right)A(u) + B_c(u) = 0. \tag{3.41}$$

De (3.41) et du fait que $P_c(u) = \frac{N-3}{N-1}A(u) + B_c(u) = 0$, on en déduit que $\frac{1}{\alpha} = -\frac{2}{N-1}$ et u satisfait (3.22). La régularité de u résulte de la Proposition 3.1. □

Dans le cas $N = 3$ la preuve suit les mêmes étapes, avec quelques difficultés techniques supplémentaires (dont la plupart sont dûes à l'invariance des fonctionnelles A et B_c par dilatations par rapport aux variables (x_2, x_3) et au fait qu'en dimension 3 on a $P_c = B_c$, donc $P_c(v)$ ne contient pas de termes $\int_{\mathbf{R}^3}\left|\frac{\partial v}{\partial x_2}\right|^2 dx$ et $\int_{\mathbf{R}^3}\left|\frac{\partial v}{\partial x_3}\right|^2 dx$).

Pour $v \in \mathcal{X}$ on note

$$D(v) = \int_{\mathbf{R}^3}\left|\frac{\partial v}{\partial x_1}\right|^2 dx + a^2\int_{\mathbf{R}^3}\left(\varphi^2(|r_0-v|) - r_0^2\right)^2 dx.$$

Il est évident que pour tout $v \in \mathcal{X}$ et $\sigma > 0$ on a

$$A(v_{1,\sigma}) = A(v), \qquad B_c(v_{1,\sigma}) = \sigma^2 B_c(v) \qquad \text{et} \qquad D(v_{1,\sigma}) = \sigma^2 D(v). \tag{3.42}$$

Contrairement au cas $N \geq 4$, (3.42) implique qu'il existe des suites $(u_n)_{n\geq 1} \subset \mathcal{C}$ telles que $E_c(u_n) \longrightarrow T_c$ et $D(u_n) \longrightarrow \infty$, et par conséquent $E_{GL}(u_n) \longrightarrow \infty$. Cependant, par (3.42) on déduit qu'il existe des suites $(u_n)_{n\geq 1} \subset \mathcal{C}$ telles que $E_c(u_n) \longrightarrow T_c$ et $D(u_n) = 1$ pour tout n.

On considère l'ensemble

$$\Lambda_c = \{\lambda \in \mathbf{R} \mid \text{ il existe une suite } (u_n)_{n\geq 1} \subset \mathcal{X} \text{ telle que } D(u_n) \geq 1,\ B_c(u_n) \longrightarrow 0 \text{ et } A(u_n) \longrightarrow \lambda \text{ lorsque } n \longrightarrow \infty\}.$$

Soit $\lambda_c = \inf \Lambda_c$. Il est facile de voir que $T_c \in \Lambda_c$, donc $\lambda_c \leq T_c$. On peut montrer que $\lambda_c \geq S_c$, où S_c est donné par (3.27) (mais on ne sait pas si $S_c = T_c$).

Le résultat principal est le suivant :

Théorème 3.14 *On suppose que $N = 3$. Soit $(u_n)_{n\geq 1} \subset \mathcal{X}$ une suite telle que*

$$D(u_n) \longrightarrow 1, \quad B_c(u_n) \longrightarrow 0 \quad \text{et} \quad A(u_n) \longrightarrow \lambda_c \quad \text{quand } n \longrightarrow \infty. \tag{3.43}$$

Il existe une sous-suite $(u_{n_k})_{k\geq 1}$, une suite de points $(x_k)_{k\geq 1} \subset \mathbf{R}^3$ et une fonction $u \in \mathcal{C}$ telles que

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad \text{et} \quad |r_0 - u_{n_k}(\cdot + x_k)|^2 - r_0^2 \longrightarrow |r_0 - u|^2 - r_0^2 \quad \text{dans } L^2(\mathbf{R}^3).$$

De plus, on a $E_c(u) = A(u) = T_c = \lambda_c$ et u est un minimiseur de E_c dans $\mathcal{C}$.

Si u est un minimiseur de E_c dans $\mathcal{C}$, comme dans la preuve de la Proposition 3.13 on montre qu'il existe $\alpha < 0$ tel que $A'(u) = \alpha B_c'(u)$. Ensuite il est facile de voir qu'il existe $\sigma > 0$ tel que $u_{1,\sigma}$ satisfait (3.22). La régularité des solutions découle de la Proposition 3.1.

Finalement remarquons que le Lemme 3.9 implique que tous les minimiseurs de E_c dans $\mathcal{C}$ sont aussi des minimiseurs de la fonctionnelle $-P_c$ sous la contrainte $A = \frac{N-1}{2} T_c$. En utilisant les résultats de [7] (décrits dans la section 2.1) on déduit que ces minimiseurs sont à symétrie axiale par rapport à Ox_1 (après une translation).

3.3 Un système de Gross-Pitaevskii-Schrödinger

Dans [5] on a étudié le système

$$\begin{cases} 2i\psi_t = -\Delta\psi + \frac{1}{\varepsilon^2}(|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi\ , \\ 2i\delta\varphi_t = -\Delta\varphi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k_M^2)\varphi\ , \end{cases} \qquad x \in \mathbf{R}^N,\ t \in \mathbf{R}, \tag{3.44}$$

où ψ et φ sont des fonctions complexes et vérifient les conditions aux limites $|\psi| \longrightarrow 1$, $\varphi \longrightarrow 0$ lorsque $x \longrightarrow \pm\infty$. Le système (3.44) modélise le mouvement d'une impureté dans un codensat de Bose. Il a été étudié par J. Grant et P. H. Roberts ([GR74]). En utilisant des développements asymptotiques formels et des calculs numériques, ils ont trouvé le rayon effectif et la masse induite de l'impureté.

Notons que la vitesse du son à l'infini associée à (3.44) est $v_s = \frac{1}{\varepsilon\sqrt{2}}$.

Les solutions de type onde progressive $\psi(x,t) = \tilde{\psi}(x_1 - ct, x')$, $\varphi(x,t) = \tilde{\varphi}(x_1 - ct, x')$ semblent jouer un rôle important dans l'étude du système (3.44). On a montré dans [8] qu'en toute dimension $N \geq 2$, ce système n'admet pas d'onde progressive de vitesse supersonique et d'énergie finie.

En dimension un d'espace, on a montré l'existence des ondes progressives et on a obtenu une déscription assez précise de la structure globale de l'ensemble de telles solutions.

Compte tenu des conditions aux limites, on a cherché des ondes progressives de la forme

$$\tilde{\psi}(x) = (1 + \tilde{r}(x))e^{i\psi_0(x)}, \quad \tilde{\varphi}(x) = \tilde{u}(x)e^{i\varphi_0(x)}.$$

Après calcul, on trouve $\psi_0' = c(1 - \frac{1}{(1+\tilde{r})^2})$, et $\varphi_0' = c\delta$. On effectue le changement d'échelle $\tilde{u}(x) = \frac{1}{\varepsilon}u(\frac{x}{\varepsilon})$ et $\tilde{r}(x) = r(\frac{x}{\varepsilon})$ et on obtient que les fonctions r et u satisfont les équations

$$(3.45) \qquad \begin{aligned} &-r'' - (1+r) + (1+r)^3 - c^2\varepsilon^2\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)u^2 = 0, \\ &-u'' + (q^2(1+r)^2 - \varepsilon^2(c^2\delta^2 + k^2))u = 0, \end{aligned}$$

avec les conditions aux limites $r(x) \longrightarrow 0$, $u(x) \longrightarrow 0$ lorsque $|x| \longrightarrow \infty$. Comme $|\psi|(x) = 1 + r(\frac{x}{\varepsilon})$, on doit avoir $r(x) \geq -1$ sur $\mathbf{R}$. On note

$$V_1 = \{r \in H^1(\mathbf{R}) \mid \inf_{x \in \mathbf{R}} r(x) > -1\}.$$

Avec la notation $g(s) = g_{2c\varepsilon}(s) = -(1+s) + (1+s)^3 - c^2\varepsilon^2\Big(1 + s - \frac{1}{(1+s)^3}\Big)$ et $\lambda = \varepsilon^2(c^2\delta^2 + k^2)$, le système (3.45) s'écrit sous la forme

$$(3.46) \qquad \begin{aligned} &-r'' + g_{2c\varepsilon}(r) + (1+r)u^2 = 0, \\ &-u'' + q^2(1+r)^2u - \lambda u = 0. \end{aligned}$$

Si $u = 0$, la première des équations (3.46) admet uniquement la solution triviale $r = 0$ pour $|c\varepsilon| \geq \frac{1}{\sqrt{2}}$. Lorsque $|c\varepsilon| < \frac{1}{\sqrt{2}}$, elle admet aussi la solution

$$(3.47) \qquad r_*(x) = r_{2c\varepsilon}(x) = -1 + \sqrt{2c^2\varepsilon^2 + (1 - 2c^2\varepsilon^2)\tanh^2(\sqrt{\tfrac{1}{2} - c^2\varepsilon^2}\,x)}.$$

On appelle $(0,0)$ et $(r_{2c\varepsilon}, 0)$ les solutions triviales de (3.46). Une solution non-triviale est un triplet (λ, r, u) qui vérifie (3.46) et tel que $u \neq 0$.

L'objectif de l'article [5] est de montrer l'existence des solutions non-triviales et d'étudier le structure de l'ensemble de telles solutions. Tout d'abord, on a le résultat de non-existence suivant :

Proposition 3.15

a) Quelque soit $\lambda \in \mathbf{R}$, le système (3.46) n'admet pas de solution $(r,u) \neq (0,0)$ si $|c| \geq \frac{1}{\varepsilon\sqrt{2}}$.

b) On suppose que $|c| < \frac{1}{\varepsilon\sqrt{2}}$ et que $(\lambda, r, u) \in \mathbf{R} \times V_1 \times H^1(\mathbf{R})$ est une solution non-triviale de (2). Alors :

i) $2c^2\varepsilon^2q^2 < \lambda \leq q^2$ et

ii) $-1 + \sqrt{2}c\varepsilon < r(x) \leq 0$ pour tout $x \in \mathbf{R}$.

Pour montrer l'existence des solutions non-triviales de (3.46) on a utilisé la théorie des bifurcations. On considère les espaces fonctionnels

$$\mathbf{H} = \{f \in H^2(\mathbf{R}) \mid f(x) = f(-x)\} \quad \text{et} \quad \mathbf{L} = \{f \in L^2(\mathbf{R}) \mid f(x) = f(-x) \ \ p.p.\}.$$

On note $V = V_1 \cap \mathbf{H}$ et on introduit les opérateurs

$$(3.48) \qquad \begin{aligned} &S : V \times \mathbf{H} \longrightarrow \mathbf{L}, && S(r,u) = -r'' + g_{2c\varepsilon}(r) + (1+r)u^2, \\ &T : \mathbf{R} \times \mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{L}, && T(\lambda, r, u) = -u'' + q^2(1+r)^2u - \lambda u. \end{aligned}$$

Il est évident que (λ, r, u) est solution de (3.46) si et seulement si $S(r,u) = 0$ et $T(\lambda, r, u) = 0$. L'égalité $T(\lambda, r, u) = 0$ exprime le fait que λ est une valeur propre de l'opérateur linéaire $-\frac{d^2}{dx^2} + q^2(1+r)^2$ et u est un vecteur propre associé.

Afin de prouver l'apparition des branches de solutions non-triviales, on étudie les opérateurs S et T dans un voisinage d'une solution triviale $(\lambda, r_{2c\varepsilon}, 0)$. On montre que les propriétés suivantes sont vérifiées :

1. L'opérateur $D_r S(r_{2c\varepsilon}, 0) = -\frac{d^2}{dx^2} + g'(r_{2c\varepsilon}) : \mathbf{H} \longrightarrow \mathbf{L}$ est inversible.

2. L'opérateur $Au = -u'' + q^2(1 + r_{2c\varepsilon})^2 u : \mathbf{H} \longrightarrow \mathbf{L}$ satisfait

   i) $A \geq 2c^2\varepsilon^2 q^2$

   ii) $\sigma_{ess}(A) = [q^2, \infty)$,

   iii) toute valeur propre $\lambda < q^2$ est simple, et le vecteur propre correspondant est à décroissance exponentielle,

   iv) le nombre de valeurs propres est strictement inférieur à $1 + (2\sqrt{2})q^2$,

   v) le nombre de valeurs propres tend vers $+\infty$ quand $q \longrightarrow \infty$.

Les propriétés ci-dessus et le théorème des fonctions implicites impliquent que pour $\lambda < q^2$ il existe des solutions non-triviales au voisinage d'une solution $(\lambda, r_{2c\varepsilon}, 0)$ si et seulement si λ est valeur propre de l'opérateur A.

En utilisant une variante du théorème de bifurcation à partir d'une valeur propre simple de Crandall et Rabinowitz [CR71], on prouve :

Théorème 3.16 *Soit λ_0 une valeur propre de A et soit u_0 un vecteur propre correspondant. Il existe une fonction*

$$s \longmapsto (\lambda(s), r(s), u(s)) \in \mathbf{R} \times \mathbf{H} \times \mathbf{H} \cap \{u_0\}^\perp$$

définie sur $(-\eta, \eta)$ telle que $r(0) = 0$, $u(0) = 0$, $\lambda(0) = \lambda_0$ et

$$\begin{array}{l} S(r_{2c\varepsilon} + sr(s), s(u_0 + u(s))) = 0, \\ T(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_0 + u(s))) = 0. \end{array}$$

De plus, il existe un voisinage U de $(\lambda_0, r_{2c\varepsilon}, 0)$ dans $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ tel que toute solution du système (3.46) dans U est soit de la forme $(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_0 + u(s)))$, soit de la forme $(\lambda, r_{2c\varepsilon}, 0)$.

Pour obtenir une information globale sur la structure de l'ensemble des solutions, on travaille dans des espaces de Sobolev à poids. Plus précisément, on choisit une fonction $W : \mathbf{R} \longrightarrow [1, \infty)$ continue, paire, croissante sur $(0, \infty)$ et qui se comporte comme $|x|^s$ au voisinage de l'infini pour un $s > 0$. On considère les espaces

$$\mathbf{L}_W = \{\varphi \in \mathbf{L} \mid W\varphi \in \mathbf{L}\} \qquad \text{et} \qquad \mathbf{H}_W = \{\varphi \in \mathbf{H} \mid W\varphi, W\varphi', W\varphi'' \in \mathbf{L}\}.$$

Le résultat de décroissance suivant montre qu'il n'y a pas de perte de solutions lorsqu'on remplace l'espace $\mathbf{H}$ par $\mathbf{H}_W$.

Lemme 3.17 *Si (λ, r, u) est solution du système (3.46) dans $(-\infty, q^2) \times V \times \mathbf{H}$, alors $r \in \mathbf{H}_W$ et $u \in \mathbf{H}_W$.*

On note $w = r - r_{2c\varepsilon}$ et on écrit le système (3.46) sous la forme

$$\begin{pmatrix} w \\ u \end{pmatrix} = -\begin{pmatrix} B & 0 \\ 0 & A_\lambda \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} - \begin{pmatrix} H_1(w,u) \\ H_2(\lambda, w, u) \end{pmatrix}, \qquad (3.49)$$

où

$$A_\lambda(u) = A(\lambda, u) = q^2\Big(-\frac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(r_{2c\varepsilon}^2 + 2r_{2c\varepsilon})u],$$

$$B(w) = \Big(-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[(g'_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(0))w]$$

sont des opérateurs linéaires, compacts de $\mathbf{H}_W$ dans $\mathbf{H}_W$ et H_1, H_2 sont continus, compacts sur les ensembles bornés de $\Omega := (-\infty, q^2) \times ((V - r_{2c\varepsilon}) \cap \mathbf{H}_W) \times \mathbf{H}_W$ et satisfont les estimations

$$||H_1(w, u)||_{\mathbf{H}_W} = o(||w||_{\mathbf{H}_W} + ||u||_{\mathbf{H}_W}),$$

respectivement

$$||H_2(\lambda, w, u)||_{\mathbf{H}_W} = o(||w||_{\mathbf{H}_W} + ||u||_{\mathbf{H}_W})$$

lorsque w, u sont proches de zéro dans $\mathbf{H}_W$, uniformément par rapport à λ lorsque $\lambda \in [d, e] \subset (-\infty, q^2)$.

En utilisant une variante du théorème de bifurcation globale de Rabinowitz ([Ra71]), on montre :

Théorème 3.18 *Soit $\mathcal{S}$ l'ensemble des solutions non-triviales du système (3.46) dans* $\mathbf{R} \times V \times \mathbf{H}$. *Pour toute valeur propre $\lambda_m < q^2$ de A, l'ensemble $\mathcal{S} \cup \{(\lambda_m, r_*, 0)\}$ possède une composante connexe $\mathcal{C}_m$ dans $(-\infty, q^2) \times \mathbf{H}_W \times \mathbf{H}_W$ qui contient $(\lambda_m, r_*, 0)$ et qui a au moins une des propriétés suivantes :*

i) $\mathcal{C}_m$ est non-bornée.

ii) $\mathcal{C}_m$ contient une suite (λ_n, r_n, u_n) telle que $\lim_{n\to\infty} \lambda_n = q^2$.

Remarquons que le nombre de branches de solutions dans le Théorème 3.18 est le même que le nombre de valeurs propres de l'opérateur A. On a donc une seule branche si q est suffisamment petit et le nombre de branches tend vers l'infini lorsque $q \longrightarrow \infty$.

Nous pensons que les informations obtenues en dimension un seront utiles dans l'étude des cas bi$-$ et tridimensionnels, plus intéressants d'un point de vue physique.

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Première partie

Symétrie des solutions des équations aux dérivées partielles

On the symmetry of minimizers

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On the symmetry of minimizers

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Dedicated to Dorel Miheţ,
for his teaching, his friendship, and the inspiration he gave to me.

Abstract

For a large class of variational problems we prove that minimizers are symmetric whenever they are C^1.

AMS subject classifications. 35A15, 35B05, 35B65, 35H30, 35J20, 35J45, 35J50, 35J60.

1 Introduction and main results

In this paper we study the symmetry of minimizers for general variational problems of the form

$$(\mathcal{P}) \qquad \begin{aligned} &\text{minimize } E(u) := \int_\Omega F(|x|, u(x), |\nabla u(x)|)\, dx \qquad \text{under } k \text{ constraints} \\ &Q_j(u) = \int_\Omega G_j(|x|, u(x), |\nabla u(x)|)\, dx = \lambda_j, \qquad j = 1, \ldots, k. \end{aligned}$$

The solutions of many partial differential equations are obtained as minimizers for problems like $(\mathcal{P})$. Knowing in advance that such solutions are symmetric is very important for their theoretical study as well as for their numerical approximation. If the minimizers of $(\mathcal{P})$ are standing or solitary waves for an evolution equation, symmetry could be very useful to investigate the stability properties of such solutions. Note also that in many problems, symmetry is the first step in proving the uniqueness of special solutions.

Given the motivation above, many important particular cases of $(\mathcal{P})$ have already been considered in the literature. In [11, 12], O. Lopes has developed his reflection method - a very efficient tool to prove symmetries for minimizers of functionals $E_1(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F_1(|x|, u)\, dx$ under the constraint $Q(u) = \int_\Omega G(|x|, u)\, dx = constant$, where Ω is a domain invariant by rotations. This method is based on a device of "reflecting" a minimizer with respect to hyperplanes that "split the constraint in two" and on the use of a unique continuation principle for the Euler-Lagrange equations satisfied by minimizers. Note that the method can be used for vector-valued minimizers whose components eventually change sign and no additional assumptions are made on the functions F_1 and G (except the usual growth and smoothness assumptions that ensure the existence and the regularity of minimizers). Up to now this method has been used for problems involving only one constraint. Its main restriction is that it can be used only when the minimizers satisfy an Euler-Lagrange system for which a unique continuation theorem is available. However, we have to mention that the reflection method has been successfully used in [13] for minimizers of some nonlocal functionals of the form $E_2(u) = \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbf{R}^N} F_2(u) \, dx$. The class of functionals considered in [13] include the generalized Choquard functional, the Hamiltonian for the generalized Davey-Stewartson equation as well as functionals involving fractional powers of the Laplacian. Instead of unique continuation results, some new and quite unexpected integral identities for nonlocal operators were used to get symmetry results.

In a recent paper [4], F. Brock studies the symmetry of minimizers of the functional $\int_{\mathbf{R}^N} \sum_{i=1}^{n} |\nabla u_i|^p + F(|x|, u_1, \ldots, u_n) \, dx$ under several constraints $\int_{\mathbf{R}^N} G_{i,j}(u_i) \, dx = c_{i,j}$. He uses two-points rearrangements and a variant of the strong maximum principle due to Pucci, Serrin and Zou ([16]) to prove symmetries. Assuming that F is nonincreasing in the first variable and that $\frac{\partial F}{\partial u_i}$ is nondecreasing in the variables u_k for $k \neq i$ (a cooperative condition), he shows that the superlevel sets $\{u_i > t\}$ for $t > 0$, respectively the sublevel sets $\{u_i < t\}$ for $t < 0$, are balls. Under more restrictive conditions (F strictly decreaing in the first variable or an assumption that depends on Lagrange multipliers associated to minimizers - assumption that could be quite difficult to check in applications, as already mentioned in [4]), he proves that any component of the minimizer is radially symmetric about 0, has constant sign and is monotone in $|x|$. Note that whenever the arguments in [4] lead to symmetry, they also imply monotonicity. On the other hand, in [4] there is an example of sign-changing minimizer for a particular functional of the type considered. It is remarkable that the results of F. Brock are valid for an arbitrary number of constraints. However, these constraints must have a special form (because they have to be preserved by rearrangements of functions). For instance, one cannot allow constraints of the form $\int_{\mathbf{R}^N} G(u_i, u_j) \, dx = constant$.

We have to mention that in a series of recent papers (see [2], [15], [17] and references therein), different new techniques were developed to study the symmetry of solutions for some classes of elliptic problems. These techniques are essentially based on foliated Schwarz rearrangements and on polarization of functions and can be used for sign-changing solutions. They also give some monotonicity properties.

The aim of the present paper is to prove symmetry of minimizers for problem $(\mathcal{P})$ under general assumptions. We use the device of reflecting minimizers with respect to hyperplanes introduced by O. Lopes, but we do not need unique continuation theorems. Instead, we use in an essential way the regularity of minimizers. (To our knowledge, symmetry results for minimizers that may be nonsmooth were obtained only in the case of convex functionals.) We are able to deal with several constraints, but each additional constraint produces the loss of one direction of symmetry; we will see later (Examples 6 and 7) that under the general conditions considered here, this is a very natural phenomenon.

In the sequel Ω denotes an open set in $\mathbf{R}^N$ invariant under rotations (and centered at the origin). It is not assumed that Ω is connected or bounded. We denote $A_\Omega = \{|x| \mid x \in \Omega\}$. We consider vector-valued minimizers $u : \Omega \longrightarrow \mathbf{R}^n$ of $(\mathcal{P})$ that belong to some function space $\mathcal{X}$. Throughout $F, G_1, \ldots G_k$ are real-valued functions defined on $A_\Omega \times \mathbf{R}^m \times [0, \infty)$ in such a way that for any $v \in \mathcal{X}$, the functions $x \longmapsto F(|x|, v(x), |\nabla v(x)|)$ and $x \longmapsto G_j(|x|, v(x), |\nabla v(x)|)$, $1 \leq j \leq k$, belong to $L^1(\Omega)$.

Let V be an affine subspace of $\mathbf{R}^N$. For $x \in \mathbf{R}^N$ we denote by $p_V(x)$ the projection of x onto V and by $s_V(x)$ the symmetric point of x with respect to V, that is $s_V(x) = 2p_V(x) - x$. We say that a function f defined on $\mathbf{R}^N$ is *symmetric* with respect to V if $f(x) = f(s_V(x))$ for any x. We say that f is *radially symmetric* with respect to V if there exists a function $\tilde{f}$ defined on $V \times [0, \infty)$ such that $f(x) = \tilde{f}(p_V(x), |x - p_V(x)|)$.

Let Π be a hyperplane in $\mathbf{R}^N$ and let Π^+ and Π^- be the two half-spaces determined by Π. Given a function f defined on $\mathbf{R}^N$, we denote

$$(1) \qquad f_{\Pi^+}(x) = \begin{cases} f(x) & \text{if} \quad x \in \Pi^+ \cup \Pi, \\ f(s_\Pi(x)) & \text{if} \quad x \in \Pi^-, \end{cases} \qquad \text{respectively}$$

$$f_{\Pi^-}(x) = \begin{cases} f(x) & \text{if} \quad x \in \Pi^- \cup \Pi, \\ f(s_\Pi(x)) & \text{if} \quad x \in \Pi^+. \end{cases}$$

If f is defined on a rotation invariant subset Ω centered at the origin, $\Omega \neq \mathbf{R}^N$, the above definition makes sense only if Π contains the origin. We say that Π splits the constraints in two for a function $v \in \mathcal{X}$ if

$$(2) \quad \int_{\Omega \cap \Pi^+} G_j(|x|, v(x), |\nabla v(x)|)\, dx = \int_{\Omega \cap \Pi^-} G_j(|x|, v(x), |\nabla v(x)|)\, dx \qquad \text{for } j = 1, \ldots, k.$$

We make the following assumptions.

A1. For any $v \in \mathcal{X}$ and any hyperplane Π containing the origin, we have $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$.

A2. Problem $(\mathcal{P})$ admits minimizers in $\mathcal{X}$ and any minimizer is a C^1 function on Ω.

We can now state our symmetry results.

Theorem 1. *Assume that $0 \leq k \leq N-2$ and* **A1, A2** *are satisfied. Let $u \in \mathcal{X}$ be a minimizer for problem $(\mathcal{P})$. There exists a $k-$dimensional vector subspace V in $\mathbf{R}^N$ such that u is radially symmetric with respect to V.*

If $\Omega = \mathbf{R}^N$ and the considered functionals are invariant by translations, Theorem 1 can be improved. More precisely, consider the following particular case of $(\mathcal{P})$:

$$(\mathcal{P}') \qquad \text{minimize } E(u) := \int_{\mathbf{R}^N} F(u(x), |\nabla u(x)|)\, dx \qquad \text{subject to } k \text{ constraints}$$

$$Q_j(u) = \int_{\mathbf{R}^N} G_j(u(x), |\nabla u(x)|)\, dx = \lambda_j, \qquad j = 1, \ldots, k.$$

In this case assumption **A1** is replaced by

A1.' For any $v \in \mathcal{X}$ and any affine hyperplane Π in $\mathbf{R}^N$ we have $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$.

The following result holds.

Theorem 2. *Assume that $1 \leq k \leq N-1$,* **A1'** *and* **A2** *are satisfied and there exists $j \in \{1, \ldots, k\}$ such that $\lambda_j \neq 0$. Let $u \in \mathcal{X}$ be a minimizer for problem $(\mathcal{P}')$. There exists a $(k-1)-$dimensional affine subspace V in $\mathbf{R}^N$ such that u is radially symmetric with respect to V.*

If $(\mathcal{P}')$ involves only one constraint, Theorem 2 implies that any minimizer is radial with respect to some point.

In applications, assumptions **A1** or **A1'** are usually easy to check. On the contrary, assumption **A2** requires much more attention. In most applications, under suitable growth and smoothness assumptions on the functions $F, G_1, \ldots, G_k$, the functionals $E, Q_1, \ldots, Q_k$ are differentiable on $\mathcal{X}$ and the minimizers satisfy Euler-Lagrange equations (however, this is not always the case: see [1] for examples of minimizers that do not satisfy Euler-Lagrange equations). Very often the Euler-Lagrange equations are, in fact, quasilinear elliptic systems. Many efforts have been made during the last 50 years, since the pioneer work of de Giorgi, Nash and Moser, to study the regularity of solutions of such systems and there is a huge literature devoted to the subject. Important progress has been made and various sufficient conditions that guarantee the regularity of solutions have been given. It would exceed the scope of the present paper to resume these works, or even to give here a significant list of conditions that ensure the regularity of minimizers. For these issues (and also for historical notes) we refer the reader to the standard books [5, 7, 8, 9, 10, 14] and references therein.

In the next section we give the proofs of Theorems 1 and 2. We end this paper by some remarks and examples which show that, under the general conditions considered here, our results are optimal even for scalar-valued minimizers.

2 Proofs

Proof of Theorem 1. Consider first the case $1 \le k \le N-2$. For $v \in \mathbf{R}^N$, $v \neq 0$, denote $\Pi_v = \{x \in \mathbf{R}^N \mid x.v = 0\}$, $\Pi_v^+ = \{x \in \mathbf{R}^N \mid x.v > 0\}$ and $\Pi_v^- = \{x \in \mathbf{R}^N \mid x.v < 0\}$. For $j = 1, \ldots, k$, we define $\varphi_j : S^{N-1} \longrightarrow \mathbf{R}$ by

$$\varphi_j(v) = \int_{\Pi_v^+ \cap \Omega} G_j(|x|, u(x), |\nabla u(x)|)\, dx - \int_{\Pi_v^- \cap \Omega} G_j(|x|, u(x), |\nabla u(x)|)\, dx.$$

It is obvious that $\varphi_j(-v) = \varphi_j(v)$ and it follows immediately from Lebesgue's dominated convergence theorem that each φ_j is continuous on S^{N-1}. We will use the following well-known result (see, e.g., [18], Theorem 9 p. 266):

Borsuk-Ulam Theorem. *Given a continuous map $f : S^{n_1} \longrightarrow \mathbf{R}^{n_2}$ with $n_1 \ge n_2 \ge 1$, there exists $x \in S^{n_1}$ such that $f(x) = f(-x)$.*

Equivalently, any continuous odd map $f : S^{n_1} \longrightarrow \mathbf{R}^{n_2}$, $n_1 \ge n_2 \ge 1$, must vanish.

We use the Borsuk-Ulam theorem for the odd continuous map $\Phi = (\varphi_1, \ldots, \varphi_k) : S^{N-1} \longrightarrow \mathbf{R}^k$ and we infer that there exists $e_1 \in S^{N-1}$ such that $\Phi(e_1) = 0$, that is Π_{e_1} splits the constraints in two for the minimizer u.

Our aim is to show that u is symmetric with respect to Π_{e_1}. We denote $u_1 = u_{\Pi_{e_1}^-}$ and $u_2 = u_{\Pi_{e_1}^+}$ the two reflected functions obtained from u as in (1). By **A1** we have $u_1,\ u_2 \in \mathcal{X}$. Since Π_{e_1} splits the constraints in two, a simple change of variables shows that $\displaystyle\int_\Omega G_j(|x|, u_1(x), |\nabla u_1(x)|)\, dx = 2\int_{\Pi_v^- \cap \Omega} G_j(|x|, u_1(x), |\nabla u_1(x)|)\, dx = \lambda_j$ for any $j \in \{1, \ldots, k\}$, that is u_1 satisfies the constraints. In the same way u_2 satisfies the constraints. Since u is a minimizer for $(\mathcal{P})$, we must have $E(u_1) \ge E(u)$ and $E(u_2) \ge E(u)$. On the other hand, we get

$$E(u_1) + E(u_2) = 2\int_{\Pi_v^- \cap \Omega} F(|x|, u_1(x), |\nabla u_1(x)|)\, dx + 2\int_{\Pi_v^+ \cap \Omega} F(|x|, u_1(x), |\nabla u_1(x)|)\, dx$$
$$= 2E(u).$$

Thus necessarily $E(u_1) = E(u_2) = E(u)$ and u_1, u_2 are also minimizers for problem $(\mathcal{P})$. Moreover, they are symmetric with respect to Π_{e_1}.

Now let us consider the minimizer u_1. We define $\psi_j : S^{N-1} \longrightarrow \mathbf{R}$ by

$$\psi_j(v) = \int_{\Pi_v^+ \cap \Omega} G_j(|x|, u_1(x), |\nabla u_1(x)|)\, dx - \int_{\Pi_v^- \cap \Omega} G_j(|x|, u_1(x), |\nabla u_1(x)|)\, dx.$$

As previously, it is not hard to see that ψ_j is a continuous odd mapping on S^{N-1}, $1 \le j \le k$. In particular, the restriction of $\Psi = (\psi_1, \dots, \psi_k)$ to $S^{N-1} \cap \Pi_{e_1}$ is a continuous odd mapping from this space to $\mathbf{R}^k$. Since $S^{N-1} \cap \Pi_{e_1}$ can be identified to S^{N-2} and $k \le N-2$, we may use the Borsuk-Ulam theorem again and we infer that there exists $e_2 \in S^{N-1} \cap \Pi_{e_1}$ such that $\Psi(e_2) = 0$, i.e. Π_{e_2} splits the constraints in two for the minimizer u_1. We denote $u_{1,1} = (u_1)_{\Pi_{e_2}^-}$ and $u_{1,2} = (u_1)_{\Pi_{e_2}^+}$ the functions obtained from u_1 by the reflection procedure (1). Arguing as previously, we infer that $u_{1,1}$ and $u_{1,2}$ belong to $\mathcal{X}$, satisfy the constraints and are minimizers for problem $(\mathcal{P})$. Moreover, they are symmetric with respect to Π_{e_1} and with respect to Π_{e_2}. Next we use the following:

Lemma 3. *Let $w \in \mathcal{X}$ be a minimizer for $(\mathcal{P})$. Assume that* **A1, A2** *are satisfied and there exists a vector subspace V of $\mathbf{R}^N$ of dimension $m \le N-2$ such that any hyperplane containing V splits the constraints in two for w. Then w is radially symmetric with respect to V.*

Proof. Let $\mathcal{B}_1 = \{b_1, \dots, b_m\}$ be an orthonormal basis in V. Fix a hyperplane Π containing V. We extend $\mathcal{B}_1$ to an orthonormal basis $\mathcal{B} = \{b_1, \dots, b_N\}$ in $\mathbf{R}^N$ in such a way that $\Pi = \Pi_{b_N} = b_N^\perp$. We denote by $(x_1, \dots, x_N)$ the coordinates of a point x with respect to $\mathcal{B}$. Let $w_1 = w_{\Pi_{b_N}^-}$ and $w_2 = w_{\Pi_{b_N}^+}$. Clearly $w_1, w_2 \in \mathcal{X}$ by **A1**. By the assumption of Lemma 3, Π_{b_N} splits the constraints in two for w and this implies that w_1 and w_2 satisfy the constraints. As before we have $E(w_1) \ge E(w)$, $E(w_2) \ge E(w)$ and $E(w_1) + E(w_2) = 2E(w)$, thus necessarily $E(w_1) = E(w_2) = E(w)$ and w_1, w_2 are also minimizers. By **A2** we have $w, w_1, w_2 \in C^1(\Omega)$. Since w_1 is symmetric with respect to the x_N variable, we have $\frac{\partial w_1}{\partial x_N}(x_1, \dots, x_{N-1}, 0) = 0$ whenever $(x_1, \dots, x_{N-1}, 0) \in \Omega$. But $w(x) = w_1(x)$ for $x_N < 0$, therefore

$$\begin{aligned} &\frac{\partial w}{\partial x_N}(x_1, \dots, x_{N-1}, 0) = \lim_{s \uparrow 0} \frac{\partial w}{\partial x_N}(x_1, \dots, x_{N-1}, s) \\ (3) \qquad &= \lim_{s \uparrow 0} \frac{\partial w_1}{\partial x_N}(x_1, \dots, x_{N-1}, s) = \frac{\partial w_1}{\partial x_N}(x_1, \dots, x_{N-1}, 0) = 0 \end{aligned}$$

for $(x_1, \dots, x_{N-1}, 0) \in \Omega$, i.e. the derivative of w in the direction orthogonal to Π vanishes on $\Omega \cap \Pi$. Thus we have proved that for any hyperplane Π containing V, we have

$$(4) \qquad \frac{\partial w}{\partial n}(x) = 0 \qquad \text{for any } x \in \Omega \cap \Pi, \text{ where } n \text{ is the unit normal to } \Pi.$$

We pass to spherical coordinates in the last $N - m$ variables in $\mathbf{R}^N$, i.e. we use variables $(r, \theta_1, \dots, \theta_{N-m-1})$ instead of $(x_{m+1}, \dots, x_N)$, where $r = \left(x_{N-m+1}^2 + \dots + x_N^2\right)^{\frac{1}{2}}$ and $\theta_1, \dots \theta_{N-m-1}$ are the angular variables. Then (4) is equivalent to $\frac{\partial w}{\partial \theta_j} = 0$ on Ω for $j = 1, \dots, N-m-1$. We infer that w does not depend on $\theta_1, \dots, \theta_{N-m+1}$, i.e. there exists some function $\tilde{w}$ depending only on $x_1, \dots, x_m, r$ such that $w(x_1, \dots, x_N) = \tilde{w}(x_1, \dots, x_m, r)$ on Ω and Lemma 3 is proved. □

Now come back to the proof of Theorem 1. Clearly, any $x \in \mathbf{R}^N$ has a unique decomposition $x = x_1 e_1 + x_2 e_2 + x'$, where $x_1, x_2 \in \mathbf{R}$ and $x' \in \{e_1, e_2\}^{\perp}$. Since $u_{1,1}$ and $u_{1,2}$ are symmetric with respect to Π_{e_1} and with respect to Π_{e_2}, we have $u_{1,i}(x_1e_1+x_2e_2+x') = u_{1,i}(x_1e_1 - x_2e_2 + x') = u_{1,i}(-x_1e_1 - x_2e_2 + x')$. Let Π be a hyperplane containing $\{e_1, e_2\}^{\perp}$. It is obvious that the transform $x_1e_1 + x_2e_2 + x' \longmapsto -x_1e_1 - x_2e_2 + x'$ is a one-to-one correspondence between Π^+ and Π^- and a simple change of variables gives

$$\int_{\Pi^+\cap\Omega} G_j(|x|, u_{1,i}(x), |\nabla u_{1,i}(x)|)\, dx = \int_{\Pi^-\cap\Omega} G_j(|x|, u_{1,i}(x), |\nabla u_{1,i}(x)|)\, dx, \quad j = 1, \dots, k,$$

hence Π splits the constraints in two for $u_{1,i}$, $i = 1, 2$. By Lemma 3, we infer that $u_{1,i}$ are radially symmetric with respect to $\{e_1, e_2\}^{\perp}$, i.e. $u_{1,i}(x_1e_1+x_2e_2+x') = \tilde{u}_{1,i}(\sqrt{x_1^2 + x_2^2}, x')$ for some functions $\tilde{u}_{1,1}$ and $\tilde{u}_{1,2}$. On the other hand, $u_{1,1}(x) = u_1(x) = u_{1,2}(x)$ for any $x \in \Pi_{e_2} \cap \Omega$, that is $\tilde{u}_{1,1}(|x_1|, x') = \tilde{u}_{1,2}(|x_1|, x')$ whenever $x_1e_1 + x' \in \Omega$. We conclude that necessarily $\tilde{u}_{1,1} = \tilde{u}_{1,2}$ and $u_{1,1}(x) = u_1(x) = u_{1,2}(x)$ for any $x \in \Omega$, thus u_1 is radially symmetric with respect to $\{e_1, e_2\}^{\perp}$.

Similarly there exists $v_2 \in S^{N-1} \cap e_1^{\perp}$ such that Π_{v_2} splits the constraints in two for u_2 and we infer that u_2 is radially symmetric with respect to $\{e_1, v_2\}^{\perp}$. We use this information together with the fact that $u_1 = u = u_2$ on $\Omega \cap \Pi_{e_1}$ to prove the symmetry of u.

If v_2 is colinear to e_2, i.e. $v_2 = \pm e_2$, we may assume that $v_2 = e_2$. Using the symmetry of u_1, u_2 and the fact that $u_1 = u = u_2$ on $\Omega \cap \Pi_{e_1}$, we obtain as above that $u_1 = u_2 = u$ on Ω, hence u is radially symmetric with respect to $\{e_1, e_2\}^{\perp}$.

If v_2 and e_2 are not colinear, Span$\{e_1, e_2, v_2\}$ is a three-dimensional subspace. Let $\{e_4, \dots, e_N\}$ be an orthonormal basis in $\{e_1, e_2, v_2\}^{\perp}$. We choose e_3 and v_3 in such a way that $\mathcal{B} = \{e_1, e_2, e_3, \dots, e_N\}$ and $\mathcal{B}' = \{e_1, v_2, v_3, e_4, \dots, e_N\}$ are orthonormal basis in $\mathbf{R}^N$ with the same orientation. Then there exists $\theta \in (0, \pi) \cup (\pi, 2\pi)$ such that $v_2 = \cos\theta\, e_2 + \sin\theta\, e_3$ and $v_3 = -\sin\theta\, e_2 + \cos\theta\, e_3$. Given a point $x \in \mathbf{R}^N$, we denote by $(x_1, x_2, \dots, x_N)$ its coordinates with respect to $\mathcal{B}$. It is clear that $(x_1, y_2 = \cos\theta\, x_2 + \sin\theta\, x_3, y_3 = -\sin\theta\, x_2 + \cos\theta\, x_3, x_4, \dots, x_N)$ are the coordinates of x with respect to $\mathcal{B}'$.

Fix $re_3 + \sum_{j=4}^N x_j e_j \in \Omega \cap e_1^{\perp}$ and denote

$$\varphi(t) = \varphi_{r,x_4,\dots,x_N}(t) = u(r\cos t\, e_2 + r\sin t\, e_3 + \sum_{j=4}^N x_j e_j).$$

Clearly, φ is C^1 and $2\pi-$periodic on $\mathbf{R}$. Since the restriction of $u = u_1$ to $\Omega \cap e_1^{\perp}$ is symmetric with respect to $\mathbf{R}e_2$, we get

$$\varphi(t) = u(-r\cos t\, e_2 + r\sin t\, e_3 + \sum_{j=4}^N x_j e_j) = \varphi(\pi - t). \tag{5}$$

The restriction of $u = u_2$ to $\Omega \cap e_1^{\perp}$ is also symmetric with respect to $\mathbf{R}v_2$, therefore

$$\begin{aligned} \varphi(t) &= u(r(\cos t\cos\theta + \sin t\sin\theta)\, v_2 + r(\sin t\cos\theta - \cos t\sin\theta)v_3 + \textstyle\sum_{j=4}^N x_j e_j) \\ &= u_2(r\cos(t-\theta)\, v_2 + r\sin(t-\theta)\, v_3 + \textstyle\sum_{j=4}^N x_j e_j) \\ &= u_2(-r\cos(t-\theta)\, v_2 + r\sin(t-\theta)\, v_3 + \textstyle\sum_{j=4}^N x_j e_j) \qquad (6) \\ &= u_2(r\cos(\pi-(t-\theta))\, v_2 + r\sin(\pi-(t-\theta))\, v_3 + \textstyle\sum_{j=4}^N x_j e_j) \\ &= \varphi(\pi + 2\theta - t) = \varphi(t - 2\theta) \qquad \text{by (5)}. \end{aligned}$$

Hence any of the functions $\varphi_{r,x_4,\dots,x_N}$ admits 2π and 2θ as periods. The following situations may occur:

Case 1: $\frac{\theta}{\pi} \in \mathbf{R}\setminus\mathbf{Q}$. The set $\{2n\theta + 2k\pi \mid n,k \in \mathbf{Z}\}$ is dense in $\mathbf{R}$ and any number in this set is a period for $\varphi_{r,x_4,\dots,x_N}$. Since $\varphi_{r,x_4,\dots,x_N}$ is continuous, we infer that it is constant. This is equivalent to $u(\sum_{j=2}^N x_j e_j) = u(\sqrt{x_2^2+x_3^2}\, e_2 + \sum_{j=4}^N x_j e_j)$ whenever $\sum_{j=2}^N x_j e_j \in \Omega \cap e_1^\perp$. With the above notation, using the symmetry properties of u_1 and u_2 we have for any $x \in \Omega$,

$$u_1(x) = u_1(\sqrt{x_1^2+x_2^2}\, e_2 + \sum_{j=3}^N x_j e_j) = u(\sqrt{x_1^2+x_2^2+x_3^2}\, e_2 + \sum_{j=4}^N x_j e_j)$$

and

$$u_2(x) = u_2(\sqrt{x_1^2+y_2^2}\, v_2 + y_3 v_3 + \textstyle\sum_{j=4}^N x_j e_j) = u(\sqrt{x_1^2+y_2^2}\, v_2 + y_3 v_3 + \sum_{j=4}^N x_j e_j)$$

$$= u(\sqrt{x_1^2+y_2^2+y_3^2}\, e_2 + \textstyle\sum_{j=4}^N x_j e_j) = u(\sqrt{x_1^2+x_2^2+x_3^2}\, e_2 + \sum_{j=4}^N x_j e_j).$$

Consequently $u = u_1 = u_2$ on Ω and u is radially symmetric with respect to $\{e_1,e_2,e_3\}^\perp$.

Case 2: $\frac{\theta}{\pi} = \frac{k}{n}$ where k,n are relatively prime integers, k is odd and n is even, say $k = 2k_1+1$ and $n = 2n_1$. Then $\pi = 2n_1\theta - 2k_1\pi$ is also a period for $\varphi_{r,x_4,\dots,x_N}$ and this implies

$$u(\sum_{j=2}^N x_j e_j) = u(-x_2e_2 - x_3e_3 + \sum_{j=4}^N x_j e_j) \qquad \text{whenever } \sum_{j=2}^N x_j e_j \in \Omega\cap e_1^\perp. \tag{7}$$

From the symmetry of u_1 and (7) we get for $x_1 \le 0$:

$$\begin{aligned} &u(\textstyle\sum_{j=1}^N x_j e_j) = u(\sqrt{x_1^2+x_2^2}\, e_2 + \sum_{j=3}^N x_j e_j) \\ &= u(-\sqrt{x_1^2+x_2^2}\, e_2 - x_3e_3 + \textstyle\sum_{j=4}^N x_j e_j) = u(x_1e_1 - x_2e_2 - x_3e_3 + \sum_{j=4}^N x_j e_j). \end{aligned} \tag{8}$$

Using the symmetry of u_2 and (7), we infer that (8) also holds for $x_1 \ge 0$. Let Π be a hyperplane containing $\{e_1, e_4, \dots, e_N\}$. It is clear that the mapping $\sum_{j=1}^N x_j e_j \longmapsto x_1e_1 - x_2e_2 - x_3e_3 + \sum_{j=4}^N x_j e_j$ is a linear isometry between Π^+ and Π^-. Then (8) and a simple change of variables show that

$$\int_{\Pi^+\cap\Omega} G_\ell(|x|, u(x), |\nabla u(x)|)\, dx = \int_{\Pi^-\cap\Omega} G_\ell(|x|, u(x), |\nabla u(x)|)\, dx,$$

for $\ell = 1,\dots,k$, i.e. Π splits the constraints in two for u. Since u is a minimizer, by Lemma 3 we infer that u is radially symmetric with respect to $\mathrm{Span}\{e_1, e_4, \dots, e_N\}$. In particular, the restriction of u to $\Omega\cap e_1^\perp$ is radially symmetric with respect to $\mathrm{Span}\{e_4,\dots,e_N\}$. As in case 1, this implies that u is radially symmetric with respect to $\mathrm{Span}\{e_4,\dots,e_N\}$.

Case 3: $\frac{\theta}{\pi} = \frac{k}{n}$ where k,n are relatively prime integers, k is even and n is odd, say $k = 2k_1$ and $n = 2n_1+1$. Then $\theta = 2k_1\pi - 2n_1\theta$ is a period for $\varphi_{r,x_4,\dots,x_N}$. By (5) we get $\varphi_{r,x_4,\dots,x_N}(t) = \varphi_{r,x_4,\dots,x_N}(\pi - t) = \varphi_{r,x_4,\dots,x_N}(\theta+\pi-t)$. This means that for $\sum_{j=2}^N x_j e_j \in \Omega$ we have

$$u(\sum_{j=2}^N x_j e_j) = u(-(x_2\cos\theta + x_3\sin\theta)e_2 + (-x_2\sin\theta + x_3\cos\theta)e_3 + \sum_{j=4}^N x_j e_j). \tag{9}$$

In other words, for fixed $x'' \in \text{Span}\{e_4, \ldots, e_N\}$, the function $x_2e_2 + x_3e_3 \longmapsto u(x_2e_2 + x_3e_3 + x'')$ is symmetric with respect to $\mathbf{R}w$, where $w = \cos(\frac{\theta+\pi}{2})e_2 + \sin(\frac{\theta+\pi}{2})e_3$. Note that the symmetry of $\text{Span}\{e_1, e_2, e_3\}$ with respect to $\mathbf{R}w$ is a linear isometry of matrix $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\theta & -\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$ with respect to the basis $\{e_1, e_2, e_3\}$. We show that for any $x \in \Omega$ we have

$$u(x) = u(Sx), \tag{10}$$

where $Sx = -x_1e_1 - (x_2\cos\theta + x_3\sin\theta)e_2 + (-x_2\sin\theta + x_3\cos\theta)e_3 + \sum_{j=4}^N x_je_j$. It suffices to consider the case $x_1 \le 0$. By using the symmetry of u_1, u_2 and (9) we get

$$u(x) = u_1(x) = u(\sqrt{x_1^2 + x_2^2}\, e_2 + \textstyle\sum_{j=3}^N x_je_j)$$

$$= u(-(\sqrt{x_1^2 + x_2^2}\cos\theta + x_3\sin\theta)e_2 + (-\sqrt{x_1^2 + x_2^2}\sin\theta + x_3\cos\theta)e_3 + \textstyle\sum_{j=4}^N x_je_j)$$

and

$$u(Sx) = u_2(Sx) = u_2(-x_1e_1 - x_2v_2 + x_3v_3 + \textstyle\sum_{j=4}^N x_je_j)$$

$$= u_2(-\sqrt{x_1^2 + x_2^2}\, v_2 + x_3v_3 + \textstyle\sum_{j=4}^N x_je_j)$$

$$= u(-\sqrt{x_1^2 + x_2^2}(\cos\theta\, e_2 + \sin\theta\, e_3) + x_3(-\sin\theta\, e_2 + \cos\theta\,{}_3) + \textstyle\sum_{j=4}^N x_je_j),$$

hence $u(x) = u(Sx)$. Let Π be a vector hyperplane containing $w, e_4, \ldots, e_N$. It is easy to see that S is a linear isometry of $\mathbf{R}^N$ mapping $\Omega \cap \Pi^-$ onto $\Omega \cap \Pi^+$. Using (10) and a change of variables, we find that Π splits the constraints in two for u. By Lemma 3 we infer that u is radially symmetric with respect to $\text{Span}\{w, e_4, \ldots, e_N\}$.

In fact, since u_1 is radially symmetric with respect to $\text{Span}\{e_3, e_4, \ldots, e_N\}$ and $\text{Span}\{w, e_4, \ldots, e_N\}$, it can be proved that u_1 is radially symmetric with respect to $\text{Span}\{e_4, \ldots, e_N\}$. Similarly u_2 is radially symmetric with respect to $\text{Span}\{e_4, \ldots, e_N\}$ and then it is clear that u has the same property. We omit the proof because we will not make use of this observation.

Case 4: $\frac{\theta}{\pi} = \frac{k}{n}$ where k, n are relatively prime odd integers, say $k = 2k_1 + 1$ and $n = 2n_1 + 1$. Then $\theta - \pi = 2k_1\pi - 2n_1\theta$ is a period for $\varphi_{r,x_4,\ldots,x_N}$. By (5) we have $\varphi_{r,x_4,\ldots,x_N}(t) = \varphi_{r,x_4,\ldots,x_N}(\pi - t) = \varphi_{r,x_4,\ldots,x_N}(\theta - t)$, that is

$$u(x) = u((x_2\cos\theta + x_3\sin\theta)e_2 + (x_2\sin\theta - x_3\cos\theta)e_3 + \sum_{j=4}^N x_je_j) \tag{11}$$

for any $x = \sum_{j=2}^N x_je_j \in \Omega \cap e_1^\perp$. Proceeding as in case 3, we prove that u is radially symmetric with respect to $\text{Span}\{w', e_4, \ldots, e_N\}$, where $w' = \cos\frac{\theta}{2}\, e_2 + \sin\frac{\theta}{2}\, e_3$. (In fact, it can be proved that u is radially symmetric with respect to $\text{Span}\{e_4, \ldots, e_N\}$).

Note that in either case it follows that u is symmetric with respect to Π_{e_1}. Thus we have proved that whenever $e_1 \in S^{N-1}$ satisfies $\Phi(e_1) = 0$, u is symmetric with respect to Π_{e_1}. Assume that $e_1, \ldots, e_\ell \in S^{N-1}$ are mutually orthogonal, satisfy $\Phi(e_1) = \ldots = \Phi(e_\ell) = 0$ and $\ell \le N - k - 1$. It is clear that $S_\ell = S^{N-1} \cap \{e_1, \ldots, e_\ell\}^\perp$ can be identified to $S^{N-\ell-1}$ and the restriction of Φ to S_ℓ is an odd, continuous function from S_ℓ to

$\mathbf{R}^k$. Using the Borsuk-Ulam theorem we infer that there exists $e_{\ell+1} \in S_\ell$ such that $\Phi(e_{\ell+1}) = 0$. By induction it follows that there exist $N-k$ mutually orthogonal vectors $e_1, \dots, e_{N-k} \in S^{N-1}$ such that $\Phi(e_1) = \dots = \Phi(e_{N-k}) = 0$. We complete this set to an orthonormal basis $\{e_1, \dots, e_N\}$ in $\mathbf{R}^N$. We already know that u is symmetric with respect to any of the hyperplanes $\Pi_{e_1}, \dots, \Pi_{e_{N-k}}$. In particular, for $x = \sum_{j=1}^N x_j e_j \in \Omega$ we have

$$u(x) = u(-x_1 e_1 + \sum_{j=2}^N x_j e_j) = \dots = u(-\sum_{j=1}^{N-k} x_j e_j + \sum_{j=N-k+1}^{N} x_j e_j). \tag{12}$$

Let Π be a (vector) hyperplane containing $e_{N-k+1}, \dots, e_N$. It is clear that the mapping $\sum_{j=1}^N x_j e_j \longmapsto -\sum_{j=1}^{N-k} x_j e_j + \sum_{j=N-k+1}^{N} x_j e_j$ is a linear isometry between Π^+ and Π^-. Using (12), we infer that Π splits the constraints in two for u. By Lemma 3, u is radially symetric with respect to $\operatorname{Span}\{e_{N-k+1}, \dots, e_N\}$.

The case $k = 0$ is much simpler. Problem $(\mathcal{P})$ consists in minimizing E on $\mathcal{X}$ without constraints. Assume that u is a minimizer. Let Π be a hyperplane containing the origin and let u_{Π^-}, u_{Π^+} be the two functions obtained from u as in (1). By **A1** we have $u_{\Pi^-}, u_{\Pi^+} \in \mathcal{X}$, thus $E(u_{\Pi^-}) \geq E(u)$ and $E(u_{\Pi^+}) \geq E(u)$. On the other hand, $E(u_{\Pi^-}) + E(u_{\Pi^+}) = 2E(u)$, thus necessarily $E(u_{\Pi^-}) = E(u_{\Pi^+}) = E(u)$ and u_{Π^-}, u_{Π^+} are also minimizers. As in the proof of Lemma 3, this implies $\frac{\partial u}{\partial n}(x) = 0$ for any $x \in \Omega \cap \Pi$, where n is the unit normal to Π. Then passing to spherical coordinates, as in Lemma 3, we see that u does not depend on the angular variables, i.e. u is a radial function. □

Proof of Theorem 2. For $v \in S^{N-1}$ and $t \in \mathbf{R}$ we denote by $\Pi_{v,t}$ the affine hyperplane $\{x \in \mathbf{R}^N \mid (x - tv).v = 0\}$ and by $\Pi_{v,t}^+ = \{x \in \mathbf{R}^N \mid (x - tv).v > 0\}$, respectively $\Pi_{v,t}^- = \{x \in \mathbf{R}^N \mid (x - tv).v < 0\}$ the two half-spaces determined by $\Pi_{v,t}$. It is clear that $\Pi_{-v,-t}^- = \Pi_{v,t}^+$. For $j = 1, \dots, k$, we define $\tilde{\psi}_j : S^{N-1} \times \mathbf{R} \longrightarrow \mathbf{R}$ by

$$\tilde{\psi}_j(v,t) = \int_{\Pi_{v,t}^+} G_j(u(x), |\nabla u(x)|)\,dx - \int_{\Pi_{v,t}^-} G_j(u(x), |\nabla u(x)|)\,dx.$$

Since $G_j(u, |\nabla u|) \in L^1(\mathbf{R}^N)$, it is a simple consequence of Lebesgue's dominated convergence theorem that $\tilde{\psi}_j$ is continuous on $S^{N-1} \times \mathbf{R}$. It is obvious that $\tilde{\psi}_j(-v,-t) = -\tilde{\psi}_j(v,t)$.

We claim that $\lim_{t \to \infty} \tilde{\psi}_j(v,t) = -\int_{\mathbf{R}^N} G_j(u(x), |\nabla u(x)|)\,dx = -\lambda_j$ uniformly with respect to $v \in S^{N-1}$. Indeed, fix $\varepsilon > 0$. There exists $R > 0$ such that

$$\int_{\mathbf{R}^N \setminus B(0,R)} |G_j(u(x), |\nabla u(x)|)|\,dx < \frac{\varepsilon}{2}.$$

For any $v \in S^{N-1}$ and $t > R$ we have $\Pi_{v,t}^+ \subset \mathbf{R}^N \setminus B(0,R)$, therefore

$$\left|\tilde{\psi}_j(v,t) + \int_{\mathbf{R}^N} G_j(u(x), |\nabla u(x)|)\,dx\right| = 2\left|\int_{\Pi_{v,t}^+} G_j(u(x), |\nabla u(x)|)\,dx\right| < \varepsilon$$

and the claim is proved. It is clear that $\lim_{t \to -\infty} \tilde{\psi}_j(v,t) = \lambda_j$ uniformly in $v \in S^{N-1}$.

We denote $P = (0, \dots, 0, 1) \in \mathbf{R}^{N+1}$, $S = (0, \dots, 0, -1) \in \mathbf{R}^{N+1}$ and we define $\psi_j : S^N \longrightarrow \mathbf{R}$ by

$$\psi_j(x_1, \dots, x_N, x_{N+1}) = \tilde{\psi}_j\left(\frac{(x_1, \dots, x_N)}{|(x_1, \dots, x_N)|}, \frac{x_{N+1}}{1 - |x_{N+1}|}\right)$$

if $(x_1,\dots,x_N,x_{N+1}) \notin \{P,S\}$, respectively $\psi_j(P) = -\lambda_j$ and $\psi_j(S) = \lambda_j$. Then ψ_j is an odd, continuous function on S^N.

Consider first the case $1 \le k \le N-2$. It follows from Theorem 1 that there exist two orthogonal vector subspaces V_1 and V_2 such that $\dim(V_1) = k$, $V_1 \oplus V_2 = \mathbf{R}^N$ and u is radially symmetric with respect to V_1. The set $\mathbf{S} = \{(y_1,\dots,y_N,y_{N+1}) \in S^N \mid (y_1,\dots,y_N) \in V_1\}$ can be identified to S^k. Since the restriction of $\Psi = (\psi_1,\dots,\psi_k)$ to $\mathbf{S} \simeq S^k$ is continuous, odd, $\mathbf{R}^k-$valued, by the Borsuk-Ulam theorem we infer that there exists $y^* = (y_1^*,\dots,y_N^*,y_{N+1}^*) \in \mathbf{S}$ such that $\psi(y^*) = 0$. We cannot have $y^* = S$ or $y^* = P$ because $\psi(S) = -\psi(P) = (\lambda_1,\dots,\lambda_N) \neq 0$. Denote $e_k = \frac{(y_1^*,\dots,y_N^*)}{|(y_1^*,\dots,y_N^*)|}$ and $t = \frac{y_{N+1}^*}{1-|y_{N+1}^*|}$. Then $e_k \in V_1$, $|e_k| = 1$ and $\tilde{\psi}_j(e_k,t) = 0$ for $j = 1,\dots,k$, i.e. $\Pi_{e_k,t}$ splits the constraints in two for u. Choose e_i, $i = 1,\dots,N$, $i \neq k$ in such a way that $\{e_1,\dots,e_{k-1},e_k\}$ and $\{e_{k+1},\dots,e_N\}$ are orthonormal basis in V_1, respectively in V_2. Denote $u_*(x) = u(x-te_k)$. It is clear that u_* is a minimizer for $(\mathcal{P}')$, it is radially symmetric with respect to V_1 and the hyperplane $e_k^\perp = \Pi_{e_k,0}$ splits the constraints in two for u_*. Arguing exactly as in the proof of Theorem 1, we see that u_* is symmetric with respect to $e_k^\perp$. Using this fact and the radial symmetry with respect to V_1, we get

$$u_*(\sum_{i=1}^N x_i e_i) = u_*(\sum_{i=1}^k x_i e_i - \sum_{i=k+1}^N x_i e_i) = u_*(\sum_{i=1}^{k-1} x_i e_i - \sum_{i=k}^N x_i e_i). \tag{13}$$

By (13) we infer that any (vector) hyperplane containing $e_1,\dots,e_{k-1}$ splits the constraints in two for u_*. Then Lemma 3 implies that u_* is radially symmetric with respect to $\mathrm{Span}\{e_1,\dots,e_{k-1}\}$, consequently u is radially symmetric with respect to the affine subspace $te_k + \mathrm{Span}\{e_1,\dots,e_{k-1}\}$.

Now consider the case $k = N-1$. As above, there exists $y^* = (y_1^*,\dots,y_N^*,y_{N+1}^*) \in S^N \setminus \{S,P\}$ such that $\psi(y^*) = 0$. Denoting $e_1 = \frac{(y_1^*,\dots,y_N^*)}{|(y_1^*,\dots,y_N^*)|}$ and $t_1 = \frac{y_{N+1}^*}{1-|y_{N+1}^*|}$, this means that Π_{e_1,t_1} splits the constraints in two for u. Let $u_1 = u_{\Pi^-_{e_1,t_1}}$ and $u_2 = u_{\Pi^+_{e_1,t_1}}$. It is clear that u_1, u_2 are also minimizers for $(\mathcal{P}')$. Since $\{(y_1,\dots,y_{N+1}) \in S^N \mid (y_1,\dots,y_N) \perp e_1\}$ is homeomorphic to S^{N-1} and there are exactly $N-1$ constraints, it is possible to restart the prevoius process with u_1 instead of u. We infer that there exists $e_2 \in e_1^\perp$, $|e_2| = 1$ and $t_2 \in \mathbf{R}$ such that Π_{e_2,t_2} splits the constraints in two for u_1. Putting $u_{1,1} = (u_1)_{\Pi^-_{e_2,t_2}}$ and $u_{1,2} = (u_1)_{\Pi^+_{e_2,t_2}}$, we see that $u_{1,1}$ and $u_{1,2}$ are minimizers for $(\mathcal{P}')$ and are symmetric with respect to Π_{e_1,t_1} and Π_{e_2,t_2}. It follows that $\tilde{u}_{1,1} = u_{1,1}(\cdot - t_1e_1 - t_2e_2)$ and $\tilde{u}_{1,2} = u_{1,2}(\cdot - t_1e_1 - t_2e_2)$ minimize $(\mathcal{P}')$ and are symmetric with respect to $e_1^\perp$ and $e_2^\perp$. Therefore any (vector) hyperplane in $\mathbf{R}^N$ containing $\{e_1,e_2\}^\perp$ splits the constraints in two for $\tilde{u}_{1,1}$ and for $\tilde{u}_{1,2}$ and using Lemma 3 we infer that $\tilde{u}_{1,1}$ and $\tilde{u}_{1,2}$ are radially symmetric with respect to $\{e_1,e_2\}^\perp$. Since $\tilde{u}_{1,1} = \tilde{u}_{1,2}$ on $\Pi_{e_2,0} = e_2^\perp$, we have necessarily $\tilde{u}_{1,1} = \tilde{u}_{1,2}$ on $\mathbf{R}^N$. Therefore $u_1 = \tilde{u}_{1,1}(\cdot + t_1e_1 + t_2e_2)$ is radially symmetric with respect to the affine subspace $t_1e_1 + t_2e_2 + \{e_1,e_2\}^\perp$.

Similarly we prove that there exist $v_2 \in e_1^\perp$, $|v_2| = 1$ and $s_2 \in \mathbf{R}$ such that u_2 is radially symmetric with respect to the affine subspace $t_1e_1 + s_2v_2 + \{e_1,v_2\}^\perp$. Of course, nothing guarantees *à priori* that $(e_2,t_2) = \pm(v_2,s_2)$. The following situations may occur:

Case 1: e_2 and v_2 are colinear. Then we may assume that $e_2 = v_2$. There are two subcases:

a) $t_2 = s_2$. Then $u_1(\cdot - t_1e_1 - t_2e_2)$ and $u_2(\cdot - t_1e_1 - t_2e_2)$ are both radially symetric with respect to $\{e_1,e_2\}^\perp$ and are equal on $e_1^\perp$. We conclude that $u_1(\cdot - t_1e_1 - t_2e_2) = u_2(\cdot - t_1e_1 - t_2e_2)$, thus $u = u_1 = u_2$ is radially symmetric with respect to $t_1e_1 + t_2e_2 + \{e_1,e_2\}^\perp$.

b) $t_2 \neq s_2$, say $s_2 > t_2$. The symmetry of u_1 and u_2 imply that there exist some functions $\tilde{u}_1$, $\tilde{u}_2$ defined on $[0,\infty) \times \{e_1, e_2\}^\perp$ such that

$$
\begin{aligned}
u_1(x_1e_1 + x_2e_2 + x') &= \tilde{u}_1(\sqrt{(x_1-t_1)^2 + (x_2-t_2)^2}\,, x') \\
u_2(x_1e_1 + x_2e_2 + x') &= \tilde{u}_2(\sqrt{(x_1-t_1)^2 + (x_2-s_2)^2}\,, x')
\end{aligned}
\tag{14}
$$

for any $x_1, x_2 \in \mathbf{R}$ and $x' \in \{e_1, e_2\}^\perp$. Since $u_1 = u_2$ on $\Pi_{e_1,t_1} = t_1e_1 + e_1^\perp$, it follows that

$$
\tilde{u}_1(|x_2 - t_2|\,, x') = \tilde{u}_2(|x_2 - s_2|\,, x') \tag{15}
$$

for any $x_2 \in \mathbf{R}$ and $x' \in \{e_1, e_2\}^\perp$. In particular, (15) implies that for fixed $x' \in \{e_1, e_2\}^\perp$, $\tilde{u}_1(\cdot, x')$ and $\tilde{u}_2(\cdot, x')$ are periodic of period $a = 2(s_2 - t_2)$. Passing to cylindrical coordinates $x_1 = t_1 + r\cos\theta$, $x_2 = t_2 + r\sin\theta$, x' and using Fubini's theorem we have

$$
\begin{aligned}
\int_{\Pi^-_{e_1,t_1}} G_j(u(x), |\nabla u(x)|)\, dx &= \int_{\Pi^-_{e_1,t_1}} G_j(u_1(x), |\nabla u_1(x)|)\, dx \\
&= \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{\{e_1,e_2\}^\perp} G_j(\tilde{u}_1(r,x'), |\nabla \tilde{u}_1(r,x')|)\, dx'\, d\theta\, r\, dr \\
&= \pi \int_0^\infty \int_{\{e_1,e_2\}^\perp} G_j(\tilde{u}_1(r,x'), |\nabla \tilde{u}_1(r,x')|)\, dx'\, r\, dr.
\end{aligned}
\tag{16}
$$

Let $h_j(r) = \displaystyle\int_{\{e_1,e_2\}^\perp} G_j(\tilde{u}_1(r,x'), |\nabla \tilde{u}_1(r,x')|)\, dx'$. The function h_j is well-defined for a.e. $r \geq 0$, measurable, periodic of period a, and $\pi \displaystyle\int_0^\infty r h_j(r)\, r = \lambda_j/2$. By periodicity we have $\displaystyle\int_{na}^{(n+1)a} r h_j(r)\, dr = na \int_0^a h_j(r)\, dr + \int_0^a r h_j(r)\, dr$, thus $\displaystyle\int_0^{na} r h_j(r)\, dr = \frac{n(n-1)}{2} a \int_0^a h_j(r)\, dr + n \int_0^a r h_j(r)\, dr$. It follows that necessarily $\displaystyle\int_0^a h_j(r)\, dr = 0$ and $\displaystyle\int_0^a r h_j(r)\, dr = 0$ and this implies $\displaystyle\int_0^\infty r h_j(r)\, dr = 0$, i.e. $\lambda_j = 0$ for any j, contrary to the assumptions of Theorem 2. Consequently the case 1 b) may never occur.

Case 2: e_2 and v_2 are not colinear. It is then clear that the space $\text{Span}\{e_1, e_2, v_2\}$ is $3-$dimensional (thus $N \geq 3$). Let $\{e_4, \ldots, e_N\}$ be an orthonormal basis of $\{e_1, e_2, v_2\}^\perp$. We choose e_3 and v_3 in such a way that $\mathcal{B} = \{e_1, \ldots, e_N\}$ and $\mathcal{B}' = \{e_1, v_2, v_3, e_4, \ldots, e_N\}$ are orthonormal basis in $\mathbf{R}^N$ with the same orientation. There exists $\theta \in (0,\pi) \cup (\pi, 2\pi)$ such that $v_2 = \cos\theta\, e_2 + \sin\theta\, e_3$ and $v_3 = -\sin\theta\, e_2 + \cos\theta\, e_3$. Since $\sin\theta \neq 0$, there exist some $\alpha, \beta \in \mathbf{R}$ such that $t_2e_2 + \alpha e_3 = s_2v_2 + \beta v_3$. Let $y = t_1e_1 + t_2e_2 + \alpha e_3$. We denote $u^* = u(\cdot - y)$, $u_1^* = u_1(\cdot - y)$ and $u_2^* = u_2(\cdot - y)$. It is obvious that u^*, u_1^* and u_2^* are minimizers for $(\mathcal{P}')$, u_1^* is radially symmetric with respect to $\text{Span}\{e_3, \ldots, e_N\}$, u_2^* is radially symmetric with respect to $\text{Span}\{v_3, e_4, \ldots, e_N\}$, $u^* = u_1^*$ on $\Pi^-_{e_1,0} \cup \Pi_{e_1,0}$ and $u^* = u_2^*$ on $\Pi^+_{e_1,0} \cup \Pi_{e_1,0}$. Proceeding as in the proof of Theorem 1 we show that either u^* is radially symmetric with respect to $\text{Span}\{e_4, \ldots, e_N\}$, or there exists $w \in \text{Span}\{e_2, e_3\}$, such that u^* is radially symmetric with respect to $\text{Span}\{w, e_4, \ldots, e_N\}$. In any case it follows that u is radially symmetric with respect to an affine subspace of dimension at most $k - 1 = N - 2$. This completes the proof of Theorem 2. □

3 Remarks and examples

Remark 4. If Ω is connected and a unique continuation principle is available for minimizers, the proofs in the preceding section can be considerably simplified. Moreover, it is possible to deal with $N-1$ constraints in Theorem 1, respectively with N constraints in Theorem 2 (but this is of quite limited interest in applications because we get only symmetry with respect to a hyperplane).

For example, consider the problem $(\mathcal{P}1)$ of minimizing

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\,dx \qquad \text{in } H^1(\Omega, \mathbf{R}^m) \quad (\text{or in } H_0^1(\Omega, \mathbf{R}^m))$$

under the constraints $Q_j(u) = \int_\Omega G_j(u)\,dx = \lambda_j$, $1 \le j \le k$ and the following standard assumptions:

H1. $F, G_1, \dots, G_k \in C^2(\mathbf{R}^m, \mathbf{R})$, $F(0) = G_j(0) = 0$, $\nabla F(0) = \nabla G_j(0) = 0$, and

$$|\nabla F(u)| \le C|u|^p, \qquad |\nabla G_j(u)| \le C|u|^p \qquad \text{for } |u| \ge 1, \text{ where } p < \frac{N+2}{N-2}.$$

H2. If $u \in H^1(\Omega, \mathbf{R}^m)$ (respectively $u \in H_0^1(\Omega, \mathbf{R}^m)$) is nonconstant and $\sum_{j=1}^k \alpha_j \nabla G_j(u) = \sum_{j=1}^k \beta_j \nabla G_j(u)$ on Ω, then $\alpha_j = \beta_j$ for $j = 1, \dots, k$.

Suppose that u is a minimizer for $(\mathcal{P}1)$ and a hyperplane Π (with $0 \in \Pi$ if $\Omega \ne \mathbf{R}^N$) splits the contraints in two for u. As before, it follows easily that the functions u_{Π^-} and u_{Π^+} are minimizers for $(\mathcal{P}1)$. Thus u and u_{Π^-} satisfy the Euler-Lagrange equations

$$(17) \qquad -\Delta u + \nabla F(u) + \sum_{j=1}^k \alpha_j \nabla G_j(u) = 0 \qquad \text{in } \Omega, \quad \text{respectively}$$

$$(18) \qquad -\Delta u_{\Pi^-} + \nabla F(u_{\Pi^-}) + \sum_{j=1}^k \beta_j \nabla G_j(u_{\Pi^-}) = 0 \qquad \text{in } \Omega$$

for some $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \mathbf{R}$. By standard regularity theory we get u, $u_{\Pi^-} \in W^{2,q}(\Omega)$ for any $q \in [2, \infty)$. In particular, u, $u_{\Pi^-} \in C^{1,\alpha}(\Omega)$ for $\alpha \in [0,1)$, and u, u_{Π^-} as well as their derivatives are bounded on Ω. If u is constant on $\Omega \cap \Pi^-$, it follows form (17) and the unique continuation principle (see [11]) that u is constant on Ω. Otherwise, from (17) and (18) we obtain $\sum_{j=1}^k \alpha_j \nabla G_j(u) = \sum_{j=1}^k \beta_j \nabla G_j(u)$ on $\Omega \cap \Pi^-$ and by **H2** we infer that $\alpha_j = \beta_j$, $j = 1, \dots, k$. Denoting $w = u - u_{\Pi^-}$, (17) and (18) imply that w satisfies

$$-\Delta w + A(x)w = 0 \qquad \text{in } \Omega,$$

where $A \in L^\infty(\Omega, M_m(\mathbf{R}))$. Since $w = 0$ in $\Omega \cap \Pi^-$, by the unique continuation principle we find $w = 0$ in Ω, i.e. $u = u_{\Pi^-}$ and u is symmetric with respect to Π. Hence we have proved that u is symmetric with respect to any hyperplane that splits the constraints in two. The rest of the proof is as in the preceding section.

Note that a nondegeneracy hypothesis like **H2** is needed to use a unique continuation principle.

Remark 5. In Theorems 1 and 2, any supplementary constraint in the minimization problem produces the loss of one direction of symmetry for minimizers. Under the general assumptions made there, this loss of symmetry cannot be avoided, as it can be seen in the following simple examples.

Example 6. i) Let Ω be either a ball or an annulus in $\mathbf{R}^N$, centered at the origin. Consider $F, G \in C^2(\mathbf{R}, \mathbf{R})$ satisfying assumption **H1** in Remark 4 and such that the problem $(\mathcal{P}_1)$ of minimizing $E_1(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\,dx$ in $H^1(\Omega)$ under the constraint $\int_\Omega G(u)\,dx = \lambda$ admits a nonconstant solution u_*. It has been shown in [12] that u_* cannot be radially symmetric about 0 (but, of course, u_* is radially symmetric with respect to a line passing through 0). Consider the problem

$$(\mathcal{P}_k) \qquad \begin{aligned} &\text{minimize } E_k(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u_1) + \ldots + F(u_k)\,dx, \\ &\text{under the constraints } \int_\Omega G(u_j)\,dx = \lambda, \qquad j = 1, \ldots, k, \end{aligned}$$

where $u = (u_1, \ldots, u_k) \in H^1(\Omega, \mathbf{R}^k)$. It is clear that $u = (u_1, \ldots, u_k)$ is a solution of $(\mathcal{P}_k)$ if and only if each u_j is a solution of $(\mathcal{P}_1)$. If $R_1, \ldots, R_k$ are rotations in $\mathbf{R}^N$, the function $u(x) = (u_*(R_1 x), \ldots, u_*(R_k x))$ is a solution of $(\mathcal{P}_k)$. We infer that there are minimizers of $(\mathcal{P}_k)$ that are not radially symmetric with respect to any $(k-1)-$dimensional vector subspace of $\mathbf{R}^N$.

ii) Consider two functions $F, G \in C^2(\mathbf{R}, \mathbf{R})$ satisfying assumption **H1** in Remark 4 and $\lambda \in \mathbf{R}^*$ such that the problem $(\mathcal{P}'_1)$ consisting in minimizing $\tilde{E}_1(u) = \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 + F(u)\,dx$ in $H^1(\mathbf{R}^N)$ under the constraint $\int_{\mathbf{R}^N} G(u)\,dx = \lambda$ admits a nonconstant solution $\tilde{u}$. It folows immediately from Theorem 2 that $\tilde{u}$ is radially symmetric with respect to a point; we may assume that it is radially symmetric about the origin. It is easy to see that $u = (u_1, \ldots, u_k) \in H^1(\mathbf{R}^N, \mathbf{R}^k)$ is a solution of the problem

$$(\mathcal{P}'_k) \qquad \begin{aligned} &\text{minimize } \tilde{E}_k(u) = \int_{\mathbf{R}^N} \frac{1}{2}|\nabla u|^2 + F(u_1) + \ldots + F(u_k)\,dx, \\ &\text{under the constraints } \int_{\mathbf{R}^N} G(u_j)\,dx = \lambda, \qquad j = 1, \ldots, k, \end{aligned}$$

in $H^1(\mathbf{R}^N, \mathbf{R}^k)$ if and only if each u_j is a solution of $(\mathcal{P}'_1)$. Therefore for any $y_1, \ldots, y_k \in \mathbf{R}^N$, the function $u = (u_1(\cdot + y_1), \ldots, u_k(\cdot + y_k))$ is a solution for $(\mathcal{P}'_k)$. Obviously, this minimizer is radially symmetric with respect to some $(k-1)-$dimensional affine subspace but, in general, it is not radially symmetric with respect to any affine subspace of lower dimension.

In Example 6, the loss of symmetry comes from the fact that problems $(\mathcal{P}_k)$ and $(\mathcal{P}'_k)$ are decoupled: they can be decomposed into k independent scalar problems, each of them being rotation (respectively translation) invariant. It is then natural to ask whether in general problems like $(\mathcal{P})$ or $(\mathcal{P}')$ the loss of directions of symmetry could exceed the number of components of minimizers. The answer is affirmative, as it can be seen in the next example which shows that, in general, the result of Theorem 2 is optimal even for scalar-valued minimizers.

Example 7. We construct here a minimization problem of the form $(\mathcal{P}')$ involving two constraints and whose real-valued minimizers are *not* radial with respect to a point (of course, these minimizers are axially symmetric). This example relies on the existence of a nonnegative minimizer with compact support for a problem involving one constraint. A similar construction has already been used in [4].

Let $f \in C(\mathbf{R}) \cap C^1(0, \infty)$ be a real-valued function satisfying the following conditions:

C1. $f(s) = 0$ on $(-\infty, 0]$ and $f(s) = s^\alpha$ for $s \in (0, 1]$, where $\alpha \in (0, 1)$.

C2. The function $F(s) := \int_0^s f(\tau)\,d\tau$ has compact support.

C3. There exists $\zeta > 0$ such that $F(\zeta) < 0$.

Let $N \geq 3$ and $\mathcal{X} = \mathcal{D}^{1,2}(\mathbf{R}^N) \cap L^{1+\alpha}(\mathbf{R}^N)$. We introduce the functionals $T(u) = \int_{\mathbf{R}^N} |\nabla u|^2 \, dx$ and $V(u) = \int_{\mathbf{R}^N} F(u(x)) \, dx$. It is clear that $F(u) \in L^1(\mathbf{R}^N)$ for any $u \in \mathcal{X}$ and T, V are well-defined, C^1 functionals on $\mathcal{X}$. We consider the minimization problem:

$(\mathcal{M}_1)$ minimize $T(u)$ in $\mathcal{X}$ subject to the constraint $V(u) = -1$.

We denote $I = \inf\{T(u) \mid u \in \mathcal{X}, V(u) = -1\}$ and we proceed in several steps.

Step 1. We have $I > 0$ and problem $(\mathcal{M}_1)$ has a minimizer $u_* \in \mathcal{X}$. The proof of this fact is a straightforward modification of the proof of Theorem 2 in [3] or of the proof of Theorem 1 in [6], so we omit it.

Step 2. Any minimizer u of $(\mathcal{M}_1)$ is nonnegative, bounded, C^1, has compact support and satisfies the equation $-\Delta u + \beta_0 f(u) = 0$ in $\mathcal{D}'(\mathbf{R}^N)$, where $\beta_0 = \frac{N-2}{2N} I$.

Let $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. Then $u^+, u^- \in \mathcal{X}$, $V(u^+) = V(u) = -1$ and $T(u) = T(u^+) + T(u^-) \geq T(u^+)$. Since u is a minimizer, we must have $T(u^+) = T(u)$ and $T(u^-) = 0$, hence $u^- = 0$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, that is $u \geq 0$ a.e. Take $C > 0$ such that $\text{supp}(F) \subset [0, C]$ and denote $u_0 = \min(u, C)$, $u_C = \max(u - C, 0)$. It is obvious that $u_0, u_C \in \mathcal{X}$, $u = u_0 + u_C$, $V(u_0) = V(u) = -1$ and $T(u) = T(u_0) + T(u_C)$. As above we infer that $T(u_C) = 0$, consequently $u_C = 0$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and $u \leq C$ a.e.

Since T and V are C^1 functionals on $\mathcal{X}$, it is easy to see that u satisfies an Euler-Lagrange equation $T'(u) + 2\beta V'(u) = 0$ in $\mathcal{X}'$ for some $\beta \in \mathbf{R}$ and this implies

$$-\Delta u + \beta f(u) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{19}$$

Since $u \in L^\infty(\mathbf{R}^N)$ and f is continuous, by standard elliptic estimates it follows that $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in (1, \infty)$, thus $u \in C^{1,\gamma}_{loc}(\mathbf{R}^N)$ for $\gamma \in [0, 1)$. In particular, u is C^1.

It is standard to prove that u satisfies the Pohozaev identity $(N-2)T(u) + 2\beta N V(u) = 0$ (to see this, it suffices to multiply (19) by $\chi(\frac{x}{n}) \sum_{i=1}^N x_i \frac{\partial u}{\partial x_i}$, where $\chi \in C_c^\infty(\mathbf{R}^N)$ is such that $\chi \equiv 1$ on $B(0, 1)$, to integrate by parts and then to pass to the limit as $n \longrightarrow \infty$). Since $V(u) = -1$ and $T(u) = I$, we find $\beta = \frac{N-2}{2N} I = \beta_0 > 0$.

Let $v(x) = u(\frac{x}{\sqrt{\beta_0}})$. Then $v \in C^1(\mathbf{R}^N)$, $v \geq 0$ and v satisfies the equation

$$-\Delta v + f(v) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

Moreover, we have $\displaystyle\int_0^1 \frac{1}{(F(s))^{\frac{1}{2}}} \, ds = (\alpha+1)^{\frac{1}{2}} \int_0^1 \frac{1}{s^{\frac{\alpha+1}{2}}} \, ds < \infty$. Thus we may use Theorem 2 p. 773 in [16] and we infer that v has compact support. Hence u has compact support.

Step 3. Any minimizer u of $(\mathcal{M}_1)$ is radially symmetric with respect to a point. Indeed, steps 1 and 2 show that $(\mathcal{M}_1)$ satisfies assumptions **A1'** and **A2** in Introduction, hence the radial symmetry of minimizers follows from Theorem 2. Note that the unique continuation principle is not valid for minimizers of $(\mathcal{M}_1)$, therefore the method in [11] cannot be used to prove their radial symmetry.

Step 4. Construction of nonradial minimizers for a minimization problem involving two constraints.

We introduce the functional $W(u) = \int_{\mathbf{R}^N} F(-u(x)) \, dx$. Clearly, W is well-defined and C^1 on $\mathcal{X}$. We consider the minimization problem:

$(\mathcal{M}_2)$ minimize $T(u)$ in $\mathcal{X}$ subject to the constraints $V(u) = -1$ and $W(u) = -1$.

We claim that $u \in \mathcal{X}$ is a solution of $(\mathcal{M}_2)$ if and only if u^+ and u^- are solutions of $(\mathcal{M}_1)$.

To see this, let u_* be a minimizer of $(\mathcal{M}_1)$, radially symmetric with respect to the origin. Let $R > 0$ be such that $\text{supp}(u_*) \in B(0, R)$. For $y \in \mathbf{R}^N \setminus B(0, 2R)$, we put $u_y(x) = u_*(x) - u_*(x + y)$. It is obvious that $V(u_y) = V(u_*) = -1$, $W(u_y) = V(u_*(\cdot + y)) = -1$ and $T(u_y) = T(u_*) + T(u_*(\cdot + y)) = 2I$.

For any $u \in \mathcal{X}$ satisfying $V(u) = W(u) = -1$ we have $V(u^+) = V(u) = -1$ and $V(u^-) = W(u) = -1$, hence $T(u^+) \geq I$ and $T(u^-) \geq I$, consequently $T(u) \geq 2I$. We conclude that for any $|y| \geq 2R$, u_y is a minimizer of $(\mathcal{M}_2)$. Moreover, a function $u \in \mathcal{X}$ can solve $(\mathcal{M}_2)$ if and only if $V(u^+) = V(u^-) = -1$ and $T(u^+) = T(u^-) = I$, i.e. if and only if u^+ and u^- solve $(\mathcal{M}_1)$.

As in step 2 we infer that all minimizers of $(\mathcal{M}_2)$ are C^1. Thus $(\mathcal{M}_2)$ satisfies the assumptions **A1'** and **A2** and Theorem 2 implies that all minimizers of $(\mathcal{M}_2)$ are axially symmetric. Since u_* is radial with respect to the origin, it is clear that any of the minimizers u_y is axially symmetric with respect to the line Oy, but is not radial about a point. Hence $(\mathcal{M}_2)$ admits nonradial minimizers.

In fact, with some extra work it can be proved that the suport of any minimizer of $(\mathcal{M}_1)$ is precisely a ball. If u is a minimizer of $(\mathcal{M}_2)$, $\text{supp}(u) = \text{supp}(u^+) \cup \text{supp}(u^-)$ is the union of two balls with disjoint interiors. Therefore no minimizer of $(\mathcal{M}_2)$ can be radially symmetric.

In some particular cases, however, minimizers may have more symmetry than provided by Theorems 1 and 2, as it can be seen in the following example.

Example 8. Consider the problem of minimizing $E(u) = \int_{\mathbf{R}} \frac{1}{2}|u'(x)|^2 + F(u(x))\, dx$ in $H^1(\mathbf{R})$, under an arbitrary number of constraints $\int_{\mathbf{R}} G_j(u(x))\, dx = \lambda_j$, $1 \leq j \leq k$. We assume that the functions $F, G_1, \ldots, G_j$ satisfy the assumption **H1** in Remark 4.

In this case Theorem 2 gives no information about the minimizers. However, if the problem above admits minimizers, any of them must be symmetric with respect to a point. Indeed, let u be a nonconstant minimizer. Then it satisfies an Euler-Lagrange equation

$$-u'' + F'(u) + \alpha_1 G_1'(u) + \ldots + \alpha_k G_k'(u) = 0 \qquad \text{in } \mathbf{R}. \tag{20}$$

It follows easily from (20) that $u \in C^2(\mathbf{R}, \mathbf{R})$. Since $u(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$, u achieves its maximum or its minimum at some point $a \in \mathbf{R}$ and consequently $u'(a) = 0$. Let $\tilde{u}(x) = u(2a - x)$. Then $\tilde{u}$ satisfies (20) and $\tilde{u}(a) = u(a)$, $\tilde{u}'(a) = u'(a) = 0$. Since the Cauchy problem associated to (20) has unique solution, we have $u = \tilde{u}$, i.e. u is symmetric about a. Moreover, we see that u must be symmetric with respect to any of its critical points. Since u cannot be periodic, we infer that there are no other critical points, thus u is monotonic on $(-\infty, a]$ and on $[a, \infty)$.

We have discussed in the first section an example of problem where arbitrarily many constraints were allowed and the symmetry properties of minimizers did not depend on the number of constraints (see [4]). This fact is due to the assumptions made on the nonlinear term (monotonicity in $|x|$ and cooperativity condition), that imply a strong coupling between the components of the minimizers and prevent situations like those in Examples 6 and 7 to occur.

Remark 9. Our results can be extended in an obvious way to minimization problems on cylinders. To be more specific, consider the problem $(\mathcal{P}_c)$ consisting in minimizing

$$E(u) = \int_A \int_\Omega F(|x|, y, u(x,y), |\nabla_x u(x,y)|, \nabla_y u(x,y), \ldots, \nabla_y^\ell(x,y))\, dxdy$$

under the constraints

$$Q_j(u) = \int_A \int_\Omega G_j(|x|, y, u(x,y), |\nabla_x u(x,y)|, \nabla_y u(x,y), \ldots, \nabla_y^\ell(x,y))\, dxdy, \quad j = 1, \ldots, k,$$

where $x \in \Omega \subset \mathbf{R}^{N_1}$, $y \in A \subset \mathbf{R}^{N_2}$, Ω is an open set invariant by rotations in $\mathbf{R}^{N_1}$ and A is a measurable set in $\mathbf{R}^{N_2}$. We assume that problem $(\mathcal{P}_c)$ admits minimizers in a functional space $\mathcal{X}$ and the following assumptions hold:

A1$_c$. For any $w \in \mathcal{X}$ and any hyperplane Π in $\mathbf{R}^{N_1}$ containing the origin, we have $w_{(\Pi \times \mathbf{R}^{N_2})^-}, w_{(\Pi \times \mathbf{R}^{N_2})^+} \in \mathcal{X}$.

A2$_c$. For any minimizer $u \in \mathcal{X}$ and any $y \in A$, the function $u(\cdot, y)$ is C^1 on Ω.

Note that the minimization problem may involve derivatives of any order in y and we do not need more regularity of minimizers with respect to y than provided by the fact that $u \in \mathcal{X}$.

We have the following results, the proofs being similar to those of Theorems 1 and 2.

Theorem 1'. *Assume that u is a minimizer for problem $(\mathcal{P}_c)$ in $\mathcal{X}$, assumptions $\mathbf{A1}_c$ and $\mathbf{A2}_c$ are satisfied and $0 \le k \le N-2$. There exists a $k-$dimensional vector subspace V of $\mathbf{R}^{N_1}$ such that u is radially symmetric with respect to $V \times \mathbf{R}^{N_2}$.*

Theorem 2'. *Assume that $\Omega = \mathbf{R}^{N_1}$, $1 \le k \le N-1$ and the functions F, G_j in $(\mathcal{P}_c)$ do not depend on x. Assume also that $\mathbf{A2}_c$ is satisfied and $\mathbf{A1}_c$ holds for any affine hyperplane Π in $\mathbf{R}^{N_1}$. If u is a minimizer for problem $(\mathcal{P}_c)$ in $\mathcal{X}$, there exists a $(k-1)-$dimensional affine subspace $V \subset \mathbf{R}^{N_1}$ such that u is radially symmetric with respect to $V \times \mathbf{R}^{N_2}$.*

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Symmetry of least energy solutions

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Symmetry and monotonicity of least energy solutions

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Abstract

We give a simple proof of the fact that for a large class of quasilinear elliptic equations and systems the solutions that minimize the corresponding energy in the set of all solutions are radially symmetric. We require just continuous nonlinearities and no cooperative conditions for systems. Thus, in particular, our results cannot be obtained by using the moving planes method. In the case of scalar equations, we also prove that any least energy solution has a constant sign and is monotone with respect to the radial variable. Our proofs rely on results in [15, 6] and answer questions from [3, 12].

1 Introduction

We consider the system of partial differential equations

$$-\mathrm{div}(|\nabla u_i|^{p-2}\nabla u_i) = g_i(u), \qquad i = 1, \ldots, m, \tag{1}$$

where $u = (u_1, \ldots, u_m) : \mathbf{R}^N \longrightarrow \mathbf{R}^m$, $1 < p \leq N$, $|(y_1, \ldots, y_N)|^p = \left(\sum_{j=1}^N y_j^2\right)^{\frac{p}{2}}$, $g_i(0) = 0$ and there exists $G \in C^1(\mathbf{R}^m \setminus \{0\}, \mathbf{R}) \cap C(\mathbf{R}^m, \mathbf{R})$ such that $g_i(u) = \frac{\partial G}{\partial u_i}(u)$ for $u \neq 0$.

Formally, solutions of (1) are critical points of the following energy functional

$$S(u) = \frac{1}{p}\int_{\mathbf{R}^N} \sum_{i=1}^m |\nabla u_i|^p \, dx - \int_{\mathbf{R}^N} G(u)\, dx.$$

The aim of this note is to prove, under general assumptions, that those solutions of (1) which minimize the energy S in the set of all solutions are radially symmetric (up to a translation in $\mathbf{R}^N$). In the scalar case we also study the sign and monotonicity of these solutions. We do not consider here the problem of existence of solutions (respectively of least energy solutions) for (1). We believe that our results cover all situations where the existence of a least energy solution is already known in the literature.

We begin with some definitions. Let Π be an affine hyperplane in $\mathbf{R}^N$, let Π^+ and Π^- be the two closed half-spaces determined by Π and s_Π the symmetry with respect

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to Π (i.e. $s_\Pi(x) = 2p_\Pi(x) - x$, where p_Π is the orthogonal projection onto Π). Given a function f defined on $\mathbf{R}^N$, we define

$$f_{\Pi^+}(x) = \begin{cases} f(x) & \text{if } x \in \Pi^+ \\ f(s_\Pi(x)) & \text{if } x \in \Pi^- \end{cases}, \quad f_{\Pi^-}(x) = \begin{cases} f(x) & \text{if } x \in \Pi^- \\ f(s_\Pi(x)) & \text{if } x \in \Pi^+. \end{cases} \tag{2}$$

For $\sigma > 0$, we denote $f_\sigma(x) = f(\frac{x}{\sigma})$. We say that a space $\mathcal{X}$ of functions defined on $\mathbf{R}^N$ is admissible if $\mathcal{X}$ is nonempty and

(i) $\mathcal{X} \subset L^1_{loc}(\mathbf{R}^N, \mathbf{R}^m)$ and measure$(\{x \mid \ |u(x)| > \alpha\}) < \infty$ for any $u \in \mathcal{X}$ and $\alpha > 0$;

(ii) $g_i(u) \in L^1_{loc}(\mathbf{R}^N)$ for any $u \in \mathcal{X}$ and $i = 1, \ldots, m$;

(iii) $\sum_{i=1}^m |\nabla u_i|^p$ and $G(u)$ belong to $L^1(\mathbf{R}^N)$ if $u \in \mathcal{X}$;

(iv) $u_\sigma \in \mathcal{X}$ for any $u \in \mathcal{X}$ and $\sigma > 0$;

(v) $u_{\Pi^+},\ u_{\Pi^-} \in \mathcal{X}$ whenever $u \in \mathcal{X}$ and Π is an affine hyperplane in $\mathbf{R}^N$.

Let $\mathcal{X}$ be an admissible function space. We note that from (i) and (iii), $G(0) = 0$. A function $u \in \mathcal{X}$ is a solution of (1) if it satisfies (1) in $\mathcal{D}'(\mathbf{R}^N)$. If (1) admits solutions in $\mathcal{X}$, we say that $\underline{u}$ is a *least energy solution* if $\underline{u}$ is a nontrivial solution of (1) and

$$S(\underline{u}) = \inf\{S(u) \mid u \in \mathcal{X} \setminus \{0\},\ u \text{ is a solution of (1)}\}.$$

We introduce the functionals

$$J(u) = \frac{1}{p} \int_{\mathbf{R}^N} \sum_{i=1}^m |\nabla u_i|^p \, dx \qquad \text{and} \qquad V(u) = \int_{\mathbf{R}^N} G(u)\, dx.$$

Clearly, these functionals are well-defined on any admissible function space. As we will see, the least energy solutions of (1) come from the following minimization problem:

$$\text{minimize } \ J(u) \ \text{ in the set } \ \{\ u \in \mathcal{X} \mid V(u) = \lambda\}. \tag{$\mathcal{P}_\lambda$}$$

We shall prove that under some general conditions (see **(C1)-(C3)** or **(D1)-(D3)** below), all least energy solutions of (1) in the set $\mathcal{X}$ are radially symmetric, up to a translation in $\mathbf{R}^N$.

It is easy to see that $J(u_\sigma) = \sigma^{N-p} J(u)$ and $V(u_\sigma) = \sigma^N V(u)$. If $V(u) > 0$ for some $u \in \mathcal{X}$, we have $V(u_\sigma) = 1$ for $\sigma = V(u)^{-\frac{1}{N}}$. Then, denoting

$$T = \inf\{J(u) \mid u \in \mathcal{X} \text{ and } V(u) = 1\},$$

we see that

$$J(v) \geq T\,(V(v))^{\frac{N-p}{N}} \qquad \text{for any } v \in \mathcal{X} \text{ satisfying } V(v) > 0. \tag{3}$$

It is clear that u is a minimizer for problem $(\mathcal{P}_\lambda)$ above $(\lambda > 0)$ if and only if u_{σ_1} is a minimizer for $(\mathcal{P}_1)$, where $\sigma_1 = \lambda^{-\frac{1}{N}}$.

We assume first that $1 < p < N$ and the following conditions are satisfied.

(C1) $T > 0$ and problem $(\mathcal{P}_1)$ has a minimizer $u_* \in \mathcal{X}$;

(C2) Any minimizer $u \in \mathcal{X}$ of $(\mathcal{P}_1)$ is a C^1 function and satisfies the Euler-Lagrange system of equations

$$-\mathrm{div}(|\nabla u_i|^{p-2}\nabla u_i) = \alpha g_i(u) \qquad \text{in } \mathcal{D}'(\mathbf{R}^N) \tag{4}$$

for $i = 1, \ldots, m$ and some $\alpha \in \mathbf{R}$;

(C3) Any solution $u \in \mathcal{X}$ of (4) (and not only any minimizer!) satisfies the Pohozaev identity

$$(N - p)J(u) = \alpha N V(u). \tag{5}$$

A few comments are in order. Clearly, the most important of the conditions above is **(C1)**. To our knowledge, the existence of a minimizer for $(\mathcal{P}_1)$, under sufficiently general assumptions on the functions g_i and for arbitrary $m \in \mathbf{N}^*$ and $p \in (1, \infty)$, is still an open problem. However, several particular cases have been extensively studied in the literature. A series of papers has been devoted to the case $p = 2$ and fairly optimal conditions on g_i that guarantee **(C1)** have been found by Berestycki-Lions [1] for $m = 1$ and by Brezis-Lieb [3] for $m \geq 1$. In the case $m = 1$ and $1 < p < N$ the existence of a minimizer for $(\mathcal{P}_1)$ has also been proved in [9] under general assumptions on $g = g_1$ (similar to the assumptions in [1]). Under the conditions considered in [1] and [2], the functionals J and V are well defined on $H^1(\mathbf{R}^N)$ and this is clearly an admissible function space. The settings in [3] and [9] also correspond to our assumptions.

If $T > 0$ and $(\mathcal{P}_1)$ admits minimizers, in most applications it is quite standard to prove that **(C2)** and **(C3)** hold. This is indeed the case under the assumptions in [1, 3, 9].

Next we consider the case $p = N$. Note that in this case the Pohozaev identity (5) becomes $\alpha N V(u) = 0$; hence any "reasonable" solution u of (1) should satisfy $V(u) = 0$. Since we are interested in nontrivial solutions, we consider the minimization problem

$$\text{minimize } \ J(u) \ \text{ in the set } \quad \{ u \in \mathcal{X} \setminus \{0\} \mid V(u) = 0 \}. \tag{$\mathcal{P}_0'$}$$

We assume that the following conditions are satisfied.

(D1) $T_0 := \inf\{J(u) \mid u \in \mathcal{X}, u \neq 0, V(u) = 0\} > 0$ and $(\mathcal{P}_0')$ admits a minimizer u_0;

(D2) Any minimizer $u \in \mathcal{X}$ of $(\mathcal{P}_0')$ is C^1 and satisfies the Euler-Lagrange equations (4) for some $\alpha > 0$;

(D3) Any solution $u \in \mathcal{X}$ of (4) (with $\alpha > 0$) satisfies the Pohozaev identity $V(u) = 0$.

For $p = N = 2$, fairly optimal conditions on g_i that guarantee **(D1)-(D3)** have been found by Berestycki-Gallouët-Kavian [2] for $m = 1$ and by Brezis-Lieb [3] for $m \geq 1$.

In the next section we show that least energy solutions are minimizers of $(\mathcal{P}_\lambda)$ for some particular choice of λ if $1 < p < N$, respectively minimizers of $(\mathcal{P}_0')$ if $p = N$. Then we obtain the radial symmetry of such solutions as a direct consequence of the general results in [15] (in the case $N = p$, we need some extra-argument in addition to the results in [15]).

In the third section we consider the scalar case $m = 1$ and we prove that least energy solutions have constant sign and, if they tend to zero at infinity, then they are monotone with respect to the radial variable.

In the final section we make some connections with related results of symmetry and monotonicity in the literature. Let us just mention that, especially in the scalar case, the symmetry and monotonicity of solutions of (1) have been studied by many authors, see e.g. [11, 16, 8, 7] and references therein. In most of these works it is assumed that the solutions are nonnegative and some further assumptions on the nonlinearity g are made. They require, at least, g to be Lipschitz continuous and to satisfy a cooperative condition in the case of systems.

In the present work, we do not make any additional assumptions on g, except those that guarantee the existence of least energy solutions (basically, we need g to be merely continuous and to satisfy some growth conditions near zero and infinity, but we do not need any sign or monotonicity assumption; see [3] and [9]). We prove that our solutions have constant sign and our results are valid as well for compactly supported solutions and for solutions that do not vanish. Of course, there is a price we have to pay: our method works only for least energy solutions, not for any nonnegative solution of (1).

2 Variational characterization and symmetry

We begin with the case $1 < p < N$.

Lemma 1 *Assume that $1 < p < N$ and the conditions* **(C1)-(C3)** *hold.*

(i) *Let u be a minimizer for $(\mathcal{P}_1)$. Then u_{σ_0} is a least action solution of (1), where $\sigma_0 = \left(\frac{N-p}{N}T\right)^{\frac{1}{p}}$, and $S(u_{\sigma_0}) = p(N-p)^{\frac{N}{p}-1}N^{-\frac{N}{p}}T^{\frac{N}{p}}$.*

(ii) *Let v be a least energy solution for (1). Then v is a minimizer for $(\mathcal{P}_\lambda)$, where $\lambda = \left(\frac{N-p}{N}T\right)^{\frac{N}{p}}$.*

Proof. **(i)** By **(C2)** we know that $u \in C^1$ and u satisfies (4) for some $\alpha \in \mathbf{R}$. Then (5) implies $(N-p)J(u) = \alpha N V(u)$, which gives $\alpha = \frac{N-p}{N}T > 0$. It is easy to see that u_{σ_0} satisfies (1) for $\sigma_0 = \alpha^{\frac{1}{p}}$ and

$$S(u_{\sigma_0}) = \sigma_0^{N-p}J(u) - \sigma_0^N V(u) = \sigma_0^{N-p}T - \sigma_0^N = p(N-p)^{\frac{N}{p}-1}N^{-\frac{N}{p}}T^{\frac{N}{p}}.$$

Let $w \in \mathcal{X}$, $w \neq 0$, be a solution of (1). By **(C3)** we have $(N-p)J(w) = NV(w)$. If $J(w) = 0$, we have $\nabla w = 0$ a.e. on $\mathbf{R}^N$, hence w must be constant. Since measure$\{x \in \mathbf{R}^N \mid |w(x) > \alpha\} < \infty$ for any $\alpha > 0$, we infer that $w = 0$, a contradiction. Thus $J(w) > 0$ and $V(w) = \frac{N-p}{N}J(w) > 0$. On the other hand, by (3) we get $J(w) \geq T\left(V(w)\right)^{\frac{N-p}{N}}$, i.e. $J(w) \geq T\left(\frac{N-p}{N}J(w)\right)^{\frac{N-p}{N}}$, which gives

$$J(w) \geq \left(\frac{N-p}{N}\right)^{\frac{N-p}{p}} T^{\frac{N}{p}}. \tag{6}$$

Combined with Pohozaev identity, this implies

$$S(w) = J(w) - V(w) = \frac{p}{N}J(w) \geq p(N-p)^{\frac{N}{p}-1}N^{-\frac{N}{p}}T^{\frac{N}{p}} = S(u_{\sigma_0}) \tag{7}$$

and we infer that u_{σ_0} is a least energy solution for (1).

(ii) Conversely, let v be a least energy solution for (1). Then $(N-p)J(v) = NV(v)$ by **(C3)**, hence $S(v) = \frac{p}{N}J(v)$. It is obvious that the inequalities (6) and (7) above are satisfied with $w = v$. On the other hand, $S(v) = S(u_{\sigma_0})$ and we infer that v must satisfy (7) with equality sign, that is,

$$J(v) = \left(\frac{N-p}{N}\right)^{\frac{N-p}{p}} T^{\frac{N}{p}} \quad \text{and} \quad V(v) = \frac{N-p}{N}J(v) = \left(\frac{N-p}{N}\right)^{\frac{N}{p}} T^{\frac{N}{p}}.$$

A simple scaling argument shows that v is a minimizer for $(\mathcal{P}_\lambda)$, where $\lambda = \left(\frac{N-p}{N}\right)^{\frac{N}{p}} T^{\frac{N}{p}}$; equivalently, v_{σ_1} is a minimizer for $(\mathcal{P}_1)$, where $\sigma_1 = \left(\frac{N-p}{N}T\right)^{-\frac{1}{p}} = \sigma_0^{-1}$. This completes the proof of Lemma 1. □

The symmetry of least energy solutions will follow from Lemma 1 and a general symmetry result in [15]. For the convenience of the reader, we recall here that result.

Theorem 2 ([15]) *Let $N \geq 2$. Assume that $u : \mathbf{R}^N \longrightarrow \mathbf{R}^m$ belongs to some function space $\mathcal{Y}$ and solves the minimization problem*

$$\begin{aligned} &\textit{minimize } \int_{\mathbf{R}^N} F(u(x), |\nabla u(x)|)\, dx \\ &\textit{in the set } \Big\{ u \in \mathcal{Y} \;\Big|\; \int_{\mathbf{R}^N} H(u(x), |\nabla u(x)|)\, dx = \lambda \neq 0 \Big\}. \end{aligned} \tag{$\mathcal{P}$}$$

Suppose that the following conditions are satisfied:

(A1) *For any $v \in \mathcal{Y}$ and any affine hyperplane Π in $\mathbf{R}^N$ we have $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{Y}$.*

(A2) *Problem ($\mathcal{P}$) admits minimizers in $\mathcal{Y}$ and any minimizer is a C^1 function on $\mathbf{R}^N$.*

Then, after a translation, u is radially symmetric.

Lemma 1 implies that least energy solutions solve the minimization problem $(\mathcal{P}_\lambda)$ for some $\lambda > 0$. Conditions **(C1)**, **(C2)** and property **(v)** in the definition of admissible spaces imply that $(\mathcal{P}_\lambda)$ satisfies the assumptions of Theorem 2. Thus we get:

Proposition 3 *Assume that $1 < p < N$ and* **(C1)-(C3)** *hold. Then (1) admits a least energy solution and each least energy solution is radially symmetric (up to a translation in $\mathbf{R}^N$).*

Now we turn our attention to the case $p = N$.

Proposition 4 *Assume that $p = N$ and* **(D1)-(D3)** *hold. Then (1) admits a least energy solution and any least energy solution solves $(\mathcal{P}_0')$.*

Moreover, if we assume that G is either negative or positive in some ball $B_{\mathbf{R}^m}(0,\varepsilon) \setminus \{0\}$ and $u \in \mathcal{X}$ is a least energy solution such that $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, then u is radially symmetric (up to a translation in $\mathbf{R}^N$).

Proof. Let u_0 be a minimizer for $(\mathcal{P}_0')$. By **(D2)** and **(D3)** we have $V(u_0) = 0$ and u_0 satisfies (4) for some $\alpha > 0$. Let $u_1 = (u_0)_\sigma$, where $\sigma = \alpha^{\frac{1}{p}}$. It is easy to see that u_1 solves (1) and $S(u_1) = J(u_1) - V(u_1) = J(u_0) - \sigma^N V(u_0) = J(u_0) = T_0$. For any solution $u \in \mathcal{X}$, $u \neq 0$ of (1) we have $V(u) = 0$ by **(D3)** and $S(u) = J(u) \geq T_0 = J(u_1)$. Hence u_1 is a least energy solution.

If v is a least energy solution, then $V(v) = 0$ by **(D3)** and $J(v) = S(v) = S(u_1) = T_0$, thus v solves $(\mathcal{P}_0')$.

Although Theorem 2 does not apply directly to minimizers of problem $(\mathcal{P}_0')$ (because the value of the constraint in $(\mathcal{P}_0')$ is zero), its proof can still be adapted to those minimizers. Indeed, the only place where the assumption $\lambda \neq 0$ is needed in Theorem 2 is to show that for any $e \in S^{N-1}$ there exists an affine hyperplane Π orthogonal to e such that

$$\int_{\Pi^-} H(u(x), |\nabla u(x)|)\, dx = \int_{\Pi^+} H(u(x), |\nabla u(x)|)\, dx = \frac{\lambda}{2}. \tag{8}$$

From (8) it follows then easily that u_{Π^+} and u_{Π^-} are also minimizers. (In fact, if $N = 2$ the assumption $\lambda \neq 0$ was also used in the proof of Theorem 2 to show that a minimizer u of $(\mathcal{P})$ could not be of the form $u(x) = \tilde{u}(|x|)$ on $\mathbf{R}^2$, with $\tilde{u} : [0, \infty) \longrightarrow \mathbf{R}^m$ periodic and nonconstant. In our setting it is clear that no minimizer u of $(\mathcal{P}_0')$ could be of this form because $J(u)$ is finite.)

In the present case we will use the fact that $G(u)$ has a constant sign in a neighborhood of ∞ to find hyperplanes that "split the constraint in two equal parts." A similar idea has already been used in [14]. Henceforth we assume that u is a least action solution, $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ and, say, $G(\xi) < 0$ for $0 < |\xi| < \varepsilon$. For $e \in S^{N-1}$ and $t \in \mathbf{R}$, we denote $\Pi_{e,t} = \{x \in \mathbf{R}^N \mid x \cdot e = t\}$, $\Pi_{e,t}^- = \{x \in \mathbf{R}^N \mid x \cdot e < t\}$ and $\Pi_{e,t}^+ = \{x \in \mathbf{R}^N \mid x \cdot e > t\}$. We claim that for any $e \in S^{N-1}$, there exists $t_e \in \mathbf{R}$ such that

$$\int_{\Pi_{e,t_e}^-} G(u(x))\,dx = \int_{\Pi_{e,t_e}^+} G(u(x))\,dx = 0 \qquad \text{and} \qquad u_{\Pi_{e,t_e}^-} \not\equiv 0,\ u_{\Pi_{e,t_e}^+} \not\equiv 0. \quad (9)$$

To see this, fix $e \in S^{N-1}$ and define $\varphi_e^{\pm}(t) = \int_{\Pi_{e,t}^{\pm}} G(u(x))\,dx$, respectively. It follows that φ_e^+ and φ_e^- are continuous because $G(u) \in L^1(\mathbf{R}^N)$. Since u is continuous, $u \not\equiv 0$, $\lim_{|x|\to\infty} u(x) = 0$ and $G < 0$ on $B_{\mathbf{R}^m}(0,\varepsilon) \setminus \{0\}$, it is not hard to see that there exist $t^-, t^+ \in \mathbf{R}$, $t^- < t^+$ such that

$$\varphi_e^-(t^-) < 0, \quad \varphi_e^+(t^+) < 0 \quad \text{and} \quad u_{\Pi_{e,t^-}^-} \not\equiv 0, \quad u_{\Pi_{e,t^+}^+} \not\equiv 0.$$

Since $\varphi_e^+(t^-) = V(u) - \varphi_e^-(t^-) = -\varphi_e^-(t^-)$, it follows that $\varphi_e^+(t^+) < 0 < \varphi_e^+(t^-)$. From the mean value property, we see that there exists $t_e \in (t^-, t^+)$ satisfying (9). It is clear that u_{Π_{e,t_e}^-}, $u_{\Pi_{e,t_e}^+} \in \mathcal{X} \setminus \{0\}$ because $\mathcal{X}$ is admissible and (9) implies that $V(u_{\Pi_{e,t_e}^-}) = V(u_{\Pi_{e,t_e}^+}) = 0$, hence $J(u_{\Pi_{e,t_e}^-}) \geq T_0$, $J(u_{\Pi_{e,t_e}^+}) \geq T_0$. On the other hand, it is easy to see that $J(u_{\Pi_{e,t_e}^-}) + J(u_{\Pi_{e,t_e}^+}) = 2J(u) = 2T_0$. Thus $J(u_{\Pi_{e,t_e}^-}) = J(u_{\Pi_{e,t_e}^+}) = T_0$ and u_{Π_{e,t_e}^-}, u_{Π_{e,t_e}^+} are also minimizers for $(\mathcal{P}_0')$. Then arguing exactly as in the proof of Theorem 2 in [15], it follows that after a translation, u is radially symmetric. □

Remark 5 The situation is different for $p > N$. The system (1) may still have solutions in some cases, and least energy solutions may also exist. For instance, if $N = 1$ and $p = 2$ it can be proved, under suitable assumptions on g, that (1) admits a finite energy solution which is unique up to translations; hence it is a least energy solution (and it is symmetric with respect to a point).

The existence and the symmetry of least energy solutions for (1) in the case $p > N \geq 2$ would be interesting problems to consider.

Note that whenever (1) admits finite energy solutions in the case $p > N$, they cannot admit a variational characterization as in Lemma 1 or Proposition 4 above. Indeed, any reasonable solution u of (1) should satisfy the Pohozaev identity $(N - p)J(u) = NV(u)$; if u is nontrivial, then necessarily $V(u) < 0$. It turns out that in any admissible function space $\mathcal{X}$, a condition like **(C1)** cannot hold for $p > N$, no matter what the nonlinearity g is. More precisely, denote

$$T_\lambda := \inf\{J(u) \mid u \in \mathcal{X} \text{ and } V(u) = \lambda\}.$$

Let $\lambda \neq 0$. We claim that either the set $\{u \in \mathcal{X} \mid V(u) = \lambda\}$ is empty (thus $T_\lambda = -\infty$), or we have $T_\lambda = 0$. To see this we argue by contradiction and we assume that there is some

$\lambda \neq 0$ such that $T_\lambda > 0$. Let $u \in \mathcal{X}$ be such that $V(u) = \lambda$ and $J(u) < 2T_\lambda$. Choose $\varepsilon > 0$ sufficiently small, so that $\varepsilon < \frac{1}{2}$ and $2\varepsilon^{\frac{p}{N}-1} < 1$. Let $\varphi(t) = \int_{\{x_1<t\}} G(u)\,dx$. The function φ is continuous, $\lim_{t\to-\infty} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = \lambda$, hence there exist $t_1,\ t_2 \in \mathbf{R}$, $t_1 < t_2$ such that $\varphi(t_1) = \frac{\varepsilon}{2}\lambda$ and $\varphi(t_2) = (1-\frac{\varepsilon}{2})\lambda$. Let $u_1(x) = \begin{cases} u(x) & \text{if } x_1 \le t_1, \\ u(2t_1 - x_1, x') & \text{if } x_1 > t_1, \end{cases}$ $u_2(x) = \begin{cases} u(x) & \text{if } x_1 > t_2, \\ u(2t_2 - x_1, x') & \text{if } x_1 \le t_2, \end{cases}$ where $x' = (x_2, \dots, x_N)$. A simple change of variables shows that $V(u_1) = 2\int_{\{x_1<t_1\}} G(u)\,dx = \varepsilon\lambda$ and $V(u_2) = 2\int_{\{x_1>t_2\}} G(u)\,dx = \varepsilon\lambda$. Since $J(u_1) + J(u_2) = \frac{2}{p}\int_{\{x_1<t_1\}\cup\{x_1>t_2\}} \sum_{i=1}^{m} |\nabla u_i|^p\,dx \le 2J(u)$, we see that $J(u_1) \le J(u)$ or $J(u_2) \le J(u)$. Assume that $J(u_1) \le J(u) < 2T_\lambda$. For $\sigma = \varepsilon^{-\frac{1}{N}}$ we have $V((u_1)_\sigma) = \sigma^N V(u_1) = \lambda$ and $J((u_1)_\sigma) = \sigma^{N-p} J(u_1) \le \sigma^{N-p} J(u) < \varepsilon^{\frac{p}{N}-1} 2T_\lambda < T_\lambda$, contradicting the definition of T_λ. Our claim is thus proved.

3 Monotonicity results

Throughout this section we assume that $m = 1$. Given a measurable function $f : \mathbf{R}^N \longrightarrow [0,\infty)$ such that measure$(\{x \in \mathbf{R}^N \mid f(x) > \alpha\})$ is finite for any $\alpha > 0$, we denote by f^* the Schwarz rearrangement of f. We consider the following additional conditions for an admissible space $\mathcal{X}$.

(vi) For any $u \in \mathcal{X}$ and $t \ge 0$, $s \le 0$, we have $\min(u,t) \in \mathcal{X}$ and $\max(u,s) \in \mathcal{X}$.

(vii) If $u \in \mathcal{X}$ and is a radial function and $u \ge 0$ (respectively $u \le 0$), then $u^* \in \mathcal{X}$ (respectively $-(-u)^* \in \mathcal{X}$).

Note that assumption **(vii)** is needed only in the proof of Theorem 8 below.

Proposition 6 *Let $\mathcal{X}$ be an admissible function space such that for any $v \in \mathcal{X}$ the functions $v_+ = \max(v,0)$ and $v_- = \min(v,0)$ belong to $\mathcal{X}$. Assume that $1 < p < N$ and* **(C1)** *holds. If $u \in \mathcal{X}$ is a solution of $(\mathcal{P}_\lambda)$ for some $\lambda > 0$, then u does not change sign.*

Proof. This is a simple consequence of scaling. Indeed, let u be as above. It is clear that $V(u_+) + V(u_-) = V(u) = \lambda$ and $J(u_+) + J(u_-) = J(u)$. If $V(u_-) < 0$, then necessarily $V(u_+) > \lambda$. For $\sigma = \left(\frac{\lambda}{V(u_+)}\right)^{\frac{1}{N}} \in (0,1)$ we have $V((u_+)_\sigma) = \sigma^N V(u_+) = \lambda$ and $J((u_+)_\sigma) = \sigma^{N-p} J(u_+) \le \sigma^{N-p} J(u) < J(u)$, contradicting the fact that u is a minimizer. Thus necessarily $V(u_-) \ge 0$. In the same way $V(u_+) \ge 0$, therefore $V(u_-), V(u_+) \in [0,\lambda]$. Using inequality (3) (which trivially holds if $V(v) = 0$), we get

$$T\lambda^{\frac{N-p}{N}} = J(u) = J(u_+) + J(u_-) \ge TV(u_+)^{\frac{N-p}{N}} + TV(u_-)^{\frac{N-p}{N}},$$

which gives

$$1 \ge \left(\frac{V(u_+)}{\lambda}\right)^{\frac{N-p}{N}} + \left(\frac{V(u_-)}{\lambda}\right)^{\frac{N-p}{N}}. \tag{10}$$

Since $V(u_+)+V(u_-) = \lambda$, (10) implies that either $V(u_+) = 0$ or $V(u_-) = 0$. If $V(u_-) = 0$ and $V(u_+) = \lambda$ we see that u_+ satisfies the constraint and

$$J(u_+) = J(u) - J(u_-) \le J(u). \tag{11}$$

Since u is a minimizer, we must have equality in (11) and this gives $J(u_-) = 0$, hence $u_- = 0$ and $u = u_+ \geq 0$. Similarly $V(u_+) = 0$ implies $u = u_- \leq 0$. $\square$

Proposition 7 *Let an admissible space $\mathcal{X}$ satisfy the condition* **(vi)**. *Assume that $p = N$ and* **(D1)** *holds. We have:*

(a) *if $G < 0$ on $[-\varepsilon, 0) \cup (0, \varepsilon]$ for some $\varepsilon > 0$, then $u \in \mathcal{X}$ is a minimizer of $(\mathcal{P}_0')$ if and only if it solves the problem*

$$\text{minimize } J(v) \text{ in the set } \{ v \in \mathcal{X} \mid v \neq 0, V(v) \geq 0\}; \qquad (\mathcal{P}_0'')$$

(b) *if $G > 0$ on $[-\varepsilon, 0) \cup (0, \varepsilon]$, then $u \in \mathcal{X}$ solves $(\mathcal{P}_0')$ if and only if it solves the problem*

$$\text{minimize } J(v) \text{ in the set } \{ v \in \mathcal{X} \mid v \neq 0, V(v) \leq 0\}. \qquad (\mathcal{P}_0''')$$

Moreover, any minimizer of $(\mathcal{P}_0'')$ or $(\mathcal{P}_0''')$ does not change sign.

Proof. It clearly suffices to prove **(a)**.

Consider $v \in \mathcal{X}$ such that $v \geq 0$ a.e. and $V(v) > 0$. For $t \geq 0$ we define $v^t(x) = \min(v(x), t)$. By **(vi)** we have $v^t \in \mathcal{X}$. We claim that there exists $t_* > 0$ such that $V(v^{t_*}) = 0$.

The continuity of G, properties **(i)** and **(iii)** in the definition of admissible spaces and the dominated convergence theorem imply that the mapping $t \mapsto V(v^t) = \int_{\mathbf{R}^N} G(v^t(x))\,dx$ is continuous on $(0, \infty)$. Since $G(v^\varepsilon(x)) < 0$ whenever $v(x) \neq 0$ and we cannot have $v(x) = 0$ a.e. because $V(v) > 0$, we infer that $V(v^\varepsilon) < 0$.

We claim that there exists $t_0 > \varepsilon$ such that $V(v^{t_0}) > 0$. Two situations may occur:

Case 1. There exists an increasing sequence $t_n \to \infty$ such that $\{G(t_n)\}_{n=1}^\infty$ is bounded from below. Let $m = \inf_{n \geq 1} G(t_n)$. By dominated convergence we get

$$V(v^{t_n}) - V(v) = \int_{\{v \geq t_n\}} G(t_n) - G(v(x))\,dx \geq \int_{\{v \geq t_n\}} m - G(v(x))\,dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty;$$

hence $V(v^{t_n}) \geq \frac{1}{2} V(v) > 0$ for n sufficiently large.

Case 2. $G(s) \longrightarrow -\infty$ as $s \longrightarrow \infty$. Then, since $v \geq 0$ a.e. and $V(v) > 0$, we see that the set $A = \{s > 0 \mid G(s) > 0\}$ is nonempty. Let $M = \sup A < \infty$. It follows that $G(s) \leq 0$ for $s \geq M$. It is clear that $M > \varepsilon$ and $V(v^M) \geq V(v) > 0$. The claim is thus proved.

Now the continuity of the mapping $t \longmapsto V(v^t)$ implies that there exists $t_* \in (\varepsilon, t_0)$ such that $V(v^{t_*}) = 0$. Similarly, if $w \in \mathcal{X}$, $w \leq 0$ a.e. and $V(w) > 0$ there is some $\tilde{t} > 0$ such that $V(-(-w)^{\tilde{t}}) = 0$.

Next let $u_0 \in \mathcal{X}$ be a minimizer of $(\mathcal{P}_0')$. Suppose $V(u) > 0$ for some $u \in \mathcal{X}$. Then at least one of the quantities $V(u_+)$ and $V(u_-)$ is positive. If $V(u_+) > 0$, take $t_* > 0$ such that $V(u_+^{t_*}) = 0$. We have $u_+^{t_*} \in \mathcal{X} \setminus \{0\}$ and

$$J(u) \geq J(u_+) \geq J(u_+^{t_*}) \geq J(u_0) = T_0. \qquad (12)$$

Hence $\inf\{J(u) \mid u \in \mathcal{X}, u \neq 0, V(u) \geq 0\} = J(u_0) = T_0$ and u_0 is a solution of $(\mathcal{P}_0'')$.

Conversely, assume that u is a solution of $(\mathcal{P}_0'')$. We prove that

$$V(u_+) = V(u_-) = V(u) = 0. \qquad (13)$$

We argue again by contradiction. If (13) does not hold, the inequality $V(u_+)+V(u_-) = V(u) \geq 0$ implies that at least one of the quantities $V(u_+)$ and $V(u_-)$ must be positive. Suppose that $V(u_+) > 0$. As above we find $t_* > 0$ such that $V(u_+^{t_*}) = 0$ and then (12) holds for u. Moreover, since u is a minimizer of $(\mathcal{P}_0'')$ we have $J(u) \leq T_0$ and therefore all inequalities in (12) are in fact equalities. But $J(u_+) = J(u_+^{t_*})$ implies $\int_{\{u>t_*\}} |\nabla u|^p \, dx = 0$, hence $\nabla u = 0$ a.e. on $\{u > t_*\}$ which gives $\nabla((u - t_*)_+) = 0$ a.e. and we infer that $(u - t_*)_+ = 0$ a.e., that is $u \leq t_*$ a.e. Then we have $u_+ = u_+^{t_*}$ and consequently $V(u_+) = V(u_+^{t_*}) = 0$, contrary to our assumption. We argue similarly if $V(u_-) > 0$ and (13) is proved. Since $V(u) = 0$ and $J(u) = T_0 = J(u_0)$, we see that u solves $(\mathcal{P}_0')$.

Lastly we show that if u is a minimizer of $(\mathcal{P}_0'')$, then either $u_+ = 0$ a.e. or $u_- = 0$ a.e. (but we cannot have $u_+ = u_- = 0$ a.e. because $J(u) = T_0 > 0$). Indeed, if $u^+ \neq 0$ and $u^- \neq 0$, (13) would imply $J(u_+) \geq T_0$ and $J(u_-) \geq T_0$ and this would give

$$T_0 = J(u) = J(u_+) + J(u_-) \geq 2T_0 > 0,$$

which is a contradiction. This completes the proof. □

Next we prove the monotonicity of scalar minimizers.

Theorem 8 *Let $\mathcal{X}$ be an admissible space satisfying the conditions* **(vi)** *and* **(vii)**. *We assume that conditions* **(C1)-(C3)** *hold if $1 < p < N$, respectively conditions* **(D1)-(D3)** *hold if $p = N$. In the case $p = N$, we also assume that there exists $\varepsilon > 0$ such that either $G > 0$ or $G < 0$ on $[-\varepsilon, 0) \cup (0, \varepsilon]$. Then any least energy solution u of (1) such that $\lim_{|x|\to\infty} u(x) = 0$ is, up to a translation, radially symmetric and monotone with respect to $r = |x| \in [0, \infty)$.*

Proof. Symmetry follows directly from Propositions 3 and 4. Hence there is a function $\tilde{u} : [0, \infty) \longrightarrow \mathbf{R}$ such that $u(x) = \tilde{u}(|x|) = \tilde{u}(r)$. From Lemma 1 and Proposition 4 we know that any least energy solution is a minimizer of $(\mathcal{P}_\lambda)$ for some $\lambda > 0$, respectively of $(\mathcal{P}_0')$. We will show that whenever $u(x) = \tilde{u}(r)$ solves one of these minimization problems and tends to zero at infinity, $\tilde{u}$ is monotone on $[0, \infty)$. The proof relies on Lemma 9 below. The first part of this Lemma is well known and the second part is a simple consequence of Lemma 3.2 p. 163 in [6].

Lemma 9 ([6]) *Let w be a nonnegative measurable function defined on $\mathbf{R}^N$ such that for any $t > 0$ the function $(w - t)_+$ belongs to $W^{1,p}(\mathbf{R}^N)$ and has compact support. Then we have*

$$\int_{\mathbf{R}^N} |\nabla w^*|^p \, dx \leq \int_{\mathbf{R}^N} |\nabla w|^p \, dx. \tag{14}$$

Moreover, if equality holds in (14) then for any $t \in (0, \sup\text{ess}(w))$ the level set $\{x \in \mathbf{R}^N \mid w(x) > t\}$ is equivalent to a ball.

Now let u be as above. From Proposition 6 and Proposition 7, we know that u has constant sign; hence we may assume that $u \geq 0$. Since $u \in C^1$ and $\lim_{|x|\to\infty} u(x) = 0$, we see that u is bounded and $(u - t)_+$ belongs to $W^{1,p}(\mathbf{R}^N)$ and has compact support for any $t > 0$. By assumption **(vii)** we have $u^* \in \mathcal{X}$. It is clear that $V(u^*) = V(u)$, $u^* \neq 0$ if $u \neq 0$, and Lemma 9 implies that $J(u^*) \leq J(u)$. Since u is a minimizer of $(\mathcal{P}_\lambda)$ (respectively of $(\mathcal{P}_0')$), we have necessarily $J(u) \leq J(u^*)$, and hence $J(u) = J(u^*)$. Using Lemma 9 again we infer that for any $t \in (0, \sup(u))$, the set $E_t = \{x \in \mathbf{R}^N \mid u(x) > t\}$ is equivalent to a ball.

If $\tilde{u}$ is not nonincreasing, there exist $0 \leq r_1 < r_2$ such that $0 < \tilde{u}(r_1) < \tilde{u}(r_2)$. Since $\tilde{u}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, there exists $r_3 > r_2$ such that $u(r_3) = u(r_1)$. Denoting $a = u(r_1)$ and $b = u(r_2)$, we see that for any $t \in (a, b)$, E_t is nonempty and is not equivalent to a ball, which is a contradiction. This completes the proof of Theorem 8. □

4 Some remarks and examples

Remark 10 In the scalar case $m = 1$ it is well known (see for example the Introduction of [5]) that if g is odd then any least energy solution has a constant sign. In Remark II.6 of [12], Lions raised the question (for $p = 2$ and $N \geq 3$) whether this remains true without assuming g odd. Proposition 6 gives an affirmative answer for any $1 < p < N$ and Proposition 7, under some mild additional assumptions, for $p = N$. Previous partial results were obtained by Brock [5], using rearrangement arguments, assuming that $1 < p \leq 2$, the minimizer u satisfies $u(x) \longrightarrow 0$ as $|x| \to \infty$ and $g \in C^{0,p-1}(\mathbf{R})$. Nothing was proved for $p > 2$.

Remark 11 If $N \geq 3$, $p = 2$, $m = 1$ and under the assumption that g is odd, the existence of least energy solutions for (1) has been proved in [1] by showing that problem $(\mathcal{P}_1)$ admits a minimizer. The minimizer found in [1] was radial by construction, but it was not known whether all least energy solutions were radially symmetric. The existence of a minimizer for $(\mathcal{P}_1)$ without the oddness assumption on g has also been proved in [12], but nothing was known about the symmetry or the sign of such minimizers. Our results imply that any least energy solution is radially symmetric, has constant sign and is monotone with respect to the radial variable, no matter whether g is odd or not.

In the case $N \geq 2$, $p = 2$, $m \in \mathbf{N}^*$, the existence of least energy solutions is also known (see [3] for general results, historical notes, comments and further references). If $N > 2$, the existence of a minimizer for $(\mathcal{P}_\lambda)$ and the existence of least energy solutions have been proved in [3] under very general assumptions on the functions g_i. It has also been shown that the solutions are smooth (Theorem 2.3 p. 105 in [3]) and satisfy the Pohozaev identity (Lemma 2.4 p. 104 in [3]). However, as already mentioned in [3] p. 99, the existence of radially symmetric least energy solutions was not clear. Indeed, the Schwarz symmetrization that lead to a radial minimizer in [1] could not be used in [3] because of the general assumptions on the nonlinearity made there. In fact, it is known that the Schwarz rearrangements may be used for systems only if the nonlinearity satisfies a cooperative condition.

Proposition 3 above implies that all least energy solutions of the system considered in [3] are radially symmetric.

If $N = 2$ and $G(\xi) < 0$ for $0 < |\xi| \leq \varepsilon$, the existence of least energy solutions and the existence of minimizers for $(\mathcal{P}'_0)$ have been proved in [2, 3]. It has also been shown that such solutions are smooth, satisfy the Pohozaev identity and tend to 0 as $|x| \longrightarrow \infty$. Therefore Proposition 4 implies that any least energy solution is radially symmetric.

We have to mention that if $p = 2$ and if the minimizers of $(\mathcal{P}_\lambda)$ satisfy a unique continuation principle, it has already been proved in [13] that any minimizer is radially symmetric (modulo translation). In [13] no cooperative condition is required when $m \geq 2$ but using a unique continuation principle require in particular g to be C^1. Our results are still valid when a unique continuation principle fails (e.g., for minimizers with compact support). Note that compactly supported minimizers may occur in some applications (cf. Theorem 3.2 (ii) p. 111 in [3]; see also [15] for such an example). In the scalar case $m = 1$, [13] does not say anything about the sign of the minimizers.

However, in the case $p = 2$ and $N \geq 3$, the symmetry, positivity and monotonicity of minimizers for problem $(\mathcal{P}_\lambda)$ have been proved in [10] in the "zero-mass case" (that is, when $g(0) = g'(0) = 0$). The proofs in [10] rely on some sharp estimates of the decay of solutions at infinity (which are a consequence of the "zero-mass" condition) and on a result in [6]. Note that in [10] it is not assumed that g is continuous on $\mathbf{R}$, but it is assumed that $g \geq 0$ on $(0, \infty)$ and $g \leq 0$ on $(-\infty, 0)$, respectively.

Remark 12 If $1 < p < N$ and $m = 1$, it has been proved in [9], under general conditions on g, that problem $(\mathcal{P}_\lambda)$ admits minimizers (thus (1) has least energy solutions). The minimizers found in [9] were radially symmetric by construction. It follows from Proposition 3 that any least energy solution is radially symmetric.

If, in addition to the assumptions of Theorem 8, it is assumed that g is locally Lipschitz on $(0, \infty)$ and non-increasing on some interval $(0, s_0)$ and $1 < p < 2$, it has been proved in [8] that any nonnegative solution of (1) is radially symmetric and that $u(x) = \tilde{u}(|x|)$ satisfies $\tilde{u}'(r) < 0$ whenever $r > 0$ and $\tilde{u}(r) > 0$. The same result is true when $p > 2$ if it is assumed in addition that the critical set of the solution u is reduced to one point (see [16]). These assumptions are not necessary for us but, of course, we only deal with least energy solutions.

Remark 13 (i) The symmetry results in Section 2 hold without any change if we replace the functional J by a functional of the form $\int_{\mathbf{R}^N} \sum_{i=1}^{m} A_i(u, \nabla u_i) dx$ where $\xi \to A_i(u, \xi)$ is p-homogeneous for any $i = 1, ..., m$.

(ii) Our method still works for more general functionals of the form

$$\tilde{J}(u) = \frac{1}{p} \int_{\mathbf{R}^N} |x|^\alpha \sum_{i=1}^{m} A_i(u) |\nabla u_i|^p \, dx \qquad \text{and} \qquad \tilde{V}(u) = \int_{\mathbf{R}^N} |x|^\beta G(u) \, dx.$$

In this case, using Theorem 1 in [15], we obtain that minimizers (and the corresponding minimum action solutions) are axially symmetric.

Functionals of this type appear, e.g., in the Caffarelli-Kohn-Nirenberg problem (which consists in minimizing $\int_{\mathbf{R}^N} |\nabla u|^q |x|^{-aq} \, dx$ under the constraint $\int_{\mathbf{R}^N} |u|^p |x|^{-bp} \, dx = const.$, where $q > 1$, $p > 1$, $a \leq b < \frac{N}{q}$ and $0 < \frac{1}{q} - \frac{1}{p} = \frac{1+a-b}{N}$). It has been proved that minimizers for this problem exist and, in general, are not radially symmetric (see [4] and references therein).

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Symmetry of minimizers for some nonlocal variational problems

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Abstract

We present a new approach to study the symmetry of minimizers for a large class of nonlocal variational problems. This approach which generalizes the Reflection method is based on the obtention of some integral identities. We study the identities that lead to symmetry results, the functionals that can be considered and the function spaces that can be used. Then we use our method to prove the symmetry of minimizers for a class of variational problems involving the fractional powers of Laplacian, for the generalized Choquard functional and for the standing waves of the Davey-Stewartson equation.

Keywords. Symmetry of minimizers, nonlocal functional, minimization under constraints, fractional powers of Laplacian, Choquard functional, Davey-Stewartson equation.

AMS subject classifications. 35J60, 35B65, 35J05, 35J50, 35Q35, 35Q51, 35Q53, 35Q55, 35R45, 35S30

1 Introduction

Many important partial differential equations arising in Physics are Euler-Lagrange equations of variational problems. Among the solutions of these equations those who correspond to a minimum of the associated functional (e.g. the "energy") subject to some constraint are of particular interest. For example in many situations the set of such solutions is orbitally stable (see [9]).

In this paper we address the general question of whether, or not, the fact that the underlying problem has some symmetries is reflected on the minimizers. Namely if a problem is invariant under the action of a group of transformations, is it true that the corresponding minimizers are also invariant under the action of this group (or, perhaps, a subgroup of it) ? As it is shown in [14], this may not be the case.

A classical approach to radial symmetry of minimizers is Schwarz symmetrization (or spherical decreasing rearrangement, see [16]). For a nonnegative function $u \in H^1(\mathbf{R}^N)$ its symmetrization u^* is a radially-decreasing function from $\mathbf{R}^N$ into $\mathbf{R}$ which has the property that $meas(\{x \in \mathbf{R}^N \mid u(x) > \lambda\} = meas(\{x \in \mathbf{R}^N \mid u^*(x) > \lambda\}$ for any $\lambda > 0$. It is well-known that u^* satisfies (among others) the following properties:

$$\int_{\mathbf{R}^N} |\nabla u^*(x)|^2\, dx \le \int_{\mathbf{R}^N} |\nabla u(x)|^2\, dx \quad \text{and} \quad \int_{\mathbf{R}^N} F(u^*(x))\, dx = \int_{\mathbf{R}^N} F(u(x))\, dx, \tag{1.1}$$

where F is, say, a smooth function from $\mathbf{R}$ into itself such that $F(u) \in L^1(\mathbf{R}^N)$ (see [16]). As a simple application of symmetrization, consider the problem of minimizing

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u(x)|^2\, dx + \int_{\mathbf{R}^N} F(u(x))\, dx$$

subject to the constraint

$$\int_{\mathbf{R}^N} G(u(x))\, dx = \lambda,$$

where F, $G \in C^1(\mathbf{R}, \mathbf{R})$ have the property that $F(u)$, $G(u) \in L^1(\mathbf{R}^N)$ whenever $u \in H^1(\mathbf{R}^N)$. If $u \in H^1(\mathbf{R}^N)$ is a nonnegative minimizer, then from (1.1) it follows that u^* also satisfies the constraint and $E(u^*) \le E(u)$; therefore, u^* is also a minimizer. To show that $u \equiv u^*$ except for translation is a more delicate question and this follows from a result in [6] and the Unique Continuation Principle.

The case of nonlocal functionals also arises in applications. For instance, the Choquard problem consists in minimizing

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u(x)|^2\, dx - \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\, dx\, dy$$

subject to

$$\int_{\mathbf{R}^3} u^2(x)\, dx = \lambda.$$

The radial symmetry of minimizers of Choquard problem has been proved in [15] by using Riesz' inequality for rearrangements :

$$\int_{\mathbf{R}^N\times\mathbf{R}^N} f(x)g(x-y)h(y)\, dx\, dy \le \int_{\mathbf{R}^N\times\mathbf{R}^N} f^*(x)g^*(x-y)h^*(y)\, dx\, dy, \tag{1.2}$$

where f, g and h are nonnegative functions. Moreover, if g is strictly symmetric-decreasing then equality holds in (1.2) if and only if $f(x) = f^*(x-y)$ and $h(x) = h^*(x-y)$ for some $y \in \mathbf{R}^N$.

In the vector case symmetrization can also be used because of the inequality

$$\int_{\mathbf{R}^N} F(u^*(x), v^*(x))\, dx \le \int_{\mathbf{R}^N} F(u(x), v(x))\, dx, \tag{1.3}$$

which holds provided that the function F is C^2 and satisfies the cooperative condition $\frac{\partial^2 F}{\partial u \partial v}(u,v) \le 0$ for $u, v \ge 0$ (see [5]). Therefore, consider the problem of minimizing

$$E(u,v) = \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u(x)|^2 + |\nabla v(x)|^2)\, dx + \int_{\mathbf{R}^N} F(u(x), v(x))\, dx$$

subject to the constraint

$$\int_{\mathbf{R}^N} (G_1(u(x)) + G_2(v(x))\, dx = \lambda,$$

where $\frac{\partial^2 F}{\partial u \partial v}(u,v) \le 0$ for $u, v \ge 0$. If (u,v) is a nonnegative minimizer, then from (1.1) and (1.3) we see that (u^*, v^*) is also a minimizer. Notice that the function defining the constraint must have a special form because we want the value of the constraint to be preserved by symmetrization.

Another tool to prove radial symmetry of minimizers is the result by Gidas, Ni and Nirenberg [11] about the radial symmetry of positive solutions of the semilinear elliptic equation

$$-\Delta u + f(u) = 0.$$

In the case of systems, an extension of that result has been proved in [7] and [25] assuming a cooperative condition for the nonlinearity. In [11] as well as in its generalizations the nonlinearities are also allowed to depend on the space variable in a radial and monotonic way.

As we can see, in the vector case, besides the need to know in advance that the components of the minimizer are positive, both methods described above require the nonlinearity to satisfy a cooperative condition and the function defining the constraint to have a special form. To avoid these two restrictions, the Reflection method has been developed in [18] and [19]. We now briefly describe this method.

Consider the problem of minimizing

$$E(u,v) = \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u(x)|^2 + |\nabla v(x)|^2)\, dx + \int_{\mathbf{R}^N} F(u(x),v(x))\, dx$$

subject to

$$\int_{\mathbf{R}^N} G(u(x),v(x))\, dx = \lambda \neq 0.$$

To show that any minimizer (u,v) is symmetric with respect to x_1 (except possibly for a translation), we first make a translation in the x_1 variable in such way that

$$\int_{\{x_1<0\}} G(u(x),v(x))\, dx = \int_{\{x_1>0\}} G(u(x),v(x))\, dx = \frac{\lambda}{2}. \tag{1.4}$$

Next, setting $x = (x_1, x')$, where $x' \in \mathbf{R}^{N-1}$, we define the functions u_1 and u_2 by

$$u_1(x) = u_1(x_1,x') = \begin{cases} u(x_1,x') \text{ if } x_1 < 0, \\ u(-x_1,x') \text{ if } x_1 \geq 0 \end{cases} \quad \text{and} \quad u_2(x) = \begin{cases} u(-x_1,x') \text{ if } x_1 < 0, \\ u(x_1,x') \text{ if } x_1 \geq 0. \end{cases}$$

In a similar way we define v_1 and v_2. According to (1.4), the pairs (u_1,v_1) and (u_2,v_2) also satisfy the constraint (i.e. they are admissible). Moreover, a change of variables shows that

$$E(u_1,v_1) + E(u_2,v_2) = 2E(u,v). \tag{1.5}$$

Thus (u_1,v_1) and (u_2,v_2) are also minimizers. This shows that there exist minimizers which are symmetric with respect to x_1. In fact, by using the Euler-Lagrange equations and the Unique Continuation Principle we can show that necessarily $(u_1,v_1) = (u,v) = (u_2,v_2)$. Clearly, this implies that any minimizer (u,v) is symmetric with respect to the first variable. Replacing the x_1-direction by any other direction in $\mathbf{R}^N$ and repeating the same argument, we can show that (u,v) is radially symmetric except for translation (details will be given later). Notice that to use this argument there is no need to know the sign of components of the minimizers.

The main point of this paper is to extend the Reflection method to a class of nonlocal functionals. To be more specific, consider the problem of minimizing

$$E(u,v) = \int_{\mathbf{R}^N} (\frac{1}{2}|(-\Delta)^{\frac{s}{2}}u|^2 + \frac{1}{2}|\nabla v|^2)\, dx + \int_{\mathbf{R}^N} F(u,v)\, dx \tag{1.6}$$

subject to the constraint

$$Q(u, v) = \int_{\mathbf{R}^N} G(u, v)\, dx = \lambda \neq 0, \tag{1.7}$$

where $0 < s < 1$. Defining

$$W(u) = \frac{1}{2} \int_{\mathbf{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\, dx$$

and (u_1, u_2) and (v_1, v_2) as above, instead of (1.5) we have

$$E(u_1, v_1) + E(u_2, v_2) - 2E(u, v) = W(u_1) + W(u_2) - 2W(u).$$

Therefore, to show that the pairs (u_1, v_1) and (u_2, v_2) are also minimizers we need to know that the following inequality holds

$$W(u_1) + W(u_2) - 2W(u) \leq 0. \tag{1.8}$$

The key to the method developed here is to show that inequality (1.8) holds true (see Theorem 2.8). Moreover, we have equality in (1.8) if and only if u is symmetric with respect to x_1. As we will see, this gives the desired radial symmetry of minimizers. More general multipliers $m(\xi)$ and more regular nonlocal functionals like the one appearing in the Choquard problem above are also considered. In this article we will use this extended Reflection method to show the symmetry of all minimizers of the following problems:

- the Hamiltonian of a coupled system between a multidimensional Korteweg-de Vries equation and a Benjamin-Ono equation (this is precisely problem (1.6)-(1.7) with $s = 1/2$). Here the minimizers correspond to solitary waves;
- the generalized Choquard problem. In this case the minimizers give rise to standing waves for the generalized Hartree equation;
- the Hamiltonian of the generalized Davey-Stewartson equation. Here again, minimizers correspond to standing waves.

The existence of minimizers for these problems can be proved by using the concentration-compactness method [17] or the alternative method presented in [20] and will not be discussed here.

Notice that the symmetrization approach, in general, does not apply to the problems above. Indeed, in the first two examples, symmetrization cannot be used to prove the existence of a radially symmetric minimizer under the general assumptions on the nonlinearities made in this paper. Furthermore, with the tools available at the present time, it is not clear how to prove the radial symmetry of *all minimizers*, even in the cases where symmetrization can be used to prove the existence of a radially symmetric minimizer. Finally, in the last example, symmetrization cannot be used because one term of the Hamiltonian of the Davey-Stewartson equation is a singular integral operator whose kernel changes sign.

This paper is organized as follows: in the next section we present some integral identities for functionals of the form $W(u) = \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\, d\xi$. These identities are first proved for functions $u \in C_c^\infty$ and are crucial for our approach to symmetry. It will also appear clearly what kind of symbols $m(\xi)$ we may consider. In section 3 we search for appropriate function spaces on which our method can be applied. It will be proved that we may work on $H^s(\mathbf{R}^N)$ or on $\dot{H}^s(\mathbf{R}^N)$ if $-\frac{1}{2} < s < \frac{3}{2}$. We will extend the integral identities obtained in section 2 to these function spaces. In section 4 we apply our results to the concrete problems presented above. We end this article with some open problems.

2 Some identities

In what follows, $x = (x_1, x_2, \ldots, x_N) = (x_1, x')$ denotes a point of $\mathbf{R}^N$, $x' = (x_2, \ldots, x_N) \in \mathbf{R}^{N-1}$, $\xi = (\xi_1, \xi_2, \ldots, \xi_N) = (\xi_1, \xi') \in \mathbf{R}^N$ with $\xi' = (\xi_2, \ldots, \xi_N) \in \mathbf{R}^{N-1}$. We denote the Fourier transform either by $\widehat{\ }$ or by $\mathcal{F}$.

The aim of this section is to prove an identity for some functionals of the type

$$W(u) = \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\, d\xi \tag{2.1}$$

which will play a very important role in proving symmetries.

Consider a function $u \in C_c^\infty(\mathbf{R}^N)$. We define the reflected functions u_1 and u_2 as follows :

$$u_1(x) = u_1(x_1, x') = \begin{cases} u(x_1, x') \text{ if } x_1 < 0, \\ u(-x_1, x') \text{ if } x_1 \geq 0 \end{cases} \quad \text{and} \quad u_2(x) = \begin{cases} u(-x_1, x') \text{ if } x_1 < 0, \\ u(x_1, x') \text{ if } x_1 \geq 0. \end{cases} \tag{2.2}$$

We also define

$$g(x) = \frac{1}{2}(u(x_1, x') + u(-x_1, x')) \quad \text{and} \quad f(x) = \frac{1}{2}(u(x_1, x') - u(-x_1, x')). \tag{2.3}$$

Clearly, $f, g \in C_c^\infty(\mathbf{R}^N)$, g is even and f is odd with respect to x_1 and $u = f + g$. Let

$$f_*(x) = \begin{cases} f(-x_1, x') = -f(x) \text{ if } x_1 < 0, \\ f(x_1, x') \text{ if } x_1 \geq 0. \end{cases} \tag{2.4}$$

Then f_* is even with respect to x_1, $u_1 = g - f_*$ and $u_2 = g + f_*$.

We want to study the quantity

$$W(u_1) + W(u_2) - 2W(u) \tag{2.5}$$

where W is given by (2.1). Later in Theorem 2.8 we specify the class of multipliers under consideration but, at this early stage, besides integrability conditions, we assume that

$$m(\xi) \text{ is real and } m(-\xi_1, \xi') = m(\xi_1, \xi'). \tag{2.6}$$

We have :

$$\begin{aligned} \widehat{g}(-\xi_1, \xi') &= \int_{\mathbf{R}^N} e^{ix_1\xi_1 - ix'.\xi'} g(x_1, x')dx = \int_{\mathbf{R}^N} e^{-iy_1\xi_1 - ix'.\xi'} g(-y_1, x')dy_1 dx' \\ &= \widehat{g}(\xi_1, \xi') \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \widehat{f}(-\xi_1, \xi') &= \int_{\mathbf{R}^N} e^{ix_1\xi_1 - ix'.\xi'} f(x_1, x')dx = \int_{\mathbf{R}^N} e^{-iy_1\xi_1 - ix'.\xi'} f(-y_1, x')dy_1 dx' \\ &= \quad -\widehat{f}(\xi_1, \xi'). \end{aligned} \tag{2.8}$$

Therefore

$$\begin{aligned} & W(u_1) + W(u_2) - 2W(u) \\ &= \int_{\mathbf{R}^N} m(\xi_1, \xi')(|\widehat{g}(\xi) - \widehat{f}_*(\xi)|^2 + |\widehat{g}(\xi) + \widehat{f}_*(\xi)|^2 - 2|\widehat{g}(\xi) + \widehat{f}(\xi)|^2)\, d\xi \\ &= \int_{\mathbf{R}^N} m(\xi_1, \xi')(2|\widehat{f}_*(\xi)|^2 - 2|\widehat{f}(\xi)|^2 - 4\mathrm{Re}(\widehat{g}(\xi)\overline{\widehat{f}(\xi)})\, d\xi \\ &= 2\int_{\mathbf{R}^N} m(\xi_1, \xi')(|\widehat{f}_*(\xi)|^2 - |\widehat{f}(\xi)|^2)\, d\xi = 2W(f_*) - 2W(f) \end{aligned} \tag{2.9}$$

because $\int_{\mathbf{R}^N} m(\xi_1,\xi')\mathrm{Re}(\widehat{g}(\xi)\overline{\widehat{f}(\xi)}d\xi = 0$ in view of (2.6), (2.7) and (2.8).
It is easy to see that

$$\begin{aligned}\widehat{f}(\xi_1,\xi') &= \int_{\mathbf{R}}\int_{\mathbf{R}^{N-1}} e^{-ix_1\xi_1-ix'.\xi'} f(x_1,x')\,dx'\,dx_1\\ &= \int_0^\infty\int_{\mathbf{R}^{N-1}} (e^{-ix_1\xi_1}-e^{ix_1\xi_1})e^{-ix'.\xi'} f(x_1,x')\,dx'\,dx_1\\ &= -2i\int_0^\infty\int_{\mathbf{R}^{N-1}} \sin(x_1\xi_1)e^{-ix'.\xi'} f(x_1,x')\,dx'\,dx_1\end{aligned}$$

and

$$\begin{aligned}\widehat{f}_*(\xi_1,\xi') &= \int_{\mathbf{R}}\int_{\mathbf{R}^{N-1}} e^{-ix_1\xi_1-ix'.\xi'} f_*(x_1,x')\,dx'\,dx_1\\ &= \int_0^\infty\int_{\mathbf{R}^{N-1}} (e^{-ix_1\xi_1}+e^{ix_1\xi_1})e^{-ix'.\xi'} f(x_1,x')\,dx'\,dx_1\\ &= 2\int_0^\infty\int_{\mathbf{R}^{N-1}} \cos(x_1\xi_1)e^{-ix'.\xi'} f(x_1,x')\,dx'\,dx_1.\end{aligned}$$

We denote by $\mathcal{F}_{N-1}$ the partial Fourier transform in the last $N-1$ variables, that is

$$\mathcal{F}_{N-1}f(x_1,\xi') = \int_{\mathbf{R}^{N-1}} e^{-ix'.\xi'} f(x_1,x')\,dx'.$$

Since $f\in C_c^\infty(\mathbf{R}^N)$ we may use Fubini's theorem to get

$$\begin{aligned}&|\widehat{f}(\xi_1,\xi')|^2 = \widehat{f}(\xi_1,\xi')\overline{\widehat{f}(\xi_1,\xi')}\\ &= 4\int_0^\infty\int_0^\infty \sin(x_1\xi_1)\sin(y_1\xi_1)(\mathcal{F}_{N-1}f)(x_1,\xi')\overline{(\mathcal{F}_{N-1}f)(y_1,\xi')}\,dx_1\,dy_1\end{aligned}$$

and similarly

$$\begin{aligned}&|\widehat{f}_*(\xi_1,\xi')|^2 = \widehat{f}_*(\xi_1,\xi')\overline{\widehat{f}_*(\xi_1,\xi')}\\ &= 4\int_0^\infty\int_0^\infty \cos(x_1\xi_1)\cos(y_1\xi_1)(\mathcal{F}_{N-1}f)(x_1,\xi')\overline{(\mathcal{F}_{N-1}f)(y_1,\xi')}\,dx_1\,dy_1.\end{aligned}$$

Consequently,
(2.10)

$$\begin{aligned}&W(f_*)-W(f) = \int_{\mathbf{R}^N} m(\xi_1,\xi')(|\widehat{f}_*(\xi_1,\xi')|^2-|\widehat{f}(\xi_1,\xi')|^2)\,d\xi\\ &= 4\int_{\mathbf{R}^N} m(\xi_1,\xi')\int_0^\infty\int_0^\infty [\cos(x_1\xi_1)\cos(y_1\xi_1)-\sin(x_1\xi_1)\sin(y_1\xi_1)]\\ &\qquad (\mathcal{F}_{N-1}f)(x_1,\xi')\overline{(\mathcal{F}_{N-1}f)(y_1,\xi')}\,dx_1\,dy_1\,d\xi\\ &= 4\int_{\mathbf{R}^N} m(\xi_1,\xi')\int_0^\infty\int_0^\infty \cos((x_1+y_1)\xi_1)(\mathcal{F}_{N-1}f)(x_1,\xi')\overline{(\mathcal{F}_{N-1}f)(y_1,\xi')}\,dx_1\,dy_1\,d\xi.\end{aligned}$$

For an arbitrary (but fixed) $\xi'\in\mathbf{R}^{N-1}$, we define $\varphi_{\xi'}(t) = (\mathcal{F}_{N-1}f)(t,\xi')$. Since $f\in C_c^\infty(\mathbf{R}^N)$, it is clear that $\varphi_{\xi'}\in C_c^\infty(\mathbf{R})$. If $\mathrm{supp}(f)\subset B_{\mathbf{R}^N}(0,R)$, then $\mathrm{supp}(\varphi_{\xi'})\subset[-R,R]$. For $z\in\mathbf{C}$, we define

(2.11) $$h_{\xi'}(z) = \int_0^\infty\int_0^\infty e^{i(x_1+y_1)z}\varphi_{\xi'}(x_1)\overline{\varphi_{\xi'}(y_1)}\,dx_1\,dy_1.$$

Since $\varphi_{\xi'}$ is bounded and has compact support, $h_{\xi'}$ is well-defined and is an holomorphic function on $\mathbf{C}$. For any $z \in \mathbf{R}$ we have

$$\overline{h_{\xi'}(z)} = \int_0^\infty \int_0^\infty e^{-i(x_1+y_1)z} \overline{\varphi_{\xi'}(x_1)} \varphi_{\xi'}(y_1)\, dx_1\, dy_1 = h_{\xi'}(-z)$$

and

$$\mathrm{Re}(h_{\xi'}(z)) = \frac{1}{2}(h_{\xi'}(z) + \overline{h_{\xi'}(z)}) = \int_0^\infty \int_0^\infty \cos((x_1+y_1)z)\varphi_{\xi'}(x_1)\overline{\varphi_{\xi'}(y_1)}\, dx_1\, dy_1.$$

From (2.6), (2.9) and (2.10) we get

$$W(u_1) + W(u_2) - 2W(u) = 2W(f_*) - 2W(f) = 8 \int_{\mathbf{R}^{N-1}} \int_{-\infty}^{\infty} m(\xi_1, \xi')h_{\xi'}(\xi_1)\, d\xi_1\, d\xi'. \tag{2.12}$$

Some properties of the function $h_{\xi'}$ are given in the next lemma. To simplify the notation, we shall write h instead of $h_{\xi'}$.

Lemma 2.1 *For any fixed ξ', the function $h = h_{\xi'}$ given by (2.11) has the following properties:*
i) h is bounded in the upper half-plane $\{z \in \mathbf{C} \mid Im(z) \geq 0\}$.
ii) There exists a constant $C > 0$ (depending on f and ξ') such that for any $z \neq 0$ with $\mathrm{Im}(z) \geq 0$ we have:

$$|h(z)| \leq \frac{C}{|z|^4} \qquad \textit{and} \tag{2.13}$$

$$|h'(z)| \leq \frac{C}{|z|^5}. \tag{2.14}$$

Proof. *i)* If $b \geq 0$ and $x \geq 0$ then $|e^{iax-bx}| \leq 1$ and we have

$$|h(a+ib)| = \left| \int_0^\infty \int_0^\infty e^{i(x_1+y_1)a-(x_1+y_1)b} \varphi_{\xi'}(x_1)\overline{\varphi_{\xi'}(y_1)}\, dx_1\, dy_1 \right|$$
$$\leq \left(\int_0^\infty |e^{iat-bt}| \cdot |\varphi_{\xi'}(t)|\, dt \right)^2 \leq \left(\int_0^\infty |\varphi_{\xi'}(t)|\, dt \right)^2.$$

ii) It is clear that

$$h(z) = \int_0^\infty e^{ix_1 z}\varphi_{\xi'}(x_1)\, dx_1 \cdot \int_0^\infty e^{iy_1 z}\overline{\varphi_{\xi'}(y_1)}\, dy_1 = \Psi_1(z)\Psi_2(z), \tag{2.15}$$

where $\Psi_1(z)$ and $\Psi_2(z)$ are defined in an obvious way. Notice that $\varphi_{\xi'}(0) = (\mathcal{F}_{N-1}f)(0,\xi') = 0$ because $f(0,x') = 0$ (recall that f is odd with respect to x_1). Moreover, for any $k \in \mathbf{N}$,

$$\frac{d^k}{dt^k}\varphi_{\xi'}(t) = \frac{d^k}{dt^k} \int_{\mathbf{R}^{N-1}} e^{-ix'.\xi'} f(t,x')\, dx'$$

$$= \int_{\mathbf{R}^{N-1}} e^{-ix'\xi'} \frac{\partial^k f}{\partial x_1^k}(t,x')\, dx' = (\mathcal{F}_{N-1}\frac{\partial^k f}{\partial x_1^k})(t,\xi')$$

is a C_c^∞ function of t, uniformly bounded for $(t,\xi') \in \mathbf{R} \times \mathbf{R}^{N-1}$. Integrating by parts, we get:

$$\Psi_1(z) = \int_0^\infty e^{itz}\varphi_{\xi'}(t)dt = \frac{1}{iz}e^{itz}\varphi_{\xi'}(t)\Big|_{t=0}^\infty - \frac{1}{iz}\int_0^\infty e^{itz}\varphi'_{\xi'}(t)\, dt$$

$$= -\frac{e^{itz}}{(iz)^2}\varphi'_{\xi'}(t)\Big|_{t=0}^\infty + \frac{1}{(iz)^2}\int_0^\infty e^{itz}\varphi''_{\xi'}(t)\, dt$$

$$= -\frac{1}{z^2}\left[\varphi'_{\xi'}(0) + \int_0^\infty e^{itz}\varphi''_{\xi'}(t)\, dt\right].$$

It is clear that a similar estimate is true for $\Psi_2(z)$; hence (2.13) holds.

In a similar way we have

$$\Psi_1'(z) = \int_0^\infty ite^{itz}\varphi_{\xi'}(t)\,dt = \frac{1}{z}e^{itz}t\varphi_{\xi'}(t)\Big|_{t=0}^\infty - \frac{1}{z}\int_0^\infty e^{itz}\frac{d}{dt}(t\varphi_{\xi'}(t))\,dt$$

$$= -\frac{1}{iz^2}e^{itz}\frac{d}{dt}(t\varphi_{\xi'}(t))\Big|_{t=0}^\infty + \frac{1}{iz^2}\int_0^\infty e^{itz}\frac{d^2}{dt^2}(t\varphi_{\xi'}'(t))\,dt$$

$$= -\frac{1}{z^3}e^{itz}\frac{d^2}{dt^2}(t\varphi_{\xi'}(t))\Big|_{t=0}^\infty + \frac{1}{z^3}\int_0^\infty e^{itz}\frac{d^3}{dt^3}(t\varphi_{\xi'}'(t))\,dt$$

$$= \frac{1}{z^3}\left[2\varphi_{\xi'}'(0) + \int_0^\infty e^{itz}\frac{d^3}{dt^3}(t\varphi_{\xi'}'(t))\,dt\right].$$

Since an analogous estimate is valid for $\Psi_2'(z)$ and $h'(z) = \Psi_1'(z)\Psi_2(z) + \Psi_1(z)\Psi_2'(z)$, inequality (2.14) holds. □

Remark 2.2 In general, $\dfrac{\partial f}{\partial x_1}(0,x')$ does not vanish identically; hence $\mathcal{F}_{N-1}f(0,\xi') \neq 0$ for some ξ', i.e. there exists ξ' such that $\varphi_{\xi'}'(0) \neq 0$. For such ξ', the functions Ψ_1 and Ψ_2 do not decay faster than $\dfrac{1}{|z|^2}$ and then the estimate (2.13) is optimal.

Remark 2.3 Note that for any $t \in \mathbf{R}$ we have

$$h(it) = \left|\int_0^\infty e^{-x_1 t}\varphi_{\xi'}(x_1)\,dx_1\right|^2 \in [0,\infty).$$

Suppose that for any fixed $\xi' \in \mathbf{R}^{N-1}$, $m(\xi_1,\xi')$ admits an holomorphic extension $z \longmapsto m(z,\xi')$ to the upper half-plane $\{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$, with possibly some singularities on the imaginary axis $\{it \mid t \in [0,\infty)\}$. If $|m(z,\xi')|$ increases more slowly than $|z|^3$ as $|z| \longrightarrow \infty$, then $\displaystyle\int_{-\infty}^\infty m(\xi_1,\xi')h(\xi_1)\,d\xi_1$ should depend only on the values of h on the singular set of $m(\cdot,\xi')$. This simple idea will enable us to prove the identities that will be crucial in symmetry problems.

In order to clarify what kind of symbols may be considered, we start with some auxiliary technical results about holomorphic functions in a half-plane and their boundary values.

Given a function $\alpha \in L^p(\mathbf{R})$, $1 \le p < \infty$, we recall that its Hilbert transform is defined by

$$(H\alpha)(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\{|y|>\varepsilon\}}\frac{\alpha(x-y)}{y}\,dy \qquad \text{or equivalently} \qquad \widehat{H\alpha}(\xi) = -i\,\mathrm{sgn}(\xi)\,\widehat{\alpha}(\xi).$$

It is well-known that H is a bounded linear mapping from $L^p(\mathbf{R})$ into $L^p(\mathbf{R})$ (see, e.g., Chapter II in [23], or inequality (2.11) p. 188 in [24]).

In the next two lemmas we collect some classical facts that will be very useful in the sequel.

Lemma 2.4 *Consider $\alpha \in L^p(\mathbf{R})$, $1 < p < \infty$, and let $\beta = H\alpha$. For $x > 0$ and $y \in \mathbf{R}$ define*

$$a(x,y) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{x}{x^2+(y-t)^2}\alpha(t)\,dt = \int_{-\infty}^\infty P(y-t,x)\alpha(t)\,dt \quad \textit{and}$$

$$b(x,y) = -\frac{1}{\pi}\int_{-\infty}^\infty \frac{y-t}{x^2+(y-t)^2}\alpha(t)\,dt = -\int_{-\infty}^\infty Q(y-t,x)\alpha(t)\,dt,$$

where $P(s,k) = \dfrac{1}{\pi}\dfrac{k}{s^2+k^2}$ *and* $Q(s,k) = \dfrac{1}{\pi}\dfrac{s}{s^2+k^2}$ *are the Poisson kernel, respectively the conjugate Poisson kernel.*

Then we have:

i) $b(x,y) = -\displaystyle\int_{-\infty}^{\infty} P(y-t,x)\beta(t)\,dt$ *for any* $x>0$ *and* $t \in \mathbf{R}$.

ii) $||a(x,\cdot)||_{L^p(\mathbf{R})} \le ||\alpha||_{L^p(\mathbf{R})}$, $||b(x,\cdot)||_{L^p(\mathbf{R})} \le ||\beta||_{L^p(\mathbf{R})}$ *and* $||a(x,\cdot)-\alpha||_{L^p(\mathbf{R})} \longrightarrow 0$, $||b(x,\cdot)+\beta||_{L^p(\mathbf{R})} \longrightarrow 0$ *as* $x \longrightarrow 0$. *Moreover,* $a(x,y) \longrightarrow \alpha(y)$ *for any* y *in the Lebesgue set of* α *(hence almost everywhere) and* $b(x,y) \longrightarrow -\beta(y)$ *for any* y *in the Lebesgue set of* β.

iii) The functions a *and* b *are harmonic in* $\{(x,y) \in \mathbf{R}^2 \mid x > 0\}$ *and* $r(z) = r(x+iy) := a(x,y) + ib(x,y)$ *is holomorphic in* $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.

iv) For any $\delta > 0$ *we have*

$$\lim_{|(x,y)|\to\infty,\, x\ge\delta} a(x,y) = 0 \qquad \textit{and} \qquad \lim_{|(x,y)|\to\infty,\, x\ge\delta} b(x,y) = 0.$$

v) Suppose in addition that α *is even and there exists* $\varepsilon > 0$ *such that* $\alpha \equiv 0$ *on* $[-\varepsilon,\varepsilon]$. *Then* a *and* b *are well-defined, bounded and harmonic in the strip* $\{(x,y) \in \mathbf{R}^2 \mid -\frac{\varepsilon}{2} < y < \frac{\varepsilon}{2}\}$, r *is well-defined and holomorphic in this strip and* $r(0) = 0$.

Proof. *i*) is exactly Lemma 1.5 p. 219 in [24] and *ii*) follows from Theorem 2.1 p. 47 in [24]. Since the Poisson kernel is a harmonic function, it is straightforward that a and b are harmonic. It is easy to check that the Cauchy-Riemann conditions $\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$ are satisfied; then r is holomorphic in $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$ and *iii*) holds.

iv) Using Lemma 2.6 p. 51 in [24] we infer that there exists a constant $A > 0$ such that

$$|a(x,y)| \le \frac{A||\alpha||_{L^p}}{x^{\frac{1}{p}}} \qquad \text{and} \qquad |b(x,y)| \le \frac{A||\alpha||_{L^p}}{x^{\frac{1}{p}}} \tag{2.16}$$

for any $x > 0$ and $y \in \mathbf{R}$.

We fix $\varepsilon > 0$. It follows from (2.16) that there exists $M > 0$ such that $|a(x,y)| < \varepsilon$ and $|b(x,y)| < \varepsilon$ for any (x,y) with $x \ge M$. Let $q \in (1,\infty)$ be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to see that $||P(\cdot,x)||_{L^1(\mathbf{R})} = 1$ and $||P(\cdot,x)||_{L^\infty(\mathbf{R})} = \frac{1}{\pi x}$; consequently, $||P(\cdot,x)||_{L^q(\mathbf{R})} \le ||P(\cdot,x)||_{L^1(\mathbf{R})}^{\frac{1}{q}} ||P(\cdot,x)||_{L^\infty(\mathbf{R})}^{\frac{1}{p}} = \pi^{-\frac{1}{p}} x^{-\frac{1}{p}}$. Also, for any $B > 0$ we have $||P(\cdot,x)||_{L^1([B,\infty))} = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan\frac{B}{x}\right)$ and $||P(\cdot,x)||_{L^\infty([B,\infty))} = \frac{1}{\pi}\frac{x}{x^2+B^2}$, hence

$$||P(\cdot,x)||_{L^q([B,\infty))} \le \left(\frac{1}{\pi}\frac{x}{x^2+B^2}\right)^{\frac{1}{p}} \left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{B}{x}\right)^{\frac{1}{q}}. \tag{2.17}$$

A similar estimate holds on $(-\infty,-B]$. For any $x \in [\delta, M]$ and any $y \ge 2B$ we have $||P(\cdot,x)||_{L^q((y-B,y+B))} \le ||P(\cdot,x)||_{L^q([B,\infty))}$ and

$$\begin{aligned}
|a(x,y)| &\le \left|\int_{-B}^{B} P(y-t,x)\alpha(t)\,dt\right| + \left|\int_{\{|t|\ge B\}} P(y-t,x)\alpha(t)\,dt\right| \\
&\le \left|\int_{y-B}^{y+B} P(s,x)\alpha(y-s)\,ds\right| + ||P(\cdot,x)||_{L^q(\mathbf{R})}\cdot||\alpha||_{L^p((-\infty,B]\cup[B,\infty))} \\
&\le ||P(\cdot,x)||_{L^q([y-B,y+B])}\cdot||\alpha||_{L^p(\mathbf{R})} + ||P(\cdot,x)||_{L^q(\mathbf{R})}\cdot||\alpha||_{L^p((-\infty,B]\cup[B,\infty))} \qquad (2.18) \\
&\le ||\alpha||_{L^p(\mathbf{R})}\left(\frac{1}{\pi}\frac{x}{x^2+B^2}\right)^{\frac{1}{p}}\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{B}{x}\right)^{\frac{1}{q}} + ||\alpha||_{L^p((-\infty,B]\cup[B,\infty))}\pi^{-\frac{1}{p}}x^{-\frac{1}{p}} \\
&\le ||\alpha||_{L^p(\mathbf{R})}\left(\frac{M}{\pi(\delta^2+B^2)}\right)^{\frac{1}{p}}\left(\frac{1}{2} - \frac{1}{\pi}\arctan\frac{B}{M}\right)^{\frac{1}{q}} + ||\alpha||_{L^p((-\infty,B]\cup[B,\infty))}\pi^{-\frac{1}{p}}\delta^{-\frac{1}{p}}.
\end{aligned}$$

We may choose $B = B(\varepsilon)$ sufficiently large so that the right-hand side term in (2.18) is less than ε. Then for any $x \in [\delta, M]$ and $y \geq 2B(\varepsilon)$ we have $|a(x,y)| < \varepsilon$. Clearly the same inequality is true if $y \leq -2B$. Therefore $|a(x,y)| < \varepsilon$ if $x \geq M$ or if $|y| \geq 2B$ and $x \in [\delta, M]$. Since ε was arbitrary, we infer that $|a(x,y)| \longrightarrow 0$ as $|(x,y)| \longrightarrow \infty$ and $x \geq \delta$. A similar proof is valid for the function b and *iv)* is proved.

v) For any $y \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and $t \in \text{supp}(a)$ we have $|t-y| \geq \frac{\varepsilon}{2}$; hence $x^2 + (y-t)^2 \geq \frac{\varepsilon^2}{4}$ and $|P(y-t,x)| = \frac{1}{\pi}|\frac{x}{x^2+(y-t)^2}| \leq \frac{1}{\pi}\frac{4}{\varepsilon^2}|x|$, therefore $|P(y-t,x)| \leq \frac{1}{\pi}\min\left(\frac{4}{\varepsilon^2}|x|, \frac{1}{2|y-t|}\right)$. Similarly $|Q(y-t,x)| = \frac{1}{\pi}|\frac{y-t}{x^2+(y-t)^2}| \leq \frac{1}{\pi}\min\left(\frac{4}{\varepsilon^2}|y-t|, \frac{1}{|y-t|}\right)$. Thus $P(y-\cdot,x)$ and $Q(y-\cdot,x)$ are uniformly bounded in $L^q(\mathbf{R})$ for $(x,y) \in [-1,1] \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. It follows that a and b are well-defined for any (x,y) with $|y| \leq \frac{\varepsilon}{2}$ and bounded near the origin. It is straightforward to check that a and b are twice continuously differentiable, $\Delta a = \Delta b = 0$ and $r(x+iy) = a(x,y)+ib(x,y)$ is holomorphic. Clearly, $a(0,y) = 0$ for any $y \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and $b(x,0) = \int_{-\infty}^{\infty} \frac{t}{t^2+x^2}\alpha(t)\,dt = 0$ for any $x \in \mathbf{R}$ because $t \longmapsto \frac{t}{t^2+x^2}$ is odd and $t \longmapsto \alpha(t)$ is even. Hence $r(0) = 0$. □

Lemma 2.5 *Let μ be a finite Borel measure on $\mathbf{R}$. For $x > 0$ and $y \in \mathbf{R}$ define*

$$a(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2}\,d\mu(t) = \int_{-\infty}^{\infty} P(y-t,x)\,d\mu(t) \quad \textit{and}$$

$$b(x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y-t}{x^2+(y-t)^2}\,d\mu(t) = -\int_{-\infty}^{\infty} Q(y-t,x)\,d\mu(t),$$

where $P(s,k)$ and $Q(s,k)$ are the Poisson kernel, respectively the conjugate Poisson kernel. Then:

i) The functions a and b are harmonic in $\{(x,y) \in \mathbf{R}^2 \mid x > 0\}$ and $r(z) = r(x+iy) := a(x,y)+ib(x,y)$ is holomorphic in the right half-plane $\{z \in \mathbf{C} \mid \text{Re}(z) > 0\}$.

ii) For any $x > 0$ and any p, $1 \leq p \leq \infty$, we have

$$||a(x,\cdot)||_{L^p(\mathbf{R})} \leq \frac{1}{\pi^{\frac{1}{q}}x^{\frac{1}{q}}}||\mu||, \tag{2.19}$$

where q is the conjugate exponent of p and $||\mu||$ is the total variation of μ. Furthermore,

$$\lim_{x\to 0}\int_{\mathbf{R}} a(x,y)\phi(y)\,dy = \int_{\mathbf{R}} \phi(y)\,d\mu(y) \tag{2.20}$$

for any function ϕ which is continuous on $\mathbf{R}$ and tends to zero at $\pm\infty$.

iii) For any $x > 0$ we have $|b(x,y)| \leq \frac{1}{2\pi x}||\mu||$.

iv) For $x > 0$ we have $b(x,\cdot) = -Ha(x,\cdot)$ and for any x_1, $x_2 > 0$,

$$a(x_1+x_2,y) = \int_{-\infty}^{\infty} P(y-t,x_1)a(x_2,t)\,d\mu(t), \tag{2.21}$$

$$b(x_1+x_2,y) = \int_{-\infty}^{\infty} P(y-t,x_1)b(x_2,t)\,d\mu(t) = -\int_{-\infty}^{\infty} Q(y-t,x_1)a(x_2,t)\,d\mu(t). \tag{2.22}$$

v) For any $p \in (1,\infty)$ there exists $A_p > 0$ such that

$$||b(x,\cdot)||_{L^p(\mathbf{R})} \leq A_p x^{-\frac{p-1}{p}}||\mu||.$$

vi) For any $\delta > 0$,

$$\lim_{|(x,y)|\to\infty,\, x\geq\delta} a(x,y) = 0 \qquad \textit{and} \qquad \lim_{|(x,y)|\to\infty,\, x\geq\delta} b(x,y) = 0.$$

vii) Suppose in addition that $\mu(S) = \mu(-S)$ and $\mu(S \cap [-\varepsilon, \varepsilon]) = 0$ for any Borel measurable set S. Then a and b are well-defined, bounded and holomorphic in the strip $\{(x, y) \in \mathbf{R}^2 \mid -\frac{\varepsilon}{2} < y < \frac{\varepsilon}{2}\}$, the function $r(x + iy) = a(x, y) + ib(x, y)$ is holomorphic in that strip and $r(0) = 0$.

Proof. *i)* If $x > 0$, the functions $t \longmapsto P(y-t, x)$ and $t \longmapsto Q(y-t, x)$ are continuous on $\mathbf{R}$ and tend to zero at $\pm\infty$; hence $a(x, y)$ and $b(x, y)$ are well-defined. Using Lebesgue's Dominated Convergence Theorem it is easy to check that a and b are twice continuously differentiable and $\Delta a = \Delta b = 0$. Moreover, a and b satisfy the Cauchy-Riemann conditions $\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$ and $\frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$, and then $r = a + ib$ is holomorphic in the right half-plane.

ii) It follows from Theorem 2.3 p. 49 in [24] that $||a(x, \cdot)||_{L^1(\mathbf{R})} \leq ||\mu||$ and that (2.20) holds. It is obvious that $||P(y - \cdot, x)||_{L^\infty(\mathbf{R})} \leq \frac{1}{\pi x}$; hence $|a(x, y)| \leq ||P(y - \cdot, x)||_{L^\infty(\mathbf{R})}||\mu|| = \frac{1}{\pi x}||\mu||$. Finally, for $1 < p < \infty$ we have $||a(x, \cdot)||_{L^p} \leq ||a(x, \cdot)||_{L^\infty}^{\frac{1}{q}} \cdot ||a(x, \cdot)||_{L^1}^{\frac{1}{p}} \leq \pi^{-\frac{1}{q}} x^{-\frac{1}{q}} ||\mu||$.

iii) It is obvious that $|Q(y - t, x)| \leq \frac{1}{2\pi x}$ and this implies
$|b(x, y)| \leq ||Q(y - \cdot, x)||_{L^\infty(\mathbf{R})}||\mu|| \leq \frac{1}{2\pi x}||\mu||$.

iv) We have just proved that a and b are harmonic in the right half-plane and bounded in each proper sub-half-plane $\{(x, y) \in \mathbf{R}^2 \mid x > \delta\}$, where $\delta > 0$. Then (2.21) and the first equality in (2.22) follow directly from Lemma 2.7 p. 51 in [24]. Fix $x_2 > 0$. We introduce the function

$$r_1(z) = r_1(x + iy) = \int_{-\infty}^{\infty} P(y - t, x)a(x_2, t)\, dt - i \int_{-\infty}^{\infty} Q(y - t, x)a(x_2, t)\, dt.$$

It is not hard to see that $a(x_2, \cdot) \in L^p(\mathbf{R})$ for any $p \in [1, \infty]$, $a(x_2, \cdot)$ is C^∞ and $Ha(x_2, \cdot)$ is continuous. It is clear that r_1 is bounded and by Lemma 2.4 *ii)* and *iii)* we infer that r_1 is holomorphic in the right half-plane, $\lim_{x \to 0} \mathrm{Re}(r_1(x, y)) = a(x_2, y)$ and $\lim_{x \to 0} \mathrm{Im}(r_1(x, y)) = -(Ha(x_2, \cdot))(y)$ for any $y \in \mathbf{R}$. Let $r_2(z) = r(x_2 + z) - r_1(z)$. It is easy to see that r_2 is well-defined, bounded and holomorphic in the right half-plane and $\lim_{x \to 0} \mathrm{Re}(r_2(x, y)) = 0$. Using Schwarz' reflection principle (see, e.g., [8] p. 75), we may extend r_2 to a holomorphic function $\tilde{r}_2$ defined in the whole complex plane so that we have $\tilde{r}_2(z) = -\overline{r_2(-\overline{z})}$ for any z with $\mathrm{Re}(z) < 0$. Since $\tilde{r}_2$ is also bounded, from Liouville's theorem it follows that $\tilde{r}_2$ is constant. From *ii)* and *iii)* we infer that $\lim_{x \to \infty} r(x) = 0$ and from Lemma 2.4, part *iv)*, we get $\lim_{x \to \infty} r_1(x) = 0$; hence $\lim_{x \to \infty} r_2(x) = 0$. Consequently $\tilde{r}_2$ is identically zero on $\mathbf{C}$, that is $r_1(z) = r(x_2 + z)$. This proves the second equality in (2.22). Moreover, we have $\mathrm{Im}(r(x_2 + iy)) = b(x_2, y)$ and $\lim_{x \to 0} \mathrm{Im}(r_1(x + iy)) = -H(a(x_2, \cdot))(y)$; we conclude that $b(x_2, \cdot) = -H(a(x_2, \cdot))$.

v) We know that there exists $C_p > 0$ such that $||H\phi||_{L^p} \leq C_p||\phi||_{L^p}$ for any $\phi \in L^p(\mathbf{R})$. Using *ii)* and *iv)* we get

$$||b(x, \cdot)||_{L^p} = ||Ha(x, \cdot)||_{L^p} \leq C_p||a(x, \cdot)||_{L^p} \leq C_p \pi^{-\frac{1}{q}} x^{-\frac{1}{q}} ||\mu||$$

for any $x > 0$, where $\frac{1}{p} + \frac{1}{q} = 1$.

vi) is a direct consequence of (2.21), (2.22) and Lemma 2.4, part *iv)*. The proof of *vii)* is very similar to the proof of part *v)* of Lemma 2.4 and we omit it. □

Remark 2.6 Under the assumptions *v)* of Lemma 2.4 (respectively *vii)* of Lemma 2.5) an easy computation gives

$$\frac{\partial a}{\partial x}(0, 0) = \frac{\partial b}{\partial y}(0, 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(t)}{t^2}\, dt, \quad \text{respectively} \quad \frac{\partial a}{\partial x}(0, 0) = \frac{\partial b}{\partial y}(0, 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2}\, d\mu(t).$$

If α is nonnegative and $\alpha \not\equiv 0$ (respectively if μ is a positive measure) we have $\frac{\partial r}{\partial z}(0) = \frac{\partial a}{\partial x}(0,0) > 0$; hence $z = 0$ is a simple zero of r.

After this preparation, we come back to the study of the integral $\int_{\mathbf{R}} m(\xi_1, \xi')h_{\xi'}(\xi_1)\, d\xi_1$ which appears in the right hand side of (2.12).

Lemma 2.7 *Suppose that for a given $\xi' \in \mathbf{R}^{N-1}$ the symbol $m(\xi_1, \xi')$ can be written as*

$$
\begin{aligned}
& m(\xi_1, \xi') = A_0(\xi') + A_1(\xi')|\xi_1| + A_2(\xi')\xi_1^2 \\
& +\frac{1}{\pi}\left[\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\, d\mu_{\xi',0}(t) + \xi_1^2 \int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\, d\mu_{\xi',1}(t) + \xi_1^4 \int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\, d\mu_{\xi',2}(t)\right] \\
& +\frac{1}{\pi}\sum_{k=0}^{4} |\xi_1|^k \int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\alpha_{\xi',k}(t)\, dt,
\end{aligned}
\tag{2.23}
$$

where :

a) $A_0(\xi'),\ \ A_1(\xi'),\ \ A_2(\xi') \in \mathbf{R}$,

b) $\mu_{\xi',i}$ are finite Borel measures on $\mathbf{R}$ such that $\mu_{\xi',i}(S) = \mu_{\xi',i}(-S)$ for any Borel measurable set $S \subset \mathbf{R}$, $i = 0,\ 1,\ 2$.

c) $\alpha_{\xi',k} \in L^{p_k}(\mathbf{R})$ for some $p_k \in (1,\infty)$ and $\alpha_{\xi',k}$ are even functions, $k = 0,\ 1,\ 2,\ 3,\ 4$.

d) There exists $\eta > 0$ such that $\alpha_{\xi',0} \equiv 0$ on $[-\eta,\eta]$ and $\mu_{\xi',0}(S) = 0$ for any Borel measurable set $S \subset [-\eta,\eta]$.

Let $\beta_{\xi',1} = H\alpha_{\xi',1}$ and $\beta_{\xi',3} = H\alpha_{\xi',3}$, where H is the Hilbert transform. If $h = h_{\xi'}$ is given by (2.11) then we have the identity:

$$
\begin{aligned}
& \frac{1}{2}\int_{-\infty}^{\infty} m(\xi_1, \xi')h(\xi_1)\, d\xi_1 = -A_1(\xi')\int_0^{\infty} t\, h(it)\, dt \\
& + \int_0^{\infty} \frac{h(it)}{t}\, d\mu_{\xi',0}(t) - \int_0^{\infty} t\, h(it)\, d\mu_{\xi',1}(t) + \int_0^{\infty} t^3 h(it)\, d\mu_{\xi',2}(t) \\
& + \int_0^{\infty}\left(\frac{\alpha_{\xi',0}(t)}{t} + \beta_{\xi',1}(t) - t\alpha_{\xi',2}(t) - t^2\beta_{\xi',3}(t) + t^3\alpha_{\xi',4}(t)\right) h(it)\, dt.
\end{aligned}
\tag{2.24}
$$

Proof. For $i = 0,\ 1,\ 2$ and $z = x + iy \in \mathbf{C}$ with $\mathrm{Re}(z) > 0$ we define

$$p_i(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{x}{x^2+(y-t)^2}\, d\mu_{\xi',i}(t) - \frac{i}{\pi}\int_{\mathbf{R}} \frac{y-t}{x^2+(y-t)^2}\, d\mu_{\xi',i}(t).$$

In view of Lemma 2.5, p_i are well-defined and holomorphic in the right half-plane $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$. Moreover, by assumption *d*) and Lemma 2.5, part *vii*), p_0 admits an holomorphic extension to the domain $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0 \text{ or } |\mathrm{Im}(z)| < \frac{\eta}{2}\}$, and $p_0(0) = 0$. Consequently, $\frac{p_0(z)}{z}$ is holomorphic in this domain and is bounded in a neighbourhood of zero.

For $k = 0,\ 1,\ 2,\ 3,\ 4$ we define

$$r_k(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{x}{x^2+(y-t)^2}\alpha_{\xi',k}(t)\, dt - \frac{i}{\pi}\int_{\mathbf{R}} \frac{y-t}{x^2+(y-t)^2}\alpha_{\xi',k}(t)\, dt.$$

It follows from Lemma 2.4 that r_k are well-defined and holomorphic in the right half-plane. Furthermore, r_0 admits an holomorphic extenion to $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0 \text{ or } |\mathrm{Im}(z)| < \frac{\eta}{2}\}$ and $r_0(0) = 0$; therefore, $\dfrac{r_0(z)}{z}$ is holomorphic in this domain and bounded near zero.

Finally, we define

$$(2.25) \qquad m_{\xi'}(z) = A_0(\xi') + A_1(\xi')z + A_2(\xi')z^2 + \frac{p_0(z)}{z} + zp_1(z) + z^3 p_2(z) + \sum_{k=0}^{4} z^{k-1} r_k(z).$$

It is obvious that $m_{\xi'}$ is well-defined and holomorphic in the right half-plane. Since $\alpha_{\xi',k}$ and $\mu_{\xi',i}$ are "even" and $t \longmapsto \frac{t}{\xi_1^2+t^2}$ is odd, for any $\xi_1 > 0$ we have $\mathrm{Im}(m_{\xi'}(\xi_1)) = 0$ and

$$m_{\xi'}(\xi_1) = \mathrm{Re}(m_{\xi'}(\xi_1)) = m(\xi_1, \xi').$$

For ε, $R > 0$, consider the closed continuous path $\gamma_{\varepsilon,R}$ composed by the following pieces :

$$\begin{array}{ll} \gamma_{1,\varepsilon,R}(t) = t, & t \in [\varepsilon, \varepsilon + R] \\ \gamma_{2,\varepsilon,R}(\theta) = \varepsilon + Re^{i\theta}, & \theta \in [0, \frac{\pi}{2}] \\ \gamma_{3,\varepsilon,R}(t) = \varepsilon + i(R - t), & t \in [0, R]. \end{array}$$

The function $z \longmapsto m_{\xi'}(z)h(z)$ being holomorphic in the right half-plane we have $\displaystyle\int_{\gamma_{\varepsilon,R}} m_{\xi'}(z)h(z)\, dz = 0$, that is

$$(2.26) \qquad \int_{\varepsilon}^{R} m(\xi_1, \xi')h(\xi_1)\, d\xi_1 + \int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z)h(z)\, dz + \int_{\gamma_{3,\varepsilon,R}} m_{\xi'}(z)h(z)\, dz = 0.$$

It follows from (2.25), Lemma 2.4 part *iv*) and Lemma 2.5 part *vi*) that $\displaystyle\lim_{|z|\to\infty,\, Re(z)\geq\varepsilon} \frac{m_{\xi'}(z)}{z^3} = 0$; hence, $\displaystyle\lim_{R\to\infty} \frac{m_{\xi'}(\varepsilon + Re^{i\theta})}{(\varepsilon + Re^{i\theta})^3} = 0$ uniformly with respect to $\theta \in [0, \frac{\pi}{2}]$. On the other hand, from Lemma 2.1 part *ii*), we have $|h(\varepsilon + Re^{i\theta})| \leq \frac{C}{|\varepsilon+Re^{i\theta}|^4}$ and then $|(\varepsilon + Re^{i\theta})^3 h(\varepsilon + Re^{i\theta}) \cdot iRe^{i\theta}| \leq \frac{CR}{|\varepsilon+Re^{i\theta}|} \leq \frac{CR}{R-\varepsilon} \leq 2C$ for any $R \geq 2\varepsilon$. We infer that $\displaystyle\lim_{R\to\infty} \int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z)h(z)\, dz = 0$.

From (2.16) and (2.19) it follows that $|m(\xi_1, \xi')| \leq C|\xi_1|^{-1+\delta_1}$ for $0 < \xi_1 < 1$ and $|m(\xi_1, \xi')| \leq C|\xi_1|^{3-\delta_2}$ for large ξ_1 and some C, δ_1, $\delta_2 > 0$. Since h is continuous and $|h(\xi_1)| \leq \frac{C}{|\xi_1|^4}$ (see(2.13)), the integral $\displaystyle\int_0^\infty m(\xi_1, \xi')h(\xi_1)\, d\xi_1$ converges absolutely.

Clearly we have

$$\int_{\gamma_{3,\varepsilon,R}} m_{\xi'}(z)h(z)\, dz = -i \int_0^R m_{\xi'}(\varepsilon + iy)h(\varepsilon + iy)\, dy.$$

Passing to the limit as $R \longrightarrow \infty$ in (2.26) we infer that $\displaystyle\int_0^\infty m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy)\, dy$ converges and

$$(2.27) \qquad \int_{\varepsilon}^{\infty} m(\xi_1, \xi')h(\xi_1)\, d\xi_1 = i \int_0^\infty m_{\xi'}(\varepsilon + iy)h(\varepsilon + iy)\, dy.$$

Since $m(\xi_1, \xi')$ is real and symmetric with respect to ξ_1 we have

$$\int_{-\infty}^{-\varepsilon} m(\xi_1, \xi')h(\xi_1)\, d\xi_1 = \int_{\varepsilon}^{\infty} m(-\xi_1, \xi')h(-\xi_1)\, d\xi_1 = \int_{\varepsilon}^{\infty} m(\xi_1, \xi')\overline{h(\xi_1)}\, d\xi_1,$$

and then, taking (2.27) into account, we get

$$\int_{-\infty}^{-\varepsilon} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 + \int_{\varepsilon}^{\infty} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 = -2\int_0^{\infty} \operatorname{Im}(m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy))\,dy; \tag{2.28}$$

hence

$$\int_{-\infty}^{\infty} m(\xi_1,\xi')h(\xi_1)\,d\xi_1 = -2\lim_{\varepsilon\to 0}\int_0^{\infty} \operatorname{Im}(m_{\xi'}(\varepsilon+iy)h(\varepsilon+iy))\,dy. \tag{2.29}$$

Since $h(iy)\in\mathbf{R}$ for $y\in[0,\infty)$, using Lemma 2.1 and the Dominated Convergence Theorem we find

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{\infty} \operatorname{Im}\left[(A_0(\xi')+A_1(\xi')(\varepsilon+iy)+A_2(\xi')(\varepsilon+iy)^2)h(\varepsilon+iy)\right]\,dy\\ &= A_1(\xi')\int_0^{\infty} y\,h(iy)\,dy.\end{aligned} \tag{2.30}$$

Let $\chi\in C_c^{\infty}(\mathbf{R},\mathbf{R}_+)$ be such that $\operatorname{supp}(\chi)\subset[-\frac{\eta}{4},\frac{\eta}{4}]$ and $\chi\equiv 1$ on $[-\frac{\eta}{8},\frac{\eta}{8}]$. Since the function $z\longmapsto\frac{p_0(z)}{z}h(z)$ is uniformly continuous on $[-1,1]\times[-\frac{\eta}{4},\frac{\eta}{4}]$ we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{\infty} \operatorname{Im}\left[\frac{p_0(\varepsilon+iy)}{\varepsilon+iy}h(\varepsilon+iy)\chi(y)\right]\,dy = \int_0^{\infty} \operatorname{Im}\left(\frac{p_0(iy)}{iy}h(iy)\chi(y)\right)\,dy\\ &= -\int_0^{\infty}\frac{\operatorname{Re}(p_0(iy))}{y}h(iy)\chi(y)\,dy = 0.\end{aligned} \tag{2.31}$$

By Lemma 2.1 we infer that there exists $C_1>0$ such that $|h(\varepsilon+iy)-h(iy)|\le\varepsilon C_1\min(1,\frac{1}{|y|^5})$ for any $y\in(0,\infty)$ and $\varepsilon\in[0,1]$. It is easy to see that $|\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy}-\frac{h(iy)}{iy}\right)(1-\chi(y))|\le C_2\varepsilon\min(\frac{1}{y^6},1)$ for any $y\in(0,\infty)$ and some $C_2>0$. Consequently there exists $C_3>0$ such that

$$\left|\left|\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy}-\frac{h(iy)}{iy}\right)(1-\chi(y))\right|\right|_{L^p(0,\infty)}\le C_3\varepsilon \qquad \text{for any } p\in[1,\infty]. \tag{2.32}$$

Using the Cauchy-Schwarz inequality, Lemma 2.5 parts *ii*) and *v*) and (2.32), we get

$$\begin{aligned}&\left|\int_0^{\infty} p_0(\varepsilon+iy)\left(\frac{h(\varepsilon+iy)}{\varepsilon+iy}-\frac{h(iy)}{iy}\right)(1-\chi(y))\,dy\right|\\ &\le\Big(||\operatorname{Re}(p_0(\varepsilon+i\cdot))||_{L^2(\mathbf{R})}+||\operatorname{Im}(p_0(\varepsilon+i\cdot))||_{L^2(\mathbf{R})}\Big)\\ &\qquad\qquad\left|\left|\left(\tfrac{h(\varepsilon+iy)}{\varepsilon+iy}-\tfrac{h(iy)}{iy}\right)(1-\chi(y))\right|\right|_{L^2(0,\infty)}\\ &\le C_4\varepsilon^{\frac{1}{2}}\longrightarrow 0 \text{ as } \varepsilon\longrightarrow 0.\end{aligned} \tag{2.33}$$

We also have by (2.20) and assumption *d*),

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{\infty} \operatorname{Im}\left[p_0(\varepsilon+iy)\frac{h(iy)}{iy}(1-\chi(y))\right]\,dy\\ &= -\lim_{\varepsilon\to 0}\int_0^{\infty}\operatorname{Re}(p_0(\varepsilon+iy))\frac{h(iy)}{y}(1-\chi(y))\,dy\\ &= -\int_0^{\infty}\frac{h(iy)}{y}(1-\chi(y))\,d\mu_{\xi',0}(y) = -\int_0^{\infty}\frac{h(iy)}{y}d\mu_{\xi',0}(y).\end{aligned} \tag{2.34}$$

From (2.31), (2.33) and (2.34) we get

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\left[\frac{p_0(\varepsilon+iy)}{\varepsilon+iy}h(\varepsilon+iy)\right]\,dy = -\int_0^\infty \frac{h(iy)}{y}d\mu_{\xi',0}(y). \tag{2.35}$$

This proof can be slightly modified to show that

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\left[\frac{r_0(\varepsilon+iy)}{\varepsilon+iy}h(\varepsilon+iy)\right]\,dy = -\int_0^\infty \frac{h(iy)}{y}\alpha_{\xi',0}(y)\,dy. \tag{2.36}$$

(All we have to do is to use Hölder's inequality to obtain an analogous of (2.33) and to use Lemma 2.4 part *ii)* instead of (2.20) to get an analogous of (2.34)). Moreover, it is easy to see that $|(\varepsilon+iy)^\ell h(\varepsilon+iy)-(iy)^\ell h(iy)| \le C_5\varepsilon\min(1,\frac{1}{y^2})$ for $y\in(0,\infty)$, $\ell\in\{0,1,2,3\}$ and $\varepsilon\in[0,1]$. Therefore, there exists $C_6>0$ such that

$$||(\varepsilon+iy)^{k-1}h(\varepsilon+iy)-(iy)^{k-1}h(iy)||_{L^p(0,\infty)} \le C_6\varepsilon \tag{2.37}$$

for any $\varepsilon\in[0,1]$, $k\in\{1,\,2,\,3,\,4\}$ and $p\in[1,\infty]$. This implies that

$$\begin{aligned}&\left|\int_0^\infty \operatorname{Im}\Big((\varepsilon+iy)^{k-1}h(\varepsilon+iy)r_k(\varepsilon+iy)\Big)\,dy - \int_0^\infty \operatorname{Im}\Big((iy)^{k-1}h(iy)r_k(\varepsilon+iy)\Big)\,dy\right|\\ &\le (||\operatorname{Re}(r_k(\varepsilon+i\cdot))||_{L^{p_k}}+||\operatorname{Im}(r_k(\varepsilon+i\cdot))||_{L^{p_k}})\,||(\varepsilon+iy)^{k-1}h(\varepsilon+iy)-(iy)^{k-1}h(iy)||_{L^{q_k}(0,\infty)}\\ &\le (||\alpha_{\xi',k}||_{L^{p_k}}+||H\alpha_{\xi',k}||_{L^{p_k}})\,C_6\varepsilon \longrightarrow 0 \qquad \text{as } \varepsilon\longrightarrow 0.\end{aligned}$$

Consequently we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\Big((\varepsilon+iy)^{k-1}r_k(\varepsilon+iy)h(\varepsilon+iy)\Big)\,dy\\ &=\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\Big((iy)^{k-1}r_k(\varepsilon+iy)h(iy)\Big)\,dy,\end{aligned} \tag{2.38}$$

where the latter limit exists by Lemma 2.4 *ii)* and (2.13). Using (2.38) and Lemma 2.4 *ii)* we obtain :

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\left(r_1(\varepsilon+iy)h(\varepsilon+iy)\right)\,dy = -\int_0^\infty (H\alpha_{\xi',1})(y)h(iy)\,dy, \tag{2.39}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\left((\varepsilon+iy)r_2(\varepsilon+iy)h(\varepsilon+iy)\right)\,dy = \int_0^\infty \alpha_{\xi',2}(y)\cdot yh(iy)\,dy, \tag{2.40}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\Big((\varepsilon+iy)^2r_3(\varepsilon+iy)h(\varepsilon+iy)\Big)\,dy = \int_0^\infty (H\alpha_{\xi',3})(y)\cdot y^2h(iy)\,dy, \tag{2.41}$$

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\Big((\varepsilon+iy)^3r_4(\varepsilon+iy)h(\varepsilon+iy)\Big)\,dy = -\int_0^\infty \alpha_{\xi',4}(y)\cdot y^3h(iy)\,dy. \tag{2.42}$$

Similarly we find

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\left((\varepsilon+iy)p_1(\varepsilon+iy)h(\varepsilon+iy)\right)\,dy = \lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\left(p_1(\varepsilon+iy)(iy)h(iy)\right)\,dy\\ &=\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Re}\left(p_1(\varepsilon+iy)yh(iy)\right)\,dy = \int_0^\infty yh(iy)\,d\mu_{\xi',1}(y)\end{aligned} \tag{2.43}$$

and

$$\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\Big((\varepsilon+iy)^3 p_2(\varepsilon+iy)h(\varepsilon+iy)\Big)\,dy$$

(2.44)
$$= \lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Im}\Big(p_2(\varepsilon+iy)(iy)^3 h(iy)\Big)\,dy$$

$$= -\lim_{\varepsilon\to 0}\int_0^\infty \operatorname{Re}\Big(p_2(\varepsilon+iy)y^3 h(iy)\Big)\,dy = -\int_0^\infty y^3 h(iy)\,d\mu_{\xi',2}(y).$$

Since $m_{\xi'}(z)$ is given by (2.25), replacing (2.30), (2.35), (2.36) and (2.39)-(2.44) into (2.29) we obtain the conclusion of the lemma. □

Now we are ready to state and prove the main result of this section.

Theorem 2.8 *Suppose that for any $\xi' \in \mathbf{R}^{N-1}$, $m(\xi_1,\xi')$ satisfies the assumptions of Lemma 2.7. For $u \in C_c^\infty(\mathbf{R}^N)$ define u_1, u_2, f and g as in (2.2)-(2.4) and for a given function $\varphi \in C_c^0(\mathbf{R}^N)$, let $W(\varphi) = \int_{\mathbf{R}^N} m(\xi)|\widehat{\varphi}(\xi)|^2\,d\xi$. Then we have the identity:*

(2.45)
$$\begin{aligned}
&\frac{\pi^2}{16}\left(W(u_1)+W(u_2)-2W(u)\right)\\
&= -\int_{R^{N-1}} A_1(\xi')\int_0^\infty t\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'\\
&+\int_{R^{N-1}}\int_0^\infty \frac{1}{t}\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',0}(t)\,d\xi'\\
&-\int_{R^{N-1}}\int_0^\infty t\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',1}(t)\,d\xi'\\
&+\int_{R^{N-1}}\int_0^\infty t^3\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 d\mu_{\xi',2}(t)\,d\xi'\\
&+\int_{R^{N-1}}\int_0^\infty \left[\frac{\alpha_{\xi',0}(t)}{t}+\beta_{\xi',1}(t)-t\alpha_{\xi',2}(t)-t^2\beta_{\xi',3}(t)+t^3\alpha_{\xi',4}(t)\right]\\
&\qquad\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'.
\end{aligned}$$

Proof. Since $\mathcal{F}_{N-1}f \in \mathcal{S}(\mathbf{R}^N)$, the integral $\int_0^\infty e^{-x_1 t}(\mathcal{F}_{N-1}f)(x_1,\xi')\,dx_1$ is well defined for all $t>0$ and $\xi' \in \mathbf{R}^{N-1}$. Using Plancherel's theorem we get

(2.46)
$$\begin{aligned}
\int_0^\infty e^{-x_1 t}(\mathcal{F}_{N-1}f)(x_1,\xi')\,dx_1 &= \langle \mathcal{F}_{N-1}f(\cdot,\xi')\,,\ e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\rangle_{L^2(\mathbf{R})}\\
&= (2\pi)^{-1}\langle \mathcal{F}_1(\mathcal{F}_{N-1}f(\cdot,\xi')),\mathcal{F}_1\left(e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\right)\rangle_{L^2(\mathbf{R})}.
\end{aligned}$$

Moreover, we have

$$\mathcal{F}_1\left(e^{-(\cdot)t}\chi_{[0,\infty)}(\cdot)\right)(\xi_1) = \int_0^\infty e^{-ix_1\xi_1}e^{-x_1 t}dx_1 = -\frac{1}{t+i\xi_1}e^{-(t+i\xi_1)x_1}\Big|_{x_1=0}^\infty = \frac{1}{t+i\xi_1}$$

and then, using (2.46) and the oddness of $\widehat{f}$ with respect to ξ_1 we get :

$$h_{\xi'}(it) = \left| \int_0^\infty e^{-x_1 t} (\mathcal{F}_{N-1} f)(x_1, \xi')\, dx_1 \right|^2 = (2\pi)^{-2} \left| \int_{-\infty}^\infty \widehat{f}(\xi_1, \xi') \cdot \frac{1}{t - i\xi_1}\, d\xi_1 \right|^2$$

$$(2.47) \qquad = (2\pi)^{-2} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \left(\frac{1}{t - i\xi_1} - \frac{1}{t + i\xi_1} \right) d\xi_1 \right|^2$$

$$= \frac{1}{\pi^2} \left| \int_0^\infty \widehat{f}(\xi_1, \xi')\ \frac{\xi_1}{t^2 + \xi_1^2}\, d\xi_1 \right|^2.$$

Identity (2.45) is a simple consequence of (2.12), (2.24) and (2.47) and Theorem 2.8 is proved. □

Remark 2.9 It is worth to note that we can prove an identity analogous to (2.45) whenever we work with a symbol $m(\xi) = m(\xi_1, \xi')$ symmetric with respect to ξ_1 and such that for any $\xi' \in \mathbf{R}^{N-1}$, $m(\cdot, \xi')$ admits an holomorphic extension $m_{\xi'}(z)$ to the domain $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > 0\}$ having the following properties :

P1 : $\lim\limits_{z \to \xi_1,\, Im(z) > 0} m_{\xi'}(z) = m(\xi_1, \xi')$.

P2 : For any $\varepsilon > 0$, $\lim\limits_{|z| \to \infty,\, Re(z) \geq \varepsilon} \dfrac{m_{\xi'}(z)}{z^3} = 0$.

P3 : $\lim\limits_{\varepsilon \to 0} \int_0^\infty m_{\xi'}(\varepsilon + it) h_{\xi'}(\varepsilon + it)\, dt$ exists (and depends on ξ' and the values taken by $h_{\xi'}$ on the imaginary axis).

Note that assumption **P1** implies that $m(\cdot, \xi')$ admits an holomorphic extension to the whole right half-plane. Indeed, it follows from Schwarz' reflection principle ([8], p. 75) that the function

$$\tilde{m}_{\xi'} = \begin{cases} m_{\xi'}(z) & \text{if} \quad \mathrm{Im}(z) \geq 0, \\ \overline{m_{\xi'}(\overline{z})} & \text{if} \quad \mathrm{Im}(z) < 0 \end{cases}$$

is holomorphic in $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.

Assumption **P2** is needed in the proof of Lemma 2.7 to show that $\lim\limits_{R \to \infty} \int_{\gamma_{2,\varepsilon,R}} m_{\xi'}(z) h_{\xi'}(z)\, dz = 0$ (where $\gamma_{2,\varepsilon,R}(\theta) = \varepsilon + Re^{i\theta}$, $\theta \in [0, \frac{\pi}{2}]$). We recall that $|h_{\xi'}(z)|$ behaves like $\frac{1}{|z|^4}$ as $|z| \longrightarrow \infty$ (see Lemma 2.1 and Remark 2.2). This assumption could be replaced by a weaker one that guarantees at least that $\lim\limits_{n \to \infty} \int_{\gamma_{2,\varepsilon,R_n}} m_{\xi'}(z) h_{\xi'}(z)\, dz = 0$ for some sequence $R_n \longrightarrow \infty$.

In Theorem 2.8 assumption **P3** is satisfied because of the special form of $m(\cdot, \xi')$ given by (2.23).

In this context, the hypotheses of Theorem 2.8 are almost optimal. Indeed, suppose that a function $m(z)$ has the properties **P1, P2, P3** above. Let $\tilde{m}$ be the holomorphic extension of m to the right half-plane and define $q(z) = \frac{\tilde{m}(z)}{z^3}$. Clearly, q is an holomorphic function in the right half-plane and $\lim\limits_{|z| \to \infty,\, Re(z) \geq \varepsilon} q(z) = 0$ for any $\varepsilon > 0$. Thus for any $x > \varepsilon$ we have the Poisson representation formulae

$$(2.48) \qquad \begin{aligned} q(x + iy) &= \frac{1}{\pi} \int_{-\infty}^\infty \frac{x - \varepsilon}{(x - \varepsilon)^2 + (t - y)^2} \mathrm{Re}(q(\varepsilon + it))\, dt \\ &+ \frac{i}{\pi} \int_{-\infty}^\infty \frac{t - y}{(x - \varepsilon)^2 + (t - y)^2} \mathrm{Re}(q(\varepsilon + it))\, dt \end{aligned}$$

and

$$q(x+iy) = \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{t-y}{(x-\varepsilon)^2+(t-y)^2}\mathrm{Im}(q(\varepsilon+it))\,dt \tag{2.49}$$
$$+\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{x-\varepsilon}{(x-\varepsilon)^2+(t-y)^2}\mathrm{Im}(q(\varepsilon+it))\,dt.$$

Multiplying (2.48) (respectively (2.49)) by $(x+iy)^3$, we find the expression of $m(x+iy)$ in terms of $\mathrm{Re}(q(\varepsilon+it))$ (respectively in terms of $\mathrm{Im}(q(\varepsilon+it))$). If $\mathrm{Re}(q(\varepsilon+it)) \longrightarrow \alpha(t)$ as $\varepsilon \longrightarrow 0$ and if it is possible to pass to the limit as $\varepsilon \longrightarrow 0$ in (2.48) then we obtain, at least formally,

$$m_{\xi'}(\xi_1) = \xi_1^3 q(\xi_1) = \frac{\xi_1^4}{\pi}\int_{-\infty}^{\infty} \frac{\alpha(t)}{\xi_1^2+t^2}\,dt.$$

However, as it will be seen later in applications, the function q may be singular at the origin. In this case it is not possible to pass to the limit as $\varepsilon \longrightarrow 0$ in (2.48) or in (2.49) in order to express the function q (hence the function m) in terms of its "boundary values" on the imaginary axis. This is the reason why we have introduced "lower order terms" in the expression of $m_{\xi'}(z)$ in (2.23).

We give now some examples illustrating several situations that may be encountered in applications. Throughout $u \in C_c^\infty(\mathbf{R}^N)$ and we keep the notation introduced in (2.2)-(2.3).

Example 2.10 If the symbol m is of the form $m(\xi_1,\xi') = A_1(\xi')|\xi_1|$, then Theorem 2.8 gives

$$W(u_1)+W(u_2)-2W(u) = -\frac{16}{\pi^2}\int_{R^{N-1}} A_1(\xi')\int_0^\infty t\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'. \tag{2.50}$$

This kind of symbol appears in problems involving operators of the type $H_1\frac{\partial}{\partial x_1}P(\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_N})$, where H_1 is the Hilbert transform with respect to the x_1 variable and P is a pseudo-differential operator in the last $N-1$ variables.

Example 2.11 *i)* Consider the symbol $m(\xi) = \dfrac{1}{|\xi|^2}$ appearing in Choquard's problem. It can be written as

$$m(\xi_1,\xi') = \frac{1}{\xi_1^2+|\xi'|^2} = \frac{1}{\pi}\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',0}(t),$$

where $\mu_{\xi',0} = \frac{\pi}{2}(\delta_{-|\xi'|}+\delta_{|\xi'|})$ and δ_a is the Dirac measure with support $\{a\}$. From Theorem 2.8 we get the identity

$$W(u_1)+W(u_2)-2W(u) = \frac{8}{\pi}\int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|}\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'. \tag{2.51}$$

The same identity could be obtained by observing that the function $m_{\xi'}(z) = \dfrac{1}{z^2+|\xi'|^2}$ is meromorphic in $\mathbf{C}$ and has exactly one pole in the upper half-plane, namely $i|\xi'|$. Using Residue's Theorem it is not hard to see that

$$\int_{-\infty}^{\infty} m_{\xi'}(z)h_{\xi'}(z)\,dz = 2\pi i\,\mathrm{Res}(m_{\xi'}h_{\xi'},\, i|\xi'|),$$

and integrating this identity over $\mathbf{R}^{N-1}$ we get (2.51).

ii) Consider the symbol $m(\xi) = \dfrac{1}{|\xi|^2+a^2} = \dfrac{1}{\xi_1^2+|\xi'|^2+a^2}$ corresponding to the operator $(-\Delta+a^2)^{-1}$. It is obvious that

$$m(\xi_1,\xi') = \frac{1}{\pi}\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',0}(t),$$

where $\mu_{\xi',0} = \frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2+a^2}} + \delta_{\sqrt{|\xi'|^2+a^2}})$. From Theorem 2.8 we get the identity

$$(2.52)\quad W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi}\int_{R^{N-1}} \frac{1}{\sqrt{|\xi'|^2+a^2}}\left| \int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{a^2+|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'.$$

The same identity could be obtained by applying Residue's Theorem to the meromorphic function $z \longmapsto \frac{1}{z^2+|\xi'|^2+a^2}h_{\xi'}(z)$.

iii) More generally, consider a symbol of the form $m(\xi_1,\xi') = \frac{c(\xi')}{\xi_1^2+r^2(\xi')}$. It can be written as

$$m(\xi_1,\xi') = \frac{1}{\pi}\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',0}(t),$$

where $\mu_{\xi',0} = \frac{\pi}{2}c(\xi')(\delta_{-r(\xi')} + \delta_{r(\xi')})$. Using Theorem 2.8 we obtain the identity

$$(2.53)\qquad W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi}\int_{R^{N-1}} \frac{c(\xi')}{r(\xi')}\cdot\left| \int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{r^2(\xi')+\xi_1^2}\,d\xi_1\right|^2 d\xi'.$$

In particular, for the symbol $m(\xi_1,\xi') = \frac{\xi_j^{2k}}{\xi_1^2+|\xi'|^2+a^2}$, $j = 2,\dots,N$ (corresponding to the operator $(-1)^k\frac{\partial^{2k}}{\partial x_j^{2k}}(-\Delta+a^2)^{-1})$), we get

$$(2.54)\quad W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi}\int_{R^{N-1}} \frac{\xi_j^{2k}}{\sqrt{|\xi'|^2+a^2}}\left| \int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{a^2+|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'.$$

iv) The symbol $m(\xi_1,\xi') = \frac{\xi_1^2}{\xi_1^2+|\xi'|^2+a^2}$ can be expressed as

$$m(\xi_1,\xi') = \frac{\xi_1^2}{\pi}\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',1}(t),$$

where $\mu_{\xi',1} = \frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2+a^2}} + \delta_{\sqrt{|\xi'|^2+a^2}})$. From Theorem 2.8 we find the identity

$$(2.55)\quad W(u_1)+W(u_2)-2W(u) = -\frac{8}{\pi}\int_{R^{N-1}} \sqrt{|\xi'|^2+a^2}\left| \int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{a^2+|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'.$$

Notice that the right-hand side in (2.55) is negative, while in (2.54) it is positive.

v) The symbol $m(\xi_1,\xi') = \frac{\xi_1^4}{\xi_1^2+|\xi'|^2+a^2}$ (corresponding to the operator $\frac{\partial^4}{\partial x_1^4}(-\Delta+a^2)^{-1}$) can be written as

$$m(\xi_1,\xi') = \frac{\xi_1^4}{\pi}\int_{\mathbf{R}} \frac{1}{\xi_1^2+t^2}\,d\mu_{\xi',2}(t),$$

where $\mu_{\xi',2} = \frac{\pi}{2}(\delta_{-\sqrt{|\xi'|^2+a^2}} + \delta_{\sqrt{|\xi'|^2+a^2}})$. By Theorem 2.8 we have the identity

$$(2.56)\quad W(u_1) + W(u_2) - 2W(u) = \frac{8}{\pi}\int_{R^{N-1}} (|\xi'|^2+a^2)^{\frac{3}{2}}\left| \int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{a^2+|\xi'|^2+\xi_1^2}\,d\xi_1\right|^2 d\xi'.$$

Obviously all the identities in (2.53)-(2.56) could be obtained by using the Residue Theorem.

Example 2.12 Consider the symbol $m(\xi) = |\xi|^{2s}$, corresponding to the operator $(-\Delta)^s$.

It is well-known that the argument of a complex number, $\arg(z)$, can be defined analytically on $\mathbf{C} \setminus (-\infty, 0]$ in such a way that

$$\begin{array}{ll} \forall t \in (0,\infty), & \arg(t) = 0, \\ \forall t \in (-\infty, 0), & \lim\limits_{\varepsilon \downarrow 0} \arg(t + i\varepsilon) = \pi \quad \text{ and } \quad \lim\limits_{\varepsilon \uparrow 0} \arg(t + i\varepsilon) = -\pi. \end{array}$$

The complex logarithm $\log(z) = \ln|z| + i\arg(z)$ is well defined and holomorphic on $\mathbf{C}\setminus(-\infty,0]$. For $z \in \Omega_{\xi'} := \mathbf{C} \setminus \{it \mid t \in (-\infty, -|\xi'|] \cup [|\xi'|, \infty)\}$, we have $z^2 + |\xi'|^2 \notin (-\infty, 0]$; hence we may define

$$m_{\xi'}(z) = e^{s\log(z^2+|\xi'|^2)} = |z^2 + |\xi'|^2|^s e^{is\arg(z^2+|\xi'|^2)}.$$

The function $m_{\xi'}$ is holomorphic in $\Omega_{\xi'}$ and $|m_{\xi'}(z)| = |z^2 + |\xi'|^2|^s$ for any $z \in \Omega_{\xi'}$.

If $s < \frac{3}{2}$ and $\xi' \neq 0$, the function $z \longmapsto \frac{m_{\xi'}(z)}{z^3}$ is holomorphic in $\Omega_{\xi'} \setminus \{0\}$, tends to zero as $|z| \longrightarrow \infty$ and has a third order pole at the origin. It is easy to see that

$$m_{\xi'}(z) = |\xi'|^{2s}\left(1 + s\frac{z^2}{|\xi'|^2} + \sum_{k=2}^{\infty} C_s^k \frac{z^{2k}}{|\xi'|^{2k}}\right), \tag{2.57}$$

where $C_s^k = \frac{s(s-1)\dots(s-k+1)}{k!}$ and the series converges in the open ball $B_{\mathbf{C}}(0, |\xi'|)$. Consider the function $r_{\xi'}(z) = \frac{1}{z^3}(m_{\xi'}(z) - |\xi'|^{2s} - s|\xi'|^{2s-2}z^2)$. According to (2.57), $r_{\xi'}$ is a holomorphic function in $\Omega_{\xi'}$. If $s < \frac{3}{2}$, we have $r_{\xi'}(z) \longrightarrow 0$ as $|z| \longrightarrow \infty$. Consequently, the Poisson representation formula (2.48) holds for $r_{\xi'}$. Since $r_{\xi'}(\overline{z}) = \overline{r_{\xi'}(z)}$, the function $t \longmapsto \mathrm{Re}(r_{\xi'}(\varepsilon + it))$ is even and we have, in particular,

$$\begin{aligned} m_{\xi'}(\xi_1) &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \xi_1^3 r_{\xi'}(\xi_1) \\ &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \frac{\xi_1^3}{\pi}\int_{-\infty}^{\infty} \frac{\xi_1 - \varepsilon}{(\xi_1-\varepsilon)^2 + (t-y)^2}\mathrm{Re}(r_{\xi'}(\varepsilon + it))\,dt. \end{aligned} \tag{2.58}$$

It is clear from the definition of $r_{\xi'}$ that for any $t \in (-|\xi'|, |\xi'|)$ we have $\lim\limits_{\varepsilon\to 0}\mathrm{Re}(r_{\xi'}(\varepsilon + it)) = \mathrm{Re}(r_{\xi'}(it)) = 0$. For any $t > |\xi'|$ we have $\lim\limits_{\varepsilon\downarrow 0} m_{\xi'}(\varepsilon + it) = (t^2 - |\xi'|^2)^s e^{is\pi}$ and $\lim\limits_{\varepsilon\downarrow 0}\mathrm{Re}(r_{\xi'}(\varepsilon + it)) = -\sin(s\pi)\dfrac{(t^2 - |\xi'|^2)^s}{t^3}$.

On the other hand, it is not hard to check that for $-1 < s < \frac{3}{2}$, there exists $p_s \in (1,\infty)$ and $C_{s,\xi'} > 0$ such that

$$||r_{\xi'}(\varepsilon + i\cdot)||_{L^{p_s}}(\mathbf{R}) \le C_{s,\xi'} \qquad \text{for any } \varepsilon \in (0, \frac{|\xi'|}{2}). \tag{2.59}$$

Indeed, since $|r_{\xi'}(\varepsilon + i\cdot)|$ is even, it suffices to show that $||r_{\xi'}(\varepsilon + i\cdot)||_{L^{p_s}}([0,\infty))$ has a bound independent of ε. Since $|r_{\xi'}(\varepsilon + it)|$ is uniformly bounded for $\varepsilon \in [0, \frac{|\xi'|}{2}]$ and $t \in [0, \frac{|\xi'|}{2}]$, it suffices to show that $||r_{\xi'}(\varepsilon + i\cdot)||_{L^{p_s}([\frac{|\xi'|}{2},\infty))} \le C'_{s,\xi'}$.

If $s \ge 0$, we have $|m_{\xi'}(z)| = |z^2 + |\xi'|^2|^s \le C_{1,s}(|z|^{2s} + |\xi'|^{2s})$. Thus for any $\varepsilon \in (0, \frac{|\xi'|}{2})$ and $t \ge \frac{|\xi'|}{2}$ we have

$$\begin{aligned} |r_{\xi'}(\varepsilon + it)| &\le \tfrac{|m_{\xi'}(\varepsilon+it)|}{|\varepsilon+it|^3} + \tfrac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \tfrac{s|\xi'|^{2s-2}}{|\varepsilon+it|} \\ &\le \tfrac{C_{1,s}}{|\varepsilon+it|^{3-2s}} + \tfrac{C_{1,s}|\xi'|^{2s}}{|\varepsilon+it|^3} + \tfrac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \tfrac{s|\xi'|^{2s-2}}{|\varepsilon+it|} \\ &\le C_{1,s}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right)^{3-2s} + (C_{1,s}+1)|\xi'|^{2s}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right)^3 + s|\xi'|^{2s-2}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right). \end{aligned}$$

Thus it suffices to take $p_s > 1$ such that $p_s(3-2s) > 1$ to obtain the desired bound.

If $s < 0$ then for $\varepsilon \in (0, \frac{|\xi'|}{2})$ and $t \geq \frac{|\xi'|}{2}$, we have $|(\varepsilon+it)+i|\xi'|\,|^s \leq |\varepsilon+it|^s$, and $|(\varepsilon+it)-i|\xi'|\,|^s \leq |t-|\xi'|\,|^s$. Since $|m_{\xi'}(\varepsilon+it)| = |(\varepsilon+it)+i|\xi'|\,|^s\,|(\varepsilon+it)-i|\xi'|\,|^s$, we find in this case

$$|r_{\xi'}(\varepsilon+it)| \leq \frac{|m_{\xi'}(\varepsilon+it)|}{|\varepsilon+it|^3} + \frac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \frac{s|\xi'|^{2s-2}}{|\varepsilon+it|} \leq \frac{|(\varepsilon+it)-i|\xi'|\,|^s}{|\varepsilon+it|^{3-s}} + \frac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \frac{s|\xi'|^{2s-2}}{|\varepsilon+it|}$$

$$\leq \frac{|t-|\xi'|\,|^s}{|\varepsilon+it|^{3-s}} + \frac{|\xi'|^{2s}}{|\varepsilon+it|^3} + \frac{s|\xi'|^{2s-2}}{|\varepsilon+it|}$$

$$\leq |t-|\xi'|\,|^s \min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right)^{3-s} + |\xi'|^{2s}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right)^3 + s|\xi'|^{2s-2}\min\left(\tfrac{2}{|\xi'|}, \tfrac{1}{t}\right).$$

Consequently it suffices to take $p_s > 1$ such that $-sp_s < 1$ (i.e. $p_s \in (1, -\frac{1}{s})$) to obtain (2.59).

It follows from (2.59) and Theorem 2.5 p. 50 in [24] that there exists $k_{\xi'} \in L^{p_s}(\mathbf{R})$ such that $\mathrm{Re}(r_{\xi'}(x+iy)) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+(y-t)^2} k_{\xi'}(t)\,dt$. Moreover, from Theorem 2.1 p. 47 in [24] we have $\lim\limits_{\varepsilon\downarrow 0} \mathrm{Re}(r_{\xi'}(\varepsilon+it)) = k_{\xi'}(t)$ for almost every $t \in \mathbf{R}$ and $||\mathrm{Re}(r_{\xi'}(\varepsilon+i\cdot)) - k_{\xi'}||_{L^{p_s}} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. In view of the pointwise convergence, we infer that $k_{\xi'}$ is even and

$$k_{\xi'}(t) = \begin{cases} 0 & \text{if} \quad t \in (-|\xi'|, |\xi'|) \\ -\sin(s\pi)\frac{(t^2-|\xi'|^2)^s}{|t|^3} & \text{if} \quad |t| > |\xi'| \end{cases}$$

a.e. on $\mathbf{R}$. Now it is clear that the symbol $m(\xi_1, \xi')$ can be written as

$$\begin{aligned} m(\xi_1,\xi') &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \xi_1^3 r_{\xi'}(\xi_1) \\ &= |\xi'|^{2s} + s|\xi'|^{2s-2}\xi_1^2 + \frac{\xi_1^4}{\pi}\int_{-\infty}^{\infty} \frac{1}{\xi_1^2+t^2} k_{\xi'}(t)\,dt. \end{aligned} \tag{2.60}$$

Thus we may apply Theorem 2.8 to get, for any $u \in C_c^\infty(\mathbf{R}^N)$ and $s \in (-1, \frac{3}{2})$,

$$\begin{aligned} W(u_1)+W(u_2)-2W(u) &= \frac{16}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_0^\infty t^3 k_{\xi'}(t)\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi' \\ &= -\frac{16\sin(s\pi)}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_{|\xi'|}^\infty \left(t^2-|\xi'|^2\right)^s \left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'. \end{aligned} \tag{2.61}$$

Similarly, if we consider the symbol $m(\xi) = (|\xi|^2+a^2)^s$ we get the identity

$$\begin{aligned} &W(u_1)+W(u_2)-2W(u) \\ &= -\frac{16\sin(s\pi)}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_{\sqrt{|\xi'|^2+a^2}}^\infty \left(t^2-|\xi'|^2-a^2\right)^s \left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'. \end{aligned} \tag{2.62}$$

3 Symmetry and function spaces

For any $u \in C_c^\infty(\mathbf{R}^N)$ we define u_1 and u_2 as in (2.1) and we put $T_1u = u_1$, $T_2u = u_2$. Clearly, T_1 and T_2 are linear continuous mappings from $C_c^\infty(\mathbf{R}^N)$ to $C_c^0(\mathbf{R}^N)$. In this section we consider the following intimately related problems :

1°. Determine significant subspaces $\mathcal{X} \subset \mathcal{D}'(\mathbf{R}^N)$ such that T_1 and T_2 can be extended to linear continuous mappings from $\mathcal{X}$ to $\mathcal{X}$. (Or, equivalently, find the subspaces $\mathcal{X}$ such that $u \in \mathcal{X}$ implies T_1u, $T_2u \in \mathcal{X}$ and $u \longmapsto T_1u$, $u \longmapsto T_2u$ are continuous for the $\mathcal{X}$ topology).

2°. If $\mathcal{X}$ is a subspace as above, how the identities proved in the previous section can be extended to $\mathcal{X}$?

The answer to these questions is of great importance in symmetry problems. For instance, suppose that a function space $\mathcal{X}$ has the two properties described above and that the solutions of the variational problem

$$\begin{aligned} &\text{minimize } E(u) := \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\,d\xi + \int_{\mathbf{R}^N} F(u)\,dx \\ &\text{under the constraint } \int_{\mathbf{R}^N} G(u)\,dx = \lambda \end{aligned} \tag{3.1}$$

belong to $\mathcal{X}$. As before, the symbol $m(\xi) = m(\xi_1, \xi')$ is assumed to be symmetric with respect to ξ_1. Defining $W(u) := \int_{\mathbf{R}^N} m(\xi)|\widehat{u}(\xi)|^2\,d\xi$, we suppose also that that an identity of type (2.45) holds for $W(u)$ and it can be extended to $\mathcal{X}$ in such a way that

$$W(T_1u) + W(T_2u) - 2W(u) < 0 \qquad \text{whenever } T_1u \neq u,\ T_2u \neq u.$$

(We will see later that most of the symbols in Examples 2.10-2.12 have this property.) Then, we claim that after a translation in the x_1 direction, any solution of (3.1) is symmetric with respect to x_1. Indeed, let u be a minimizer. After a translation in the x_1 direction, we may assume that $\int_{\{x_1<0\}} G(u(x))\,dx = \int_{\{x_1>0\}} G(u(x))\,dx = \frac{\lambda}{2}$. This implies $\int_{\mathbf{R}^N} G(u_1(x))\,dx = 2\int_{\{x_1<0\}} G(u(x))\,dx = \lambda$ and $\int_{\mathbf{R}^N} G(u_2(x))\,dx = 2\int_{\{x_1>0\}} G(u(x))\,dx = \lambda$; consequently u_1 and u_2 (which belong to $\mathcal{X}$) also satisfy the constraint. It is obvious that $\int_{\mathbf{R}^N} F(u_1(x))\,dx + \int_{\mathbf{R}^N} F(u_2(x))\,dx = 2\int_{\mathbf{R}^N} F(u(x))\,dx$. Suppose by contradiction that u is not symmetric with respect to x_1. Then we get

$$E(u_1) + E(u_2) - 2E(u) = W(u_1) + W(u_2) - 2W(u) < 0,$$

and this implies that either $E(u_1) < E(u)$ or $E(u_2) < E(u)$. Therefore u cannot be a minimizer and this proves the claim.

Given the motivation above, we will study the behavior of T_1 and T_2 from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$, respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$, where

$$H^s(\mathbf{R}^N) = \{u \in \mathcal{S}'(\mathbf{R}^N) \mid \widehat{u} \in L^1_{loc}(\mathbf{R}^N) \text{ and } \int_{\mathbf{R}^N} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2\,d\xi < \infty\},$$

$$\dot{H}^s(\mathbf{R}^N) = \{u \in \mathcal{S}'(\mathbf{R}^N) \mid \widehat{u} \in L^1_{loc}(\mathbf{R}^N) \text{ and } \int_{\mathbf{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2\,d\xi < \infty\}.$$

Consider $\varphi \in C_c^\infty(\mathbf{R})$, φ odd, such that $\varphi'(0) = 1$. It is obvious that $T_1\varphi(x) = -\text{sgn}(x)\varphi(x)$ and $(T_1\varphi)'(x) = \begin{cases} \varphi'(x) \text{ if } x < 0, \\ -\varphi'(x) \text{ if } x > 0 \end{cases}$ and we have (in the distributional sense) $(T_1\varphi)'' = -\text{sgn}(x)\varphi''(x) - 2\delta_0$. Since $(T_1\varphi)'' \notin L^2(\mathbf{R})$, we conclude that T_1 and T_2 are not well-defined from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$ if $s \geq 2$. In fact, T_1 and T_2 are not well-defined from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $s \geq \frac{3}{2}$, as it can be seen in the following example.

Example 3.1 Define $\varphi : \mathbf{R} \longrightarrow \mathbf{R}$, $\varphi(x) = xe^{-|x|}$. An easy computation shows that $\widehat{\varphi}(\xi) = \frac{-4i\xi}{(1+\xi^2)^2}$, hence $\varphi \in H^s(\mathbf{R})$ for any $s < \frac{5}{2}$ and $\varphi \in \dot{H}^s(\mathbf{R})$ for any $s \in (-\frac{3}{2}, \frac{5}{2})$. It is clear that $(T_1\varphi)(x) = -|x|e^{-|x|}$ and $\widehat{T_1\varphi}(\xi) = \frac{2(\xi^2-1)}{(1+\xi^2)^2}$. Consequently, $T_1\varphi \in H^s(\mathbf{R})$ for $s < \frac{3}{2}$ (respectively $T_1\varphi \in \dot{H}^s(\mathbf{R})$ for $-\frac{1}{2} < s < \frac{3}{2}$), but $T_1\varphi \notin H^s(\mathbf{R})$ and $T_1\varphi \notin \dot{H}^s(\mathbf{R})$ for $s \geq \frac{3}{2}$.

In dimension $N \geq 2$ it suffices to take $\psi(x) = \varphi(x_1)\varphi_1(x_2, \dots, x_N)$, where $\varphi_1 \in C_c^\infty(\mathbf{R}^{N-1})$, to see that T_1 and T_2 are not well-defined from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $\frac{3}{2} \leq s < \frac{5}{2}$.

If $s < 0$, the elements of $H^s(\mathbf{R}^N)$ or $\dot{H}^s(\mathbf{R}^N)$ are not necessarily measurable functions. In this case we extend T_1 and T_2 to $H^s(\mathbf{R}^N)$ or $\dot{H}^s(\mathbf{R}^N)$ by duality. For u, $\varphi \in C_c^\infty(\mathbf{R}^N)$ we have

$$\langle T_1 u, \varphi\rangle_{\mathcal{S}', \mathcal{S}} = \int_{\mathbf{R}^N} (T_1 u)(x)\varphi(x)\,dx = \int_{\{x_1<0\}} u(x)\varphi(x)\,dx + \int_{\{x_1>0\}} u(-x_1, x')\varphi(x)\,dx$$

$$= \int_{\{x_1<0\}} u(x)\varphi(x)\,dx + \int_{\{x_1<0\}} u(x_1, x')\varphi(-x_1, x')\,dx = \langle u, T_1^*\varphi\rangle_{L^2, L^2},$$

where $(T_1^*\varphi)(x) = \chi_{\{x_1<0\}}(\varphi(x_1, x') + \varphi(-x_1, x'))$. Hence, for $u \in H^s(\mathbf{R}^N)$ with $s < 0$ we should define $T_1 u$ by

$$\langle T_1 u, \varphi\rangle_{H^s, H^{-s}} = \langle u, T_1^*\varphi\rangle_{H^s, H^{-s}}$$

for any test function $\varphi \in C_c^\infty(\mathbf{R}^N)$. However, the operator T_1^* does not map $H^k(\mathbf{R}^N)$ into $H^k(\mathbf{R}^N)$ if $k \geq \frac{1}{2}$ (as it can be easily seen by taking the function $\eta(x) = e^{-|x|}$ in one dimension, respectively $\eta(x_1)\eta_1(x_2, \dots, x_N)$, where $\eta_1 \in C_c^\infty(\mathbf{R}^{N-1})$ in dimension $N \geq 2$). This shows that we cannot define T_1 and T_2 on $H^s(\mathbf{R}^N)$ and on $\dot{H}^s(\mathbf{R}^N)$ if $s \leq -\frac{1}{2}$.

Example 3.2 Consider the tempered distribution u defined by $u = p.v.(\frac{1}{x})$, that is

$$\langle u, \varphi\rangle_{\mathcal{S}', \mathcal{S}} = \lim_{\varepsilon \to 0} \int_{\{|x|>\varepsilon\}} \frac{1}{x}\varphi(x)\,dx \qquad \text{for any } \varphi \in \mathcal{S}(\mathbf{R}).$$

It is well-known (and easy to check) that $\widehat{u}(\xi) = -i\pi\,\mathrm{sgn}(\xi)$; hence $u \in H^s(\mathbf{R})$ for any $s < -\frac{1}{2}$. However, $T_1 u = -\frac{1}{|x|}$ and $T_2 u = \frac{1}{|x|}$ do not define distributions on $\mathbf{R}$!

Our next goal is to prove that the operators T_1 and T_2 are well-defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) if $-\frac{1}{2} < s < \frac{3}{2}$. It is obvious that T_1 and T_2 are well-defined and continuous from $L^2(\mathbf{R}^N)$ to $L^2(\mathbf{R}^N)$. It is well-known that $H^1(\mathbf{R}^N) = W^{1,2}(\mathbf{R}^N) = \{\varphi \in L^2(\mathbf{R}^N) \mid \frac{\partial\varphi}{\partial x_i} \in L^2(\mathbf{R}^N),\ i = 1, \dots, N\}$ and that T_1, $T_2 : W^{1,2}(\mathbf{R}^N) \longrightarrow W^{1,2}(\mathbf{R}^N)$ are well-defined and continuous. Using interpolation theory we conclude that T_1 and T_2 are well-defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ if $0 \leq s \leq 1$. However, interpolation gives no information if either $s < 0$ or $s > 1$. Our next result deals with some values of s in this range.

Theorem 3.3 *The operators T_1 and T_2 are well-defined and continuous from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ and from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$ for any $s \in (-\frac{1}{2}, \frac{3}{2})$.*

Proof. We will prove that there exists $C_s > 0$ such that for any $u \in C_c^\infty(\mathbf{R}^N)$ we have

$$(3.2) \qquad ||T_i u||_{H^s} \leq C_s ||u||_{H^s}, \qquad \text{respectively} \qquad ||T_i u||_{\dot{H}^s} \leq C_s ||u||_{\dot{H}^s}, \quad s = 1, 2,$$

and then the theorem will follow by density.

Therefore, suppose $u \in C_c^\infty(\mathbf{R}^N)$. If $N \geq 2$ we have by (2.61) and (2.62)

$$(3.3) \qquad \begin{aligned} &||T_1 u||_{\dot{H}^s}^2 + ||T_2 u||_{\dot{H}^s}^2 - 2||u||_{\dot{H}^s}^2 \\ &= -\frac{16\sin(s\pi)}{\pi^2} \int_{\mathbf{R}^{N-1}} \int_{|\xi'|}^\infty \left(t^2 - |\xi'|^2\right)^s \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2}\,d\xi_1 \right|^2 dt\,d\xi', \end{aligned}$$

respectively

$$
\begin{aligned}
&||T_1 u||^2_{H^s} + ||T_2 u||^2_{H^s} - 2||u||^2_{H^s} \\
&= -\frac{16\sin(s\pi)}{\pi^2}\int_{\mathbf{R}^{N-1}}\int_{\sqrt{|\xi'|^2+1}}^{\infty}\left(t^2-|\xi'|^2-1\right)^s\left|\int_0^\infty \widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2 dt\,d\xi'.
\end{aligned}
\tag{3.4}
$$

If $N = 1$ we have

$$
||T_1 u||^2_{\dot{H}^s} + ||T_2 u||^2_{\dot{H}^s} - 2||u||^2_{\dot{H}^s} = -\frac{16\sin(s\pi)}{\pi^2}\int_0^\infty t^{2s}\left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2 dt,
\tag{3.5}
$$

respectively

$$
||T_1 u||^2_{H^s} + ||T_2 u||^2_{H^s} - 2||u||^2_{H^s} = -\frac{16\sin(s\pi)}{\pi^2}\int_1^\infty \left(t^2-1\right)^s\left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2 dt.
\tag{3.6}
$$

We begin by proving that T_1 and T_2 are bounded from $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$, $-\frac{1}{2} < s < \frac{3}{2}$. The integral in the right-hand side of (3.5) can be formally written as

$$
\int_0^\infty\int_0^\infty\int_0^\infty t^{2s}\frac{\xi}{t^2+\xi^2}\cdot\frac{\eta}{t^2+\eta^2}\widehat{f}(\xi)\overline{\widehat{f}(\eta)}\,d\xi\,d\eta\,dt.
\tag{3.7}
$$

Our strategy is as follows: first we compute explicitly the integral

$$
I_s(\xi,\eta) = \int_0^\infty t^{2s}\frac{\xi}{t^2+\xi^2}\cdot\frac{\eta}{t^2+\eta^2}\,dt = \xi\eta\int_0^\infty t^{2s}\frac{1}{t^2+\xi^2}\cdot\frac{1}{t^2+\eta^2}\,dt.
\tag{3.8}
$$

Observe that $I_s(\xi,\eta) > 0$ if $\xi > 0$, $\eta > 0$. Then we will prove that for any $s \in (-\frac{1}{2}, \frac{3}{2})$ and any φ, $\psi \in L^2(0,\infty)$ we have

$$
\left|\int_0^\infty\int_0^\infty \xi^{-s}\eta^{-s}I_s(\xi,\eta)\varphi(\xi)\psi(\eta)\,d\xi\,d\eta\right| \le C(s)||\varphi||_{L^2(0,\infty)}\cdot||\psi||_{L^2(0,\infty)}.
$$

This will be done in Lemma 3.4. Thereafter it will be clear that for any $f \in \dot{H}^s(\mathbf{R})$ we have

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty I_s(\xi,\eta)|\widehat{f}(\xi)|\cdot|\overline{\widehat{f}(\eta)}|\,d\xi\,d\eta \\
&= \int_0^\infty\int_0^\infty \xi^{-s}\eta^{-s}I_s(\xi,\eta)|\xi^s\widehat{f}(\xi)|\cdot|\eta^s\overline{\widehat{f}(\eta)}|\,d\xi\,d\eta \\
&\le C(s)||\,|\cdot|^s\widehat{f}||^2_{L^2(0,\infty)} \le C(s)||f||^2_{\dot{H}^s(\mathbf{R})}.
\end{aligned}
\tag{3.9}
$$

This justifies the use of Fubini's Theorem in evaluating (3.7) and proves that the right-hand side of (3.5) is less than $C_1(s)||f||_{\dot{H}^s(\mathbf{R})}$, where $C_1(s)$ is a constant depending only on s. Thus we infer that there exists $C_s > 0$ such that $||T_1 u||_{\dot{H}^s(\mathbf{R})} \le C_s||u||_{\dot{H}^s(\mathbf{R})}$ and $||T_2 u||_{\dot{H}^s(\mathbf{R})} \le C_s||u||_{\dot{H}^s(\mathbf{R})}$ for any $u \in C_c^\infty(\mathbf{R})$. Consequently, T_1 and T_2 can be extended as continuous linear mappings form $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$, $-\frac{1}{2} < s < \frac{3}{2}$, as claimed.

To carry out the first step of this strategy, we come back to $I_s(\xi,\eta)$ given by (3.8). The complex logarithm can be defined analytically on $\mathbf{C}\setminus\{it \mid t \in (-\infty,0]\}$. Hence, we may define the holomorphic function $z \longmapsto z^{2s} := e^{2s\log(z)} = |z|^{2s}e^{2is\arg(z)}$ on $\mathbf{C}\setminus\{it \mid t \in (-\infty,0]\}$. With this definition the function $k(z) = \dfrac{z^{2s}}{(z^2+\xi^2)(z^2+\eta^2)}$ is meromorphic on $\mathbf{C}\setminus\{it \mid t \in (-\infty,0]\}$. If $\xi \ne \eta$, k has four simple poles, namely $\pm i\xi$ and $\pm i\eta$; if $\xi = \eta$ it has two double poles at

$\pm i\xi$. For $0 < \varepsilon < \min(\xi, \eta)$, and $R > \max(\xi, \eta)$, consider the closed path $\beta_{\varepsilon,R}$ composed by the following pieces :

$$\begin{array}{ll} \beta_{1,\varepsilon,R}(t) = t, & t \in [-R, -\varepsilon] \\ \beta_{2,\varepsilon}(\theta) = \varepsilon e^{i(\pi-\theta)}, & \theta \in [0, \pi] \\ \beta_{3,\varepsilon,R}(t) = t, & t \in [\varepsilon, R] \\ \beta_{4,R}(\theta) = Re^{i\theta}, & \theta \in [0, \pi]. \end{array}$$

Using the Residue Theorem we get

$$\int_{\beta\varepsilon,R} k(z)\,dz = 2\pi i[\mathrm{Res}(k, i\xi) + \mathrm{Res}(k, i\eta)] = \pi e^{is\pi}\left[\frac{\xi^{2s}}{\xi(\eta^2-\xi^2)} + \frac{\eta^{2s}}{\eta(\xi^2-\eta^2)}\right]. \tag{3.10}$$

Since $s > -\frac{1}{2}$ we have $\lim\limits_{\varepsilon\to 0}\int_{\beta_{2,\varepsilon}} k(z)\,dz = 0$. We have also $\lim\limits_{R\to\infty}\int_{\beta_{4,R}} k(z)\,dz = 0$ because $s < \frac{3}{2}$. Passing to the limit as $\varepsilon \longrightarrow 0$ in (3.10) and then passing to the limit as $R \longrightarrow \infty$ in the resulting equation, we get $\int_{-\infty}^{0} k(z)\,dz + \int_0^\infty k(z)\,dz = \pi e^{is\pi}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}$, that is $(e^{2is\pi}+1)\int_0^\infty \frac{t^{2s}}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \pi e^{is\pi}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}$. For $s \neq \frac{1}{2}$ we obtain

$$\int_0^\infty \frac{t^{2s}}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \frac{\pi}{2\cos(s\pi)}\frac{\xi^{2s-1}-\eta^{2s-1}}{\eta^2-\xi^2}. \tag{3.11}$$

For $s = \frac{1}{2}$ we compute directly

$$\begin{aligned} &\int_0^\infty \frac{t}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \frac{1}{\eta^2-\xi^2}\int_0^\infty \frac{t}{t^2+\xi^2} - \frac{t}{t^2+\eta^2}\,dt \\ &= \frac{1}{2}\frac{1}{\eta^2-\xi^2}\Big(\ln(t^2+\xi^2) - \ln(t^2+\eta^2)\Big)\Big|_{t=0}^{\infty} = \frac{\ln\eta - \ln\xi}{\eta^2-\xi^2}. \end{aligned} \tag{3.12}$$

Notice that $\lim\limits_{\eta\to\xi}\int_0^\infty \frac{t^{2s}}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \frac{\pi(1-2s)}{4\cos(s\pi)}\xi^{2s-3}$ if $s \neq \frac{1}{2}$ and $\lim\limits_{\eta\to\xi}\int_0^\infty \frac{t}{(t^2+\xi^2)(t^2+\eta^2)}\,dt = \frac{1}{2\xi^2}$. Hence

$$I_s(\xi,\eta) = \frac{\pi}{2\cos(s\pi)}\frac{\xi\eta(\xi^{2s-1}-\eta^{2s-1})}{\eta^2-\xi^2} \quad \text{if } s \neq \frac{1}{2}, \quad \text{and} \quad I_{\frac{1}{2}}(\xi,\eta) = \frac{\xi\eta(\ln\eta - \ln\xi)}{\eta^2-\xi^2}. \tag{3.13}$$

This gives $\xi^{-s}\eta^{-s}I_s(\xi,\eta) = \frac{\pi}{2\cos(s\pi)}\frac{\xi^s\eta^{1-s}-\xi^{1-s}\eta^s}{\eta^2-\xi^2}$ if $s \neq \frac{1}{2}$ and $\xi^{-\frac{1}{2}}\eta^{-\frac{1}{2}}I_{\frac{1}{2}}(\xi,\eta) = \xi^{\frac{1}{2}}\eta^{\frac{1}{2}}\frac{\ln\eta-\ln\xi}{\eta^2-\xi^2}$. An interesting property of these functions is given by the next lemma.

Lemma 3.4 *Let* $K_s(\xi,\eta) = \dfrac{\xi^s\eta^{1-s}-\xi^{1-s}\eta^s}{\eta^2-\xi^2}$ *if* $s \neq \frac{1}{2}$, *respectively* $K_{\frac{1}{2}}(\xi,\eta) = \xi^{\frac{1}{2}}\eta^{\frac{1}{2}}\dfrac{\ln\eta-\ln\xi}{\eta^2-\xi^2}$. *For any* $s \in (-\frac{1}{2}, \frac{3}{2})$ *there exists a constant* $C(s)$ *(depending only on* s*) such that for any* φ, $\psi \in L^2(0,\infty)$ *we have*

$$\left|\int_0^\infty\int_0^\infty \varphi(\xi)K_s(\xi,\eta)\psi(\eta)\,d\xi\,d\eta\right| \le C(s)||\varphi||_{L^2(0,\infty)}||\psi||_{L^2(0,\infty)}.$$

Proof. Using polar coordinates we write $\xi = r\cos(\theta)$, $\eta = r\sin(\theta)$, where $r = \sqrt{\xi^2+\eta^2}$ and $\theta = \arctan\frac{\eta}{\xi}$. It is easy to see that $K_s(\xi,\eta) = \dfrac{1}{r}L_s(\theta)$, where

$L_s(\theta) = \dfrac{(\sin\theta)^s(\cos\theta)^{1-s} - (\cos\theta)^s(\sin\theta)^{1-s}}{\cos^2\theta - \sin^2\theta}$ if $s \neq \frac{1}{2}$ and
$L_{\frac{1}{2}}(\theta) = \dfrac{-\ln\tan\theta}{(1-\tan^2\theta)\cos^2\theta}(\sin\theta)^{\frac{1}{2}}(\cos\theta)^{\frac{1}{2}}$. By a change of variables we get

$$\int_0^\infty \int_0^\infty \left|\varphi(\xi)K_s(\xi,\eta)\psi(\eta)\right| d\xi\, d\eta = \int_0^{\frac{\pi}{2}} \int_0^\infty \left|\varphi(r\cos\theta)\psi(r\sin\theta)\right| dr\, |L_s(\theta)|\, d\theta.$$

Using the Cauchy-Schwarz inequality we have

$$\int_0^\infty \left|\varphi(r\cos\theta)\psi(r\sin\theta)\right| dr \leq ||\varphi(\cdot\cos\theta)||_{L^2(0,\infty)}||\psi(\cdot\sin\theta)||_{L^2(0.\infty)} = \frac{||\varphi||_{L^2(0,\infty)}||\psi||_{L^2(0,\infty)}}{\sqrt{\cos\theta\cdot\sin\theta}}.$$

Consequently,

$$\int_0^\infty \int_0^\infty \left|\varphi(\xi)K_s(\xi,\eta)\psi(\eta)\right| d\xi\, d\eta \leq ||\varphi||_{L^2(0,\infty)}||\psi||_{L^2(0,\infty)} \int_0^{\frac{\pi}{2}} \frac{|L_s(\theta)|}{\sqrt{\cos\theta\cdot\sin\theta}}\, d\theta. \tag{3.14}$$

The lemma will be proved if we show that the last integral in (3.14) is finite. If $s \neq \frac{1}{2}$ we have

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{|L_s(\theta)|}{\sqrt{\cos\theta\cdot\sin\theta}}\, d\theta &= \int_0^{\frac{\pi}{2}} \left|\frac{(\sin\theta)^{s-\frac{1}{2}}(\cos\theta)^{\frac{1}{2}-s} - (\cos\theta)^{s-\frac{1}{2}}(\sin\theta)^{\frac{1}{2}-s}}{\cos^2\theta - \sin^2\theta}\right| d\theta \\ &= \int_0^{\frac{\pi}{2}} \left|\frac{(\tan\theta)^{s-\frac{1}{2}} - (\tan\theta)^{\frac{1}{2}-s}}{1-\tan^2\theta}\right| \cdot \frac{1}{\cos^2\theta}\, d\theta = \int_0^\infty \left|\frac{t^{s-\frac{1}{2}} - t^{\frac{1}{2}-s}}{1-t^2}\right| dt. \end{aligned} \tag{3.15}$$

Using l'Hôspital's rule it is easy to see that $\lim\limits_{t\to 1} \frac{t^{s-\frac{1}{2}} - t^{\frac{1}{2}-s}}{1-t^2} = \frac{1}{2} - s$; hence the function $t \longmapsto \frac{t^{s-\frac{1}{2}} - t^{\frac{1}{2}-s}}{1-t^2}$ is bounded near 1. Since $s - \frac{1}{2} \in (-1,1)$, the last integral in (3.15) converges.

If $s = \frac{1}{2}$ we have

$$\int_0^{\frac{\pi}{2}} \frac{|L_{\frac{1}{2}}(\theta)|}{\sqrt{\cos\theta\cdot\sin\theta}}\, d\theta = \int_0^{\frac{\pi}{2}} \left|\frac{-\ln\tan\theta}{1-\tan^2\theta}\right| \cdot \frac{1}{\cos^2\theta}\, d\theta = \int_0^\infty \left|\frac{\ln y}{y^2-1}\right| dy. \tag{3.16}$$

Note that $\lim\limits_{y\to 1} \frac{\ln y}{y^2-1} = \frac{1}{2}$ and this implies easily that that the last integral in (3.16) converges. This completes the proof of Lemma 3.4. □

In view of (3.5), (3.7), (3.9), (3.13) and Lemma 3.4, it follows that T_1 and T_2 are well-defined and continuous from $\dot{H}^s(\mathbf{R})$ to $\dot{H}^s(\mathbf{R})$, $-\frac{1}{2} < s < \frac{3}{2}$.

Next we estimate the integral in the right-hand side of (3.6). If $s \in [0, \frac{3}{2})$ we have by (3.7), (3.8) and (3.9)

$$\begin{aligned} \int_1^\infty \left(t^2-1\right)^s \left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\, d\xi\right|^2 dt &\leq \int_0^\infty t^{2s}\left|\int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\, d\xi\right|^2 dt \\ &\leq C(s)||f||^2_{\dot{H}^s} \leq C(s)||f||^2_{H^s}. \end{aligned} \tag{3.17}$$

If $s \in (-\frac{1}{2}, 0)$, using the change of variable $\tau = \sqrt{t^2-1}$ and (3.11) we get

$$\begin{aligned} \int_1^\infty \frac{(t^2-1)^s}{(t^2+\xi^2)(t^2+\eta^2)}\, dt &= \int_0^\infty \frac{\tau^{2s}}{(\tau^2+1+\xi^2)(t^2+1+\eta^2)} \cdot \frac{\tau}{\sqrt{\tau^2+1}}\, d\tau \\ &\leq \int_0^\infty \frac{\tau^{2s}}{(\tau^2+1+\xi^2)(t^2+1+\eta^2)}\, d\tau = \frac{\pi}{2\cos(s\pi)} \cdot \frac{(1+\xi^2)^{\frac{2s-1}{2}} - (1+\eta^2)^{\frac{2s-1}{2}}}{\eta^2-\xi^2}. \end{aligned} \tag{3.18}$$

Consequently,

$$\int_1^\infty \left(t^2-1\right)^s \left| \int_0^\infty \widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2 dt$$

$$\le \int_0^\infty\int_0^\infty |\widehat{f}(\xi)|\cdot|\overline{\widehat{f}(\eta)}|\int_1^\infty (t^2-1)^s\frac{\xi\eta}{(t^2+\xi^2)(t^2+\eta^2)}\,dt\,d\xi\,d\eta$$

$$\le \frac{\pi}{2\cos(s\pi)}\int_0^\infty\int_0^\infty |\widehat{f}(\xi)|\cdot|\overline{\widehat{f}(\eta)}|\cdot\xi\eta\frac{(1+\xi^2)^{\frac{2s-1}{2}}-(1+\eta^2)^{\frac{2s-1}{2}}}{\eta^2-\xi^2}\,d\xi\,d\eta \tag{3.19}$$

$$= \frac{\pi}{2\cos(s\pi)}\int_0^\infty\int_0^\infty (1+\xi^2)^{\frac{s}{2}}|\widehat{f}(\xi)|\cdot(1+\eta^2)^{\frac{s}{2}}|\overline{\widehat{f}(\eta)}|$$
$$\cdot\frac{\xi\eta}{\eta^2-\xi^2}\cdot\frac{(1+\xi^2)^{\frac{2s-1}{2}}-(1+\eta^2)^{\frac{2s-1}{2}}}{(1+\xi^2)^{\frac{s}{2}}(1+\eta^2)^{\frac{s}{2}}}\,d\xi\,d\eta.$$

We claim that for any ξ, $\eta>0$, $\xi\neq\eta$ we have

$$\frac{\xi\eta}{\eta^2-\xi^2}\cdot\frac{(1+\xi^2)^{\frac{2s-1}{2}}-(1+\eta^2)^{\frac{2s-1}{2}}}{(1+\xi^2)^{\frac{s}{2}}(1+\eta^2)^{\frac{s}{2}}}\le\frac{\xi^s\eta^{1-s}-\xi^{1-s}\eta^s}{\eta^2-\xi^2}=K_s(\xi,\eta). \tag{3.20}$$

We may suppose without loss of generality that $\eta>\xi$. Then (3.20) is equivalent to

$$(1+\xi^2)^{\frac{s}{2}-\frac{1}{2}}(1+\eta^2)^{-\frac{s}{2}}-(1+\eta^2)^{\frac{s}{2}-\frac{1}{2}}(1+\xi^2)^{-\frac{s}{2}}\le\xi^{s-1}\eta^{-s}-\eta^{s-1}\xi^{-s}. \tag{3.21}$$

Let $\alpha=\frac{\eta}{\xi}>1$, $\eta_1=\sqrt{1+\eta^2}$, $\xi_1=\sqrt{1+\xi^2}$, $\alpha_1=\frac{\eta_1}{\xi_1}>1$. It is clear that $\alpha>\alpha_1$ (because $\alpha^2-1=\frac{\eta^2-\xi^2}{\xi^2}>\frac{\eta^2-\xi^2}{\xi^2+1}=\alpha_1^2-1$). Inequality (3.21) can be written as

$$\xi_1^{s-1}\eta_1^{-s}-\eta_1^{s-1}\xi_1^{-s}\le\xi^{s-1}\eta^{-s}-\eta^{s-1}\xi^{-s},$$

or equivalently

$$\frac{1}{\eta_1}(\alpha_1^{1-s}-\alpha_1^s)\le\frac{1}{\eta}(\alpha^{1-s}-\alpha^s). \tag{3.22}$$

Since $s<0$, the function $x\longmapsto x^{1-s}-x^s$ is increasing on $(0,\infty)$ and then $\alpha^{1-s}-\alpha^s>\alpha_1^{1-s}-\alpha_1^s>0$. It is obvious that $\frac{1}{\eta}>\frac{1}{\eta_1}>0$ and this implies (3.22). This proves our claim.

Coming back to (3.19) and using Lemma 3.4 we obtain

$$\int_1^\infty\left(t^2-1\right)^s\left|\int_0^\infty\widehat{f}(\xi)\frac{\xi}{t^2+\xi^2}\,d\xi\right|^2dt\le\frac{\pi C(s)}{2\cos(s\pi)}||(1+|\cdot|^2)^{\frac{s}{2}}\widehat{f}||^2_{L^2(0,\infty)}\le C'(s)||f||^2_{H^s}. \tag{3.23}$$

From (3.6) and (3.17) if $s\in[0,\frac{3}{2})$, respectively from (3.6) and (3.23) if $s\in(-\frac{1}{2},0)$, we infer that T_1 and T_2 can be extended as linear continuous operators from $H^s(\mathbf{R})$ to $H^s(\mathbf{R})$.

Now we prove Theorem 3.3 in the case $N\geq 2$.

If $s\in[0,\frac{3}{2})$, arguing as in (3.7)-(3.9) and using Lemma 3.4 we have

$$\int_{|\xi'|}^\infty\left(t^2-|\xi'|^2\right)^s\left|\int_0^\infty\widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2dt\le\int_0^\infty t^{2s}\left|\int_0^\infty\widehat{f}(\xi_1,\xi')\frac{\xi_1}{t^2+\xi_1^2}\,d\xi_1\right|^2dt$$

$$\le\int_0^\infty\int_0^\infty|\widehat{f}(\xi_1,\xi')|\xi_1^s\cdot|\overline{\widehat{f}(\eta_1,\xi')}|\eta_1^s\cdot\left(\xi_1^{-s}\eta_1^{-s}I_s(\xi_1,\eta_1)\right)\,d\xi_1\,d\eta_1 \tag{3.24}$$

$$\le C(s)||\,|\cdot|^s\widehat{f}(\cdot,\xi')||^2_{L^2(0,\infty)}\le C(s)\int_{-\infty}^\infty\left(\xi_1^2+|\xi'|^2\right)^s|\widehat{f}(\xi_1,\xi')|^2\,d\xi_1.$$

If $s \in (-\frac{1}{2}, 0)$, using the change of variable $\tau = \sqrt{t^2 - |\xi'|^2}$, arguing as in the proof of (3.18), then taking (3.11) into account we obtain

$$\int_{|\xi'|}^{\infty} \frac{(t^2 - |\xi'|^2)^s}{(t^2 + \xi^2)(t^2 + \eta^2)} \, dt = \int_0^{\infty} \frac{\tau^{2s}}{(\tau^2 + |\xi'|^2 + \xi_1^2)(\tau^2 + |\xi'|^2 + \eta_1^2)} \cdot \frac{\tau}{\sqrt{\tau^2 + |\xi'|^2}} \, d\tau$$

$$\leq \int_0^{\infty} \frac{\tau^{2s}}{(\tau^2 + |\xi'|^2 + \xi_1^2)(\tau^2 + |\xi'|^2 + \eta_1^2)} \, d\tau = \frac{\pi}{2\cos(s\pi)} \cdot \frac{(|\xi'|^2 + \xi_1^2)^{\frac{2s-1}{2}} - (|\xi'|^2 + \eta_1^2)^{\frac{2s-1}{2}}}{\eta_1^2 - \xi_1^2}.$$

We also have

$$\frac{\xi_1 \eta_1}{\eta_1^2 - \xi_1^2} \cdot \frac{(\xi_1^2 + |\xi'|^2)^{\frac{2s-1}{2}} - (\eta_1^2 + |\xi'|^2)^{\frac{2s-1}{2}}}{(\xi_1^2 + |\xi'|^2)^{\frac{s}{2}} (\eta_1^2 + |\xi'|^2)^{\frac{s}{2}}} \leq K_s(\xi_1, \eta_1)$$

(the proof being the same as the proof of (3.20)). Arguing as in (3.19), using the two previous inequalities and Lemma 3.4 we get

$$\begin{aligned} &\int_{|\xi'|}^{\infty} \left(t^2 - |\xi'|^2\right)^s \left| \int_0^{\infty} \widehat{f}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 \right|^2 dt \\ &\leq \frac{\pi C(s)}{2\cos(s\pi)} ||(|\xi'|^2 + |\cdot|^2)^{\frac{s}{2}} \widehat{f}(\cdot, \xi')||^2_{L^2(0,\infty)} \leq C'(s) \int_{-\infty}^{\infty} \left(\xi_1^2 + |\xi'|^2\right)^s |\widehat{f}(\xi_1, \xi')|^2 \, d\xi_1. \end{aligned} \tag{3.25}$$

Integrating (3.24), respectively (3.25), over $\mathbf{R}^{N-1}$ we infer that the integral in the right-hand side of (3.3) is less than $C''(s)||f||^2_{\dot{H}^s}$. This proves that T_1 and T_2 can be extended by continuity from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$ for $s \in (-\frac{1}{2}, \frac{3}{2})$.

In a similar way we show that T_1 and T_2 can be extended by continuity from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ for $s \in (-\frac{1}{2}, \frac{3}{2})$. Theorem 3.3 is now proved. □

For a measurable function u defined on $\mathbf{R}^N$, we define its antisymmetric part in the x_1 direction by $Au(x_1, x') = \frac{1}{2}(u(x_1, x') - u(-x_1, x'))$. If u is a tempered distribution, we define Au by $\langle Au, \phi \rangle_{\mathcal{S}', \mathcal{S}} = \langle u, A\phi \rangle_{\mathcal{S}', \mathcal{S}}$ for any $\phi \in \mathcal{S}$. Obviously, Au is odd with respect to x_1 (for distributions, this means that $\langle Au, \phi(-x_1, x') \rangle_{\mathcal{S}', \mathcal{S}} = -\langle Au, \phi \rangle_{\mathcal{S}', \mathcal{S}}$). It is clear from the definition that A defines a linear continuous map from $H^s(\mathbf{R}^N)$ to $H^s(\mathbf{R}^N)$ (respectively from $\dot{H}^s(\mathbf{R}^N)$ to $\dot{H}^s(\mathbf{R}^N)$) for any s. Moreover, for any tempered distribution u, the distribution $\mathcal{F}(Au)$ is odd with respect to x_1.

It follows from the proof of Theorem 3.3 that for any $s \in (-\frac{1}{2}, \frac{3}{2})$, the following complex bilinear forms are continuous :

$$B_{1,s} : \dot{H}^s(\mathbf{R}) \times \dot{H}^s(\mathbf{R}) \longrightarrow \mathbf{C},$$
$$B_{1,s}(u, v) = \int_0^{\infty} t^{2s} \int_0^{\infty} \widehat{Au}(\xi) \frac{\xi}{t^2 + \xi^2} \, d\xi \cdot \int_0^{\infty} \overline{\widehat{Av}(\eta)} \frac{\eta}{t^2 + \eta^2} \, d\eta \, dt,$$

$$\tilde{B}_{1,s} : H^s(\mathbf{R}) \times H^s(\mathbf{R}) \longrightarrow \mathbf{C},$$
$$\tilde{B}_{1,s}(u, v) = \int_1^{\infty} (t^2 - 1)^s \int_0^{\infty} \widehat{Au}(\xi) \frac{\xi}{t^2 + \xi^2} \, d\xi \cdot \int_0^{\infty} \overline{\widehat{Av}(\eta)} \frac{\eta}{t^2 + \eta^2} \, d\eta \, dt,$$

$$B_{N,s} : \dot{H}^s(\mathbf{R}^N) \times \dot{H}^s(\mathbf{R}^N) \longrightarrow \mathbf{C},$$
$$B_{N,s}(u, v) = \int_{\mathbf{R}^{N-1}} \int_{|\xi'|}^{\infty} \left(t^2 - |\xi'|^2\right)^s \int_0^{\infty} \widehat{Au}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 \int_0^{\infty} \overline{\widehat{Av}(\eta_1, \xi')} \frac{\eta_1}{t^2 + \eta_1^2} \, d\eta_1 \, dt \, d\xi',$$

$$\tilde{B}_{N,s} : H^s(\mathbf{R}^N) \times H^s(\mathbf{R}^N) \longrightarrow \mathbf{C},$$
$$\tilde{B}_{N,s}(u, v)$$
$$= \int_{\mathbf{R}^{N-1}} \int_{\sqrt{|\xi'|^2+1}}^{\infty} \left(t^2 - |\xi'|^2 - 1\right)^s \int_0^{\infty} \widehat{Au}(\xi_1, \xi') \frac{\xi_1}{t^2 + \xi_1^2} \, d\xi_1 \int_0^{\infty} \overline{\widehat{Av}(\eta_1, \xi')} \frac{\eta_1}{t^2 + \eta_1^2} \, d\eta_1 \, dt \, d\xi'.$$

Moreover, from (3.3) - (3.6) we have the identities

$$||T_1 u||^2_{\dot{H}^s(\mathbf{R}^N)} + ||T_1 u||^2_{\dot{H}^s(\mathbf{R}^N)} - 2||u||^2_{\dot{H}^s(\mathbf{R}^N)} = -\frac{16\sin(s\pi)}{\pi^2} B_{N,s}(Au, Au), \tag{3.26}$$

$$||T_1 u||^2_{H^s(\mathbf{R}^N)} + ||T_1 u||^2_{H^s(\mathbf{R}^N)} - 2||u||^2_{H^s(\mathbf{R}^N)} = -\frac{16\sin(s\pi)}{\pi^2} \tilde{B}_{N,s}(Au, Au) \tag{3.27}$$

for any $u \in C_c^\infty(\mathbf{R}^N)$. From Theorem 3.3, the continuity of $B_{N,s}$ and of $\tilde{B}_{N,s}$ and the density of $C_c^\infty(\mathbf{R}^N)$ in $\dot{H}^s(\mathbf{R}^N)$ and in $H^s(\mathbf{R}^N)$ we infer that we have the following :

Corollary 3.5 *Let $s \in (-\frac{1}{2}, \frac{3}{2})$. The identity (3.26) holds for any $u \in \dot{H}^s(\mathbf{R}^N)$ and (3.27) holds for any $u \in H^s(\mathbf{R}^N)$.*

Our next aim is to show that the quadratic forms $B_{N,s}$ and $\tilde{B}_{N,s}$ define norms in some spaces of odd functions. We start with the following proposition :

Lemma 3.6 *Assume that $g : \mathbf{R} \longrightarrow \mathbf{R}$ is measurable, odd and*
- *either $g \in L^p(\mathbf{R})$ for some $p \in (1, \infty)$,*
- *or $(\alpha^2 + \xi^2)^{\frac{s}{2}} g(\xi) \in L^2(\mathbf{R})$ (respectively $|\xi|^s g(\xi) \in L^2(\mathbf{R})$) for some $s \in (-\frac{1}{2}, \frac{3}{2})$.*

Suppose that the set $A = \{x > 0 \mid \int_0^\infty \frac{\xi}{x^2+\xi^2} g(\xi)\, d\xi = 0\}$ has a limit point $x_0 > 0$.

Then $g = 0$ almost everywhere on $\mathbf{R}$.

In particular, if $\int_0^\infty \frac{\xi}{x^2+\xi^2} g(\xi)\, d\xi = 0$ for almost every x in some open interval, then $g \equiv 0$.

Proof. We may suppose without loss of generality that g is real (otherwise we carry out the proof for its real and imaginary parts).

First we deal with the much simpler case $g \in L^p(\mathbf{R})$ for some p, $1 < p < \infty$. We define the Poisson integrals for g,

$$a(x,y) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{x}{x^2+(y-t)^2} g(t)\, dt \quad \text{and} \quad b(x,y) = -\frac{1}{\pi}\int_{-\infty}^\infty \frac{y-t}{x^2+(y-t)^2} g(t)\, dt. \tag{3.28}$$

It follows from Lemma 2.4 iii) that the functions a and b are well-defined and harmonic in the right half-plane and $r(x+iy) := a(x,y) + ib(x,y)$ is holomorphic in $\{z \in \mathbf{C} \mid \text{Re}(z) > 0\}$. Since g is odd, we have $a(x,0) = 0$ for any $x > 0$. If $x \in A$, we have also $b(x,0) = 0$. Consequently, $r(x) = 0$ for any $x \in A$. But r is holomorphic and A has a limit point $x_0 > 0$, thus necessarily $r \equiv 0$. By Lemma 2.4 ii) we know that $a(x,y) \longrightarrow g(y)$ as $x \longrightarrow 0$ for almost every y, hence $g = 0$ a.e. on $\mathbf{R}$.

Suppose that $(\alpha^2 + |\cdot|^2)^{\frac{s}{2}} g \in L^2(\mathbf{R})$ for some $s \in (-\frac{1}{2}, \frac{3}{2})$. We may assume that $\alpha = 1$. If $s \in [0, \frac{3}{2})$, then obviously $g \in L^2(\mathbf{R})$ and the conclusion of the lemma follows from the above considerations. If $s \in (-\frac{1}{2}, 0)$, then for any $x > 0$ and $y \in \mathbf{R}$ the functions $\varphi_{x,y}(t) = (1+t^2)^{-\frac{s}{2}} \frac{x}{x^2+(y-t)^2}$ and $\psi_{x,y}(t) = (1+t^2)^{-\frac{s}{2}} \frac{y-t}{x^2+(y-t)^2}$ belong to $L^2(\mathbf{R})$. We may write $\int_{-\infty}^\infty \frac{x}{x^2+(y-t)^2} g(t)\, dt = \int_{-\infty}^\infty \varphi_{x,y}(t)(1+t^2)^{\frac{s}{2}} g(t)\, dt$ and $\int_{-\infty}^\infty \frac{y-t}{x^2+(y-t)^2} g(t)\, dt = \int_{-\infty}^\infty \psi_{x,y}(t)(1+t^2)^{\frac{s}{2}} g(t)\, dt$. Using the Cauchy-Schwarz inequality, we see that the functions a and b in (3.28) are well-defined in the right half-plane (in particular, $\int_0^\infty \frac{\xi}{x^2+\xi^2} g(\xi)\, d\xi$ exists for any $x > 0$). Clearly the function $r(x+iy) := a(x,y) + ib(x,y)$ is holomorphic and, as above we have $r(x) = 0$ for $x \in A$. Since A has a limit point $x_0 > 0$, we infer that $r \equiv 0$. Next, we claim that $\lim_{x\downarrow 0} a(x,y) = g(y)$ whenever y is a Lebesgue point of g. This obviously implies $g = 0$ a.e., as desired. Let y be a Lebesgue point of g and fix $\varepsilon > 0$. Then there exists $\delta = \delta(\varepsilon) > 0$ such that $\frac{1}{r}\int_{-r}^{r} |g(y-\tau) - g(y)|\, d\tau < \varepsilon$ for any $r \in (0, \delta]$. We have :

$$|a(x,y) - g(y)| = \frac{1}{\pi}\left| \int_{-\infty}^{\infty} \frac{x}{x^2+t^2}(g(y-t) - g(y))\, dt \right|$$

$$(3.29) \qquad \le \frac{1}{\pi}\int_{-\delta}^{\delta} \frac{x}{x^2+t^2}|g(y-t) - g(y)|\, dt + \frac{1}{\pi}\int_{|t|>\delta} \frac{x}{x^2+t^2}|g(y-t)|\, dt$$

$$+\frac{1}{\pi}\int_{|t|>\delta} \frac{x}{x^2+t^2}|g(y)|\, dt = I_1 + I_2 + I_3.$$

Let $G(r) = \int_{-r}^{r} |g(y-\tau) - g(y)|\, d\tau$. It is obvious that G is nondecreasing on $[0,\infty)$ and we have $G'(r) = |g(y-r) - g(y)| + |g(y+r) - g(y)|$ almost everywhere. Using integration by parts and the fact that $0 \le G(t) \le \varepsilon t$ for any $t \in [0, \delta]$, we get :

$$I_1 = \frac{1}{\pi}\int_0^{\delta} \frac{x}{x^2+t^2}|g(y-r) - g(y)| + |g(y+r) - g(y)|\, dt = \frac{1}{\pi}\int_0^{\delta} \frac{x}{x^2+t^2}G'(t)\, dt$$

$$(3.30) \qquad = \frac{1}{\pi}\frac{x}{x^2+\delta^2}G(\delta) + \frac{2x}{\pi}\int_0^{\delta} \frac{t}{(x^2+t^2)^2}G(t)\, dt \le \frac{G(\delta)}{2\pi\delta} + \frac{2x}{\pi}\int_0^{\delta} \frac{\varepsilon t^2}{(x^2+t^2)^2}\, dt$$

$$\le \frac{\varepsilon}{2\pi} + \frac{2x\varepsilon}{\pi}\int_0^{\delta} \frac{1}{x^2+t^2}\, dt \le \frac{\varepsilon}{2\pi} + \frac{2\varepsilon}{\pi}\arctan\frac{\delta}{x} \le \frac{\varepsilon}{2\pi} + \varepsilon.$$

Using the Cauchy-Schwarz inequality we have :

$$I_2 = \frac{1}{\pi}\int_{|t|>\delta} \frac{x}{x^2+t^2}(1+|y-t|^2)^{-\frac{s}{2}}(1+|y-t|^2)^{\frac{s}{2}}|g(y-t)|\, dt$$

$$(3.31) \qquad \le \frac{x}{\pi}||(1+|\cdot|^2)^{\frac{s}{2}}g||_{L^2(\mathbf{R})}\left(\int_{|t|>\delta} \frac{(1+|y-t|^2)^{-s}}{t^4}\, dt\right)^{\frac{1}{2}}.$$

Since $s > -\frac{1}{2}$, the last integral in (3.31) converges. Let $K(y,\delta)$ be its value. We have proved that

$$(3.32) \qquad I_2 \le \frac{x}{\pi}K(y,\delta)||(1+|\cdot|^2)^{\frac{s}{2}}g||_{L^2(\mathbf{R})} \qquad \text{for any } x > 0.$$

Finally, the integral I_3 is easy to compute :

$$(3.33) \qquad I_3 = \frac{|g(y)|}{\pi}(\pi - 2\arctan\frac{\delta}{x}).$$

For x sufficiently small, the right-hand side terms in (3.32) and (3.33) are less than ε. From (3.29), (3.30), (3.32) and (3.33) we infer that $|a(x,y) - g(y)| \le 4\varepsilon$ if x is sufficiently small. Consequently $a(x,y) \longrightarrow g(y)$ as $y \longrightarrow 0$ and the claim is proved.

In the case $|\cdot|^s g \in L^2(\mathbf{R})$ and $s \in (-\frac{1}{2}, \frac{1}{2})$, we may repeat almost word by word the proof above (we have only to replace the functions $\varphi_{x,y}$ and $\psi_{x,y}$ by $t \longmapsto t^{-s}\frac{x}{x^2+(y-t)^2}$, respectively by $t \longmapsto t^{-s}\frac{y-t}{x^2+(y-t)^2}$).

If $|\cdot|^s g \in L^2(\mathbf{R})$ and $s \in [\frac{1}{2}, \frac{3}{2})$, the integrals defining a and b in (3.28) do not necessarily converge. In this case we define

$$(3.34)\qquad \begin{aligned} a_1(x,y) &= \frac{1}{\pi}\int_0^\infty \frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]} g(t)\,dt \quad \text{and} \\ b_1(x,y) &= \frac{1}{\pi}\int_0^\infty \frac{2t(t^2+x^2-y^2)}{[x^2+(y-t)^2][x^2+(y+t)^2]} g(t)\,dt. \end{aligned}$$

Notice that if $g \in L^1_{loc}(\mathbf{R})$ is odd and $\frac{g(t)}{t} \in L^1([1,\infty))$, then $a = a_1$ and $b = b_1$. It is obvious that for fixed $x > 0$, $y \in \mathbf{R}$ and $s \in (-\frac{1}{2}, \frac{3}{2})$, the functions $\varphi_1(t) = t^{-s}\frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]}$ and $\psi_1(t) = t^{-s}\frac{2t(t^2+x^2-y^2)}{[x^2+(y-t)^2][x^2+(y+t)^2]}$ belong to $L^2((0,\infty))$ and this implies that a_1 and b_1 are well-defined. It is straightforward that $r_1(x+iy) := a_1(x,y) + b_1(x,y)$ is holomorphic in the right half-plane. Obviously $a_1(x,0) = 0$ for any $x > 0$ and $b_1(x,0) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{t}{x^2+t^2} g(t)\,dt = 0$ for $x \in A$. Consequently $r = 0$ on A. Since A has a limit point $x_0 > 0$, we infer that $r \equiv 0$ in the right half-plane. The lemma will be proved if we show that $a_1(x,y) \longrightarrow g(y)$ as $x \longrightarrow 0$ for almost every y.

Let $y > 0$ be a Lebesgue point of g. Note that $\displaystyle\int_0^\infty \frac{4xyt}{[x^2+(y-t)^2][x^2+(y+t)^2]}\,dt = 2\arctan\frac{y}{x}$. Proceeding as in (3.29)-(3.33), we may show that $|a_1(x,y) - \frac{2}{\pi}(\arctan\frac{y}{x})g(y)| \longrightarrow 0$ as $x \longrightarrow 0$, hence $\lim\limits_{x\downarrow 0} a_1(x,y) = g(y)$ and the lemma is proved. □

We set

$$\begin{aligned} H^s_{1,odd}(\mathbf{R}^N) &= \{f \in H^s(\mathbf{R}^N) \mid f \text{ is odd with respect to } x_1\} = \{f \in H^s(\mathbf{R}^N) \mid f = Af\}, \\ \dot H^s_{1,odd}(\mathbf{R}^N) &= \{f \in \dot H^s(\mathbf{R}^N) \mid f \text{ is odd with respect to } x_1\} = \{f \in \dot H^s(\mathbf{R}^N) \mid f = Af\}, \end{aligned}$$

where, as before, Af is the antisymmetric part of f in the x_1 direction. For $f \in \dot H^s_{1,odd}(\mathbf{R}^N)$ we define $N_s(f) = (B_{N,s}(f,f))^{\frac{1}{2}}$ and for $f \in H^s_{1,odd}(\mathbf{R}^N)$ we define $\tilde N_s(f) = (\tilde B_{N,s}(f,f))^{\frac{1}{2}}$.

Theorem 3.7 *$\tilde N_s$ is a norm on $H^s_{1,odd}(\mathbf{R}^N)$, continuous with respect to the usual H^s norm, and N_s is a norm on $\dot H^s_{1,odd}(\mathbf{R}^N)$, continuous with respect to the $\dot H^s$ norm.*

Endowed with these norms, $H^s_{1,odd}(\mathbf{R}^N)$ and $\dot H^s_{1,odd}(\mathbf{R}^N)$ are pre-Hilbert spaces.

Proof. It is clear that $\tilde B_{N,s}$ and $B_{N,s}$ are complex-symmetric bilinear forms on $H^s(\mathbf{R}^N)$ (respectively on $\dot H^s(\mathbf{R}^N)$) and that $\tilde B_{N,s}(f,f) \geq 0$ and $B_{N,s}(f,f) \geq 0$ for any f (thus, in particular, $\tilde N_s$ and N_s are well-defined). Suppose, for instance, that $f \in H^s_{1,odd}(\mathbf{R}^N)$ and $\tilde B_{N,s}(f,f) = 0$. This implies that for almost every $\xi' \in \mathbf{R}^{N-1}$ we have : $\widehat f(\cdot,\xi')$ is odd, $(|\cdot|^2 + |\xi'|^2)^{\frac{s}{2}}\widehat f(\cdot,\xi') \in L^2(\mathbf{R})$ and $\displaystyle\int_{\sqrt{|\xi'|^2+1}}^\infty \left(t^2 - |\xi'|^2 - 1\right)^s \left|\int_0^\infty \widehat f(\xi_1,\xi')\frac{\xi_1}{t^2+|\xi'|^2}\,d\xi_1\right| dt = 0$. For such ξ' we must have $\displaystyle\int_0^\infty \widehat f(\xi_1,\xi')\frac{\xi_1}{t^2+|\xi'|^2}\,d\xi_1 = 0$ for almost every $t \in (\sqrt{|\xi'|^2+1},\infty)$ and using Lemma 3.6 we infer that $\widehat f(\cdot,\xi') = 0$ a.e. on $\mathbf{R}$, hence $\displaystyle\int_{\mathbf{R}}\left(\xi_1^2+|\xi'|^2\right)^s|\widehat f(\xi_1,\xi')|^2\,d\xi_1 = 0$. Consequently $||f||^2_{H^s} = \displaystyle\int_{\mathbf{R}^{N-1}}\int_{\mathbf{R}}\left(\xi_1^2+|\xi'|^2\right)^s|\widehat f(\xi_1,\xi')|^2\,d\xi_1\,d\xi' = 0$, i.e. $f = 0$ a.e. The proof is the same for $f \in \dot H^s(\mathbf{R}^N)$. Finally, the continuity of $\tilde N_s$ and N_s with respect to the usual norms follows from Theorem 3.3 and Corollary 3.5. □

4 Applications

In this section we illustrate how the results in Sections 2 and 3 can be used to prove the symmetry of minimizers in some concrete examples.

4.1 We start with two scalar variational problems.

Theorem 4.1 *Let $s \in (0,1)$ and assume that $F, G : \mathbf{R} \to \mathbf{R}$ are such that $u \to F(u)$ and $u \to G(u)$ map $\dot{H}^s(\mathbf{R}^N)$ (or $H^s(\mathbf{R}^N)$) into $L^1(\mathbf{R}^N)$. Suppose that either*

Case A. $u \in \dot{H}^s(\mathbf{R}^N)$ and u is a solution of the minimization problem

$$\begin{aligned} &\textit{minimize} \quad E(u) := \int_{\mathbf{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbf{R}^N} F(u(x))\, dx \\ &\textit{under the constraint } I(u) = \int_{\mathbf{R}^N} G(u(x))\, dx = \lambda, \qquad \textit{or} \end{aligned}$$

Case B. $u \in H^s(\mathbf{R}^N)$ and u is a solution of the minimization problem

$$\begin{aligned} &\textit{minimize} \quad E(u) := \int_{\mathbf{R}^N} \left(1 + |\xi|^2\right)^s |\widehat{u}(\xi)|^2 \, d\xi + \int_{\mathbf{R}^N} F(u(x))\, dx \\ &\textit{under the constraint } I(u) = \int_{\mathbf{R}^N} G(u(x))\, dx = \lambda. \end{aligned}$$

Then, after a translation in $\mathbf{R}^N$, u is radially symmetric.

Proof. Let us prove first that u is symmetric with respect to x_1. Making a translation in the x_1 direction if necessary, we may assume that $\displaystyle\int_{\{x_1<0\}} G(u(x))\, dx = \int_{\{x_1>0\}} G(u(x))\, dx = \frac{\lambda}{2}$. Let $u_1 = T_1 u$ and $u_2 = T_2 u$. It follows from Theorem 3.3 that $u_1, u_2 \in \dot{H}^s(\mathbf{R}^N)$ in case A, respectively $u_1, u_2 \in H^s(\mathbf{R}^N)$ in case B. It is obvious that we have $\displaystyle\int_{\mathbf{R}^N} G(u_1(x))\, dx = 2\int_{\{x_1<0\}} G(u(x))\, dx = \lambda$ and $\displaystyle\int_{\mathbf{R}^N} G(u_2(x))\, dx = 2\int_{\{x_1>0\}} G(u(x))\, dx = \lambda$; hence u_1 and u_2 also satisfy the constraint. From (3.26) and (3.27) we have

$$E(u_1) + E(u_2) - 2E(u) = -\tfrac{16\sin(s\pi)}{\pi^2} N_s^2(Au) \qquad \text{in case A, respectively}$$

$$E(u_1) + E(u_2) - 2E(u) = -\tfrac{16\sin(s\pi)}{\pi^2} \tilde{N}_s^2(Au) \qquad \text{in case B,}$$

where, as before, $Au(x_1, x') = \frac{1}{2}(u(x_1, x') - u(-x_1, x'))$ is the antisymmetric part of u in the x_1 direction. If $Au \not\equiv 0$, then Theorem 3.7 implies $N_s^2(Au) > 0$ (respectively $\tilde{N}_s^2(Au) > 0$) and we infer that $E(u_1) + E(u_2) - 2E(u) < 0$, contradicting the fact that u is a minimizer. Thus necessarily $Au \equiv 0$ and this means that u is symmetric with respect to x_1.

Arguing similarly with the remaining variables $x_2, \dots, x_N$, we find a new origin O' such that u is symmetric with respect to any of the variables $x_1, \dots, x_N$; in particular, $u(-x) = u(x)$ a.e. on $\mathbf{R}^N$. Now let Π be any hyperplane containing the new origin O' and let Π_+ and Π_- be the halfspaces determined by Π. Since the transformation $x \longmapsto -x$ maps Π_- into Π_+, we see that $\displaystyle\int_{\Pi_-} G(u(x))\, dx = \int_{\Pi_+} G(u(x))\, dx = \frac{\lambda}{2}$. Arguing as above we conclude that u must be symmetric with respect to Π. This implies that u is radially symmetric with respect to the new origin O'. □

An application of Theorem 4.1 concerns the solitary waves to the generalized Benjamin-Ono equation

$$A_t + \alpha A A_x - \beta(-\Delta)^{\frac{1}{2}} A_x = 0, \qquad (x, y) \in \mathbf{R}^2, \ t \in \mathbf{R},$$

where α, $\beta > 0$. Solitary waves are solutions of the form $A(t,x,y) = u(x-ct,y)$. After a scale change, a solitary wave $u(x,y)$ satisfies the equation

$$u + (-\Delta)^{\frac{1}{2}} u = u^2 \qquad \text{in } \mathbf{R}^2.$$

The existence of solitary waves was proved in [21] by minimizing the functional

$$V(u) = \frac{1}{2}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} u|^2\,dx + \int_{\mathbf{R}^2} u^2\,dx = \frac{1}{2(2\pi)^2}\int_{\mathbf{R}^2} |\xi||\widehat{u}(\xi)|^2\,d\xi + \int_{\mathbf{R}^2} u^2\,dx$$

under the constraint $I(u) = \frac{1}{3}\int_{\mathbf{R}^2} u^3\,dx = constant$. It has been shown in [21] that any solution u_* of the above problem also minimizes

$$E(v) := \frac{1}{2}\int_{\mathbf{R}^2} |(-\Delta)^{\frac{1}{4}} v|^2\,dx - \frac{1}{3}\int_{\mathbf{R}^2} v^3\,dx$$

under the constraint $Q(v) = Q(u_*)$, where $Q(v) = \frac{1}{2}\int_{\mathbf{R}^2} |u|^2\,dx$.

It follows directly from Theorem 4.1 that, except for translation, any minimizer of these problems is radially symmetric.

4.2 Next we apply our method to a variational problem involving two unknown functions (the vector case). Consider the functionals

$$E(u,v) = \frac{1}{2}\int_{\mathbf{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |\nabla v|^2)\,dx + \int_{\mathbf{R}^N} F(u,v)\,dx$$

where $0 < s < 1$, and

$$Q(u,v) = \int_{\mathbf{R}^N} G(u,v)\,dx.$$

We make the following assumptions:

A1 : $F, G : \mathbf{R}^2 \longrightarrow \mathbf{R}$ are C^2 functions satisfying $F(0,0) = \partial_1 F(0,0) = \partial_2 F(0,0) = 0$, $G(0,0) = \partial_1 G(0,0) = \partial_2 G(0,0) = 0$ and the growth conditions

$$|\partial_i F(u,v)| \leq C(|u|^{p-1} + |v|^{q-1}) \quad \text{and} \quad |\partial_i G(u,v)| \leq C(|u|^{p-1} + |v|^{q-1}) \quad \text{if } |(u,v)| \geq 1,$$

where $i \in \{1, 2\}$, C is a positive constant, $2 < p < \frac{2N}{N-2s}$ and $2 < q < \frac{2N}{N-2}$.

A2 : If $(u,v) \in H^s(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ and $(u,v) \not\equiv (0,0)$, then either $\partial_1 G(u,v) \not\equiv 0$ or $\partial_2 G(u,v) \not\equiv 0$ (a manifold condition).

Theorem 4.2 *Under assumptions* **A1** *and* **A2**, *any minimizer* $(u,v) \in H^s(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ *of* $E(u,v)$ *subject to the constraint* $Q(u,v) = \lambda$ *is radially symmetric (except for translation).*

Proof. First we prove that after a translation, (u,v) is symmetric with respect to x_1. In fact, after possibly a translation in the x_1 direction we may assume that

$$\int_{\{x_1<0\}} G(u,v)\,dx = \int_{\{x_1>0\}} G(u,v)\,dx = \frac{\lambda}{2}. \tag{4.1}$$

We put $u_1 = T_1 u$, $u_2 = T_2 u$, $v_1 = T_1 v$ and $v_2 = T_2 v$. By Theorem 3.3, the pairs (u_1, v_1) and (u_2, v_2) belong to $H^s(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$ and in view of (4.1) they also satisfy the constraint

$Q(u_1, v_1) = Q(u_2, v_2) = \lambda$. Moreover, defining $W(\varphi) = \displaystyle\int_{\mathbf{R}^N} |\xi|^{2s} |\widehat{\varphi}(\xi)|^2 \, d\xi$ and using (3.26) we see that

$$E(u_1, v_1) + E(u_2, v_2) - 2E(u, v) = \frac{1}{2} \frac{1}{(2\pi)^N} \left(W(u_1) + W(u_2) - 2W(u) \right)$$

$$= -\frac{1}{(2\pi)^N} \frac{8 \sin(s\pi)}{\pi^2} B_{N,s}(Au, Au) \leq 0.$$

We conclude that (u_1, v_1) and (u_2, v_2) are also minimizers and we must have $B_{N,s}(Au, Au) = 0$. By Theorem 3.7 we infer that $Au = 0$, that is u is symmetric with respect to x_1, i.e. $u = u_1 = u_2$.

Since (u, v) and $(u_1, v_1) = (u, v_1)$ are minimizers, they satisfy the Euler-Lagrange equations

$$\begin{cases} (-\Delta)^s u + \partial_1 F(u, v) + \alpha \partial_1 G(u, v) = 0, \\ -\Delta v + \partial_2 F(u, v) + \alpha \partial_2 G(u, v) = 0, \end{cases} \tag{4.2}$$

respectively

$$\begin{cases} (-\Delta)^s u + \partial_1 F(u, v_1) + \beta \partial_1 G(u, v_1) = 0, \\ -\Delta v_1 + \partial_2 F(u, v_1) + \beta \partial_2 G(u, v_1) = 0. \end{cases} \tag{4.3}$$

From (4.2), **A1**, the elliptic regularity for the Laplacian and its fractional powers and the usual boot-strap argument we get $u \in H^{2s}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $v \in H^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Of course that the same conclusion holds for (u, v_1). Notice that the L^p elliptic regularity for fractional powers of the Laplacian and for $1 < p < \infty$ follows from the fact that the multiplier $m(\xi) = \dfrac{(1 + |\xi|^2)^s}{1 + |\xi|^{2s}}$ satisfies the estimate $|D^\alpha m(\xi)| \leq \dfrac{B(\alpha)}{|\xi|^\alpha}$ and from the theorem of Mihlin-Hörmander.

We recall the following well-known result :

Unique Continuation Principle: *Assume that $\Phi \in H^2(\mathbf{R}^N, \mathbf{R}^m)$ solves the linear system*

$$-\Delta \Phi + A(x)\Phi(x) = 0 \qquad \text{in } \mathbf{R}^N, \tag{4.4}$$

where $A(x)$ is an $m \times m$ matrix whose elements belong to $L^\infty(\mathbf{R}^N)$. If $\Phi \equiv 0$ in some open set $\omega \subset \mathbf{R}^N$, then $\Phi \equiv 0$ in $\mathbf{R}^N$.

A proof for the Unique Continuation Principle is given in [13], Chapter VIII in the scalar case and in the appendix of [18] in the vector case. Notice that the Unique Continuation Principle is essentially a local result. Although it is stated for functions $\Phi \in H^2(\mathbf{R}^N)$, it is also valid for functions $\Phi \in W^{2,p}(\mathbf{R}^N)$ with $p > 2$ because $W^{2,p}_{loc}(\mathbf{R}^N) \subset H^2_{loc}(\mathbf{R}^N)$. This observation will be useful later.

Now let us come back to the proof of Theorem 4.2.

If $(u_1, v_1) = (0, 0)$, then obviously $u = 0$ in $\mathbf{R}^N$. By assumption **A2** and the regularity of v we have $\partial_2 F(0, v) = a_1(x)v$ and $\partial_2 G(0, v) = b_1(x)v$, where a_1, $b_1 \in L^\infty(\mathbf{R}^N)$. Using the second equation (4.2), the fact that $v = v_1$ in the half-space $\{x_1 < 0\}$ and the Unique Continuation Principle, we infer that $v = 0$ in $\mathbf{R}^N$, thus (u, v) is radially symmetric in a trivial way. It is obvious that this situation cannot occur if $\lambda \neq 0$.

If $(u_1, v_1) \neq (0, 0)$, it follows from **A2** that there exists $(x_1, x') \in (-\infty, 0) \times \mathbf{R}^{N-1}$ such that $\partial_1 G(u_1, v_1)(x_1, x') \neq 0$ or $\partial_2 G(u_1, v_1)(x_1, x') \neq 0$. Since $v = v_1$ for $x_1 < 0$, we infer from (4.2) and (4.3) that $\alpha = \beta$. Moreover, using the regularity of u, v, v_1 we get $\partial_2 F(u,v) - \partial_2 F(u,v_1) = b(x)(v(x) - v_1(x))$ and $\partial_2 G(u,v) - \partial_2 G(u,v_1) = c(x)(v(x) - v_1(x))$ where b, $c \in L^\infty(\mathbf{R}^N)$. Let $w(x) = v(x) - v_1(x)$. Using the second components of (4.2) and (4.3) and the fact that $\alpha = \beta$, we see that w satisfies the linear equation $-\Delta w(x) + a(x)w(x) = 0$ in $\mathbf{R}^N$, where $a = b + \alpha c \in L^\infty(\mathbf{R}^N)$. Since w vanishes on a half-space, by the Unique Continuation Principle we conclude that w vanishes everywhere, and this implies $v = v_1$ in $\mathbf{R}^N$. Thus we have shown that (u,v) is symmetric with respect to x_1.

Repeating this argument with the variables $x_2, \dots, x_N$, we find a new origin O' such that (u,v) is symmetric with respect to $x_1, \dots, x_N$. Then as in the proof of Theorem 4.1 we show that (u,v) is symmetric with respect to any hyperplane Π containing O', consequently (u,v) is radially symmetric with respect to the new origin O'. □

Remark 4.3 Symmetrization inequalities for functions in the space $H^{1/2}(\mathbf{R}^N)$ have been proved in [3]. Therefore if $s = \frac{1}{2}$, the function F in Theorem 4.2 satisfies the cooperative condition $\partial^2_{1,2} F(u,v) \leq 0$ (see [5]), G has a special form and it is known in advance that the components u, v of the minimizer are nonnegative, then using symmetrization one can conclude that *there exists* a radially symmetric minimizer.

Remark 4.4 In the case $F(u,v) = u^2 + v^2$, $G(u,v) = u^2 v$, by using symmetrization and Riesz' inequality it has been proved in [3] that *there exists* a radially symmetric minimizer. The fact that F and G are homogeneous plays a crucial role in their proof.

As an example of application for Theorem 4.2, we consider the Hamiltonian system :

$$(4.5) \qquad \begin{cases} \dfrac{\partial u}{\partial t} &=& \dfrac{\partial}{\partial x_1}((-\Delta)^{1/2} u + \partial_1 F(u,v)) \\ \\ \dfrac{\partial v}{\partial t} &=& \dfrac{\partial}{\partial x_1}(-\Delta v + \partial_2 F(u,v)). \end{cases}$$

The generalized multidimensional Benjamin-Ono equation

$$(4.6) \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}((-\Delta)^{1/2} u + g(u))$$

with $g(u) = u^2$ and the generalized multidimensional Korteweg-deVries equation

$$(4.7) \qquad \frac{\partial v}{\partial t} = \frac{\partial}{\partial x_1}(-\Delta v + f(v))$$

have been considered in [21] and in [4], respectively; in these papers, references giving the physical motivation for the above equations can also be found. System (4.5) can be considered a Hamiltonian coupling between (4.6) and (4.7).

Formally, system (4.5) has the following conserved quantities:

$$E(u,v) = \frac{1}{2}\int_{R^N} |(-\Delta)^{1/4} u|^2 + |\nabla v|^2 \, dx + \int_{R^N} F(u,v)\, dx \quad \text{and} \quad Q(u,v) = \frac{1}{2}\int_{R^N} (u^2 + v^2)\, dx.$$

If we minimize $E(u,v)$ subject to the constraint $Q(u,v) = \lambda$, where $\lambda > 0$, then according to [9] the set S_λ containing the elements of $H^{\frac{1}{2}}(R^N) \times H^1(R^N)$ where the minimum is achieved is invariant and orbitally stable with respect to (4.5). Since any element $(\phi, \psi) \in S_\lambda$ satisfies the Euler-Lagrange system

$$\begin{cases} (-\Delta)^{1/2}\phi + \partial_1 F(\phi,\psi) + c\phi &=& 0, \\ -\Delta\psi + \partial_2 F(\phi,\psi) + c\psi &=& 0, \end{cases}$$

we see that (ϕ, ψ) gives rise to a travelling wave solution of (4.5) of the form $(u(t,x), v(t,x)) = (\phi(x_1 - ct, x'), \psi(x_1 - ct, x'))$, $x' \in \mathbf{R}^{N-1}$. As a consequence of Theorem 4.2, the elements (ϕ, ψ) obtained in this way are radially symmetric (after a translation).

4.3 Next we consider the problem of minimizing the generalized Choquard functional

$$(4.8)\quad E(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} F(u(x))\frac{1}{|x-y|^{N-2}}F(u(y))\,dx\,dy + \int_{\mathbf{R}^N} H(u(x))\,dx$$

subject to the constraint $Q(u) = \int_{\mathbf{R}^N} G(u(x))\,dx = constant.$

It is worth to note that the complex version of E,

$$\tilde{E}(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx - \int_{R^N}\int_{R^N} F_1(|u(x)|^2)\frac{1}{|x-y|^{N-2}}F_1(|u(y)|^2)\,dx\,dy + \int_{R^N} H_1(|u(x)|^2)\,dx$$

is the Hamiltonian for the generalized Hartree equation

$$(4.9)\qquad iu_t + \Delta u + 4\left(\int_{\mathbf{R}^N} \frac{F_1(|u(y)|^2)}{|x-y|^{N-2}}\,dy\right) F_1'(|u|^2)(x)u(x) - 2H_1'(|u(x)|^2)u(x) = 0,$$

and $\tilde{Q}(u) = \int_{\mathbf{R}^N} |u^2(x)|\,dx$ is a conserved quantity for this evolution equation. The critical points of $\tilde{E} + \omega\tilde{Q}$ give rise to standing waves for (4.9). As far as minimization is concerned, using an argument of T. Cazenave and P.-L. Lions (see the proof of Theorem II.1 p. 555 in [9]), we can restrict ourselves to the real functionals $E(u)$ and $Q(u)$.

In the case $N = 3$, $F(u) = G(u) = u^2$ and $H(u) = 0$, the problem of minimizing $E(u)$ subject to $Q(u) = \lambda$ has been studied in [15], where the existence, the radial symmetry and the uniqueness of the minimizer have been proved. The symmetry was proved by using a sharp inequality for spherical rearrangements. This can still be used in our case if we konw that the minimizer is nonnegative and if we assume assume that F is increasing on $[0, \infty)$ (because the equality $F(u^*) = (F(u))^*$ is needed). Using the results in sections 2 and 3, we will show the radial symmetry of minimizers in dimension $N \geq 3$ under more general assumptions on F, G and H.

We begin by studying some properties of the nonlocal term appearing in (4.8) :

Lemma 4.5 *Let $N \geq 3$ and let $F : \mathbf{R} \longrightarrow \mathbf{R}$ be a function of class C^2 satisfying $F(0) = F'(0) = 0$ and*

$$|F'(x)| \leq C|x|^\sigma \qquad \text{for } |x| \geq 1,$$

where $C > 0$ is a constant and $\sigma < \frac{4}{N-2}$. Then the singular integral operator

$$I(\varphi)(x) = \int_{\mathbf{R}^N} \frac{1}{|x-y|^{N-2}}\varphi(y)\,dy$$

and the functional

$$M(\varphi) = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} F(\varphi(x))\frac{1}{|x-y|^{N-2}}F(\varphi(y))\,dx\,dy$$

have the following properties :

i) I is continuous from $L^p(\mathbf{R}^N)$ to $L^q(\mathbf{R}^N)$ if $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{2}{N}$.

ii) If $1 \leq p_1 < \frac{N}{2} < p_2 \leq \infty$, then I is continuous from $L^{p_1}(\mathbf{R}^N) \cap L^{p_2}(\mathbf{R}^N)$ to $L^\infty(\mathbf{R}^N) \cap C^0(\mathbf{R}^N)$.

iii) If $1 \le r_1 < \dfrac{2N}{N+2} < r_2 \le 2$ *and* $\varphi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$*, then*

$$\widehat{I(\varphi)}(\xi) = \frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)} \cdot \frac{1}{|\xi|^2}\widehat{\varphi}(\xi) \qquad \text{in } \mathcal{S}'(\mathbf{R}^N).$$

iv) M *is well-defined and differentiable on* $H^1(\mathbf{R}^N)$ *and*

$$M'(u).\varphi = 2\int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\frac{F(u(y))}{|x-y|^{N-2}}\,dy\right)F'(u(x))\varphi(x)\,dx.$$

v) For any $u \in H^1(\mathbf{R}^N)$ *we have*

$$M(u) = c_N\int_{\mathbf{R}^N}\frac{1}{|\xi|^2}|\widehat{F(u)}(\xi)|^2\,d\xi, \qquad \text{where } c_N = \frac{1}{2^{N-2}\pi^{\frac{N}{2}}\Gamma(\frac{N}{2}-1)}.$$

Proof. *i)* follows directly from Theorem 1 pp. 119-120 in [23].

ii) We write $\frac{1}{|x|^{N-2}}$ as $a_1(x)+a_2(x)$, where $a_1(x) = \frac{1}{|x|^{N-2}}\chi_{\{|x|>1\}}$ and $a_2(x) = \frac{1}{|x|^{N-2}}\chi_{\{|x|\le 1\}}$. Then we have $I(\varphi) = a_1 * \varphi + a_2 * \varphi$. It is obvious that $a_1 \in L^q(\mathbf{R}^N)$ for $q \in (\frac{N}{N-2},\infty]$ and $a_2 \in L^q(\mathbf{R}^N)$ for $q \in [1,\frac{N}{N-2})$. Let p_1' and p_2' be the conjugate exponents of p_1 and p_2. Then $p_1' > \frac{N}{N-2}$ and $p_2' < \frac{N}{N-2}$, so that $a_1 \in L^{p_1'}(\mathbf{R}^N)$ and $a_2 \in L^{p_2'}(\mathbf{R}^N)$. We infer that $I(\varphi)$ is continuous and by Young's inequality we get

$$||I(\varphi)||_{L^\infty}|| \le ||a_1||_{L^{p_1'}} \cdot ||\varphi||_{L^{p_1}} + ||a_2||_{L^{p_2'}} \cdot ||\varphi||_{L^{p_2}}.$$

iv) First we consider the bilinear form

$$P(\varphi,\psi) = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\varphi(x)\frac{1}{|x-y|^{N-2}}\overline{\psi(y)}\,dx\,dy.$$

Notice that P is well-defined and continuous on $L^{\frac{2N}{N+2}}(\mathbf{R}^N) \times L^{\frac{2N}{N+2}}(\mathbf{R}^N)$. Indeed, it follows from *i)* that I is well-defined and continuous from $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ to $L^{\frac{2N}{N-2}}(\mathbf{R}^N)$ and we have

$$|P(\varphi,\psi)| = \left|\int_{\mathbf{R}^N} I(\varphi)(x)\overline{\psi(x)}\,dx\right| \le ||I(\varphi)||_{L^{\frac{2N}{N-2}}} \cdot ||\psi||_{L^{\frac{2N}{N+2}}} \le A_N||\varphi||_{L^{\frac{2N}{N+2}}}||\psi||_{L^{\frac{2N}{N+2}}}.$$

Without loss of generality we may assume that $\sigma > \frac{2}{N}$. From the assumptions on F we have $|F(u)| \le C|u|^2$ if $|u| \le 1$ and $|F(u)| \le C|u|^{1+\sigma}$ if $|u| > 1$. It is well-known that $H^1(\mathbf{R}^N)$ is continuously embedded in $L^p(\mathbf{R}^N)$ for $p \in [2,\frac{2N}{N-2}]$ and then it is standard (see, e.g. [26], Appendix A) that $u \longmapsto F(u)$ is continuously differentiable from $H^1(\mathbf{R}^N)$ to $L^q(\mathbf{R}^N)$ for $q \in [\max(1,\frac{2}{1+\sigma}),\frac{2N}{(N-2)(1+\sigma)}]$. In particular, $u \longmapsto F(u)$ is continuously differentiable from $H^1(\mathbf{R}^N)$ to $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ (because $\frac{2}{1+\sigma} < \frac{2N}{N+2} < \frac{2N}{(N-2)(1+\sigma)}$). Since $M(u) = P(F(u),F(u))$, *iv)* follows.

iii) and *v)* Let $K(x) = \frac{1}{|x|^{N-2}}$. Then $K \in \mathcal{S}'(\mathbf{R}^N)$ and it follows from Theorem 4.1 p. 160 in [24] or from Lemma 1 p. 117 in [23] that $\widehat{K}(\xi) = \frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)} \cdot \frac{1}{|\xi|^2}$. From Lemma 1 p. 117 in [23] we have

$$P(\varphi,\psi) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N}\widehat{I(\varphi)}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi = c_N\int_{\mathbf{R}^N}\frac{1}{|\xi|^2}\widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}\,d\xi \qquad (4.10)$$

whenever φ, $\psi \in \mathcal{S}(\mathbf{R}^N)$. We claim that (4.10) holds for any φ, $\psi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ with $1 \le r_1 < \frac{2N}{N+2} < r_2 \le 2$. This assertion implies both *iii)* and *v)*.

Now let us prove the claim. Since (4.10) holds on $\mathcal{S} \times \mathcal{S}$, the bilinear form P is continuous on $L^{\frac{2N}{N+2}}(\mathbf{R}^N) \times L^{\frac{2N}{N+2}}(\mathbf{R}^N)$ and $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ is continuously embedded into $L^{\frac{2N}{N+2}}(\mathbf{R}^N)$, all we have to do is to show that the quadratic form

$$P_1(\varphi, \psi) = \int_{\mathbf{R}^N} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}\, d\xi$$

is continuous on $\Big(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)\Big) \times \Big(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)\Big)$; then the claim follows by density of $\mathcal{S}$ in $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$.

Let r_1', r_2' be the conjugate exponents of r_1, r_2 and let q_1, q_2 be such that $\frac{1}{r_1'} + \frac{1}{q_1} = \frac{1}{2}$, respectively $\frac{1}{r_2'} + \frac{1}{q_2} = \frac{1}{2}$. Let $b_1(\xi) = \frac{1}{|\xi|} \chi_{\{|\xi| \le 1\}}$ and $b_2(\xi) = \frac{1}{|\xi|} \chi_{\{|\xi| > 1\}}$. We have $2 \le q_1 < N$ and $q_2 > N$, so that $b_1 \in L^{q_1}(\mathbf{R}^N)$ and $b_2 \in L^{q_2}(\mathbf{R}^N)$. Since the Fourier transform maps continuously $L^{r_1}(\mathbf{R}^N)$ into $L^{r_1'}(\mathbf{R}^N)$ and $L^{r_2}(\mathbf{R}^N)$ into $L^{r_2'}(\mathbf{R}^N)$, we have :

$$|P_1(\varphi, \psi)| \le \left| \int_{\{|\xi| \le 1\}} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}\, d\xi \right| + \left| \int_{\{|\xi| > 1\}} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)}\, d\xi \right|$$

$$\le ||b_1 \widehat{\varphi}||_{L^2} \cdot ||b_1 \widehat{\psi}||_{L^2} + ||b_2 \widehat{\varphi}||_{L^2} \cdot ||b_2 \widehat{\psi}||_{L^2}$$

$$\le ||b_1||^2_{L^{q_1}} ||\widehat{\varphi}||_{L^{r_1'}} ||\widehat{\psi}||_{L^{r_1'}} + ||b_2||^2_{L^{q_2}} ||\widehat{\varphi}||_{L^{r_2'}} ||\widehat{\psi}||_{L^{r_2'}}$$

$$\le C(N, r_1, r_2) \left(||\varphi||_{L^{r_1}} ||\psi||_{L^{r_1}} + ||\varphi||_{L^{r_2}} ||\psi||_{L^{r_2}} \right).$$

This proves the continuity of P_1 and our claim. Thus the proof of Lemma 4.5 is complete. □

Theorem 4.6 *Let $N \ge 3$ and let F, G, $H : \mathbf{R} \longrightarrow \mathbf{R}$ be C^2 functions satisfying the following assumptions :*

a) $F(0) = F'(0) = 0$ and there exists $\sigma < \frac{4}{N-2}$ and $C > 0$ such that

$$|F'(u)| \le C|u|^\sigma \qquad \text{if } |u| \ge 1.$$

b) There exists $\sigma_1 \in [1, \frac{N+2}{N-2})$ and $C_1 > 0$ such that

$$|G'(u)| \le C_1 |u|^{\sigma_1} \qquad \text{and} \qquad |H'(u)| \le C_1 |u|^{\sigma_1} \qquad \text{for any } u \in \mathbf{R}.$$

Moreover, if $\sigma_1 < 2$ then we assume that $\sigma_1 \ge \max(\frac{(N-2)(1+2\sigma)-4}{N}, 1)$.

c) For any $\varepsilon > 0$, $G' \not\equiv 0$ on $(-\varepsilon, 0)$ and on $(0, \varepsilon)$.

Then any minimizer $u \in H^1(\mathbf{R}^N)$ of the functional E given by (4.8) subject to the constraint $Q(u) = \lambda$ is radially symmetric (after a translation in $\mathbf{R}^N$).

Proof. First of all, notice that the functionals E and Q are well-defined and of class C^1 on $H^1(\mathbf{R}^N)$. Let $u \in H^1(\mathbf{R}^N)$ be a minimizer. We will show that, except for translation, u is symmetric with respect to x_1. The same proof is valid for any other direction in $\mathbf{R}^N$ and the radial symmetry of u follows as in the proof of Theorem 4.1.

After a translation in the x_1 direction we may suppose that

$$\int_{\{x_1 < 0\}} G(u(x))\, dx = \int_{\{x_1 > 0\}} G(u(x))\, dx = \frac{\lambda}{2}.$$

As before, we define $u_1 = T_1 u$ and $u_2 = T_2 u$. We know that $u_1, u_2 \in H^1(\mathbf{R}^N)$. In view of assumption *a*), it is obvious that $F(u) \in L^1(\mathbf{R}^N)$ and we have $T_1(F(u)) = F(u_1)$, $T_2(F(u)) = F(u_2)$, $Q(u_1) = Q(u_2) = \lambda$. Defining $W(\varphi) = \displaystyle\int_{\mathbf{R}^N} \frac{1}{|\xi|^2} |\widehat{\varphi}(\xi)|^2 \, d\xi$, from Lemma 4.5 *v*) we get

$$
\begin{aligned}
E(u_1) + E(u_2) - 2E(u) &= -[M(u_1) + M(u_2) - 2M(u)] \\
&= -c_N [W(T_1(F(u))) + W(T_2(F(u))) - 2W(F(u))].
\end{aligned}
\tag{4.11}
$$

Recall that by (2.51) we have for any $\varphi \in C_c^\infty(\mathbf{R}^N)$,

$$
W(T_1\varphi) + W(T_2\varphi) - 2W(\varphi) = \frac{8}{\pi} \int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \left| \int_0^\infty \widehat{A\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 \right|^2 d\xi'. \tag{4.12}
$$

To show that this identity also holds for $F(u)$ we need the following lemma :

Lemma 4.7 *Let $N \geq 3$ and let r_1, r_2 be such that $1 < r_1 < \frac{2N}{N+2} < r_2 < 2$. The bilinear form*

$$
R(\varphi, \psi) = \int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}}(\eta_1, \xi') \frac{\eta_1}{|\xi'|^2 + \eta_1^2} \, d\eta_1 \, d\xi'
$$

is continuous on $\Big(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)\Big) \times \Big(L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)\Big)$.

Proof. Consider $\varphi, \psi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. Then $\widehat{\varphi}, \widehat{\psi} \in L^{r_1'}(\mathbf{R}^N) \cap L^{r_2'}(\mathbf{R}^N)$, where r_1' and r_2' are the conjugate exponents of r_1 and r_2. Using Hölder's inequality and the change of variable $\xi_1 = t|\xi'|$, we get for $\xi' \neq 0$ and $i = 1, 2$,

$$
\begin{aligned}
\left| \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 \right| &\leq \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_i'} \, d\xi_1 \right)^{\frac{1}{r_i'}} \left(\int_0^\infty \frac{\xi_1^{r_i}}{(|\xi'|^2 + \xi_1^2)^{r_i}} \, d\xi_1 \right)^{\frac{1}{r_i}} \\
&= |\xi'|^{\frac{1-r_i}{r_i}} \left(\int_0^\infty \frac{t^{r_i}}{(1+t^2)^{r_i}} \, dt \right)^{\frac{1}{r_i}} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_i'} \, d\xi_1 \right)^{\frac{1}{r_i'}} \\
&= C_i |\xi'|^{\frac{1-r_i}{r_i}} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_i'} \, d\xi_1 \right)^{\frac{1}{r_i'}}.
\end{aligned}
\tag{4.13}
$$

A similar estimate holds for ψ. Let q_i be the conjugate exponent of $\frac{r_i'}{2}$, i.e. $q_i = \frac{r_i}{2-r_i}$. Using (4.13), Hölder's inequality and the estimate $||\widehat{\varphi}||_{L^{r_i'}} \leq A_i ||\varphi||_{L^{r_i}}$ we have

$$
\begin{aligned}
&\left| \int_{B_{\mathbf{R}^{N-1}}(0,1)} \frac{1}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}}(\eta_1, \xi') \frac{\eta_1}{|\xi'|^2 + \eta_1^2} \, d\eta_1 \, d\xi' \right| \\
&\leq C_1^2 \int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{2-2r_1}{r_1} - 1} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_1'} \, d\xi_1 \right)^{\frac{1}{r_1'}} \left(\int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^{r_1'} \, d\eta_1 \right)^{\frac{1}{r_1'}} d\xi' \\
&\leq C_1^2 \left(\int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{q_1(2-3r_1)}{r_1}} \, d\xi' \right)^{\frac{1}{q_1}} \left(\int_{B_{\mathbf{R}^{N-1}}(0,1)} \int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_1'} \, d\xi_1 \, d\xi' \right)^{\frac{1}{r_1'}} \\
&\qquad \left(\int_{B_{\mathbf{R}^{N-1}}(0,1)} \int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^{r_1'} \, d\eta_1 \, d\xi' \right)^{\frac{1}{r_1'}} \\
&\leq C_1^2 A_1^2 \left(\int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{q_1(2-3r_1)}{r_1}} \, d\xi' \right)^{\frac{1}{q_1}} ||\varphi||_{L^{r_1}} ||\psi||_{L^{r_1}}
\end{aligned}
\tag{4.14}
$$

and

$$\Big| \int_{\{|\xi'|>1\}} \frac{1}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}}(\eta_1, \xi') \frac{\eta_1}{|\xi'|^2 + \eta_1^2} \, d\eta_1 \, d\xi' \Big|$$

$$\leq C_2^2 \int_{\{|\xi'|>1\}} |\xi'|^{\frac{2-2r_2}{r_2} - 1} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_2'} \, d\xi_1 \right)^{\frac{1}{r_2'}} \left(\int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^{r_2'} \, d\eta_1 \right)^{\frac{1}{r_2'}} d\xi'$$

$$(4.15) \qquad \leq C_2^2 \left(\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_1(2-3r_2)}{r_2}} \, d\xi' \right)^{\frac{1}{q_2}} \left(\int_{\{|\xi'|>1\}} \int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^{r_2'} \, d\xi_1 \, d\xi' \right)^{\frac{1}{r_2'}} \left(\int_{\{|\xi'|>1\}} \int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^{r_2'} \, d\eta_1 \, d\xi' \right)^{\frac{1}{r_2'}}$$

$$\leq C_2^2 A_2^2 \left(\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_2(2-3r_2)}{r_2}} \, d\xi' \right)^{\frac{1}{q_2}} ||\varphi||_{L^{r_2}} ||\psi||_{L^{r_2}}.$$

Since $1 < r_1 < \frac{2N}{N+2} < r_2 < 2$, a direct computation shows that $\displaystyle\int_{B_{\mathbf{R}^{N-1}}(0,1)} |\xi'|^{\frac{q_1(2-3r_1)}{r_1}} \, d\xi'$ and $\displaystyle\int_{\{|\xi'|>1\}} |\xi'|^{\frac{q_2(2-3r_2)}{r_2}} \, d\xi'$ are finite. From (4.14) and (4.15) we have

$$|R(\varphi, \psi)| \leq K \left(||\varphi||_{L^{r_1}} ||\psi||_{L^{r_1}} + ||\varphi||_{L^{r_2}} ||\psi||_{L^{r_2}} \right)$$

and Lemma 4.7 is proved. □

Let r_1 and r_2 be as in Lemma 4.7. Since the maps $\varphi \longmapsto T_1\varphi$ and $\varphi \longmapsto T_2\varphi$ are obviously continuous from $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$ into itself and we have shown in the proof of Lemma 4.5 that the bilinear form $P_1(\varphi, \psi) = \displaystyle\int_{\mathbf{R}^N} \frac{1}{|\xi|^2} \widehat{\varphi}(\xi) \overline{\widehat{\psi}(\xi)} \, d\xi$ is continuous on this space, it follows that the left-hand side of (4.12) is continuous on $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. By Lemma 4.7, the right-hand side of (4.12) also defines a continuous functional on $L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. Since (4.12) is valid for any $\varphi \in C_c^\infty(\mathbf{R}^N)$, by density we infer that (4.12) holds for any $\varphi \in L^{r_1}(\mathbf{R}^N) \cap L^{r_2}(\mathbf{R}^N)$. Recall that $u \in H^1(\mathbf{R}^N)$ and by the Sobolev embedding and assumption *a)* we have $F(u) \in L^q(\mathbf{R}^N)$ for any $q \in [\max(1, \frac{2}{1+\sigma}), \frac{2N}{(N-2)(1+\sigma)}]$; hence (4.12) is valid for $F(u)$.

Since u is a minimizer, we must have $E(u_1) + E(u_2) - 2E(u) \geq 0$. From (4.11) and (4.12) we infer that necessarily

$$(4.16) \qquad \int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \Big| \int_0^\infty \mathcal{F}(A(F(u)))(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 \Big|^2 d\xi' = 0.$$

Contrary to our previous examples, (4.16) *does not* imply directly $AF(u) \equiv 0$. To see this, consider a function $\psi \in C_c^\infty(0, \infty)$ such that $\text{supp}(\psi) \subset [1, \infty)$, $\psi \not\equiv 0$ and $\displaystyle\int_0^\infty \frac{t}{1+t^2} \psi(t) \, dt = 0$. (Such a function exists: for example, take two nonnegative functions ψ_0, $\psi_1 \in C_c^\infty(1, \infty)$ with disjoint supports and put $\psi_\tau = (1-\tau)\psi_0 - \tau\psi_1$. There is some $\tau \in (0,1)$ such that $\displaystyle\int_0^\infty \frac{t}{1+t^2} \psi_\tau(t) \, dt = 0$.) Extend ψ to an odd function defined on $\mathbf{R}$. Take $\alpha \in C_c^\infty(\mathbf{R}^{N-1})$ such that $\alpha \not\equiv 0$ and $\text{supp}(\alpha) \subset \mathbf{R}^{N-1} \setminus B(0,1)$ and put $\widehat{f}(\xi_1, \xi') = \alpha(\xi')\psi(\frac{\xi_1}{|\xi'|})$. Then $\widehat{f} \in C_c^\infty(\mathbf{R}^N)$ (hence $f \in \mathcal{S}$), $f \not\equiv 0$ and f is odd with respect to the first variable. However, we have

$\int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 = 0$ for any $\xi' \neq 0$ and consequently

$$\int_{\mathbf{R}^{N-1}} \frac{1}{|\xi'|} \left| \int_0^\infty \widehat{f}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2} \, d\xi_1 \right|^2 d\xi' = 0.$$

To show that u is symmetric with respect to x_1, we argue as follows: since u and u_1 minimize E under the constraint $Q = \lambda$, these functions satisfy the Euler-Lagrange equations $E'(u) + \alpha Q'(u) = 0$, respectively $E'(u_1) + \beta Q'(u_1) = 0$ for some constants α and β, that is

$$-\Delta u - 2I(F(u))F'(u) + H'(u) + \alpha G'(u) = 0 \qquad \text{in } \mathbf{R}^N, \tag{4.17}$$

$$-\Delta u_1 - 2I(F(u_1))F'(u_1) + H'(u_1) + \beta G'(u_1) = 0 \qquad \text{in } \mathbf{R}^N. \tag{4.18}$$

We will show in the next lemma that u and u_1 are smooth functions. Then we prove that $I(F(u))(x) = I(F(u_1))(x)$ in the half-space $\{x_1 < 0\}$. Together with assumption *c*), this implies that $\alpha = \beta$ in (4.17)-(4.18). Then we will be able to apply the Unique Continuation Principle to prove that $u = u_1$.

Lemma 4.8 *Let $u \in H^1(\mathbf{R}^N)$ be a solution of (4.17), where $F, G, H \in C^2(\mathbf{R})$ satisfy the assumptions a) and b) in Theorem 4.6. Then $u \in W^{3,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. In particular, $u \in C^2(\mathbf{R}^N)$ and $D^\alpha u$ are continuous and bounded on $\mathbf{R}^N$ if $\alpha \in \mathbf{N}^N$, $|\alpha| \leq 2$.*

Proof. The proof is rather classical and relies on a boot-strap argument. For the convenience of the reader, we give it here.

We show first that $u \in L^\infty(\mathbf{R}^N)$. By the Sobolev embedding we have $u \in L^q(\mathbf{R}^N)$ for $q \in [2, \frac{2N}{N-2}]$. We will improve this estimate by a bootstrap argument to get the desired conclusion.

Let us consider first the case $N = 3$. We may assume without loss of generality that $3 \leq \sigma < \frac{4}{N-2} = 4$ (if $\sigma < 3$, we replace σ by 3 and this gives no supplementary constraint on σ_1 in assumption *b*)). Suppose that $u \in L^q(\mathbf{R}^3)$ for any $q \in [2, \beta]$, where $\beta \geq 6$. Together with assumption *a*), this implies $F(u) \in L^q(\mathbf{R}^3)$ for $q \in [1, \frac{\beta}{1+\sigma}]$. We distinguish two cases :

Case A. If $\frac{\beta}{1+\sigma} > \frac{3}{2}$, then Lemma 4.5 *i*)-*ii*) implies $I(F(u)) \in L^q(\mathbf{R}^N)$ for $q \in (3, \infty]$. By assumption *a*) we have $F'(u)\chi_{\{|u|\leq 1\}} \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, hence $I(F(u))F'(u)\chi_{\{|u|\leq 1\}} \in L^q(\mathbf{R}^N)$ for $q \in (\frac{6}{5}, \infty]$ and $F'(u)\chi_{\{|u|>1\}} \in L^1(\mathbf{R}^N) \cap L^{\frac{\beta}{\sigma}}(\mathbf{R}^N)$, thus $I(F(u))F'(u)\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{\beta}{\sigma}]$. Consequently, $I(F(u))F'(u) \in L^q(\mathbf{R}^N)$ for $q \in (\frac{6}{5}, \frac{\beta}{\sigma}]$. Assumption *b*) implies that $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}]$. Note that $\frac{\beta}{\sigma_1} \geq \frac{6}{\sigma_1} > \frac{6}{5}$ and $\frac{\beta}{\sigma} \geq \frac{6}{\sigma} \geq \frac{2}{\sigma_1}$ by the second part of assumption *b*). From equation (4.17) we find $\Delta u \in L^q(\mathbf{R}^N)$ for any $q \in (\frac{6}{5}, \min(\frac{\beta}{\sigma}, \frac{\beta}{\sigma_1})]$ if $\frac{2}{\sigma_1} \leq \frac{6}{5}$, respectively for any $q \in [\frac{2}{\sigma_1}, \min(\frac{\beta}{\sigma}, \frac{\beta}{\sigma_1})]$ if $\frac{2}{\sigma_1} > \frac{6}{5}$. Let $q_* := \min(\frac{\beta}{\sigma}, \frac{\beta}{\sigma_1})$. If $q_* \geq 2$, we have $\Delta u \in L^{q_*}(\mathbf{R}^3)$ and obviously $u \in L^{q_*}(\mathbf{R}^N)$ (because $2 \leq q_* \leq \frac{\beta}{\sigma_1} \leq \beta$), hence $u \in W^{2,q_*}(\mathbf{R}^3)$ and by the Sobolev embedding we infer that $u \in L^\infty(\mathbf{R}^3)$. If $\frac{3}{2} < q_* < 2$, again by the Sobolev embedding we have $|\nabla u| \in L^{p_*}(\mathbf{R}^N)$, where $\frac{1}{p_*} = \frac{1}{q_*} - \frac{1}{3}$ (thus $p_* \in (3,6)$), hence $u \in W^{1,p_*}(\mathbf{R}^3) \subset L^\infty(\mathbf{R}^3)$. If $q_* = \frac{3}{2}$, we obtain $\Delta u \in L^{\frac{3}{2}}(\mathbf{R}^3)$, which implies $|\nabla u| \in L^3(\mathbf{R}^3)$, hence $u \in W^{1,3}(\mathbf{R}^3)$ so that $u \in L^q(\mathbf{R}^3)$ for any $q \in [2, \infty)$. Repeating the above argument for some $\tilde{\beta} > \beta$, we get $u \in L^\infty(\mathbf{R}^3)$. It remains to study the case $q_* < \frac{3}{2}$. It is clear that in this case we have $q_* = \frac{\beta}{\sigma_1}$ (because $\frac{\beta}{\sigma} > \frac{3}{2}$). Since $\Delta u \in L^{q_*}(\mathbf{R}^3)$, by the Sobolev embedding we get $u \in L^{\beta_1}(\mathbf{R}^3)$, where $\frac{1}{\beta_1} = \frac{1}{q_*} - \frac{2}{3}$. Notice that $\frac{1}{\beta_1} - \frac{1}{\beta} = \frac{\sigma_1 - 1}{\beta} - \frac{2}{3} \leq \frac{\sigma_1 - 5}{6} < 0$, hence $\beta_1 > \beta$. We repeat the previous reasoning with β_1 instead of β. We obtain that either $u \in L^\infty(\mathbf{R}^3)$, or $u \in L^{\beta_2}(\mathbf{R}^3)$, where $\beta_2 > \beta_1$ and

$\frac{1}{\beta_2} - \frac{1}{\beta_1} \le \frac{\sigma_1 - 5}{6} < 0$. In the latter case we continue with β_2 instead of β and we get that either $u \in L^\infty(\mathbf{R}^3)$, or $u \in L^{\beta_3}(\mathbf{R}^3)$, where $\beta_3 > \beta_2$ and $\frac{1}{\beta_3} - \frac{1}{\beta_2} \le \frac{\sigma_1 - 5}{6}$, and so on. After a finite number of steps we get $u \in L^\infty(\mathbf{R}^3)$ (since otherwise we would obtain a positive increasing sequence $(\beta_n)_{n\ge 1}$ such that $\frac{1}{\beta_n} - \frac{1}{\beta} \le \frac{n(\sigma_1 - 5)}{6} \longrightarrow -\infty$, which is impossible).

Case B. If $\frac{\beta}{1+\sigma} \le \frac{3}{2}$, we may suppose that $\frac{\beta}{1+\sigma} < \frac{3}{2}$ (otherwise we take β a little bit smaller). By Lemma 4.5 *i)* we have $I(F(u)) \in L^q(\mathbf{R}^3)$ for $q \in (3, \left(\frac{1+\sigma}{\beta} - \frac{2}{3}\right)^{-1}]$. As in case A we obtain $I(F(u))F'(u)\chi_{\{|u|\le 1\}} \in L^q(\mathbf{R}^N)$ for $q \in (\frac{6}{5}, \left(\frac{1+\sigma}{\beta} - \frac{2}{3}\right)^{-1}]$ and $I(F(u))F'(u)\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ for $q \in \left[1, \left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1}\right]$, so that $I(F(u))F'(u) \in L^q(\mathbf{R}^3)$ for $q \in \left(\frac{6}{5}, \left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1}\right]$. Notice that $\left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1} > \frac{6}{5}$ (because $\beta \ge 6$ and $\sigma < 4$) and $\left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1} \ge \frac{2}{\sigma_1}$ by assumption *b)*. Since obviously $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}]$, using equation (4.17) we infer that $\Delta u \in L^q(\mathbf{R}^3)$ for any $q \in \left[\max(\frac{6}{5}, \frac{2}{\sigma_1}), \min\left(\left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1}, \frac{\beta}{\sigma_1}\right)\right]$, $q \ne \frac{6}{5}$. Let $q_2 = \min\left(\left(\frac{1+2\sigma}{\beta} - \frac{2}{3}\right)^{-1}, \frac{\beta}{\sigma_1}\right)$. If $q_2 \ge \frac{3}{2}$, arguing as in case A we get $u \in L^\infty(\mathbf{R}^3)$. If $q_2 < \frac{3}{2}$, by the Sobolev embedding we have $u \in L^{\beta_1}(\mathbf{R}^3)$, where $\frac{1}{\beta_1} = \frac{1}{q_2} - \frac{2}{3}$, hence $\frac{1}{\beta_1} - \frac{1}{\beta} \le \max\left(\frac{\sigma-4}{3}, \frac{\sigma_1-5}{6}\right) < 0$, so that $\beta_1 > \beta$. Repeating the preceeding arguments for β_1 we obtain either $u \in L^\infty(\mathbf{R}^3)$, or $\frac{\beta_1}{1+\sigma} > \frac{3}{2}$ (so that we are in case A, consequently $u \in L^\infty(\mathbf{R}^3)$), or $u \in L^{\beta_2}(\mathbf{R}^3)$, where $\beta_2 > \beta_1$ and $\frac{1}{\beta_2} - \frac{1}{\beta_1} \le \max\left(\frac{\sigma-4}{3}, \frac{\sigma_1-5}{6}\right)$. In the latter case we repeat the same reasoning, and so on. As in case A, after a finite number of steps we get $u \in L^\infty(\mathbf{R}^3)$.

Now we consider the case $N \ge 4$ and we assume that $u \in L^q(\mathbf{R}^N)$ for any $q \in [2, \beta]$, where $\beta \ge \frac{2N}{N-2}$. It is clear that $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in \left[\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}\right]$ and $F(u) \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{\beta}{1+\sigma}]$. Once again, we distinguish two cases :

Case A. If $\frac{\beta}{1+\sigma} > \frac{N}{2}$, then $I(F(u)) \in L^q(\mathbf{R}^N)$ for any $q \in (\frac{N}{N-2}, \infty]$. We have $F'(u)\chi_{\{|u|\le 1\}} \in L^q(\mathbf{R}^N)$ for $q \in [2, \infty]$, hence $I(F(u))F'(u))\chi_{\{|u|\le 1\}} \in L^q(\mathbf{R}^N)$ for $q \in (1, \infty]$ if $N = 4$, respectively for $q \in [1, \infty]$ if $N \ge 5$ and $F'(u)\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ for $q \in [1, \frac{\beta}{\sigma}]$, hence $I(F(u))F'(u))\chi_{\{|u|>1\}} \in L^q(\mathbf{R}^N)$ if $q \in [1, \frac{\beta}{\sigma}]$. Consequently $I(F(u))F'(u)) \in L^q(\mathbf{R}^N)$ for $q \in (1, \frac{\beta}{\sigma}]$ if $N = 4$, respectively for $q \in [1, \frac{\beta}{\sigma}]$ if $N \ge 5$. Notice that $\beta \ge \frac{2N}{N-2}$ and the second part of assumption b) imply $\frac{\beta}{\sigma} \ge \frac{2}{\sigma_1}$. Using equation (4.17) we infer that $\Delta u \in L^q(\mathbf{R}^N)$ for $q \in \left[\max(1, \frac{2}{\sigma_1}), \min(\frac{\beta}{\sigma_1}, \frac{\beta}{\sigma})\right]$, $q \ne 1$ if $N = 4$. Let $q_3 = \min(\frac{\beta}{\sigma_1}, \frac{\beta}{\sigma})$. Notice that $q_3 \le \beta$ because $\sigma_1 \ge 1$ and $\Delta u \in L^{q_3}(\mathbf{R}^N)$. If $q_3 > \frac{N}{2} \ge 2$, then $u \in L^{q_3}(\mathbf{R}^N)$, hence $u \in W^{2,q_3}(\mathbf{R}^N)$ and by the Sobolev embedding we get $u \in L^\infty(\mathbf{R}^N)$. If $q_3 = \frac{N}{2}$, then $u \in W^{2,\frac{N}{2}}(\mathbf{R}^N)$, consequently $u \in L^q(\mathbf{R}^N)$ for any $q \in [2, \infty)$ and repeating the above proof with $\tilde{\beta} > \beta$ we find $u \in L^\infty(\mathbf{R}^N)$. If $q_3 < \frac{N}{2}$, then necessarily $q_3 = \frac{\beta}{\sigma_1}$ (recall that $\frac{\beta}{\sigma} > \frac{\beta}{1+\sigma} > \frac{N}{2}$ because we are in case A). By the Sobolev embedding we get $u \in L^{\beta_1}(\mathbf{R}^N)$, where $\frac{1}{\beta_1} = \frac{1}{q_3} - \frac{2}{N} = \frac{\sigma_1}{\beta} - \frac{2}{N}$, thus $\frac{1}{\beta_1} - \frac{1}{\beta} = \frac{\sigma_1 - 1}{\beta} - \frac{2}{N} \le \frac{(\sigma_1-1)(N-2)-4}{2N} < 0$ by b). Repeating the previous arguments with β replaced by β_1, we find that either $u \in L^\infty(\mathbf{R}^N)$ or $u \in L^{\beta_2}(\mathbf{R}^N)$, where $\beta_2 > \beta_1$ and $\frac{1}{\beta_2} - \frac{1}{\beta_1} \le \frac{(\sigma_1-1)(N-2)-4}{2N}$, and so on. As previously, after a finite number of steps we get $u \in L^\infty(\mathbf{R}^N)$.

Case B. If $\frac{\beta}{1+\sigma} \le \frac{N}{2}$, we may suppose that $\frac{\beta}{1+\sigma} < \frac{N}{2}$. By Lemma 4.5 *i)*, $I(F(u)) \in L^q(\mathbf{R}^N)$ for $q \in \left(\frac{N}{N-2}, \left(\frac{1+\sigma}{\beta} - \frac{2}{N}\right)^{-1}\right]$. As in case A we get $I(F(u))F'(u) \in L^q(\mathbf{R}^N)$ for

$q \in \left[1, \left(\frac{1+2\sigma}{\beta} - \frac{2}{N}\right)^{-1}\right]$, $q \neq 1$ if $N = 4$. By a), b) and the fact that $\beta \geq \frac{2N}{N-2}$ we have $\left(\frac{1+2\sigma}{\beta} - \frac{2}{N}\right)^{-1} \geq \frac{2}{\sigma_1}$. Since $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \frac{\beta}{\sigma_1}]$, using (4.17) we get $\Delta u \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), q_4]$, $q \neq 1$ if $N = 4$, where $q_4 = \min\left(\frac{\beta}{\sigma_1}, \left(\frac{1+2\sigma}{\beta} - \frac{2}{N}\right)^{-1}\right)$. If $q_4 \geq \frac{N}{2}$ then, as above, we obtain $u \in L^\infty(\mathbf{R}^N)$. Otherwise by the Sobolev embedding we find $u \in L^{\beta_1}(\mathbf{R}^N)$, where $\frac{1}{\beta_1} = \frac{1}{q_4} - \frac{2}{N}$, thus $\frac{1}{\beta_1} - \frac{1}{\beta} \leq \max\left(\frac{(\sigma_1-1)(N-2)-4}{2N}, \frac{\sigma(N-2)-4}{N}\right) < 0$. Then we restart the process with β_1 instead of β. Continuing in this way, after a finite number of steps we obtain $u \in L^\infty(\mathbf{R}^N)$.

Up to now we have proved that $u \in L^q(\mathbf{R}^N)$ for any $q \in [2, \infty]$. Thus $F(u) \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $I(F(u)) \in L^q(\mathbf{R}^N)$ for $q \in (\frac{N}{N-2}, \infty]$, $F'(u) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, hence $I(F(u))F'(u) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Clearly $G'(u)$, $H'(u) \in L^q(\mathbf{R}^N)$ for $q \in [\max(1, \frac{2}{\sigma_1}), \infty]$. Using (4.17) we have $\Delta u \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, thus $u \in W^{2,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. In particular, $u \in C^1(\mathbf{R}^N)$ and $\frac{\partial u}{\partial x_i}$ are continuous and bounded on $\mathbf{R}^N$. Differentiating (4.17) with respect to x_i we get

$$-\Delta(\frac{\partial u}{\partial x_i}) - 2I(F'(u)\frac{\partial u}{\partial x_i})F'(u) - 2I(F(u))F''(u)\frac{\partial u}{\partial x_i} + G''(u)\frac{\partial u}{\partial x_i} + \alpha H''(u)\frac{\partial u}{\partial x_i} = 0 \quad \text{in } \mathbf{R}^N.$$

It follows that $-\Delta(\frac{\partial u}{\partial x_i}) \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Since obviously $\frac{\partial u}{\partial x_i} \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, we get $\frac{\partial u}{\partial x_i} \in W^{2,p}(\mathbf{R}^N)$, which implies $u \in W^{3,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. □

It follows from Lemma 4.8 that $F(u) \in C^2(\mathbf{R}^N)$ and $F(u) \in W^{2,p}(\mathbf{R}^N)$ for $p \in [1, \infty]$. Using Lemma 4.5 *i)* and *ii)*, it is easy to check that $I(F(u)) \in C^2(\mathbf{R}^N)$ and $I(F(u)) \in W^{2,p}(\mathbf{R}^N)$ for $p \in (\frac{N}{N-2}, \infty]$. In particular, $I(F(u)) \in \mathcal{S}'(\mathbf{R}^N)$ and Lemma 4.5 *iii)* implies $\mathcal{F}(I(F(u)))(\xi) = d_N \frac{1}{|\xi|^2} \widehat{F(u)}(\xi)$, where $d_N = \frac{4\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}-1)}$. Setting $U = I(F(u))$ we have $-\Delta U = d_N F(u)$.

Next we show that $\frac{\partial U}{\partial x_1}(0, x') = \frac{\partial}{\partial x_1} I(F(u))(0, x') = 0$ for any $x' \in \mathbf{R}^{N-1}$. From (4.16) we infer that $\int_0^\infty \mathcal{F}(A(F(u)))(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 = 0$ for almost every $\xi' \in \mathbf{R}^{N-1}$, that is $\int_{-\infty}^\infty \widehat{F(u)}(\xi_1, \xi') \frac{\xi_1}{|\xi'|^2 + \xi_1^2}\, d\xi_1 = 0$ a.e. $\xi' \in R^{N-1}$, or equivalently

$$\int_{-\infty}^\infty \xi_1 \mathcal{F}(I(F(u)))(\xi_1, \xi')\, d\xi_1 = 0 \qquad \text{for almost every } \xi' \in R^{N-1}. \tag{4.19}$$

If $\frac{\partial}{\partial x_1} I(F(u))$ and $\mathcal{F}(\frac{\partial}{\partial x_1} I(F(u)))$ are in $L^1(\mathbf{R}^N)$, by the Fourier inversion theorem (4.19) is equivalent to $\frac{\partial}{\partial x_1} I(F(u))(0, x') = 0$, as desired.

Since we do not know whether $\frac{\partial}{\partial x_1} I(F(u)) \in L^1(\mathbf{R}^N)$ and $\mathcal{F}(\frac{\partial}{\partial x_1} I(F(u))) \in L^1(\mathbf{R}^N)$, we argue as follows : we take an arbitrary test function $\psi \in \mathcal{S}(\mathbf{R}^{N-1})$ and we put $\varphi_n(x_1) = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2 x_1^2}{2}}$. Clearly, $\varphi_n(x_1) = n\varphi_1(nx_1)$, $||\varphi_n||_{L^1(\mathbf{R})} = 1$ and $\widehat{\varphi}_n(\xi_1) = e^{-\frac{\xi_1^2}{2n^2}}$. On one hand we have, by using Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbf{R}^N} \varphi_n(x_1)\psi(x') \left[\frac{\partial}{\partial x_1} I(F(u))\right](x_1, x')\, dx \\ &= \lim_{n\to\infty} \int_{\mathbf{R}^N} \varphi_1(y_1)\psi(x') \left[\frac{\partial}{\partial x_1} I(F(u))\right](\frac{y_1}{n}, x')\, dy_1\, dx' \\ &= \int_{\mathbf{R}^{N-1}} \psi(x') \frac{\partial}{\partial x_1}(I(F(u)))(0, x')\, dx'. \end{aligned} \tag{4.20}$$

On the other hand we have

$$\int_{\mathbf{R}^N} \varphi_n(x_1)\psi(x') \left[\frac{\partial}{\partial x_1} I(F(u))\right](x_1,x')\,dx = \langle \frac{\partial}{\partial x_1}(I(F(u))), \varphi_n(x_1)\psi(x')\rangle_{\mathcal{S}',\mathcal{S}}$$

$$(4.21)\qquad = \langle \mathcal{F}\left(\frac{\partial}{\partial x_1} I(F(u))\right), \mathcal{F}^{-1}\left(\varphi_n(x_1)\psi(x')\right)\rangle_{\mathcal{S}',\mathcal{S}}$$

$$= \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} \frac{i d_N \xi_1}{|\xi|^2}\widehat{F(u)}(\xi) e^{-\frac{\xi_1^2}{2n^2}}\widehat{\psi}(-\xi')\,d\xi_1\,d\xi'.$$

Since $F(u) \in L^2(\mathbf{R}^N)$, for almost every $\xi' \in \mathbf{R}^{N-1}$ we have $\widehat{F(u)}(\cdot,\xi') \in L^2(\mathbf{R})$. For any such ξ', arguing as in (4.13) we get

$$\int_{\mathbf{R}} \left| e^{-\frac{\xi_1^2}{2n^2}} \cdot \frac{\xi_1}{|\xi|^2}\widehat{F(u)}(\xi_1,\xi')\right| d\xi_1 \le \int_{\mathbf{R}} \left|\frac{\xi_1}{\xi_1^2+|\xi'|^2}\widehat{F(u)}(\xi_1,\xi')\right| d\xi_1 \le \frac{C}{|\xi'|^{\frac{1}{2}}}||\widehat{F(u)}(\cdot,\xi')||_{L^2(\mathbf{R})},$$

where C does not depend on ξ'. Moreover, Cauchy-Schwarz inequality gives

$$\int_{\mathbf{R}^{N-1}} \frac{C|\widehat{\psi}(-\xi')|}{|\xi'|^{\frac{1}{2}}}||\widehat{F(u)}(\cdot,\xi')||_{L^2(\mathbf{R})}\,d\xi' \le C\left(\int_{\mathbf{R}^{N-1}} \frac{|\widehat{\psi}(-\xi')|^2}{|\xi'|}\,d\xi'\right)^{\frac{1}{2}} ||\widehat{F(u)}||_{L^2(\mathbf{R}^N)} < \infty.$$

By the Dominated Convergence Theorem, we have for almost any $\xi' \in \mathbf{R}^{N-1}$

$$\int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2+|\xi'|^2}\widehat{F(u)}(\xi_1,\xi')e^{-\frac{\xi_1^2}{2n^2}}\,d\xi_1 \longrightarrow \int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2+|\xi'|^2}\widehat{F(u)}(\xi_1,\xi')\,d\xi_1 = 0 \qquad \text{as } n \longrightarrow \infty.$$

Thus we may use Fubini's Theorem, then the Dominated Convergence Theorem on $\mathbf{R}^{N-1}$ to obtain

$$\int_{\mathbf{R}^N} \frac{\xi_1}{|\xi|^2}\widehat{F(u)}(\xi_1,\xi')e^{-\frac{\xi_1^2}{2n^2}}\psi(-\xi')\,d\xi_1\,d\xi'$$

$$(4.22)\qquad = \int_{\mathbf{R}^{N-1}} \psi(-\xi')\int_{\mathbf{R}} \frac{\xi_1}{\xi_1^2+|\xi'|^2}\widehat{F(u)}(\xi_1,\xi')e^{-\frac{\xi_1^2}{2n^2}}\,d\xi_1\,d\xi'$$

$$\longrightarrow \int_{\mathbf{R}^{N-1}} \psi(-\xi')\cdot 0\,d\xi' = 0. \qquad \text{as } n \longrightarrow \infty.$$

From (4.20), (4.21) and (4.22) we infer that $\int_{\mathbf{R}^{N-1}} \psi(x')\frac{\partial}{\partial x_1}(I(F(u)))(0,x')\,dx' = 0$. Since $\psi \in \mathcal{S}(\mathbf{R}^{N-1})$ was arbitrary, we have $\frac{\partial}{\partial x_1}(I(F(u)))(0,\cdot) = 0$ in $\mathcal{S}'(\mathbf{R}^{N-1})$, hence $\frac{\partial}{\partial x_1}(I(F(u)))(0,x') = 0$ for any $x' \in \mathbf{R}^{N-1}$ because $\frac{\partial}{\partial x_1}(I(F(u)))$ is a continuous function.

We know that $F(u_1)$ is symmetric with respect to x_1 and a simple change of variables shows that the function $U_1 := I(F(u_1))$ is also symmetric with respect to x_1. Clearly U_1 also belongs to $C^2(\mathbf{R}^N)$ and satisfies $-\Delta U_1 = -\Delta(I(F(u_1))) = d_N F(u_1)$. By symmetry we have $\frac{\partial U_1}{\partial x_1}(0,x') = 0$ for any $x' \in \mathbf{R}^{N-1}$. Since $u_1(x_1,x') = u(x_1,x')$ if $x_1 < 0$, we have proved that the functions U and U_1 are both solutions of the problem

$$(4.23)\qquad \begin{cases} -\Delta W = d_N F(u) & \text{in } \{(x_1,x') \in \mathbf{R}^N \mid x_1 < 0\} \\ W \in C^2(\mathbf{R}^N)\cap W^{2,p}(\mathbf{R}^N) & \text{for } p > \frac{N}{N-2}, \\ \frac{\partial W}{\partial x_1}(0,x') = 0 & \text{for any } x' \in \mathbf{R}^{N-1}. \end{cases}$$

It is not hard to see that the solution of (4.23) is unique. Consequently, $U(x_1, x') = U_1(x_1, x')$ if $x_1 < 0$. It is obvious that (u, U) and (u_1, U_1) solve the system

$$\text{(4.24)} \qquad \begin{cases} -\Delta u - 2UF'(u) + H'(u) + \alpha G'(u) = 0 \\ \\ -\Delta U - d_N F(u) = 0 \end{cases} \qquad \text{in } \mathbf{R}^N,$$

respectively

$$\text{(4.25)} \qquad \begin{cases} -\Delta u_1 - 2U_1 F'(u_1) + H'(u_1) + \beta G'(u_1) = 0 \\ \\ -\Delta U_1 - d_N F(u_1) = 0 \end{cases} \qquad \text{in } \mathbf{R}^N.$$

Next we show that if $u \equiv 0$ in the half-space $\{x_1 < 0\}$, then $u \equiv 0$ in $\mathbf{R}^N$. Indeed, if $u = 0$ in $\{x_1 < 0\}$, then from (4.23) it follows that $U = 0$ on that half-space. Now from (4.24) and the Unique Continuation Principle we infer that $(u, U) = (0, 0)$ on $\mathbf{R}^N$. In this case u trivially has a radial symmetry. Clearly, we cannot have $u \equiv 0$ if $\lambda \neq 0$.

If $u \not\equiv 0$ in $(-\infty, 0) \times \mathbf{R}^{N-1}$, then $u((-\infty, 0) \times \mathbf{R}^{N-1}) = u_1((-\infty, 0) \times \mathbf{R}^{N-1})$ contains an interval of the form $(-\varepsilon, 0)$ or $(0, \varepsilon)$ for some $\varepsilon > 0$. Now assumption *c*), (4.24), (4.25) and the fact that $(u, U) = (u_1, U_1)$ on $(-\infty, 0) \times \mathbf{R}^{N-1}$ imply that $\alpha = \beta$ in (4.24)-(4.25). As a consequence, we see that $(u - u_1, U - U_1)$ solves a linear system whose coefficients belong to $L^\infty(\mathbf{R}^N)$. Since $(u, U) = (u_1, U_1)$ for $x_1 < 0$ and (u, U), $(u_1, U_1) \in W^{2,p}(\mathbf{R}^N, \mathbf{R}^2)$ if $p \geq 2$ and $p > \frac{N}{N-2}$, by using the Unique Continuation Principle we infer that $u = u_1$ (and $U = U_1$) in $\mathbf{R}^N$, that is u is symmetric with respect to x_1.

Similarly we show that u is symmetric with respect to any other hyperplane Π which has the property that $\int_{\Pi_-} G(u(x))\, dx = \int_{\Pi_+} G(u(x))\, dx$, where Π_- and Π_+ are the two half-spaces determined by Π. As in the proof of Theorem 4.1 it follows that after a translation, u is radially symmetric. The proof of Theorem 4.6 is complete. □

4.4 Our last application concerns the Davey-Stewartson system

$$\text{(4.26)} \qquad \begin{cases} iu_t + \Delta u &= f(|u|^2)u - uv_{x_1}, \\ \\ \Delta v &= (|u|^2)_{x_1} \end{cases} \qquad \text{in } \mathbf{R}^3,$$

which can be written as

$$\text{(4.27)} \qquad iu_t = -\Delta u + f(|u|^2)u + R_1^2(|u|^2)u,$$

where R_1 is the Riesz transform defined by $\widehat{R_1\varphi} = \dfrac{i\xi_1}{|\xi|}\widehat{\varphi}(\xi)$. Let $F_1(t) = \displaystyle\int_0^t f(\tau)\, d\tau$. It is easy to check that

$$\tilde{E}(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\, dx + \frac{1}{2}\int_{\mathbf{R}^3} F_1(|u|^2)\, dx - \frac{1}{4}\int_{\mathbf{R}^3} |R_1(|u|^2)|^2\, dx$$

is a Hamiltonian for (4.27) and $\tilde{Q}(u) = \displaystyle\int_{\mathbf{R}^3} |u(x)|^2\, dx$ is a conserved quantity for the same equation. The standing waves for (4.27) are precisely the critical points of $\tilde{E} + \omega\tilde{Q}$. As in the previous example, when we minimize $\tilde{E}(u)$ subject to $\tilde{Q}(u) = constant$, we may restrict ourselves to real functions u and to the real version of $\tilde{E}$,

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\, dx + \int_{\mathbf{R}^3} F(u)\, dx - \frac{1}{4}\int_{\mathbf{R}^3} |R_1(u^2)|^2\, dx.$$

We will consider a more general functional than $\tilde{Q}$, namely $Q(u) = \int_{\mathbf{R}^3} G(u)\,dx$. If $G(u) = u^2$, in order to guarantee the boundedness from below of the functional E on the set of functions satisfying $Q(u) = \lambda$, the function $F(u)$ is required to behave as $a|u|^\gamma$ for u large, with $a > 0$ and $\gamma > 4$. In the case $F(u) = a|u|^\gamma$, the Cauchy problem for the evolution equation (4.27) has been analysed in [12]. The global existence of solutions was proved in the case $a > 0$ and $\gamma > 4$, while in the case $\gamma = 4$ the global existence was proved if a is sufficiently large.

Still in the case of pure power $F(u) = a|u|^\gamma$, with $a > 0$ and $\gamma > 4$, the existence of minimizers of E subject to the constraint $Q(u) = \int_{\mathbf{R}^3} |u|^2\,dx = \lambda$ can be proved by using the Concentration-Compactness Principle (see [17]) if λ is large enough (this assumption is needed to prevent vanishing).

In [10] the existence of ground states related to the problem (4.26) has been studied. However, our method cannot be used to prove the symmetry of these ground states because the nonlocal term appears in the constraint.

It is well-known that R_1 is a linear continuous map from $L^p(\mathbf{R}^3)$ to $L^p(\mathbf{R}^3)$ for $1 < p < \infty$ (see [23]). If $u^2 \in L^2(\mathbf{R}^3)$, then $R_1(u^2) \in L^2(\mathbf{R}^3)$ and by Plancherel's theorem we get

$$\int_{\mathbf{R}^3} |R_1(u^2)|^2\,dx = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} |\widehat{R_1(u^2)}(\xi)|^2\,d\xi = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}|\widehat{u^2}(\xi)|^2\,d\xi. \tag{4.28}$$

We have the following symmetry result :

Theorem 4.9 *Let $u \in H^1(\mathbf{R}^3)$ be a solution of the minimization problem*

$$\textit{minimize } E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \int_{\mathbf{R}^3} F(u)\,dx - \frac{1}{4}\int_{\mathbf{R}^3} |R_1(u^2)|^2\,dx$$

$$\textit{subject to } Q(u) = \int_{\mathbf{R}^3} G(u(x))\,dx = \lambda$$

under the following assumptions :

a) $F, G : \mathbf{R} \longrightarrow \mathbf{R}$ are C^2 functions, $F(0) = F'(0) = 0$, $G(0) = G'(0) = 0$ and there exist $C > 0$, $\sigma < 5$ such that

$$|F'(u)| \le C|u|^\sigma \qquad \textit{and} \qquad |G'(u)| \le C|u|^\sigma \qquad \textit{for } |u| \ge 1.$$

b) For any $\varepsilon > 0$, $G' \ne 0$ on $(-\varepsilon, 0)$ and on $(0, \varepsilon)$.

Then, after a translation, u is radially symmetric in the variables (x_2, x_3) (i.e. u is axially symmetric).

Proof. Making a translation in the x_2 direction if necessary, we may assume that $\int_{\{x_2<0\}} G(u(x))\,dx = \int_{\{x_2>0\}} G(u(x))\,dx = \frac{\lambda}{2}$. As before, we define u_1 and u_2 by

$$u_1(x_1,x_2,x_3) = \begin{cases} u(x_1,x_2,x_3) & \text{if } x_2 < 0,\\ u(x_1,-x_2,x_3) & \text{if } x_2 \ge 0 \end{cases} \qquad u_2(x_1,x_2,x_3) = \begin{cases} u(x_1,-x_2,x_3) & \text{if } x_2 < 0,\\ u(x_1,x_2,x_3) & \text{if } x_2 \ge 0. \end{cases}$$

It is obvious that $Q(u_1) = Q(u_2) = \lambda$. Moreover, using (4.28) we get

$$\begin{aligned} &E(u_1) + E(u_2) - 2E(u) \\ &= -\frac{1}{4}\frac{1}{(2\pi)^3}\left[\int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}|\widehat{u_1^2}(\xi)|^2\,d\xi + \int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}|\widehat{u_2^2}(\xi)|^2\,d\xi - 2\int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2}|\widehat{u^2}(\xi)|^2\,d\xi\right]. \end{aligned} \tag{4.29}$$

Recall that by (2.53) and (2.54) we have the equality

$$
\begin{aligned}
&\int_{\mathbf{R}^N} \frac{\xi_j^2}{|\xi|^2} |\widehat{T_1\varphi}(\xi)|^2 \, d\xi + \int_{\mathbf{R}^N} \frac{\xi_j^2}{|\xi|^2} |\widehat{T_2\varphi}(\xi)|^2 \, d\xi - 2 \int_{\mathbf{R}^N} \frac{\xi_j^2}{|\xi|^2} |\widehat{\varphi}(\xi)|^2 \, d\xi \\
&= \frac{8}{\pi} \int_{\mathbf{R}^{N-1}} \frac{\xi_j^2}{|\xi'|} \left| \int_0^\infty \widehat{A\varphi}(\xi_1, \xi') \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \, d\xi_1 \right|^2 d\xi'
\end{aligned}
\tag{4.30}
$$

for any $\varphi \in C_c^\infty(\mathbf{R}^N)$, where $j \in \{2, \ldots, , N\}$. It is obvious that the left-hand side of (4.30) defines a continuous functional on $L^2(\mathbf{R}^N)$. By the next lemma, it follows that the right-hand side of (4.30) also defines a continuous functional on $L^2(\mathbf{R}^N)$. Then the density of $C_c^\infty(\mathbf{R}^N)$ in $L^2(\mathbf{R}^N)$ implies that (4.30) holds for any $\varphi \in L^2(\mathbf{R}^N)$.

Lemma 4.10 *Let $j \in \{2, \ldots, N\}$. The bilinear form*

$$
S_1(\varphi, \psi) = \int_{\mathbf{R}^{N-1}} \frac{\xi_j^2}{|\xi'|} \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \, d\xi_1 \cdot \int_0^\infty \overline{\widehat{\psi}(\eta_1, \xi')} \frac{\eta_1}{\eta_1^2 + |\xi'|^2} \, d\eta_1 \, d\xi'
$$

is continuous on $L^2(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$.

Proof. As in (4.13) we have

$$
\left| \int_0^\infty \widehat{\varphi}(\xi_1, \xi') \frac{\xi_1}{\xi_1^2 + |\xi'|^2} \, d\xi_1 \right| \leq K \frac{1}{|\xi'|^{\frac{1}{2}}} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^2 \, d\xi_1 \right)^{\frac{1}{2}},
$$

where $K = \left(\int_0^\infty \frac{t^2}{(1+t^2)^2} \, dt \right)^{\frac{1}{2}}$. Consequently

$$
\begin{aligned}
|S_1(\varphi, \psi)| &\leq K^2 \int_{\mathbf{R}^{N-1}} \frac{\xi_j^2}{|\xi'|^2} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^2 \, d\xi_1 \right)^{\frac{1}{2}} \left(\int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^2 \, d\eta_1 \right)^{\frac{1}{2}} d\xi' \\
&\leq K^2 \int_{\mathbf{R}^{N-1}} \left(\int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^2 \, d\xi_1 \right)^{\frac{1}{2}} \left(\int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^2 \, d\eta_1 \right)^{\frac{1}{2}} d\xi' \\
&\leq K^2 \left(\int_{\mathbf{R}^{N-1}} \int_0^\infty |\widehat{\varphi}(\xi_1, \xi')|^2 \, d\xi_1 \, d\xi' \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbf{R}^{N-1}} \int_0^\infty |\widehat{\psi}(\eta_1, \xi')|^2 \, d\eta_1 \, d\xi' \right)^{\frac{1}{2}} \\
&\leq K_1 ||\varphi||_{L^2(\mathbf{R}^N)} ||\psi||_{L^2(\mathbf{R}^N)}.
\end{aligned}
$$

□

Since u^2, u_1^2, $u_2^2 \in L^2(\mathbf{R}^3)$ (recall that $H^1(\mathbf{R}^3) \subset L^2(\mathbf{R}^3) \cap L^6(\mathbf{R}^3)$), by exchanging the roles of x_1 and x_2 and using (4.29) and (4.30) we find

$$
\begin{aligned}
&E(u_1) + E(u_2) - 2E(u) \\
&= -\frac{1}{4} \frac{1}{(2\pi)^3} \frac{8}{\pi} \int_{\mathbf{R}^2} \frac{\xi_1^2}{\sqrt{\xi_1^2 + \xi_3^2}} \left| \int_0^\infty \widehat{A_2(u^2)}(\xi_1, \xi_2, \xi_3) \frac{\xi_2}{\xi_1^2 + \xi_2^2 + \xi_3^2} \, d\xi_2 \right|^2 d\xi_1 \, d\xi_3,
\end{aligned}
\tag{4.31}
$$

where $A_2\varphi = \frac{1}{2}(\varphi(x_1, x_2, x_3) - \varphi(x_1, -x_2, x_3))$.

Since u is a minimizer, we must have $E(u_1) + E(u_2) - 2E(u) \geq 0$, consequently the integral in the right-hand side of (4.31) must be zero, which is equivalent to

$$
\int_0^\infty \widehat{A_2(u^2)}(\xi_1, \xi_2, \xi_3) \frac{\xi_2}{\xi_1^2 + \xi_2^2 + \xi_3^2} \, d\xi_2 = 0 \qquad \text{a.e. } (\xi_1, \xi_3) \in \mathbf{R}^2. \tag{4.32}
$$

In particular, u_1 and u_2 are also minimizers. However, as in the previous example, (4.32) is not sufficient to prove that $A_2(u^2) = 0$. In order to accomplish this task, we will use the Euler-Lagrange equation of u : since u minimizes E under the constraint $Q(u) = \lambda$, there exists a constant α such that $E'(u) + \alpha Q'(u) = 0$, that is

$$-\Delta u + F'(u) + R_1^2(u^2)u + \alpha G'(u) = 0. \tag{4.33}$$

Lemma 4.11 *If F and G satisfy assumption a) in Theorem 4.9 and $u \in H^1(\mathbf{R}^3)$ is a solution of (4.33), then $u \in W^{3,p}(\mathbf{R}^3)$ for any $p \in [2, \infty)$. In particular, $u \in C^2(\mathbf{R}^3)$.*

Since R_1 and R_1^2 are linear continuous mappings from $L^p(\mathbf{R}^3)$ to $L^p(\mathbf{R}^3)$ for $1 < p < \infty$, the proof of Lemma 4.11 is standard, so we omit it.

Let $I(\varphi)(x) = \int_{\mathbf{R}^3} \frac{\varphi(y)}{|x-y|}\, dy$. Using Lemma 4.5 it is easy to see that $I(u^2) \in W^{2,p}(\mathbf{R}^3)$ for any $p \in (3, \infty]$ and $I(u^2)$ is a C^2 function. Moreover, we have

$$\mathcal{F}(R_1^2(u^2))(\xi) = -\frac{\xi_1^2}{|\xi|^2}\widehat{u^2}(\xi) = -\frac{1}{d_3}\xi_1^2 \widehat{I(u^2)}(\xi),$$

where $d_3 = \frac{4\pi^{\frac{3}{2}}}{\Gamma(\frac{1}{2})}$, thus $R_1^2(u^2) = \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2} I(u^2)$. Equation (4.33) can be written as

$$-\Delta u + F'(u) + \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\left(I(u^2)\right) u + \alpha G'(u) = 0. \tag{4.34}$$

Arguing exactly as in the proof of Theorem 4.6, (4.32) implies that $\frac{\partial}{\partial x_2}\left(I(u^2)\right)(x_1, 0, x_3) = 0$ for any $(x_1, x_3) \in \mathbf{R}^2$.

Since u_1 is also a minimizer, it satisfies the Euler-Lagrange equation

$$-\Delta u_1 + F'(u_1) + \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\left(I(u_1^2)\right) u_1 + \beta G'(u_1) = 0. \tag{4.35}$$

The conclusion of Lemma 4.11 is obviously valid for u_1. Since u_1 is symmetric with respect to x_2, $I(u_1^2)$ is also symmetric with respect to x_2 and, consequently, $\frac{\partial}{\partial x_2}\left(I(u_1^2)\right)(x_1, 0, x_3) = 0$ for any $(x_1, x_3) \in \mathbf{R}^2$. We set $U = I(u^2)$ and $U_1 = I(u_1^2)$. Recall that $u(x_1, x_2, x_3) = u_1(x_1, x_2, x_3)$ if $x_2 < 0$; thus U and U_1 are both solutions of

$$\begin{cases} -\Delta W = u^2 & \text{in } \mathbf{R} \times (-\infty, 0) \times \mathbf{R}, \\ W \in C^2(\mathbf{R}^3) \cap W^{2,p}(\mathbf{R}^3) & \text{for } 3 < p \le \infty \\ \frac{\partial W}{\partial x_2}(x_1, 0, x_3) = 0 & \text{for any } (x_1, x_3) \in \mathbf{R}^2. \end{cases} \tag{4.36}$$

It is not hard to see that the solution of (4.36) is unique. Hence we must have $I(u^2) = I(u_1^2)$ in $\mathbf{R} \times (-\infty, 0] \times \mathbf{R}$. In the same way we obtain $I(u^2) = I(u_2^2)$ in $\mathbf{R} \times [0, \infty) \times \mathbf{R}$.

Now we focus our attention on u_1. Making a translation in the x_3 direction if necessary, we may assume that $\int_{\{x_3<0\}} G(u_1(x))\, dx = \int_{\{x_3>0\}} G(u_1(x))\, dx = \frac{\lambda}{2}$. We define

$$w_1(x_1, x_2, x_3) = \begin{cases} u_1(x_1, x_2, x_3) & \text{if } x_3 < 0, \\ u_1(x_1, x_2, -x_3) & \text{if } x_3 \ge 0, \end{cases}$$

$$w_2(x_1,x_2,x_3) = \begin{cases} u_1(x_1,x_2,-x_3) & \text{if } x_3 < 0, \\ u_1(x_1,x_2,x_3) & \text{if } x_3 \geq 0. \end{cases}$$

It is obvious that $Q(w_1) = Q(w_2) = \lambda$. Proceeding as above, we find the identity

$$\begin{aligned} &E(w_1) + E(w_2) - 2E(u_1) \\ (4.37) \qquad &= -\frac{1}{4}\frac{1}{(2\pi)^3}\frac{8}{\pi}\int_{\mathbf{R}^2} \frac{\xi_1^2}{\sqrt{\xi_1^2+\xi_2^2}}\left|\int_0^\infty \widehat{A_3(u_1^2)}(\xi_1,\xi_2,\xi_3)\frac{\xi_3}{\xi_1^2+\xi_2^2+\xi_3^2}\,d\xi_3\right|^2 d\xi_1\,d\xi_2, \end{aligned}$$

where $A_3\varphi = \frac{1}{2}(\varphi(x_1,x_2,x_3) - \varphi(x_1,x_2,-x_3))$. Since u_1 is a minimizer, it follows from (4.37) that w_1 and w_2 are also minimizers of E under the constraint $Q = \lambda$; hence w_1 and w_2 satisfy the conclusion of Lemma 4.11 and $I(w_1)$, $I(w_2) \in C^2(\mathbf{R}^3) \cap W^{2,p}(\mathbf{R}^3)$ for $p \in (3,\infty]$. Moreover, the integral in the right-hand side of (4.37) must be zero. As previously, this gives $\frac{\partial}{\partial x_3} I(u_1^2)(x_1,x_2,0) = 0$ for any $(x_1,x_2) \in \mathbf{R}^2$. Proceeding as above, we find $I(u_1^2) = I(w_1^2)$ in $\mathbf{R}^2 \times (-\infty,0]$ and $I(u_1^2) = I(w_2^2)$ in $\mathbf{R}^2 \times [0,\infty)$.

Now let us consider the function w_1. It is clear that $w_1(x_1,-x_2,-x_3) = w_1(x_1,-x_2,x_3) = w_1(x_1,x_2,x_3)$, i.e. w_1 is symmetric with respect to x_2 and with respect to x_3. Consider a plane Π in $\mathbf{R}^3$ containing the line $\{(x_1,0,0) \mid x_1 \in \mathbf{R}\}$ and let Π_+ and Π_- be the two half-spaces determined by Π. Since $(x_1,x_2,x_3) \longmapsto (x_1,-x_2,-x_3)$ maps Π_+ onto Π_-, using the symmetry of w_1 we get $\displaystyle\int_{\Pi_+} G(w_1(x))\,dx = \int_{\Pi_-} G(w_1(x))\,dx = \frac{\lambda}{2}$. Let s_Π denote the symmetry in $\mathbf{R}^3$ with respect to Π. We define

$$r_1(x) = \begin{cases} w_1(x) & \text{if } x \in \Pi_-, \\ w_1(s_\Pi(x)) & \text{if } x \in \Pi_+ \end{cases} \qquad \text{and} \qquad r_2(x) = \begin{cases} w_1(s_\Pi(x)) & \text{if } x \in \Pi_-, \\ w_1(x) & \text{if } x \in \Pi_+. \end{cases}$$

Repeating the above arguments we obtain an integral identity analogous to (4.31) and (4.37) which implies that r_1 and r_2 also minimize E subject to the constraint $Q = \lambda$. Furthermore, using the fact that the integral in the right-hand side of this identity must vanish we find

$$(4.38) \qquad \frac{\partial}{\partial n} I(w_1^2)(x_1,x_2,x_3) = 0 \qquad \text{whenever } (x_1,x_2,x_3) \in \Pi,$$

where n is the unit normal to Π. Passing to cylindrical coordinates we write $I(w_1^2)(x_1,x_2,x_3) = I(w_1^2)(x_1, r\cos\theta, r\sin\theta) = \Phi(x_1,r,\theta)$, where $r = \sqrt{x_2^2+x_3^2}$. Since $I(w_1^2)$ is a C^2 function and (4.38) is valid for any plane Π containing $\{(x_1,0,0) \mid x_1 \in \mathbf{R}\}$, (4.38) is equivalent to $\dfrac{\partial\Phi}{\partial\theta} = 0$, that is Φ does not depend on θ, i.e. $I(w_1^2)(x_1,x_2,x_3) = \Phi_1(x_1,\sqrt{x_2^2+x_3^2})$ for some function Φ_1. In other words, we have proved that $I(w_1^2)$ is radially symmetric in the variables (x_2,x_3). In the same way we show that $I(w_2^2)(x_1,x_2,x_3) = \Phi_2(x_1,\sqrt{x_2^2+x_3^2})$ for some function Φ_2. Since $I(u_1^2)$ is continuous on $\mathbf{R}^3$, $I(u_1^2) = I(w_1^2)$ in the half-space $\{x_3 < 0\}$ and $I(u_1^2) = I(w_2^2)$ in the half-space $\{x_3 > 0\}$, we have necessarily $\Phi_1 = \Phi_2$, and then $I(u_1^2)$ is radially symmetric in the variables (x_2,x_3). Similarly, $I(u_2^2)$ is radially symmetric in (x_2,x_3). Recall that $I(u^2) = I(u_1^2)$ in the half-space $\{x_2 < 0\}$ and $I(u^2) = I(u_2^2)$ in the half-space $\{x_2 > 0\}$. But $I(u^2)$ is a continuous function on $\mathbf{R}^3$, thus we must have $I(u^2) = I(u_1^2) = I(u_2^2)$ on $\mathbf{R}^3$, consequently $I(u^2)$ is radially symmetric with respect to (x_2,x_3).

If $u \equiv 0$ in the half-space $\{x_2 < 0\}$, it follows that $u_1 \equiv 0$ in $\mathbf{R}^3$ and then $I(u_1^2) \equiv 0$ which implies $I(u^2) = 0$ in $\mathbf{R}^3$. In this case (4.34) becomes $-\Delta u + F'(u) + \alpha G'(u) = 0$ and from the Unique Continuation Principle we infer that $u \equiv 0$ in $\mathbf{R}^3$, thus u is radially symmetric in a trivial way. Obviously, the case $u \equiv 0$ is excluded if $\lambda \neq 0$.

If $u \not\equiv 0$ in the half-space $\{x_2 < 0\}$, by assumption $b)$ there exists $(x_1, x_2, x_3) \in \mathbf{R}^3$, $x_2 < 0$ such that $G'(u(x_1, x_2, x_3)) \neq 0$. Since $u = u_1$ on $\{x_2 < 0\}$ and $I(u^2) = I(u_1^2)$ on $\mathbf{R}^3$, from (4.34) and (4.35) we infer that $\alpha = \beta$. Let $a(x) = \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\left(I(u^2)\right)(x) = \frac{1}{d_3}\frac{\partial^2}{\partial x_1^2}\left(I(u_1^2)\right)(x)$. We know that a is a continuous and bounded function on $\mathbf{R}^3$. The functions u and u_1 both satisfy the equation $-\Delta w + F'(w) + a(x)w + \alpha G'(w) = 0$ in $\mathbf{R}^3$ and using the Unique Continuation Principle again we conclude that $u \equiv u_1$ in $\mathbf{R}^3$, i.e. u is symmetric with respect to x_2.

In the same way we prove that u is symmetric with respect to x_3 (after possibly a translation). Proceeding as in the proof of Theorem 4.1 we can show that u is symmetric with respect to any plane containing the line $\{(x_1, 0, 0) \mid x_1 \in \mathbf{R}\}$, consequently u is radially symmetric with respect to (x_2, x_3) variables. □

Remark 4.12 *i*) We have stated and proved Theorem 4.9 in dimension $N = 3$ only for simplicity. Replacing the term $\int_{\mathbf{R}^3} |R_1(u^2)|^2(x)\,dx$ in $E(u)$ by $\int_{\mathbf{R}^N} |R_1(H(u))|^2(x)\,dx$ and making suitable assumptions on the function H, this result admits a straightforward generalization to $\mathbf{R}^N$, $N \geq 3$.

ii) We do not know whether the minimizers in Theorem 4.9 are symmetric or not with respect to x_1. Recall that by (2.55) we have

$$\begin{aligned}&\int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2}|\widehat{T_1\varphi}(\xi)|^2\,d\xi + \int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2}|\widehat{T_2\varphi}(\xi)|^2\,d\xi - 2\int_{\mathbf{R}^N} \frac{\xi_1^2}{|\xi|^2}|\widehat{\varphi}(\xi)|^2\,d\xi \\ &= -\frac{8}{\pi}\int_{\mathbf{R}^{N-1}} |\xi'| \left| \int_0^\infty \widehat{A\varphi}(\xi)\frac{\xi_1}{\xi_1^2 + |\xi'|^2}\,d\xi_1 \right|^2 d\xi'\end{aligned} \tag{4.39}$$

for any $\varphi \in C_c^\infty(\mathbf{R}^N)$. Clearly, the left-hand side of (4.39) is continuous on $L^2(\mathbf{R}^N)$. Proceeding as in Lemma 4.10, it is easy to see that the right-hand side of (4.39) also defines a continuous functional on $L^2(\mathbf{R}^N)$. Consequently, (4.39) holds for any $\varphi \in L^2(\mathbf{R}^N)$. Using (4.28) and (4.39) we have

$$E(T_1u) + E(T_2u) - 2E(u) = \frac{2}{\pi}\frac{1}{(2\pi)^N}\int_{\mathbf{R}^{N-1}} |\xi'| \left| \int_0^\infty \mathcal{F}(A(H(u)))(\xi)\frac{\xi_1}{|\xi|^2}\,d\xi_1 \right|^2 d\xi'. \tag{4.40}$$

The right-hand side in this integral identity is always nonnegative and (4.40) does not imply the symmetry of minimizers with respect to x_1.

iii) Let us change the sign of the nonlocal term appearing in Theorem 4.9, i.e. let us consider the minimization problem

$$\begin{aligned}&\text{minimize } E_*(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \int_{\mathbf{R}^3} F(u)\,dx + \frac{1}{4}\int_{\mathbf{R}^3} |R_1(u^2)|^2\,dx \\ &\text{under the constraint } Q(u) := \int_{\mathbf{R}^3} G(u(x))\,dx = \lambda.\end{aligned} \tag{4.41}$$

The minimizers of this problem (when they exist) give rise to standing waves for equation (4.27) where the sign of the nonlocal term $R_1^2(|u|^2)u$ has been reversed. Clearly, the integral identities that we have do not imply the symmetry of solutions of (4.41) with respect to x_2 and x_3.

The symmetry of minimizers of (4.41) with respect to x_1 is also an open problem. As above, in this case we have the identity

$$E_*(T_1u) + E_*(T_2u) - 2E_*(u) = -\frac{2}{\pi}\frac{1}{(2\pi)^3}\int_{\mathbf{R}^2} |\xi'| \left| \int_0^\infty \mathcal{F}(A(u^2))(\xi)\frac{\xi_1}{|\xi|^2}\,d\xi_1 \right|^2 d\xi_2\,d\xi_3. \tag{4.42}$$

If u is a minimizer, the right-hand side of (4.42) must vanish. As in the proof of Theorem 4.9, this implies $\frac{\partial}{\partial x_1} I(u^2)(0, x_2, x_3) = 0$ for any $(x_2, x_3) \in \mathbf{R}^2$. Repeating the argument already used in Theorem 4.9 we get $I(u^2) = I((T_1 u)^2)$ on $\{x_1 \le 0\}$ and $I(u^2) = I((T_2 u)^2)$ on $\{x_1 \ge 0\}$. Moreover, if $\lambda \neq 0$ then u and $u_1 := T_1 u$ satisfy the same Euler-Lagrange equation, namely

$$-\Delta w + F'(w) - \frac{1}{d_3} \frac{\partial^2}{\partial x_1^2} \left(I(w^2) \right) w + \alpha G'(w) = 0. \tag{4.43}$$

Equivalently, defining $U = I(u^2)$ and $U_1 = I(u_1^2)$, we see that (u, U) and (u_1, U_1) are both solutions to the system

$$\begin{cases} -\Delta w + F'(w) - \frac{1}{d_3} \frac{\partial^2 W}{\partial x_1^2} w + \alpha G'(w) = 0, \\ -\Delta W = w^2. \end{cases} \tag{4.44}$$

Moreover, $(u, U) = (u_1, U_1)$ on $\{x_1 \le 0\}$ and u, u_1 satisfy the conclusion of Lemma 4.11. We do not know whether this information together with the boundary condition $\frac{\partial U}{\partial x_1}(0, x_2, x_3) = \frac{\partial U_1}{\partial x_1}(0, x_2, x_3) = 0$ imply that $u \equiv u_1$.

Remark 4.13 If $N = 3$, the nonlocal term in Theorem 4.9 can be written as

$$\int_{\mathbf{R}^3} |R_1(u^2)|^2 \, dx = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \frac{\xi_1^2}{|\xi|^2} |\widehat{u^2}(\xi)|^2 \, d\xi = -\frac{1}{d_3 (2\pi)^3} \int_{\mathbf{R}^3} \mathcal{F} \left(\frac{\partial^2}{\partial x_1^2} I(u^2) \right) (\xi) \overline{\widehat{u^2}(\xi)} \, d\xi$$

$$= -\frac{1}{d_3} \int_{\mathbf{R}^3} \frac{\partial^2}{\partial x_1^2} I(u^2)(x) \overline{u^2(x)} \, dx = -\frac{1}{d_3} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} u^2(x) K(x-y) u^2(y) \, dx \, dy,$$

where $K(x) = \frac{\partial^2}{\partial x_1^2} \left(\frac{1}{|x|} \right) = \frac{2x_1^2 - x_2^2 - x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{\frac{5}{2}}}$. Since this kernel changes sign, spherical rearrangements in the variables (x_2, x_3) combined with Riesz' inequality cannot be used to prove the symmetry of minimizers.

Remark 4.14 It is worth to note the following simple idea : let u_* be a minimizer for a variational problem like those studied in this paper. Suppose that one can prove that u_* is a C^1 function and that $\dfrac{\partial u_*}{\partial n} = 0$ whenever $x \in \Pi$, where Π is any hyperplane in $\mathbf{R}^N$ having the property $\displaystyle\int_{\Pi_-} G(u_*(x)) \, dx = \int_{\Pi_+} G(u_*(x)) \, dx$ (here Π_- and Π_+ are the two half-spaces determined by Π, n is the unit normal to Π and G is the function appearing in the constraint). Proceeding as we did for $I(u^2)$ in in the proof of Theorem 4.9, one can prove that after a translation, u_* is radially symmetric. This method should be useful in problems where the integral identities that one can obtain are not sufficient to deduce the symmetry of minimizers and an unique continuation theorem is unavailable. Unfortunately it cannot give symmetry with respect to only one direction.

5 Some open problems

We close this paper speaking about several problems for which the methods described above (including ours) seem to fail.

First, let us come back to the two minimization problems considered in Theorem 4.1. As before, if u is a minimizer of any of these problems, we may assume that $\displaystyle\int_{\{x_1<0\}} G(u) \, dx =$

$\int_{\{x_1>0\}} G(u)\, dx$ and we set $u_1 = T_1 u$ and $u_2 = T_2 u$. Assume that $s \in (1, \frac{3}{2})$. Then the identities (3.26) and (3.27) are still valid (see Corollary 3.5) and we get

$$E(u_1) + E(u_2) - 2E(u) = -\frac{16\sin(s\pi)}{\pi^2} N_s^2(Au) \geq 0 \quad \text{in case A, respectively}$$

$$E(u_1) + E(u_2) - 2E(u) = -\frac{16\sin(s\pi)}{\pi^2} \tilde{N}_s^2(Au) \geq 0 \quad \text{in case B.}$$

It is easy to see that these integral identities work in the wrong direction. Are the minimizers still radially symmetric for $s \in (1, \frac{3}{2})$?

Another problem is to study the symmetry of minimizers of

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{1}{|x-y|} u(x)^2 u(y)^2\, dx\, dy + \int_{\mathbf{R}^3} F(u(x))\, dx$$

subject to the constraint

$$\int_{\mathbf{R}^3} u^2(x)\, dx = \lambda > 0.$$

In the particular case $F(u) = -C|u|^{8/3}$, this problem arises in connection with the Schrödinger-Poisson-Slater system ([22]). Due to the repulsive effect of the nonlocal term, Riesz' inequality as well as the Reflection method work in the wrong direction.

A last problem concerns the symmetry of minimizers of

$$E(u) = \int_{-\infty}^{+\infty} (u_x^2(x) + u^3(x))\, dx - \gamma \int_{-\infty}^{+\infty} |\xi||\hat{u}(\xi)|^2\, d\xi,$$

where $\gamma > 0$, subject to the constraint $\int_{-\infty}^{+\infty} u^2(x)\, dx = \lambda > 0$. These two functionals are conserved quantities for the Benjamin equation (see [1]). Symmetrization and reflection cannot be used due to the sign of the nonlocal term. Oscillating travelling waves for this equation have been found numerically; perhaps this is an indication that the minimizers of the problem above may change sign.

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Deuxième partie

Ondes progressives pour des équations de Schrödinger non-linéaires avec des conditions non-nulles à l'infini

Nonexistence of supersonic traveling waves for nonlinear Schrödinger equations with non-zero conditions at infinity

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Nonexistence of supersonic travelling-waves for nonlinear Schrödinger equations with nonzero conditions at infinity

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Abstract

We prove that the non-existence of supersonic finite-energy travelling-waves for nonlinear Schrödinger equations with nonzero conditions at infinity is a general phenomenon, which holds for a large class of equations. The same is true for sonic travelling-waves in dimension two. In higher dimensions we prove that sonic travelling-waves, if they exist, must approach their limit at infinity in a very rigid way. In particular, we infer that there are no sonic travelling-waves with finite energy and finite momentum.

Keywords. nonlinear Schrödinger equation, nonzero conditions at infinity, travelling-wave, integral identities, Gross-Pitaevskii equations and systems, cubic-quintic NLS.

AMS subject classifications. 35Q51, 35Q55, 35Q40, 35B65, 35J15, 35J20, 35J50, 37K40, 37K05.

1 Introduction

The aim of this paper is to study travelling-wave solutions for nonlinear Schrödinger equations

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + F(x, |\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N, \tag{1.1}$$

where F is a real-valued function defined on $\mathbf{R}^N \times \mathbf{R}_+$, Φ is a complex-valued function on $\mathbf{R}^N$ satisfying the "boundary condition" $|\Phi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, and r_0 is a positive constant verifying $\lim\limits_{|x|\to\infty,\, s\to r_0^2} F(x,s) = 0$.

The above equation with the considered non-zero conditions at infinity arise in a large variety of physical problems, such as superconductivity, superfluidity in Helium II, phase transitions and Bose-Einstein condensate. Two important particular cases of (1.1) have been extensively studied both by physicists and by mathematicians : the Gross-Pitaevskii equation (where $F(x,s) = 1 - s$) and the so-called "cubic-quintic" Schrödinger equation (where $F(x,s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, α_1, α_3, α_5 are positive and $\frac{3}{16} < \frac{\alpha_1\alpha_5}{\alpha_3^2} < \frac{1}{4}$).

Equation (1.1) has a Hamiltonian structure: denoting $V(x,s) = \int_s^{r_0^2} F(x,\tau)\,d\tau$, it is easy to see that, at least formally, the "energy"

$$E(\Phi) = \int_{\mathbf{R}^N} |\nabla\Phi|^2\,dx + \int_{\mathbf{R}^N} V(x, |\Phi|^2)\,dx \tag{1.2}$$

is a conserved quantity. There is another important (vector) quantity associated to (1.1), namely the momentum. It is given by

$$P(\Phi) = (P_1(\Phi), \dots, P_N(\Phi)), \qquad \text{where } P_k(\Phi) = \int_{\mathbf{R}^N} (i\frac{\partial \Phi}{\partial x_k}, \Phi)\,dx = \int_{\mathbf{R}^N} \mathrm{Re}(i\frac{\partial \Phi}{\partial x_k}\overline{\Phi})\,dx. \tag{1.3}$$

Note that, in general, the momentum is not well-defined for any solution Φ of finite energy. In the case where F does not depend on the variable x_k, the momentum with respect to the x_k-direction, P_k, is conserved by those solutions of (1.1) for which it can be well-defined.

It is worth to note that equation (1.1) can be put into a hydrodynamical form by using Madelung's transformation $\Phi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$ (which is singular when $\Phi = 0$). A straightforward computation shows that, in the region where $\Phi \neq 0$, the functions $\rho = |\Phi|^2$ and θ satisfy the system

$$\rho_t + 2\text{div}(\rho \nabla \theta) = 0, \tag{1.4}$$

$$\theta_t + |\nabla \theta|^2 - \frac{\Delta \rho}{2\rho} + \frac{|\nabla \rho|^2}{4\rho} - F(x,\rho) = 0. \tag{1.5}$$

Equation (1.4) and the derivatives with respect to $x_1, \ldots, x_N$ of (1.5) are, respectively, the equation of conservation of mass and Euler's equations for a compressible inviscid fluid of density ρ and velocity $2\nabla\theta$.

Let us assume that F admits a partial derivative with respect to the last variable (in the sequel, this derivative will be denoted by $\partial_{N+1}F$ or by $\frac{\partial F}{\partial s}$) and that $\lim\limits_{|x|\to\infty,\ \rho\to r_0^2} \partial_{N+1}F(x,\rho) = -L$, where L is a positive constant. Taking the derivative with respect to t of (1.5) and substituting ρ_t from (1.4) we obtain

$$\theta_{tt} + 2\partial_{N+1}F(x,\rho)(\rho\Delta\theta + \nabla\rho.\nabla\theta) + \frac{\partial}{\partial t}\left(|\nabla\theta|^2 - \frac{\Delta\rho}{2\rho} + \frac{|\nabla\rho|^2}{4\rho}\right) = 0. \tag{1.6}$$

For a small oscillatory motion (i.e. a sound wave), all nonlinear terms in (1.6), except $2\rho\Delta\theta$, may be neglected. In view of the behavior of ρ and $\partial_{N+1}F(x,\rho)$ for large $|x|$, we find that in a neighborhood of infinity, the velocity potential θ essentially obeys the wave equation $\theta_{tt} - 2r_0^2 L\Delta\theta = 0$. It is well-known that the solutions of the wave equation propagate with a finite speed; in the present situation, we infer that the velocity of sound waves at infinity is $r_0\sqrt{2L}$. In what follows we will always assume that $\partial_{N+1}F(x,\rho) \longrightarrow -L$ as $|x| \longrightarrow \infty$ and $\rho \longrightarrow r_0^2$ (the convergence being in a sense to be defined) and we will denote by $v_s = r_0\sqrt{2L}$ the sound velocity at infinity.

For a fixed $y \in S^{N-1}$, a travelling-wave for (1.1) moving with velocity c in direction y is a solution of the form $\Phi(x,t) = \psi(x - cty)$. Without loss of generality we will assume that $y = (1,0,\ldots,0)$, i.e. travelling-waves move in the x_1-direction. The travelling-wave profile satisfies the equation

$$-ic\frac{\partial\psi}{\partial x_1} + \Delta\psi + F(x,|\psi|^2)\psi = 0 \qquad \text{in } \mathbf{R}^N. \tag{1.7}$$

In a series of papers, J. Grant, C.A. Jones, S.J. Putterman, P.H. Roberts et al. studied formally and numerically travelling-waves for the Gross-Pitaevskii equation and related systems (see, e.g., [16], [19], [21], [22], [7] and references therein). In particular, they conjectured that such solutions exist if and only if their speed c belongs to the interval $(-v_s, v_s)$. For the cubic-quintic nonlinear Schrödinger equation, the existence of subsonic travelling-waves in one dimension has been proved in [2] and their stability has been studied in [1]. The non-existence of such solutions for sonic and supersonic speeds has also been conjectured in any space dimension. In the case of the Gross-Pitaevskii equation, it has been shown in [17] that any travelling-wave of finite energy and speed $c > v_s$ must be constant. It has also been proved in [18] that the same result is true if $N = 2$ and $c^2 = v_s^2$. The proofs in [17], [18] strongly depend on the special algebraic structure of the nonlinearity in the Gross-Pitaevskii equation. In the present paper we show that the nonexistence of finite energy travelling-waves moving faster than the sound velocity is a general phenomenon, which holds for a large class of equations and systems of the form (1.1). We also prove that there are no finite energy sonic travelling-waves in space dimension two. In higher dimensions we show that any finite-energy sonic travelling-wave ψ must satisfy $|\psi|^2 - r_0^2 \in L^p(\mathbf{R}^N)$ for any $p > \frac{2N-1}{2N-3}$. On the other hand, if a sonic travelling-wave satisfies $|\psi|^2 - r_0^2 \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$, then it must be constant.

This article is organized as follows: in the next section we prove that travelling-waves, whenever they exist, are smooth functions. If their speed is supersonic (or sonic, provided they converge sufficiently fast at infinity), then they must satisfy a special integral identity. This will be proved in Section 3. In section 4 we show how this identity implies, under general assumptions, the non-existence of travelling-waves with finite energy. We apply our results to the Gross-Pitaevskii equation, to the cubic-quintic Schrödinger equation and to a Gross-Pitaevskii-Schrödinger system which describes the motion of an uncharged impurity in a Bose condensate. In the last section we describe all supersonic and sonic travelling-waves (with finite or infinite energy) for one-dimensional equations with nonlinearities independent on the space variable.

2 Basic properties of travelling-waves

We keep the previous notation and we consider the following set of assumptions:

- **(H1)** $F : \mathbf{R}^N \times [0,\infty) \longrightarrow \mathbf{R}$ is a measurable function which has the following properties:

  a) for any $s \in [0,\infty)$, $F(\cdot, s)$ is measurable;

  b) for any $x \in \mathbf{R}^N$, $F(x, \cdot)$ is continuous;

  c) F is bounded on bounded subsets of $\mathbf{R}^N \times [0,\infty)$.

- **(H2)** There exist $\alpha > 0$, $C > 0$ and $r_* > 0$ such that for any $x \in \mathbf{R}^N$ and for any $s \geq r_*$ we have $F(x,s) \leq -Cs^\alpha$.

- **(H3)** $\lim\limits_{|x|\to\infty} F(x, r_0^2) = 0$ and $F(\cdot, r_0^2) \in L^1(\mathbf{R}^N)$.

- **(H4)** F admits a partial derivative with respect to the last variable and $\partial_{N+1}F$ is bounded on bounded subsets of $\mathbf{R}^N \times [0,\infty)$. Moreover, $\lim\limits_{|x|\to\infty} \partial_{N+1}F(x, r_0^2) = -L$, where $L > 0$ and $\partial_{N+1}F(\cdot, r_0^2) + L \in L^{p_0}(\mathbf{R}^N)$ for some $p_0 \in [1,2]$.

- **(H5)** There are some positive constants R_0, η, M such that $\partial_{N+1}^2 F$ exists on $(\mathbf{R}^N \setminus \overline{B}(0,R_0)) \times (r_0^2 - \eta, r_0^2 + \eta)$ and

$$|\partial_{N+1}^2 F(x,s)| \leq M \qquad \text{for all } (x,s) \in (\mathbf{R}^N \setminus \overline{B}(0,R_0)) \times (r_0^2 - \eta, r_0^2 + \eta).$$

Definition 2.1 A travelling-wave (of speed c) for (1.1) is a function $\psi \in L^1_{loc}(\mathbf{R}^N)$ that satisfies (1.7) in $\mathcal{D}'(\mathbf{R}^N)$ together with the "boundary condition" $|\psi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$.

In view of (1.2), we say that a travelling-wave ψ has finite energy if $\nabla\psi \in L^2(\mathbf{R}^N)$ and $V(\cdot, |\psi|^2) \in L^1(\mathbf{R}^N)$.

We have the following result concerning the regularity of tavelling-waves:

Proposition 2.2 *Let ψ be a finite-energy travelling-wave for (1.1).*

i) Assume that $F : \mathbf{R}^N \times \mathbf{R}_+ \longrightarrow \mathbf{R}$ is measurable and satisfies **(H1a)**, **(H1b)**, **(H2)**, *the function $x \longmapsto \int_{r_0^2}^{r_*} F(x,\tau)\, d\tau$ belongs to $L^1_{loc}(\mathbf{R}^N)$ (where r_* is given by* **(H2)***) and $F(\cdot, |\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Then $\psi \in L^\infty(\mathbf{R}^N)$.*

If, in addition, F satisfies **(H1c)**, *then $\psi \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$. In particular, $\psi \in C^{1,\alpha}(\mathbf{R}^N)$ for any $\alpha \in [0, 1)$.*

ii) Suppose that $F \in C^k(\mathbf{R}^N \times [0, \infty))$ for some $k \in \mathbf{N}^*$, **(H2)** *holds, and $F(\cdot, |\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Then $\psi \in W^{k+2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$. In particular, if F is C^∞, then $\psi \in C^\infty(\mathbf{R}^N)$.*

Proof. i) The proof relies upon the ideas and methods developed by A. Farina in [13, 14]. By **(H2)** we have

$$V(x, s) = -\int_{r_0^2}^{s} F(x, \tau)\, d\tau \geq -\int_{r_0^2}^{r_*} F(x, \tau)\, d\tau + \int_{r_*}^{s} C\tau^\alpha\, d\tau = -\int_{r_0^2}^{r_*} F(x, \tau)\, d\tau + \frac{C}{\alpha+1}(s^{\alpha+1} - r_*^{\alpha+1}).$$

Consequently, for any $s \geq r_*$ we get $s^{\alpha+1} \leq r_*^{\alpha+1} + \frac{\alpha+1}{C}\Big(V(x, s) + \int_{r_0^2}^{r_*} F(x, \tau)\, d\tau\Big)$, so that

$$|\psi|^{2\alpha+2}(x) \leq \max\Big(r_*^{\alpha+1}, r_*^{\alpha+1} + \frac{\alpha+1}{C}\Big(V(x, |\psi|^2(x)) + \int_{r_0^2}^{r_*} F(x, \tau)\, d\tau\Big)\Big).$$

Since $V(\cdot, |\psi|^2)$ and $\int_{r_0^2}^{r_*} F(\cdot, \tau)\, d\tau$ belong to $L^1_{loc}(\mathbf{R}^N)$, we infer that $\psi \in L^{2\alpha+2}_{loc}(\mathbf{R}^N)$.

We will use a well-known inequality of T. Kato (see Lemma A p. 138 in [23]):

If $u \in L^1_{loc}(\mathbf{R}^N)$ is a real-valued function and $\Delta u \in L^1_{loc}(\mathbf{R}^N)$, then

$$\Delta(u^+) \geq \mathrm{sgn}^+(u)\Delta u \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{2.1}$$

Let $\varphi(x) = e^{-\frac{icx_1}{2}}\psi(x)$. Then $\varphi \in L^{2\alpha+2}_{loc}(\mathbf{R}^N) \subset L^1_{loc}(\mathbf{R}^N)$ and an easy computation shows that φ satisfies

$$\Delta\varphi + \Big(F(x, |\varphi|^2) + \frac{c^2}{4}\Big)\varphi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{2.2}$$

It is clear that $F(\cdot, |\varphi|^2)\varphi \in L^1_{loc}(\mathbf{R}^N)$ (because $F(x, |\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$ by hypothesis) and it follows from (2.2) that $\Delta\varphi \in L^1_{loc}(\mathbf{R}^N)$. Choose $\tilde{r} \geq r_*$ and $C_1 > 0$ such that $Cs^{2\alpha} - \frac{c^2}{4} \geq C_1(s - \tilde{r})^{2\alpha}$ for any $s \geq \tilde{r}$. Denoting $\varphi_1 = \mathrm{Re}(\varphi)$, $\varphi_2 = \mathrm{Im}(\varphi)$ and using Kato's inequality for $\varphi_i - \tilde{r}$, $i = 1, 2$, then using (2.2) and **(H2)** we get

$$\begin{aligned}
\Delta(\varphi_i - \tilde{r})^+ &\geq \mathrm{sgn}^+(\varphi_i - \tilde{r})\Delta(\varphi_i - \tilde{r}) = \mathrm{sgn}^+(\varphi_i - \tilde{r})[-(F(x, |\varphi|^2) + \tfrac{c^2}{4})\varphi_i] \\
&\geq \mathrm{sgn}^+(\varphi_i - \tilde{r})[C|\varphi|^{2\alpha} - \tfrac{c^2}{4}]\varphi_i \geq \mathrm{sgn}^+(\varphi_i - \tilde{r})[C|\varphi_i|^{2\alpha} - \tfrac{c^2}{4}]\varphi_i \\
&\geq C_1 \mathrm{sgn}^+(\varphi_i - \tilde{r})(\varphi_i - \tilde{r})^{2\alpha+1} = C_1[(\varphi_i - \tilde{r})^+]^{2\alpha+1}.
\end{aligned} \tag{2.3}$$

Next we use the following result of H. Brézis (Lemma 2 p. 273 in [9]):

Lemma 2.3 ([9]) *Let $p \in (1, \infty)$. Assume that $u \in L^p_{loc}(\mathbf{R}^N)$ satisfies*

$$-\Delta u + |u|^{p-1}u \leq 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

Then $u \leq 0$ a.e. on $\mathbf{R}^N$.

It follows from (2.3) that the function $u_i = (C_1)^{\frac{1}{2\alpha}}(\varphi_i - \tilde{r})^+$ satisfies $-\Delta u_i + |u_i|^{2\alpha}u_i \leq 0$ in $\mathcal{D}'(\mathbf{R}^N)$. Since $u_i \in L^{2\alpha+1}_{loc}(\mathbf{R}^N)$, we may use Lemma 2.3 and we get $u_i \leq 0$ a.e. in $\mathbf{R}^N$, that is $\varphi_i \leq \tilde{r}$ a.e. in $\mathbf{R}^N$.

It is obvious that both φ and $-\varphi$ satisfy (2.2). Repeating the above argument for $-\varphi$, we infer that $-\varphi_i \leq \tilde{r}$ a.e. on $\mathbf{R}^N$. Therefore we have $|\varphi_i| \leq \tilde{r}$ a.e. on $\mathbf{R}^N$, $i = 1, 2$, which implies that $\varphi \in L^\infty(\mathbf{R}^N)$. Since $|\varphi| = |\psi|$, we have proved that $\psi \in L^\infty(\mathbf{R}^N)$.

Using **(H1c)** and (2.2) we infer that $\Delta\varphi \in L^\infty(B(x, 2R)) \subset L^p(B(x, 2R))$ for any $x \in \mathbf{R}^N$, $R > 0$ and $p \geq 1$. By standard elliptic estimates we obtain $\varphi \in W^{2,p}(B(x, R))$ for any $x \in \mathbf{R}^N$, $R > 0$ and $p \in (1, \infty)$. Thus $\psi = e^{\frac{icx_1}{2}}\varphi \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in (1, \infty)$, consequently ψ belongs to $C^{1,\alpha}_{loc}(\mathbf{R}^N)$ for any $\alpha \in [0, 1)$ by the Sobolev embedding theorem.

ii) Assume $F \in C^1(\mathbf{R}^N \times [0, \infty))$. Differentiating (1.7) with respect to x_k we get

$$(2.4) \quad -ic\psi_{x_1x_k} + \Delta\psi_{x_k} + \frac{\partial F}{\partial x_k}(x, |\psi|^2)\psi + 2\partial_{N+1}F(x, |\psi|^2)(\psi.\frac{\partial\psi}{\partial x_k})\psi + F(x, |\psi|^2)\frac{\partial\psi}{\partial x_k} = 0$$

in $\mathcal{D}'(\mathbf{R}^N)$. Hence $\Delta\psi_{x_k} \in L^p_{loc}(\mathbf{R}^N)$ for $1 \leq p < \infty$. By standard elliptic regularity theory we get $\psi_{x_k} \in W^{2,p}_{loc}(\mathbf{R}^N)$ for $1 < p < \infty$, $1 \leq k \leq N$, therefore $\psi \in W^{3,p}_{loc}(\mathbf{R}^N)$ for $1 \leq p < \infty$. If $F \in C^k(\mathbf{R}^N \times [0, \infty))$ we may differentiate (2.4) further and repeat the above arguments. After an easy induction, we get $\psi \in W^{k+2,p}_{loc}(\mathbf{R}^N)$ for any $p \in (1, \infty)$. □

Lemma 2.4 *Assume that* **(H1), (H3), (H4), (H5)** *hold and* $u \in L^4_{loc}(\mathbf{R}^N, \mathbf{C})$ *satisfies* $|u(x)| \longrightarrow r_0$ *as* $|x| \longrightarrow \infty$ *and* $V(\cdot, |u|^2) \in L^1(\mathbf{R}^N)$.

Then $|u|^2 - r_0^2 \in L^2(\mathbf{R}^N)$.

Proof. Let R_0, η, M be as in **(H5)**. From **(H4)** and the fact that $|u(x)| \longrightarrow r_0$ as $|x| \longrightarrow \infty$ it follows that there exists $R_1 > R_0$ such that

$$\partial_{N+1}F(x, r_0^2) < -\frac{L}{2} \quad \text{and} \quad |u(x)|^2 \in (r_0^2 - \eta, r_0^2 + \eta) \quad \text{for any } x \text{ satisfying } |x| \geq R_1.$$

For $(x, s) \in (\mathbf{R}^N \setminus B(0, R_1)) \times (r_0^2 - \eta, r_0^2 + \eta)$ we get, by Taylor's formula with respect to the $(N+1)^{th}$ variable,

$$V(x, s) = -(s - r_0^2)F(x, r_0^2) - \frac{1}{2}(s - r_0^2)^2\partial_{N+1}F(x, r_0^2) - \frac{1}{2}\int_{r_0^2}^{s}(s - \tau)^2\partial^2_{N+1}F(x, \tau)\,d\tau.$$

In particular, for $s = |u(x)|^2$ we obtain

$$(2.5) \quad \begin{aligned} &-\tfrac{1}{2}(|u(x)|^2 - r_0^2)^2\partial_{N+1}F(x, r_0^2) \\ &= V(x, |u(x)|^2) + (|u(x)|^2 - r_0^2)F(x, r_0^2) + \tfrac{1}{2}\int_{r_0^2}^{|u(x)|^2}(|u(x)|^2 - \tau)^2\partial^2_{N+1}F(x, \tau)\,d\tau. \end{aligned}$$

For $x \in \mathbf{R}^N \setminus B(0, R_1)$ we get by **(H5)**

$$\left|\int_{r_0^2}^{|u(x)|^2}(|u(x)|^2 - \tau)^2\partial^2_{N+1}F(x, \tau)\,d\tau\right| \leq M\left|\int_{r_0^2}^{|u(x)|^2}(|u(x)|^2 - \tau)^2\,d\tau\right| = \frac{M}{3}|(|u(x)|^2 - r_0^2)|^3.$$

It is clear that there exists $R_2 \geq R_1$ such that $\frac{M}{3}\Big||u(x)|^2 - r_0^2\Big| \leq \frac{L}{4}$ on $\mathbf{R}^N \setminus B(0, R_2)$. Using **(H4)** and (2.5) we infer that

$$\begin{aligned} &\tfrac{L}{4}(|u(x)|^2 - r_0^2)^2 \leq -\tfrac{1}{2}(|u(x)|^2 - r_0^2)^2\partial_{N+1}F(x, r_0^2) \\ &\leq V(x, |u(x)|^2) + (|u(x)|^2 - r_0^2)F(x, r_0^2) + \tfrac{1}{2}\cdot\tfrac{M}{3}\Big|\,|u(x)|^2 - r_0^2\Big|^3 \\ &\leq V(x, |u(x)|^2) + (|u(x)|^2 - r_0^2)F(x, r_0^2) + \tfrac{L}{8}\Big|\,|u(x)|^2 - r_0^2\Big|^2 \qquad \text{on } \mathbf{R}^N \setminus B(0, R_2). \end{aligned}$$

Consequently

$$(2.6)\qquad \frac{L}{8}(|u(x)|^2 - r_0^2)^2 \le V(x, |u(x)|^2) + (|u(x)|^2 - r_0^2)F(x, r_0^2) \qquad \text{on } \mathbf{R}^N \setminus B(0, R_2).$$

Since $F(\cdot, r_0^2) \in L^1(\mathbf{R}^N)$ by **(H3)**, $V(\cdot, |u|^2) \in L^1(\mathbf{R}^N)$ and $|\,|u(x)|^2 - r_0^2| \le \frac{3L}{4M}$ on $\mathbf{R}^N \setminus B(0, R_2)$, using (2.6) we get $(|u|^2 - r_0^2)^2 \in L^1(\mathbf{R}^N \setminus B(0, R_2))$. It is obvious that $(|u|^2 - r_0^2)^2 \in L^1(B(0, R_2))$ because $u \in L^4_{loc}(\mathbf{R}^N)$. Hence $(|u|^2 - r_0^2)^2 \in L^1(\mathbf{R}^N)$ and Lemma 2.4 is proved. □

Proposition 2.5 *Assume that* **(H1)-(H5)** *hold and let ψ be a finite-energy travelling-wave for (1.1) (in the sense of Definition 2.1) such that $F(\cdot, |\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Then:*

i) $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$.

ii) Let $R_* \ge 0$ be such that $|\psi(x)| \ge \frac{r_0}{2}$ for $|x| \ge R_*$. There exists a real-valued function θ such that $\theta \in W^{2,p}_{loc}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p < \infty$, $\nabla\theta \in W^{1,p}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p \in [2, \infty)$ and

$$\psi(x) = |\psi(x)|e^{i\theta(x)} \qquad \text{on } \mathbf{R}^N \setminus B(0, R_*).$$

Proof. i) We already know by Proposition 2.2 i) and Lemma 2.4 that ψ is bounded, $\psi \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$ and $|\psi|^2 - r_0^2 \in L^2(\mathbf{R}^N)$.

Let R_0, η, M be as in **(H5)**. Choose $R_1 > R_0$ such that $|\psi|^2(x) \in (r_0^2 - \eta, r_0^2 + \eta)$ for $x \in \mathbf{R}^N \setminus B(0, R_1)$.

By using Taylor's formula with respect to the last variable for the function F we get

$$(2.7)\qquad F(x, s) = F(x, r_0^2) + (s - r_0^2)\partial_{N+1}F(x, r_0^2) + \int_{r_0^2}^{s} (s - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau$$

if $(x, s) \in (\mathbf{R}^N \setminus \overline{B}(0, R_0)) \times (r_0^2 - \eta, r_0^2 + \eta)$, hence

$$(2.8)\qquad \begin{aligned} F(x, |\psi|^2(x))\psi(x) &= F(x, r_0^2)\psi(x) + (|\psi|^2(x) - r_0^2)\partial_{N+1}F(x, r_0^2)\psi(x) \\ &\quad + \psi(x)\int_{r_0^2}^{|\psi|^2(x)} (|\psi|^2(x) - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau \quad \text{for any } |x| \ge R_1. \end{aligned}$$

We analyze the three terms in the right-hand side of (2.8). Assumptions **(H1)** and **(H3)** imply $F(\cdot, r_0^2) \in L^1 \cap L^\infty(\mathbf{R}^N)$. Since $\psi \in L^\infty(\mathbf{R}^N)$, it follows that $F(\cdot, r_0^2)\psi \in L^1 \cap L^\infty(\mathbf{R}^N)$.

We may write $(|\psi|^2 - r_0^2)\partial_{N+1}F(\cdot, r_0^2)\psi = -L(|\psi|^2 - r_0^2)\psi + (|\psi|^2 - r_0^2)(L + \partial_{N+1}F(\cdot, r_0^2))\psi$. We know that $\psi \in L^\infty(\mathbf{R}^N)$, $|\psi|^2 - r_0^2 \in L^2 \cap L^\infty(\mathbf{R}^N)$ and by **(H4)** we have $L + \partial_{N+1}F(\cdot, r_0^2) \in L^{p_0} \cap L^\infty(\mathbf{R}^N)$ for some $p_0 \in [1, 2]$, so we infer that $(|\psi|^2 - r_0^2)\partial_{N+1}F(\cdot, r_0^2)\psi \in L^2 \cap L^\infty(\mathbf{R}^N)$.

As in the proof of Lemma 2.4, for $x \in \mathbf{R}^N \setminus B(0, R_1)$ we have

$$(2.9)\quad \left|\int_{r_0^2}^{|\psi|^2(x)} (|\psi|^2(x) - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau\right| \le M\left|\int_{r_0^2}^{|\psi|^2(x)} \left|\,|\psi|^2(x) - \tau\right| d\tau\right| = \frac{M}{2}(|\psi|^2(x) - r_0^2)^2.$$

Consequently the function $x \longmapsto \displaystyle\int_{r_0^2}^{|\psi|^2(x)} (|\psi|^2(x) - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau$ belongs to $L^1 \cap L^\infty(\mathbf{R}^N \setminus B(0, R_1))$.

Summing up, we have proved that $F(\cdot, |\psi|^2)\psi \in L^2 \cap L^\infty(\mathbf{R}^N \setminus B(0, R_1))$. From **(H1)** and the fact that ψ is bounded on $\mathbf{R}^N$ it follows that $F(\cdot, |\psi|^2)\psi$ is bounded on $B(0, R_1)$, hence $F(\cdot, |\psi|^2)\psi \in L^2 \cap L^\infty(\mathbf{R}^N)$.

We have $\frac{\partial\psi}{\partial x_k} \in L^2(\mathbf{R}^N)$ because ψ has finite energy. Coming back to (1.7), we get

$$\Delta\psi = ic\frac{\partial\psi}{\partial x_1} - F(\cdot, |\psi|^2)\psi \in L^2(\mathbf{R}^N).$$

It is well-known that $\Delta\psi \in L^p(\mathbf{R}^N)$ with $1 < p < \infty$ implies $\frac{\partial^2\psi}{\partial x_j \partial x_k} \in L^p(\mathbf{R}^N)$ for any $j, k \in \{1, \dots, N\}$ (this follows, e.g., from the fact that $\frac{\xi_j \xi_k}{|\xi|^2}$ is a Fourier multiplier on $L^p(\mathbf{R}^N)$ if $1 < p < \infty$; see Theorem 3 p. 96 in [27]). Therefore all second derivatives of ψ are in $L^2(\mathbf{R}^N)$, so that $\frac{\partial\psi}{\partial x_k} \in H^1(\mathbf{R}^N) = W^{1,2}(\mathbf{R}^N)$ for $k = 1, \dots, N$.

The rest of the proof is an easy bootstrap argument. Assume that $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for some $p \geq 2$. In case $p < N$, it follows from the Sobolev embedding theorem that $\nabla\psi \in L^{p^*}(\mathbf{R}^N)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. From (1.7) we have $\Delta\psi = ic\frac{\partial\psi}{\partial x_1} - F(\cdot, |\psi|^2)\psi \in L^{p^*}(\mathbf{R}^N)$ and we infer as previously that $\nabla\psi \in W^{1,p^*}(\mathbf{R}^N)$. Repeating this argument if necessary, after a finite number of steps we get $\nabla\psi \in W^{1,q}(\mathbf{R}^N)$ for some $q \geq N$. Then by Sobolev embedding we get $\nabla\psi \in L^r(\mathbf{R}^N)$ for any $r \in [q, \infty)$. From (1.7) we obtain $\Delta\psi \in L^p(\mathbf{R}^N)$ for $p \in [2, \infty)$ and we infer that $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$.

ii) Take $R_* > 0$ such that $|\psi(x)| \geq \frac{r_0}{2}$ on $\mathbf{R}^N \setminus B(0, R_*)$ and denote $\tilde{\psi}(x) = \frac{\psi(x)}{|\psi(x)|}$. It is then standard to prove that $\tilde{\psi} \in W^{2,p}_{loc}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for $p \in [1, \infty)$ and $\nabla\tilde{\psi} \in W^{1,p}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p \in [2, \infty)$ (see, e.g., Lemma C1 p. 66 in [10]).

Let us consider first the case $N \geq 3$. For $R_* \leq R_1 < R_2$, the domain $\Omega_{R_1,R_2} = B(0, R_2) \setminus \overline{B}(0, R_1)$ is simply connected in $\mathbf{R}^N$. It follows from Theorem 3 p. 38 in [10] that there exists a real-valued function $\theta_{R_1,R_2} \in W^{2,p}(\Omega_{R_1,R_2})$ $(1 < p < \infty)$ such that $\tilde{\psi} = e^{i\theta_{R_1,R_2}}$ on Ω_{R_1,R_2}. If $R_* \leq R_1 < R_2$, $R_* \leq R_3 < R_4$ and $(R_1, R_2) \cap (R_3, R_4) \neq \emptyset$, then $\tilde{\psi} = e^{i\theta_{R_1,R_2}} = e^{i\theta_{R_3,R_4}}$ on $\Omega_{R_1,R_2} \cap \Omega_{R_3,R_4}$, thus $\theta_{R_3,R_4} - \theta_{R_1,R_2} \in 2\pi\mathbf{Z}$ on $\Omega_{R_1,R_2} \cap \Omega_{R_3,R_4}$. Since functions in $W^{s,p}(\Omega_{R_1,R_2} \cap \Omega_{R_3,R_4})$ with values in $\mathbf{Z}$ are constant when $sp \geq 1$ (see Theorem B1 p. 65 in [10]), there exists $k \in \mathbf{Z}$ such that $\theta_{R_3,R_4} - \theta_{R_1,R_2} = 2\pi k$ on $\Omega_{R_1,R_2} \cap \Omega_{R_3,R_4}$. Let $(R_n)_{n \geq 1}$ be an increasing sequence such that $R_* < R_1$ and $R_n \longrightarrow \infty$. Let $k_n \in \mathbf{Z}$ be such that $\theta_{R_*,R_n} = \theta_{R_*,R_1} + 2\pi k_n$ on Ω_{R_*,R_1}. Define $\theta(x) = \theta_{R_*,R_n}(x) - 2\pi k_n$ for $x \in \Omega_{R_*,R_n}$. It is clear that θ is well-defined on $\mathbf{R}^N \setminus \overline{B}(0, R_*)$, $\tilde{\psi} = e^{i\theta}$ and $\theta \in W^{2,p}_{loc}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p \in [1, \infty)$.

Next we consider the case $N = 2$. Since ψ is C^1 and $|\psi| \geq \frac{r_0}{2}$ on $\mathbf{R}^2 \setminus \overline{B}(0, R_*)$, the topological degree $deg(\psi, \partial B(0, R))$ is well-defined for any $R \geq R_*$ and does not depend on R. It is well-known that ψ admits a C^1 lifting θ (i.e. $\psi = |\psi|e^{i\theta}$) on $\mathbf{R}^2 \setminus \overline{B}(0, R_*)$ if and only if $deg(\psi, \partial B(0, R)) = 0$ for $R \geq R_*$. Denoting by $\tau = (-\sin\zeta, \cos\zeta)$ the unit tangent vector at $\partial B(0, R)$ at a point $Re^{i\zeta}$, we get

$$\begin{aligned} |deg(\psi, \partial B(0, R))| &= \left| \frac{1}{2i\pi} \int_0^{2\pi} \frac{\frac{\partial}{\partial\zeta}(\psi(Re^{i\zeta}))}{\psi(Re^{i\zeta})} \, d\zeta \right| = \left| \frac{R}{2i\pi} \int_0^{2\pi} \frac{\frac{\partial\psi}{\partial\tau}(Re^{i\zeta})}{\psi(Re^{i\zeta})} \, d\zeta \right| \\ &\leq \frac{R}{2\pi} \int_0^{2\pi} \frac{2}{r_0} |\nabla\psi(Re^{i\zeta})| \, d\zeta \leq \frac{R}{\pi r_0} \sqrt{2\pi} \left(\int_0^{2\pi} |\nabla\psi(Re^{i\zeta})|^2 \, d\zeta \right)^{\frac{1}{2}}. \end{aligned} \tag{2.10}$$

On the other hand,

$$\int_{\mathbf{R}^2 \setminus \overline{B}(0, R_*)} |\nabla\psi(x)|^2 \, dx = \int_{R_*}^{\infty} R \int_0^{2\pi} |\nabla\psi(Re^{i\zeta})|^2 \, d\zeta \, dR.$$

We have $\int_{\mathbf{R}^2 \setminus \overline{B}(0, R_*)} |\nabla\psi(x)|^2 \, dx < \infty$ (because ψ has finite energy) and we infer that there exists $R_1 > R_*$ such that $R_1 \int_0^{2\pi} |\nabla\psi(R_1 e^{i\zeta})|^2 \, d\zeta < \frac{\pi r_0^2}{8} \frac{1}{R_1}$. From (2.10) we get

$$|deg(\psi, \partial B(0, R_1))| < \frac{R_1}{\pi r_0} \sqrt{2\pi} \left(\frac{\pi r_0^2}{8} \frac{1}{R_1^2} \right)^{\frac{1}{2}} = \frac{1}{2}.$$

Since the topological degree is an integer, we have necessarily $deg(\psi, \partial B(0, R_1)) = 0$. Consequently $deg(\psi, \partial B(0, R)) = 0$ for any $R \geq R_*$ and ψ admits a C^1 lifting θ. In fact, $\theta \in W^{2,p}_{loc}(\mathbf{R}^2 \setminus \overline{B}(0, R_*))$ because $\psi \in W^{2,p}_{loc}(\mathbf{R}^2 \setminus \overline{B}(0, R_*))$ (see Theorem 3 p. 38 in [10]).

If $N = 1$, the existence of a lifting $\psi = |\psi| e^{i\theta}$ follows immediately from Theorem 1 p. 27 in [10].

Finally, it is easy to see that $|\frac{\partial \tilde{\psi}}{\partial x_j}| = |\frac{\partial \theta}{\partial x_j}|$ and $|\frac{\partial^2 \tilde{\psi}}{\partial x_j \partial x_k}|^2 = |\frac{\partial^2 \theta}{\partial x_j \partial x_k}|^2 + |\frac{\partial \theta}{\partial x_j}|^2 |\frac{\partial \theta}{\partial x_k}|^2 \geq |\frac{\partial^2 \theta}{\partial x_j \partial x_k}|^2$, and i) implies $\nabla \theta \in W^{1,p}(\mathbf{R}^N \setminus \overline{B}(0, R_*))$ for any $p \in [2, \infty)$. □

3 An integral identity

The main result of this section is given by the next theorem.

Theorem 3.1 *Assume that* **(H1) - (H5)** *hold. Let $\psi = \psi_1 + i\psi_2$ be a finite-energy travelling-wave for (1.1) such that $F(\cdot, |\psi|^2) \in L^1_{loc}(\mathbf{R}^N)$. Let R_* be sufficiently big, so that $|\psi| \geq \frac{r_0}{2}$ on $\mathbf{R}^N \setminus B(0, R_*)$ and let θ be the lifting given by Proposition 2.5 ii). Let $\chi \in C^\infty(\mathbf{R}^N)$ be a cut-off function such that $\chi = 0$ on $B(0, 2R_*)$ and $\chi = 1$ on $\mathbf{R}^N \setminus B(0, 3R_*)$. Then:*

i) The functions $F(\cdot, |\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2)$ and $G_j = \psi_1 \frac{\partial \psi_2}{\partial x_j} - \psi_2 \frac{\partial \psi_1}{\partial x_j} - r_0^2 \frac{\partial}{\partial x_j}(\chi\theta)$, $j = 1, \ldots, N$, belong $L^1 \cap L^\infty(\mathbf{R}^N)$. (We always extend $\chi\theta$ by zero on $\overline{B}(0, R_*)$).

ii) If $N \geq 2$ and $c^2 > v_s^2$ we have the following identity:

$$\int_{\mathbf{R}^N} |\nabla \psi|^2 - F(x, |\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2)\, dx$$
$$(3.1) \qquad = c\left(1 - \frac{v_s^2}{c^2}\right) \int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial}{\partial x_1}(\chi\theta)\, dx.$$

iii) Identity (3.1) holds if $c^2 = v_s^2$ and
- *either $N = 2$*
- *or $N \geq 3$ and we assume in addition that $\psi_1 \dfrac{\partial \psi_2}{\partial x_1} - \psi_2 \dfrac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$.*

Proof. i) Let R_0, η, M be as in **(H5)** and take $R_1 > R_0$ such that $|\psi|^2(x) \in (r_0^2 - \eta, r_0^2 + \eta)$ for $x \in \mathbf{R}^N \setminus B(0, R_1)$. Using (2.7) and the fact that $v_s^2 = 2Lr_0^2$ we get

$$F(x, |\psi|^2(x))|\psi|^2(x) + \tfrac{v_s^2}{2}(|\psi|^2(x) - r_0^2) = F(x, r_0^2)|\psi|^2(x)$$
$$(3.2) \qquad +(|\psi|^2(x) - r_0^2)[\partial_{N+1}F(x, r_0^2) + L]|\psi|^2(x) - L\,(|\psi|^2(x) - r_0^2)^2$$
$$+|\psi|^2(x) \int_{r_0^2}^{|\psi|^2(x)} (|\psi|^2(x) - \tau)\partial^2_{N+1}F(x, \tau)\, d\tau \qquad \text{for any } |x| \geq R_1.$$

Since $\psi \in L^\infty(\mathbf{R}^N)$ by Proposition 2.2 i) and $F(\cdot, r_0^2) \in L^1 \cap L^\infty(\mathbf{R}^N)$ by **(H1)** and **(H3)**, we infer that $F(\cdot, r_0^2)|\psi|^2 \in L^1 \cap L^\infty(\mathbf{R}^N)$.

We have $\psi \in L^\infty(\mathbf{R}^N)$, $\partial_{N+1}F(\cdot, r_0^2) + L \in L^{p_0} \cap L^\infty(\mathbf{R}^N)$ by **(H4)** and $|\psi|^2 - r_0^2 \in L^2 \cap L^\infty(\mathbf{R}^N)$ by Lemma 2.4, hence $(|\psi|^2 - r_0^2)[\partial_{N+1}F(\cdot, r_0^2) + L]|\psi|^2 \in L^1 \cap L^\infty(\mathbf{R}^N)$.

From Proposition 2.2 i), Lemma 2.4 and (2.9) it follows that the last two terms in the right-hand side of (3.2) are in $L^1 \cap L^\infty(\mathbf{R}^N \setminus \overline{B}(0, R_1))$. Hence $F(\cdot, |\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) \in L^1 \cap L^\infty(\mathbf{R}^N \setminus \overline{B}(0, R_1))$. Clearly, the function $F(\cdot, |\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2)$ is bounded on $\overline{B}(0, R_1)$, therefore this function belongs to $L^1 \cap L^\infty(\mathbf{R}^N)$.

Since $\psi_1 = |\psi| \cos\theta$ and $\psi_2 = |\psi| \sin\theta$, a straightforward computation gives

$$(3.3) \qquad \psi_1 \frac{\partial \psi_2}{\partial x_j} - \psi_2 \frac{\partial \psi_1}{\partial x_j} = (\psi_1^2 + \psi_2^2)\frac{\partial \theta}{\partial x_j} \qquad \text{on } \mathbf{R}^N \setminus \overline{B}(0, R_*).$$

Therefore

$$\psi_1 \frac{\partial \psi_2}{\partial x_j} - \psi_2 \frac{\partial \psi_1}{\partial x_j} - r_0^2 \frac{\partial}{\partial x_j}(\chi\theta) = (|\psi|^2 - r_0^2)\frac{\partial \theta}{\partial x_j} \qquad \text{on } \mathbf{R}^N \setminus \overline{B}(0, 3R_*). \tag{3.4}$$

From Lemma 2.4, Proposition 2.5 ii) and the Sobolev embedding theorem we have $|\psi|^2 - r_0^2 \in L^2 \cap L^\infty(\mathbf{R}^N)$ and $\frac{\partial \theta}{\partial x_j} \in L^2 \cap L^\infty(\mathbf{R}^N \setminus \overline{B}(0, R_*))$, respectively. Identity (3.4) implies $G_j \in L^1 \cap L^\infty(\mathbf{R}^N \setminus \overline{B}(0, 3R_*))$. Since G_j is continuous on $\mathbf{R}^N$, we conclude that $G_j \in L^1 \cap L^\infty(\mathbf{R}^N)$.

ii) Equation (1.7) is equivalent to the system

$$c\frac{\partial \psi_2}{\partial x_1} + \Delta\psi_1 + F(x, |\psi|^2)\psi_1 = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N), \tag{3.5}$$

$$-c\frac{\partial \psi_1}{\partial x_1} + \Delta\psi_2 + F(x, |\psi|^2)\psi_2 = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{3.6}$$

In view of Proposition 2.2 i), equalities (3.5) and (3.6) hold in $L^p_{loc}(\mathbf{R}^N)$ for $1 \leq p < \infty$. Multiplying (3.5) by ψ_2 and (3.6) by ψ_1, then substracting the resulting equalities we get

$$\frac{c}{2}\frac{\partial}{\partial x_1}(|\psi|^2 - r_0^2) = \operatorname{div}(\psi_1 \nabla \psi_2 - \psi_2 \nabla \psi_1). \tag{3.7}$$

We multiply (3.5) by ψ_1 and (3.6) by ψ_2, then we add the corresponding equalities to obtain

$$|\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(x, |\psi|^2)|\psi|^2 - c(\psi_1\frac{\partial \psi_2}{\partial x_1} - \psi_2\frac{\partial \psi_1}{\partial x_1}) = \frac{1}{2}\Delta(|\psi|^2 - r_0^2). \tag{3.8}$$

From (3.7) and (3.8) we get

$$\frac{c}{2}\frac{\partial}{\partial x_1}(|\psi|^2 - r_0^2) = \operatorname{div}(\psi_1 \nabla \psi_2 - \psi_2 \nabla \psi_1 - r_0^2 \nabla(\chi\theta)) + r_0^2 \Delta(\chi\theta), \tag{3.9}$$

respectively

$$\begin{aligned} \frac{1}{2}\Delta(|\psi|^2 - r_0^2) - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) &= |\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(x, |\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) \\ &-c(\psi_1\frac{\partial \psi_2}{\partial x_1} - \psi_2\frac{\partial \psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta)) - cr_0^2\frac{\partial}{\partial x_1}(\chi\theta). \end{aligned} \tag{3.10}$$

Since $\psi \in W^{2,p}_{loc}(\mathbf{R}^N)$, equalities (3.7)-(3.10) hold in $L^p_{loc}(\mathbf{R}^N)$ for $1 \leq p < \infty$. We denote

$$H = |\nabla\psi_1|^2 + |\nabla\psi_2|^2 - F(x, |\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2) - c(\psi_1\frac{\partial \psi_2}{\partial x_1} - \psi_2\frac{\partial \psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta)).$$

We take the derivative of (3.9) with respect to x_1 (in $\mathcal{D}'(\mathbf{R}^N)$) and we multiply it by c, then we take the Laplacian of (3.10) (in $\mathcal{D}'(\mathbf{R}^N)$). Summing up the resulting equalities we obtain

$$\frac{1}{2}\Big(\Delta^2 - v_s^2\Delta + c^2\frac{\partial^2}{\partial x_1^2}\Big)(|\psi|^2 - r_0^2) = \Delta H + c\frac{\partial}{\partial x_1}(\operatorname{div}(G)) \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{3.11}$$

From i) we have $H, G_1, \ldots, G_N \in L^1 \cap L^\infty(\mathbf{R}^N)$ and we know from Lemma 2.4 that $|\psi|^2 - r_0^2 \in L^2 \cap L^\infty(\mathbf{R}^N)$. Therefore $H, G_1, \ldots, G_N, |\psi|^2 - r_0^2 \in \mathcal{S}'(\mathbf{R}^N)$ and we infer that, in fact, equality (3.11) holds in $\mathcal{S}'(\mathbf{R}^N)$. Taking the Fourier transform of (3.11) we get

$$\frac{1}{2}(|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2)\mathcal{F}(|\psi|^2 - r_0^2) = -|\xi|^2\widehat{H} - c\sum_{k=1}^N \xi_1\xi_k\widehat{G_k} \qquad \text{in } \mathcal{S}'(\mathbf{R}^N). \tag{3.12}$$

We have $\widehat{H},\ \widehat{G_k} \in L^\infty \cap C^0(\mathbf{R}^N)$ because $H,\ G_k \in L^1(\mathbf{R}^N)$. Thus the right-hand side of (3.12) is a continuous function on $\mathbf{R}^N$. Since $|\psi|^2 - r_0^2 \in L^2(\mathbf{R}^N)$, we have $\mathcal{F}(|\psi|^2 - r_0^2) \in L^2(\mathbf{R}^N)$ and we infer that the left-hand side of (3.12) belongs to $L^2_{loc}(\mathbf{R}^N)$ and (3.12) holds a.e. on $\mathbf{R}^N$.

We denote

$$\Gamma = \{\xi \in \mathbf{R}^N \mid |\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2 = 0\}.$$

If $c^2 \leq v_s^2$ we have $\Gamma = \{0\}$. If $c^2 > v_s^2$, it is easy to see that Γ is a nontrivial submanifold of $\mathbf{R}^N$. In the latter case, we claim that

$$|\xi|^2\widehat{H}(\xi) + c\sum_{k=1}^N \xi_1\xi_k\widehat{G_k}(\xi) = 0 \qquad \text{for any } \xi \in \Gamma. \tag{3.13}$$

To prove this claim, we argue by contradiction and suppose that there exists $\xi^0 \in \Gamma$ such that $|\xi^0|^2\widehat{H}(\xi^0) + c\sum_{k=1}^N \xi_1^0\xi_k^0\widehat{G_k}(\xi^0) \neq 0$. By continuity, there exist $m > 0$ and a neighborhood U of ξ_0 such that $\left| |\xi|^2\widehat{H} + c\sum_{k=1}^N \xi_1\xi_k\widehat{G_k}\right| \geq m$ on U. From (3.12) we infer that

$$|\mathcal{F}(|\psi|^2 - r_0^2)(\xi)| \geq \frac{2m}{|\,|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2|} \qquad \text{a.e. on } U \setminus \Gamma.$$

Since 0 and $(\sqrt{c^2 - v_s^2}, 0, \dots, 0)$ are not isolated points of Γ, we may assume that $\xi^0 \neq 0$ and $\xi^0 \neq (\sqrt{c^2 - v_s^2}, 0, \dots, 0)$. A straightforward computation (details can be found in [17], p. 98 in the case $v_s^2 = 2$; the general case is similar) shows that

$$\int_{U\setminus\Gamma} \frac{1}{|\,|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2|^2}\, d\xi = \infty,$$

consequently $\int_{U\setminus\Gamma} |\mathcal{F}(|\psi|^2 - r_0^2)(\xi)|^2\, d\xi = \infty$. But this is in contradiction with $\mathcal{F}(|\psi|^2 - r_0^2) \in L^2(\mathbf{R}^N)$ and the claim is proved.

It is not hard to see that $\Gamma = \{(\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{N-1} \mid |\xi'|^2 = \frac{1}{2}(-v_s^2 - 2\xi_1^2 + \sqrt{v_s^4 + 4c^2\xi_1^2})\}$. Let $f(t) = \sqrt{\frac{1}{2}\left(-v_s^2 - 2t^2 + \sqrt{v_s^4 + 4c^2t^2}\right)}$. The function f is well-defined for $t \in [-\sqrt{c^2 - v_s^2}, \sqrt{c^2 - v_s^2}]$, $f(0) = 0$ and $\lim_{t\to 0}\frac{f^2(t)}{t^2} = -1 + \frac{c^2}{v_s^2}$. Fix $j \in \{2, \dots, N\}$. For $t \in (0, \sqrt{c^2 - v_s^2}]$, let $\xi(t) = (t, 0, \dots, 0, f(t), 0, \dots, 0)$ and $\tilde{\xi}(t) = (t, 0, \dots, 0, -f(t), 0, \dots, 0)$, where $f(t)$, respectively $-f(t)$, stand at the j^{th} place. It is obvious that $\xi(t), \tilde{\xi}(t) \in \Gamma$. From (3.13) we obtain

$$(t^2 + f^2(t))\widehat{H}(\xi(t)) + ct^2\widehat{G_1}(\xi(t)) + ctf(t)\widehat{G_j}(\xi(t)) = 0, \qquad \text{respectively} \tag{3.14}$$

$$(t^2 + f^2(t))\widehat{H}(\tilde{\xi}(t)) + ct^2\widehat{G_1}(\tilde{\xi}(t)) - ctf(t)\widehat{G_j}(\tilde{\xi}(t)) = 0. \tag{3.15}$$

We multiply (3.14) and (3.15) by $\frac{1}{t^2}$, then pass to the limit as $t \downarrow 0$ to obtain

$$\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) + c\sqrt{-1 + \frac{c^2}{v_s^2}}\widehat{G_j}(0) = 0, \text{ respectively} \tag{3.16}$$

$$\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) - c\sqrt{-1 + \frac{c^2}{v_s^2}}\widehat{G_j}(0) = 0. \tag{3.17}$$

From (3.16) and(3.17) we infer that $\frac{c^2}{v_s^2}\widehat{H}(0) + c\widehat{G_1}(0) = 0$ and $\widehat{G_j}(0) = 0$, that is $\int_{\mathbf{R}^N} H(x) + \frac{v_s^2}{c} G_1(x)\, dx = 0$ and $\int_{\mathbf{R}^N} G_j(x)\, dx = 0$. The first of these integral identities is exactly (3.1) and the latter can be written as

$$\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_j} - \psi_2 \frac{\partial \psi_1}{\partial x_j} - r_0^2 \frac{\partial}{\partial x_j}(\chi\theta)\, dx = 0 \qquad \text{for } j = 2, \dots, N. \tag{3.18}$$

iii) Assume that $c^2 = v_s^2$. Then (3.1) is equivalent to $\widehat{H}(0) + c\widehat{G_1}(0) = 0$. Denoting $\xi = (\xi_1, \xi')$, where $\xi' = (\xi_2, \dots, \xi_N)$, identity (3.12) implies

$$\begin{aligned} \mathcal{F}(|\psi|^2 - r_0^2)(\xi) &= -2\frac{\xi_1^2}{|\xi|^4 + c^2|\xi'|^2}(\widehat{H}(\xi) + c\widehat{G_1}(\xi)) \\ &-2c\sum_{k=2}^{N} \frac{\xi_1\xi_k}{|\xi|^4 + c^2|\xi'|^2}\widehat{G_k}(\xi) - 2\frac{|\xi'|^2}{|\xi|^4 + c^2|\xi'|^2}\widehat{H}(\xi) \qquad \text{a.e. } \xi \in \mathbf{R}^N. \end{aligned} \tag{3.19}$$

For $\varepsilon \in (0, 1]$, we denote $\Omega_\varepsilon = \{(\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{N-1} \mid \xi_1 \in [0, \varepsilon],\ 0 \le |\xi'| \le \xi_1\}$. We will use the following

Lemma 3.2 *Let $N \ge 2$ and $k \in \{2, \dots, N\}$.*

i) The function $\xi \longmapsto \frac{\xi_1^2}{\xi_1^4 + c^2|\xi'|^2}$ belongs to $L^p(\Omega_\varepsilon)$ if and only if $p < N - \frac{1}{2}$.

ii) The function $\xi \longmapsto \frac{\xi_1\xi_k}{\xi_1^4 + c^2|\xi'|^2}$ belongs to $L^p(\Omega_\varepsilon)$ for any $p \in [1, 2N-1)$.

Proof of Lemma 3.2. i) Using Fubini's theorem for positive functions, then passing to spherical coordinates in $\mathbf{R}^{N-1}$ and making the change of variables $r = \xi_1^2 t$ we get

$$\begin{aligned} \int_{\Omega_\varepsilon} \left(\frac{\xi_1^2}{\xi_1^4 + c^2|\xi'|^2}\right)^p d\xi &= \int_0^\varepsilon \xi_1^{2p} \int_{\{|\xi'| \le \xi_1\}} \frac{1}{(\xi_1^4 + c^2|\xi'|^2)^p}\, d\xi'\, d\xi_1 \\ &= \int_0^\varepsilon \xi_1^{2p} |S^{N-2}| \int_0^{\xi_1} \frac{r^{N-2}}{(\xi_1^4 + c^2 r^2)^p}\, dr\, d\xi_1 \\ &= |S^{N-2}| \int_0^\varepsilon \xi_1^{2p} \int_0^{\frac{1}{\xi_1}} \frac{(\xi_1^2 t)^{N-2}}{(\xi_1^4 + c^2\xi_1^4 t^2)^p}\xi_1^2\, dt\, d\xi_1 \qquad \text{(change of variables } r = \xi_1^2 t) \\ &= |S^{N-2}| \int_0^\varepsilon \xi_1^{2(N-1-p)} \int_0^{\frac{1}{\xi_1}} \frac{t^{N-2}}{(1 + c^2t^2)^p}\, dt\, d\xi_1. \end{aligned} \tag{3.20}$$

Assume that $p < N - \frac{1}{2}$. Obviously $\frac{t^{N-2}}{(1+c^2t^2)^p} \le 1$ for $t \in [0, 1]$ and $\frac{t^2}{1+c^2t^2} \le \frac{1}{c^2}$, thus we have

$$\int_0^{\frac{1}{\xi_1}} \frac{t^{N-2}}{(1 + c^2t^2)^p}\, dt \le 1 + \frac{1}{c^{2p}} \int_1^{\frac{1}{\xi_1}} t^{N-2p-2}\, dt = \begin{cases} C_1 + \frac{C_2}{\xi_1^{N-2p-1}} & \text{if } p \ne \frac{N-1}{2}, \\ C_3 + C_4 \ln \xi_1 & \text{if } p = \frac{N-1}{2}, \end{cases}$$

where C_j are some positive constants. This estimate implies that the right-hand side of (3.20) is finite if $p < N - \frac{1}{2}$.

If $p \ge N - \frac{1}{2}$, denote $c_p = \displaystyle\int_0^1 \frac{t^{N-2}}{(1 + c^2t^2)^p}\, dt > 0$. Since $\frac{1}{\xi_1} > 1$ for $\xi_1 \in (0, \varepsilon)$, the right-hand side of (3.20) is greater than $|S^{N-2}| c_p \int_0^\varepsilon \xi_1^{2(N-1-p)}\, d\xi_1 = \infty$.

ii) Proceeding as above, we have

$$\int_{\Omega_\varepsilon}\left|\frac{\xi_1\xi_k}{\xi_1^4+c^2|\xi'|^2}\right|^p d\xi \le \int_{\Omega_\varepsilon}\frac{\xi_1^p|\xi'|^p}{(\xi_1^4+c^2|\xi'|^2)^p}\,d\xi = \int_0^\varepsilon \xi_1^p|S^{N-2}|\int_0^{\xi_1}\frac{r^{p+N-2}}{(\xi_1^4+c^2r^2)^p}\,dr\,d\xi_1$$

$$(3.21)\qquad = |S^{N-2}|\int_0^\varepsilon \xi_1^p\int_0^{\frac{1}{\xi_1}}\frac{(\xi_1^2t)^{p+N-2}}{(\xi_1^4+c^2\xi_1^4t^2)^p}\xi_1^2\,dt\,d\xi_1 \qquad \text{(change of variables } r=\xi_1^2t\text{)}$$

$$= |S^{N-2}|\int_0^\varepsilon \xi_1^{2N-p-2}\int_0^{\frac{1}{\xi_1}}\frac{t^{p+N-2}}{(1+c^2t^2)^p}\,dt\,d\xi_1.$$

As previously,

$$\int_0^{\frac{1}{\xi_1}}\frac{t^{p+N-2}}{(1+c^2t^2)^p}\,dt < \frac{1}{c^{2p}}\int_0^{\frac{1}{\xi_1}}t^{N-p-2}\,dt = \frac{1}{c^{2p}(N-p-1)}\frac{1}{\xi_1^{N-p-1}} \qquad \text{if } N-p-1>0.$$

Therefore in the case $p<N-1$, the right-hand side of (3.21) is less than $C\int_0^\varepsilon \xi_1^{N-1}\,d\xi_1<\infty$. If $p>N-1$, the integral $\displaystyle\int_0^\infty\frac{t^{p+N-2}}{(1+c^2t^2)^p}\,dt$ converges. Let a_p be its value. If $N-1<p<2N-1$, by (3.21) we get $\displaystyle\int_{\Omega_\varepsilon}\left|\frac{\xi_1\xi_k}{\xi_1^4+c^2|\xi'|^2}\right|^p d\xi \le |S^{N-2}|a_p\int_0^\varepsilon \xi_1^{2N-2-p}d\xi_1<\infty.$ □

Remark. It can be proved that the function $\xi \longmapsto \frac{\xi_1\xi_k}{\xi_1^4+c^2|\xi'|^2}$ does not belong to $L^p(\Omega_\varepsilon)$ if $p\ge 2N-1$, but we will not make use of this fact here.

Now we come back to the proof of Theorem 3.1. All we have to do is to show that $\widehat{H}(0)+c\widehat{G_1}(0)=0$. We argue by contradiction and assume that $\widehat{H}(0)+c\widehat{G_1}(0)\neq 0$. Since the functions $\widehat{H}$ and $\widehat{G_j}$ are continuous, there exists $\varepsilon\in(0,1)$ such that $|\widehat{H}(\xi)+c\widehat{G_1}(\xi)|\ge \frac12|\widehat{H}(0)+c\widehat{G_1}(0)|$ for any $\xi\in\Omega_\varepsilon$. Taking a smaller ε if necessary, we may also assume that $|\xi|^4+c^2|\xi'|^2\le 2(\xi_1^4+c^2|\xi'|^2)$ for any $\xi\in\Omega_\varepsilon$. By (3.19) we have

$$\frac12\frac{\xi_1^2}{\xi_1^4+c^2|\xi'|^2}|\widehat{H}(0)+c\widehat{G_1}(0)| \le 2\frac{\xi_1^2}{|\xi|^4+c^2|\xi'|^2}|\widehat{H}(\xi)+c\widehat{G_1}(\xi)|$$

$$(3.22)\qquad \le |\mathcal{F}(|\psi|^2-r_0^2)(\xi)| + 2|c|\sum_{k=2}^N\frac{|\xi_1\xi_k|}{\xi_1^4+c^2|\xi'|^2}|\widehat{G_k}(\xi)| + 2\frac{|\xi'|^2}{|\xi|^4+c^2|\xi'|^2}|\widehat{H}(\xi)| \quad \text{a.e. on } \Omega_\varepsilon.$$

Consider first the case $N=2$. We know that $\mathcal{F}(|\psi|^2-r_0^2)\in L^2(\mathbf{R}^2)$, consequently $\mathcal{F}(|\psi|^2-r_0^2)\in L^p(\Omega_\varepsilon)$ for any $p\in[1,2]$. Since $\widehat{G_k}$ are continuous and bounded, by Lemma 3.2 ii) we infer that the functions $\xi\longmapsto \frac{\xi_1\xi_k}{\xi_1^4+c^2|\xi'|^2}\widehat{G_k}(\xi)$ belong to $L^p(\Omega_\varepsilon)$ for any $p\in[1,3)$. It is obvious that $\frac{|\xi'|^2}{|\xi|^4+c^2|\xi'|^2}|\widehat{H}(\xi)|\le\frac{1}{c^2}|\widehat{H}(\xi)|$ and $\widehat{H}$ is continuous and bounded on $\mathbf{R}^N$. We conclude that the right-hand side of (3.22) belongs to $L^p(\Omega_\varepsilon)$ for any $p\in[1,2]$. Then (3.22) implies that $\xi\longmapsto\frac{\xi_1^2}{\xi_1^4+c^2|\xi'|^2}$ belongs to $L^2(\Omega_\varepsilon)$, which contradicts Lemma 3.2 i). This contradiction proves that $\widehat{H}(0)+c\widehat{G_1}(0)=0$.

Next we assume that $N\ge 3$ and $\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}\in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$. Equation (3.8) can be written as

$$-\frac12\Delta(|\psi|^2-r_0^2)+\frac{v_s^2}{2}(|\psi|^2-r_0^2)$$

$$(3.23)\qquad = -|\nabla\psi_1|^2-|\nabla\psi_2|^2+F(x,|\psi|^2)|\psi|^2+\frac{v_s^2}{2}(|\psi|^2-r_0^2)+c(\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}).$$

We have already proved that $F(\cdot,|\psi|^2)|\psi|^2+\frac{v_s^2}{2}(|\psi|^2-r_0^2)\in L^1\cap L^\infty(\mathbf{R}^N)$. From Proposition 2.5 i) we have $|\nabla\psi|^2\in L^p(\mathbf{R}^N)$ for any $p\in[1,\infty]$. Using the assumption $\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}\in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$, we infer that the right-hand side of (3.23) belongs to $L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$. By the Hausdorff-Young inequality, for any function $f\in L^p(\mathbf{R}^N)$ with $1\le p\le 2$ we have $\mathcal{F}(f)\in L^{p'}(\mathbf{R}^N)$, where $\frac{1}{p}+\frac{1}{p'}=1$ (see, e.g., Theorem 1.2.1 p. 6 in [4]). Passing to Fourier transforms in (3.23) we get

$$\begin{aligned}\mathcal{F}(|\psi|^2-r_0^2)(\xi) &= \tfrac{2}{|\xi|^2+v_s^2}\mathcal{F}\Big[-|\nabla\psi|^2+(F(\cdot,|\psi|^2)|\psi|^2+\tfrac{v_s^2}{2}(|\psi|^2-r_0^2))\\ &\quad +c(\psi_1\tfrac{\partial\psi_2}{\partial x_1}-\psi_2\tfrac{\partial\psi_1}{\partial x_1})\Big](\xi) \qquad \text{a.e. } \xi\in\mathbf{R}^N.\end{aligned} \tag{3.24}$$

We obtain from (3.24) that $\mathcal{F}(|\psi|^2-r_0^2)\in L^{N-\frac{1}{2}}(\mathbf{R}^N)$. Combined with the fact that $\widehat{H}$, $\widehat{G_j}$ and $\xi\longmapsto\frac{|\xi'|^2}{|\xi|^4+c^2|\xi'|^2}$ are bounded and Lemma 3.2 ii), this implies that the last expression in (3.22) is in $L^{N-\frac{1}{2}}(\Omega_\varepsilon)$. We infer that the function $\xi\longmapsto\frac{\xi_1^2}{\xi_1^4+c^2|\xi'|^2}|\widehat{H}(0)+c\widehat{G_1}(0)|$ must be in $L^{N-\frac{1}{2}}(\Omega_\varepsilon)$ for any sufficiently small ε. If $\widehat{H}(0)+c\widehat{G_1}(0)\neq 0$, this contradicts Lemma 3.2 i). Thus necessarily $\widehat{H}(0)+c\widehat{G_1}(0)=0$ and the proof of Theorem 3.1 is complete. □

It is an open problem whether any finite energy travelling-wave ψ of (1.1) moving with speed $c=\pm v_s$ satisfies $\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}\in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$. Even for very particular cases of (1.1), such as the Gross-Pitaevskii equation, the answer to this question is not known. However, we have the following:

Proposition 3.3 *Assume that* **(H1) - (H5)** *hold and let $\psi=\psi_1+i\psi_2$ be a finite-energy travelling-wave for (1.1) such that $F(\cdot,|\psi|^2)\in L^1_{loc}(\mathbf{R}^N)$. Let R_* be sufficiently big, so that $|\psi|\ge\frac{r_0}{2}$ on $\mathbf{R}^N\setminus B(0,R_*)$, let θ be the lifting given by Proposition 2.5 ii) and let $\chi\in C^\infty(\mathbf{R}^N)$ be a cut-off function as in Theorem 3.1. Then:*

i) Let $p\in(1,\infty)$. The following assertions are equivalent:

a) $\nabla(\chi\theta)\in L^p(\mathbf{R}^N)$;

b) $\psi_1\frac{\partial\psi_2}{\partial x_j}-\psi_2\frac{\partial\psi_1}{\partial x_j}\in L^p(\mathbf{R}^N)$ for any $j\in\{1,\dots,N\}$;

c) $\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}\in L^p(\mathbf{R}^N)$;

d) $|\psi|^2-r_0^2\in W^{2,p}(\mathbf{R}^N)$;

e) $|\psi|^2-r_0^2\in L^p(\mathbf{R}^N)$.

ii) If $N\ge 3$, there exists $\theta_0\in\mathbf{R}$ such that $\chi\theta-\theta_0\in W^{2,q}(\mathbf{R}^N)$ for any $q\in[\frac{2N}{N-2},\infty)$.

Moreover, if $c^2=v_s^2$ we have:

iii) $|\psi|^2-r_0^2\in L^p(\mathbf{R}^N)$ and $\psi_1\frac{\partial\psi_2}{\partial x_j}-\psi_2\frac{\partial\psi_1}{\partial x_j}\in L^p(\mathbf{R}^N)$ for any $p>\frac{2N-1}{2N-3}$ and $j\in\{1,\dots,N\}$.

iv) $\nabla(|\psi|^2-r_0^2)\in L^p(\mathbf{R}^N)$ for any $p>\frac{2N-1}{2N-2}$.

v) $\partial^2_{j,k}(|\psi|^2-r_0^2)\in L^p(\mathbf{R}^N)$ for any $p\in(1,\infty)$.

Proof. i) Since $\psi\in L^\infty(\mathbf{R}^N)$ and (3.3) holds, the equivalence a) $\Leftrightarrow$ b) is clear. It is also obvious that b) $\Rightarrow$ c).

From the classical Marcinkiewicz Theorem (see Theorem 3 p. 96 in [27]) it follows that the functions $\frac{1}{|\xi|^2+v_s^2}$, $\frac{\xi_j}{|\xi|^2+v_s^2}$ and $\frac{\xi_j\xi_k}{|\xi|^2+v_s^2}$ are L^p-multipliers for $1<p<\infty$. Assume that $\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}\in L^p(\mathbf{R}^N)$. Since $|\nabla\psi|^2\in L^1\cap L^\infty(\mathbf{R}^N)$ and $F(\cdot,|\psi|^2)|\psi|^2+\frac{v_s^2}{2}(|\psi|^2-r_0^2)\in L^1\cap L^\infty(\mathbf{R}^N)$

by Theorem 3.1 i), we have $-|\nabla\psi|^2+(F(\cdot,|\psi|^2)|\psi|^2+\frac{v_s^2}{2}(|\psi|^2-r_0^2))+c(\psi_1\frac{\partial\psi_2}{\partial x_1}-\psi_2\frac{\partial\psi_1}{\partial x_1}) \in L^p(\mathbf{R}^N)$ and we infer from (3.24) that $|\psi|^2 - r_0^2 \in W^{2,p}(\mathbf{R}^N)$. Hence c) $\Rightarrow$ d). It is obvious that d) $\Rightarrow$ e).

It follows from Proposition 2.5 ii) that $\partial_k(\chi\theta) \in \mathcal{S}'(\mathbf{R}^N)$. It is then clear that all terms appearing in (3.9) belong to $\mathcal{S}'(\mathbf{R}^N)$. We take the derivative of (3.9) with respect to x_k (in $\mathcal{S}'(\mathbf{R}^N)$), then we take the Fourier transform of the resulting equality to obtain

$$\mathcal{F}(\frac{\partial}{\partial x_k}(\chi\theta)) = -\sum_{j=1}^{N}\frac{\xi_j\xi_k}{|\xi|^2}\widehat{G_j} + \frac{c}{2}\frac{\xi_1\xi_k}{|\xi|^2}\mathcal{F}(|\psi|^2 - r_0^2)$$

or equivalently

$$\frac{\partial}{\partial x_k}(\chi\theta) = \sum_{j=1}^{N} R_jR_k(G_j) - \frac{c}{2}R_1R_k(|\psi|^2 - r_0^2), \tag{3.25}$$

where R_j is the Riesz transform, $R_j\phi = \mathcal{F}^{-1}(i\frac{\xi_j}{|\xi|}\widehat{\phi})$. It is well-known that the Riesz transform maps continuously $L^p(\mathbf{R}^N)$ into $L^p(\mathbf{R}^N)$ for $1 < p < \infty$ (see, e.g., Theorem 3 p. 96 and Example (iii) p. 95 in [27]). From Theorem 3.1 i) we have $G_j \in L^1 \cap L^\infty(\mathbf{R}^N)$, therefore $R_jR_k(G_j) \in L^q(\mathbf{R}^N)$ for any $q \in (1,\infty)$. Assume that $|\psi|^2 - r_0^2 \in L^p(\mathbf{R}^N)$ for some $p \in (1,\infty)$. Then $R_1R_k(|\psi|^2 - r_0^2) \in L^p(\mathbf{R}^N)$ and from (3.25) we infer that $\frac{\partial}{\partial x_k}(\chi\theta) \in L^p(\mathbf{R}^N)$ for any $k \in \{1,\dots,N\}$. Thus e) $\Rightarrow$ a) and i) is proved.

ii) It is well-known that for any function ϕ satisfying $\nabla\phi \in L^p(\mathbf{R}^N)$ with $p < N$, there exists a constant λ such that $\phi - \lambda \in L^{p^*}(\mathbf{R}^N)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ (see Theorem 4.5.9 in [20] or Lemma 7 and Remark 4.2 in [15] p. 774-775 for a different proof). From Proposition 2.5 ii) we have $\nabla(\chi\theta) \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$. If $N \geq 3$, we infer that there exists $\theta_0 \in \mathbf{R}$ such that $\chi\theta - \theta_0 \in L^q(\mathbf{R}^N)$ for $q \in [\frac{2N}{N-2},\infty)$. Therefore $\chi\theta - \theta_0 \in W^{2,q}(\mathbf{R}^N)$ for any $q \in [\frac{2N}{N-2},\infty)$ and, in particular, $\chi\theta - \theta_0 \longrightarrow 0$ as $|x| \longrightarrow \infty$.

iii) We will use the following result due to Lizorkin (see Theorem 8 p. 288 in [24]):

Theorem 3.4 ([24]) *Let $\beta \in [0,1)$ and let $K \in L^\infty(\mathbf{R}^N) \cap C^N(\mathbf{R}^N \setminus \{0\})$. Assume that*

$$\Big(\prod_{j=1}^{N}\xi_j^{k_j+\beta}\Big)\partial_1^{k_1}\dots\partial_N^{k_N}K \in L^\infty(\mathbf{R}^N) \qquad \text{for any } k_1,\dots,k_N \in \{0,1\}.$$

Then K is a Fourier multiplier from $L^p(\mathbf{R}^N)$ to $L^{\frac{p}{1-\beta p}}(\mathbf{R}^N)$ for any $p \in (1,\frac{1}{\beta})$.

Let $K(\xi) = \frac{|\xi|^2}{|\xi|^4+c^2|\xi'|^2}$, where $\xi' = (\xi_2,\dots,\xi_N)$. A straightforward but tedious computation shows that K satisfies the assumptions of Lizorkin's theorem for $\beta = \frac{1}{2N-1}$. From (3.19) we obtain

$$|\psi|^2 - r_0^2 = 2R_1^2\left(\mathcal{F}^{-1}\left(K(\widehat{H}+c\widehat{G_1})\right)\right) + 2c\sum_{j=2}^{N}R_1R_j\left(\mathcal{F}^{-1}(K\widehat{G_j})\right) + 2\sum_{j=2}^{N}R_j^2\left(\mathcal{F}^{-1}(K\widehat{H})\right), \tag{3.26}$$

where R_j's denote Riesz transforms. Since $H, G_1,\dots,G_N \in L^1 \cap L^\infty(\mathbf{R}^N)$, by (3.26) and Lizorkin's theorem we infer that $|\psi|^2 - r_0^2 \in L^p(\mathbf{R}^N)$ for any $p \in (\frac{2N-1}{2N-3},\infty)$. The rest of iii) follows from part i), b) $\Leftrightarrow$ e).

iv) and v) From iii) and i), d) $\Leftrightarrow$ e) it follows immediately that $|\psi|^2 - r_0^2 \in W^{2,p}(\mathbf{R}^N)$ for any $p \in (\frac{2N-1}{2N-3}, \infty)$. Using (3.19) we obtain

$$\partial^2_{k\ell}\left(|\psi|^2 - r_0^2\right) = 2R_k R_\ell R_1^2 \left(\mathcal{F}^{-1}\left(|\xi|^2 K(\widehat{H} + c\widehat{G_1})\right)\right)$$

$$(3.27) \qquad +2c\sum_{j=2}^{N} R_k R_\ell R_1 R_j \left(\mathcal{F}^{-1}(|\xi|^2 K\widehat{G_j})\right)$$

$$+2\sum_{j=2}^{N} R_k R_\ell R_j^2 \left(\mathcal{F}^{-1}(|\xi|^2 K\widehat{H})\right) \qquad \text{in } \mathcal{S}'(\mathbf{R}^N).$$

It can be proved by direct computation that the function $|\xi|^2 K$ satisfies the assumptions of Lizorkin's theorem for $\beta = 0$. Consequently $|\xi|^2 K$ is an L^p-multiplier for $1 < p < \infty$. Since $H, G_j \in L^1 \cap L^\infty(\mathbf{R}^N)$, it follows from (3.27) that $\partial^2_{k\ell}\left(|\psi|^2 - r_0^2\right) \in L^p(\mathbf{R}^N)$ for $1 < p < \infty$.

By using the Gagliardo-Nirenberg inequality

$$||\nabla\phi||^2_{L^p} \leq C||\phi||_{L^q}||\nabla^2\phi||_{L^r} \qquad \text{if} \quad \frac{1}{p} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{r}\right),$$

we infer that $\nabla(|\psi|^2 - r_0^2) \in L^p(\mathbf{R}^N)$ for any $p > \frac{2N-1}{2N-2}$. □

Corollary 3.5 *Under the assumptions of Theorem 3.1, assume that $N \geq 3$, $c^2 = v_s^2$ and the momentum of ψ with respect to the x_1-direction is well-defined, that is $\psi_1 \frac{\partial\psi_2}{\partial x_1} - \psi_2 \frac{\partial\psi_1}{\partial x_1} \in L^1(\mathbf{R}^N)$. Then ψ satisfies (3.1).*

Proof. From Proposition 3.4 iii) and i) we have $\psi_1 \frac{\partial\psi_2}{\partial x_1} - \psi_2 \frac{\partial\psi_1}{\partial x_1} \in L^p(\mathbf{R}^N)$ for $p \in (\frac{2N-1}{2N-3}, \infty)$. Then the assumption $\psi_1 \frac{\partial\psi_2}{\partial x_1} - \psi_2 \frac{\partial\psi_1}{\partial x_1} \in L^1(\mathbf{R}^N)$ implies $\psi_1 \frac{\partial\psi_2}{\partial x_1} - \psi_2 \frac{\partial\psi_1}{\partial x_1} \in L^p(\mathbf{R}^N)$ for any $p \in [1, \infty)$. Now the conclusion follows from Theorem 3.1 iii). □

4 Nonexistence results

In this section we show how Theorem 3.1 may be used to prove nonexistence of supersonic and sonic travelling-waves with finite energy for some equations of type (1.1).

1. We consider the equation

$$(4.1) \qquad i\frac{\partial\Phi}{\partial t} + \Delta\Phi + G(|\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N.$$

We assume that the function $G : [0, \infty) \longrightarrow \mathbf{R}$ satisfies the following asumptions:

- **(A1)** $G \in C^2([0, \infty), \mathbf{R})$ and there exists $r_0 > 0$ such that $G(r_0^2) = 0$ and $G'(r_0^2) < 0$.
- **(A2)** There exists $\alpha > 0$ such that $\limsup_{s\longrightarrow\infty} \frac{G(s)}{s^\alpha} < 0$.

Obviously, equation (4.1) is of the form (1.1). As previously, we associate to (4.1) the "boundary condition" $|\Phi| \longrightarrow r_0^2$ as $|x| \longrightarrow \infty$. In this context, the sound velocity at infinity is $v_s = r_0\sqrt{-2G'(r_0^2)}$. The energy corresponding to (4.1) is $E(\Phi) = \int_{\mathbf{R}^N} |\nabla\Phi|^2\, dx + \int_{\mathbf{R}^N} V(|\Phi|^2)\, dx$, where $V(s) = \int_s^{r_0^2} G(\tau)\, d\tau$. Let ψ be a finite-energy travelling-wave for (4.1) (in the sense of Definition 2.1) moving with speed c. Then ψ satisfies the equation

$$(4.2) \qquad -ic\frac{\partial\psi}{\partial x_1} + \Delta\psi + G(|\psi|^2)\psi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N), \qquad |\psi| \longrightarrow r_0 \quad \text{as } |x| \longrightarrow \infty.$$

If G satisfies **(A1)-(A2)**, it is easy to see that $F(x,s) := G(s)$ satisfies the assumptions **(H1)-(H5)** in section 2 (with $L = -G'(r_0^2)$). It is then clear that the conclusions of Propositions 2.2, 2.5 and Theorem 3.1 i) are valid for ψ. Moreover, we have:

Proposition 4.1 *(Pohozaev identities) Let ψ be as above. Choose $R_* > 0$ such that $|\psi| \geq \frac{r_0}{2}$ on $\mathbf{R}^N \setminus B(0,R_*)$. Let θ be the lifting of $\frac{\psi}{|\psi|}$ on $\mathbf{R}^N \setminus B(0,R_*)$ (as given by Proposition 2.5 ii)) and let χ be a cut-off function as in Theorem 3.1. The following identities hold:*

$$(4.3)\qquad -\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_1}\right|^2 dx + \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left|\frac{\partial \psi}{\partial x_j}\right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2)\, dx = 0 \qquad \textit{and}$$

$$(4.4)\qquad \begin{aligned} &-\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx + \int_{\mathbf{R}^N} \sum_{j=1,\, j\neq k}^{N} \left|\frac{\partial \psi}{\partial x_j}\right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2)\, dx \\ &-c\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial}{\partial x_1}(\chi\theta)\, dx = 0 \qquad \textit{for } k = 2, \dots, N. \end{aligned}$$

It is worth to note that Proposition 4.1 is valid for any speed $c \in \mathbf{R}$.

Proof. Since the arguments are rather classical, we only sketch the proof.

Formally, travelling-waves are critical points of the functional $E_c = E + cP_1$, where E is the energy and P_1 is the momentum with respect to the x_1-direction (see (1.3)). Identities (4.3) and (4.4) are simple consequences of the behavior of E_c with respect to dilations in $\mathbf{R}^N$. To be more precise, define $\psi_{k,t}(x) = \psi(x_1, \dots, x_{k-1}, tx_k, x_{k+1}, \dots, x_N)$ and $g_k(t) = E_c(\psi_{k,t})$. If ψ is a critical point of E_c, one would expect that $g_k'(1) = \frac{d}{dt}(E_c(\psi_{k,t}))|_{t=1} = 0$ and this is precisely (4.3) if $k = 1$, respectively (4.4) if $k \geq 2$. However, this argument is not rigorous for at least two reasons. First, it is not clear what function space one should consider to define E_c (and this could not be a vector space because of the boundary conditions at infinity). Second, even if an appropriate function space is found, we do not know whether $\frac{d}{dt}(\psi_{k,t})|_{t=1} = x_k \frac{\partial \psi}{\partial x_k}$ belong to the tangent space at ψ of the considered function space.

The most convenient way to prove Pohozaev identities is to use a truncation argument. Fix a function $\eta \in C_c^\infty(\mathbf{R}^N)$ such that $\eta = 1$ on $B(0,1)$ and $\eta = 0$ on $\mathbf{R}^N \setminus B(0,2)$. For $n \geq 1$, define $\eta_n(x) = \eta(\frac{x}{n})$. We take the scalar product of (4.2) by $x_k \eta_n(x) \frac{\partial \psi}{\partial x_k}$ and we integrate by parts the resulting equality. It is standard (see, e.g., Proposition 1 p. 320 in [3] or Lemma 2.4 p. 104 in [11]) to prove that

$$(4.5)\qquad \lim_{n\to\infty} \int_{\mathbf{R}^N} (\Delta\psi, x_k \eta_n(x) \frac{\partial \psi}{\partial x_k})\, dx = -\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx + \frac{1}{2}\int_{\mathbf{R}^N} |\nabla \psi|^2\, dx \qquad \text{and}$$

$$(4.6)\qquad \lim_{n\to\infty} \int_{\mathbf{R}^N} (G(|\psi|^2)\psi, x_k \eta_n(x) \frac{\partial \psi}{\partial x_k})\, dx = \frac{1}{2}\int_{\mathbf{R}^N} V(|\psi|^2)\, dx.$$

It is obvious that $(ic\frac{\partial \psi}{\partial x_1}, \eta_n(x) x_1 \frac{\partial \psi}{\partial x_1}) = c\eta_n(x) x_1 (i\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_1}) = 0$. Thus taking the scalar product of (4.2) by $x_1 \eta_n(x) \frac{\partial \psi}{\partial x_1}$, integrating and using (4.5) and (4.6) we get (4.3).

By (3.3) we have $(-i\frac{\partial \psi}{\partial x_j}, \psi) = \psi_1 \frac{\partial \psi_2}{\partial x_j} - \psi_2 \frac{\partial \psi_1}{\partial x_j} = |\psi|^2 \frac{\partial \theta}{\partial x_j}$ on $\mathbf{R}^N \setminus \overline{B}(0,R_*)$. Using the convention $\partial^\alpha(\chi\theta) = 0$, $(\partial^\alpha \chi)\theta = 0$ on $B(0, 2R_*)$, we have

$$(4.7)\qquad \begin{aligned} (-i\tfrac{\partial \psi}{\partial x_j}, \psi) &= (1-\chi)(-i\tfrac{\partial \psi}{\partial x_j}, \psi) + \chi|\psi|^2 \tfrac{\partial \theta}{\partial x_j} \\ &= (1-\chi)(-i\tfrac{\partial \psi}{\partial x_j}, \psi) + |\psi|^2 \tfrac{\partial(\chi\theta)}{\partial x_j} - |\psi|^2 \theta \tfrac{\partial \chi}{\partial x_j} \qquad \text{on } \mathbf{R}^N. \end{aligned}$$

Therefore we get for $k = 2, \dots, N$:

$$\int_{\mathbf{R}^N} (-ic\frac{\partial \psi}{\partial x_1}, x_k \eta_n(x) \frac{\partial \psi}{\partial x_k})\, dx$$

$$= \frac{c}{2} \int_{\mathbf{R}^N} x_k \eta_n(x) \Big[\frac{\partial}{\partial x_1}(-i\psi, \frac{\partial \psi}{\partial x_k}) + \frac{\partial}{\partial x_k}(-i\frac{\partial \psi}{\partial x_1}, \psi)\Big]\, dx$$

$$= -\frac{c}{2} \int_{\mathbf{R}^N} x_k \frac{\partial \eta_n}{\partial x_1}(x)(-i\psi, \frac{\partial \psi}{\partial x_k}) + \Big(\eta_n(x) + x_k \frac{\partial \eta_n}{\partial x_k}(x)\Big)(-i\frac{\partial \psi}{\partial x_1}, \psi)\, dx$$

$$\begin{aligned} &= \frac{c}{2} \int_{\mathbf{R}^N} x_k \frac{\partial \eta_n}{\partial x_1}(x)\Big[(1-\chi)(-i\frac{\partial \psi}{\partial x_k}, \psi) + |\psi|^2 \frac{\partial(\chi\theta)}{\partial x_k} - |\psi|^2 \theta \frac{\partial \chi}{\partial x_k}\Big]\, dx \\ &\quad -\frac{c}{2} \int_{\mathbf{R}^N} \eta_n(x)(-i\frac{\partial \psi}{\partial x_1}, \psi)\, dx \\ &\quad -\frac{c}{2} \int_{\mathbf{R}^N} x_k \frac{\partial \eta_n}{\partial x_k}(x)\Big[(1-\chi)(-i\frac{\partial \psi}{\partial x_1}, \psi) + |\psi|^2 \frac{\partial(\chi\theta)}{\partial x_1} - |\psi|^2 \theta \frac{\partial \chi}{\partial x_1}\Big]\, dx \end{aligned} \tag{4.8}$$

$$= \frac{c}{2} \int_{\mathbf{R}^N} x_k |\psi|^2 \Big(\frac{\partial \eta_n}{\partial x_1} \frac{\partial(\chi\theta)}{\partial x_k} - \frac{\partial \eta_n}{\partial x_k} \frac{\partial(\chi\theta)}{\partial x_1}\Big) - \eta_n(x)\Big(-i\frac{\partial \psi}{\partial x_1}, \psi\Big)\, dx \qquad \text{if } n > 3R_*$$

because $\operatorname{supp}(1-\chi) \subset \overline{B}(0, 3R_*)$ and $\operatorname{supp} \nabla \eta_n \subset \overline{B}(0, 2n) \setminus B(0, n)$, consequently $(1-\chi)\frac{\partial \eta_n}{\partial x_j} = 0$ and $\frac{\partial \chi}{\partial x_\ell} \frac{\partial \eta_n}{\partial x_j} = 0$ on $\mathbf{R}^N$ for $n > 3R_*$.

It is obvious that

$$\begin{aligned} &\int_{\mathbf{R}^N} x_k \Big(\frac{\partial \eta_n}{\partial x_1} \frac{\partial(\chi\theta)}{\partial x_k} - \frac{\partial \eta_n}{\partial x_k} \frac{\partial(\chi\theta)}{\partial x_1}\Big)\, dx \\ &= \int_{\mathbf{R}^N} x_k \Big[\frac{\partial}{\partial x_1}\Big(\eta_n \frac{\partial(\chi\theta)}{\partial x_k}\Big) - \frac{\partial}{\partial x_k}\Big(\eta_n \frac{\partial(\chi\theta)}{\partial x_1}\Big)\Big]\, dx = \int_{\mathbf{R}^N} \eta_n \frac{\partial(\chi\theta)}{\partial x_1}\, dx. \end{aligned} \tag{4.9}$$

Since $|\psi|^2 - r_0^2$ and $\nabla(\chi\theta)$ belong to $L^2(\mathbf{R}^N)$, using the Dominated Convergence Theorem we obtain

$$\begin{aligned} &\Big| \int_{\mathbf{R}^N} x_k (|\psi|^2 - r_0^2) \Big(\frac{\partial \eta_n}{\partial x_1} \frac{\partial(\chi\theta)}{\partial x_k} - \frac{\partial \eta_n}{\partial x_k} \frac{\partial(\chi\theta)}{\partial x_1}\Big)\, dx \Big| \\ &\le 2||\nabla \eta||_{L^\infty(\mathbf{R}^N)} \int_{B(0,2n)\setminus B(0,n)} |\, |\psi|^2 - r_0^2| \Big(\Big|\frac{\partial(\chi\theta)}{\partial x_1}\Big| + \Big|\frac{\partial(\chi\theta)}{\partial x_k}\Big|\Big)\, dx \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{4.10}$$

Recall that $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial(\chi\theta)}{\partial x_1} \in L^1(\mathbf{R}^N)$ by Theorem 3.1 i) and by dominated convergence we get

$$\begin{aligned} &\int_{\mathbf{R}^N} \eta_n \Big[\Big(-i\frac{\partial \psi}{\partial x_1}, \psi\Big) - r_0^2 \frac{\partial(\chi\theta)}{\partial x_1}\Big]\, dx = \int_{\mathbf{R}^N} \eta_n \Big[\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial(\chi\theta)}{\partial x_1}\Big]\, dx \\ &\longrightarrow \int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial(\chi\theta)}{\partial x_1}\, dx \qquad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.11}$$

Combining (4.8)-(4.11) we find

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} (-ic\frac{\partial \psi}{\partial x_1}, x_k \eta_n(x) \frac{\partial \psi}{\partial x_k})\, dx = -\frac{c}{2} \int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - r_0^2 \frac{\partial(\chi\theta)}{\partial x_1}\, dx. \tag{4.12}$$

Taking the scalar product of (4.2) by $\eta_n(x)x_k\frac{\partial\psi}{\partial x_k}$, integrating over $\mathbf{R}^N$ and using (4.5), (4.6) and (4.12) we obtain (4.4). □

Theorem 4.2 *Assume that $N \geq 2$,* **(A1), (A2)** *hold and let ψ be a finite-energy travelling-wave for (3.1) such that $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Suppose that*

- *either $c^2 > v_s^2$, where $v_s = r_0\sqrt{-2G'(r_0^2)}$ is the sound velocity at infinity,*
- *or $N = 2$ and $c^2 = v_s^2$,*
- *or $N \geq 3$ and $c^2 = v_s^2$ and $\psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$.*

Moreover, assume that G satisfies

- **(A3)** *there exists $\alpha \in [-1 + \frac{N-3}{N-1}(1 - \frac{v_s^2}{c^2}), \frac{v_s^2}{c^2}]$ such that*

$$sG(s) + \frac{v_s^2}{2}(s - r_0^2) + (1 - \alpha - \frac{v_s^2}{c^2})V(s) \leq 0 \qquad \textit{for any } s \geq 0.$$

Then ψ is constant.

Proof. It follows from Propositions 2.2 and 2.5 that ψ is smooth and Proposition 4.1 implies that ψ satisfies (4.3) and (4.4). Summing up the identities (4.4) for $k = 2, \ldots, N$ we get

$$\begin{aligned} &\int_{\mathbf{R}^N} \left|\frac{\partial\psi}{\partial x_1}\right|^2 + \frac{N-3}{N-1}\sum_{k=2}^N \left|\frac{\partial\psi}{\partial x_k}\right|^2 dx + \int_{\mathbf{R}^N} V(|\psi|^2)\,dx \\ &-c\int_{\mathbf{R}^N} \psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial(\chi\theta)}{\partial x_1}\,dx = 0. \end{aligned} \tag{4.13}$$

On the other hand, from Theorem 3.1 we have

$$\begin{aligned} &\int_{\mathbf{R}^N} |\nabla\psi|^2 - G(|\psi|^2)|\psi|^2 - \frac{v_s^2}{2}(|\psi|^2 - r_0^2)\,dx \\ &-c(1 - \frac{v_s^2}{c^2})\int_{\mathbf{R}^N} \psi_1\frac{\partial\psi_2}{\partial x_1} - \psi_2\frac{\partial\psi_1}{\partial x_1} - r_0^2\frac{\partial}{\partial x_1}(\chi\theta)\,dx = 0. \end{aligned} \tag{4.14}$$

We multiply (4.13) by $-1 + \frac{v_s^2}{c^2}$ and we add the resulting equality to (4.14) to get

$$\begin{aligned} &\int_{\mathbf{R}^N} \frac{v_s^2}{c^2}\left|\frac{\partial\psi}{\partial x_1}\right|^2 + \Big(1 - (1 - \frac{v_s^2}{c^2})\frac{N-3}{N-1}\Big)\sum_{k=2}^N \left|\frac{\partial\psi}{\partial x_k}\right|^2 dx \\ &-\int_{\mathbf{R}^N} G(|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + (1 - \frac{v_s^2}{c^2})V(|\psi|^2)\,dx = 0. \end{aligned} \tag{4.15}$$

Let α satisfy **(A3)**. Multiplying (4.3) by α and adding it to (4.15) we obtain

$$\begin{aligned} &\int_{\mathbf{R}^N} \Big(\frac{v_s^2}{c^2} - \alpha\Big)\left|\frac{\partial\psi}{\partial x_1}\right|^2 + \Big(\alpha + 1 - (1 - \frac{v_s^2}{c^2})\frac{N-3}{N-1}\Big)\sum_{k=2}^N \left|\frac{\partial\psi}{\partial x_k}\right|^2 dx \\ &= \int_{\mathbf{R}^N} G(|\psi|^2)|\psi|^2 + \frac{v_s^2}{2}(|\psi|^2 - r_0^2) + (1 - \alpha - \frac{v_s^2}{c^2})V(|\psi|^2)\,dx. \end{aligned} \tag{4.16}$$

By **(A3)**, the right-hand side of (4.16) is less than or equal to zero. If $\alpha \in (-1+(1-\frac{v_s^2}{c^2})\frac{N-3}{N-1}, \frac{v_s^2}{c^2})$, it follows from (4.16) that $\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_k}\right|^2 dx = 0$ for $k = 1, \ldots, N$, which implies $\nabla \psi = 0$ on $\mathbf{R}^N$, i.e. ψ is constant. If $\alpha = -1 + (1 - \frac{v_s^2}{c^2})\frac{N-3}{N-1}$, we infer from (4.16) that $\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_1}\right|^2 dx = 0$, consequently $\frac{\partial \psi}{\partial x_1} = 0$ on $\mathbf{R}^N$ which implies that ψ does not depend on x_1. Since $\int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx$ is finite, we have necessarily $\nabla \psi = 0$ on $\mathbf{R}^N$, which means that is ψ is constant. A similar argument shows that ψ is constant in the case $\alpha = \frac{v_s^2}{c^2}$. □

Remark. Let α, C_1 and $\tilde{r}$ be positive constants satisfying $G(s^2) + \frac{c^2}{4} \leq -C_1(s - \tilde{r})^{2\alpha}$ for any $s \geq \tilde{r}$ (such constants exist by assumption **(A2)**). Let ψ be as in Theorem 4.2. It follows from the proof of Proposition 2.2 i) that $|\psi(x)| \leq \tilde{r}\sqrt{2}$ for any x. Therefore the proof of Theorem 4.2 is still valid if the inequality in **(A3)** only holds for all $s \in [0, 2\tilde{r}^2]$.

If $c^2 = v_s^2$, $N \geq 3$ and ψ is as above, we already know from Proposition 3.3 iii) that $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^p(\mathbf{R}^N)$ for any $p \in (\frac{2N-1}{2N-3}, \infty)$. Therefore we have:

Corollary 4.3 *Assume that* **(A1), (A2), (A3)** *hold, $N \geq 3$ and $c^2 = v_s^2$. Let ψ be a travelling-wave for (4.1) having finite energy, finite momentum with respect to the x_1-direction (i.e. $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^1(\mathbf{R}^N)$) and such that $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Then ψ is constant.*

Example 4.4 The Gross-Pitaevskii equation is of type (4.1) with $G(s) = 1 - s$. In this case we have $r_0 = 1$, $V(s) = \frac{1}{2}(s-1)^2$ and $v_s = \sqrt{2}$. For any finite-energy function ψ we have $\int_{\mathbf{R}^N} (|\psi|^2 - 1)^2 \, dx < \infty$, hence $\psi \in L^4_{loc}(\mathbf{R}^N)$ and consequently $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. Assumptions **(A1)** and **(A2)** are clearly satisfied. We find $sG(s)+\frac{v_s^2}{2}(s-r_0^2)+(1-\alpha-\frac{v_s^2}{c^2})V(s) = -(\frac{1}{2} + \alpha + \frac{v_s^2}{c^2})(1-s)^2$. The last expression is nonpositive for any s if $\alpha \geq -\frac{1}{2} - \frac{v_s^2}{c^2}$, thus assumption **(A3)** is also satisfied. Hence the conclusion of Theorem 4.2 holds for the Gross-Pitaevskii equation. In particular, we recover the non-existence results in [17], [18].

Example 4.5 The cubic-quintic Schrödinger equation is of the form (4.1) with $G(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, where α_1, α_3, α_5 are positive and $\frac{3}{16} < \frac{\alpha_1 \alpha_5}{\alpha_3^2} < \frac{1}{4}$. The nonlinearity G can be written as $G(s) = -\alpha_5(s - r_1^2)(s - r_0^2)$, where $0 < r_1 < r_0$. In this case we have $v_s^2 = -2r_0^2 G'(r_0) = 2\alpha_5 r_0^2(r_0^2 - r_1^2)$ and $V(s) = \frac{\alpha_5}{3}(s - r_0^2)^2(s + \frac{1}{2}r_0^2 - \frac{3}{2}r_1^2)$. For any function ψ with finite energy we have $V(|\psi|^2) \in L^1(\mathbf{R}^N)$, which implies $\psi \in L^6_{loc}(\mathbf{R}^N)$ and consequently $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R}^N)$. It is obvious that G satisfies **(A1)** and **(A2)**. If $c^2 \geq v_s^2$ we have $-\frac{v_s^2}{c^2} \in [-1 + \frac{N-3}{N-1}(1 - \frac{v_s^2}{c^2}), \frac{v_s^2}{c^2}]$ and an easy computation shows that $sG(s) + \frac{v_s^2}{2}(s - r_0^2) + V(s) = -\frac{\alpha_5}{6}(4s + 5r_0^2 - 3r_1^2) \leq 0$ for any $s \geq 0$. Hence assumption **(A3)** holds for $\alpha = -\frac{v_s^2}{c^2}$, therefore the conclusion of Theorem 4.2 is valid for the cubic-quintic Schrödinger equation.

Remark. The proof of nonexistence of supersonic and sonic travelling-waves for equations of type (1.1) relies on identity (3.1), combined with Pohozaev identities. We have proved (3.1) in an "indirect" way, starting from (3.11), using the Fourier transform and analyzing the behavior near the origin of the symbols of the differential operators involved. A natural question is whether (3.1) could be proved "directly", by multiplying the equations by appropriate functions and integrating by parts (and it is very tempting to try to do so because of the form of equations (3.7) and (3.8)!). We suspect that it is not possible to find such a proof, a heuristical reason being the following: if a "direct" proof of (3.1) could be found, it should be valid for any value of c. Since Pohozaev identities are also valid for any c, one could infer that, for any c, equation (4.1) and the system (4.17)-(4.18) below do not admit nontrivial finite-energy travelling-waves. However, in the case of the Gross-Pitaevskii equation the existence of nontrivial, finite-energy travelling-waves moving with sufficiently small speed c has been proved in [7] in dimension $N = 2$, respectively in [6] and [12] in dimension $N = 3$. In a recent work [5], existence of travelling-waves has been proved in space dimensions $N = 2$ and $N = 3$ for a wider range of speeds, including speeds c close to (and less than) v_s if $N = 2$. For Schrödinger equations of cubic-quintic type, the existence of small velocity travelling-waves has been proved in [25] in any space dimension $N \geq 4$. Even for these particular cases, the question whether such solutions exist for any speed $c \in (-v_s, v_s)$ is, to our knowledge, still open.

2. Our second application concerns the system

$$i\frac{\partial \Psi}{\partial t} + \Delta\Psi - \frac{1}{\varepsilon^2}(|\Psi|^2 + \frac{1}{\varepsilon^2}|\Phi|^2 - 1)\Psi = 0 \qquad \text{in } \mathbf{R}^N, \tag{4.17}$$

$$i\delta\frac{\partial \Phi}{\partial t} + \Delta\Phi - \frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N, \tag{4.18}$$

which describes the motion of an uncharged impurity in a Bose condensate (see [16]). Here Ψ and Φ are the wavefunctions for bosons, respectively for the impurity, and ε, δ, q, k are dimensionless physical constants. Assuming that the condensate is at rest at infinity, the functions Ψ and Φ must satisfy the "boundary conditions" $|\Psi| \longrightarrow 1$ and $|\Phi| \longrightarrow 0$ as $|x| \longrightarrow \infty$.

The system (4.17)-(4.18) has a Hamiltonian structure, the associated energy is

$$E(\Psi, \Phi) = \int_{\mathbf{R}^N} |\nabla\Psi|^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\Phi|^2 + \frac{1}{2\varepsilon^2}(|\Psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\Psi|^2|\Phi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\Phi|^2\, dx. \tag{4.19}$$

We are interested in travelling-wave solutions for (4.17)-(4.18), i.e. solutions of the form $\Psi(x,t) = \psi(x_1 - ct, x_2, \dots, x_N)$, $\Phi(x,t) = \varphi(x_1 - ct, x_2, \dots, x_N)$. Such solutions must satisfy the equations

$$-ic\frac{\partial \psi}{\partial x_1} + \Delta\psi - \frac{1}{\varepsilon^2}(|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi = 0, \tag{4.20}$$

$$-ic\delta\frac{\partial \varphi}{\partial x_1} + \Delta\varphi - \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi = 0, \tag{4.21}$$

together with the boundary conditions $|\psi| \longrightarrow 1$ and $|\varphi| \longrightarrow 0$ as $|x| \longrightarrow \infty$.

Equation (4.17) is of type (1.1). In view of the analysis in the Introduction, the associated sound velocity at infinity is $\frac{\sqrt{2}}{\varepsilon}$.

In space dimension one, the system (4.20)-(4.21) with the considered boundary conditions has been studied in [26]. It was proved that it admits nontrivial solutions if c is less than the sound velocity at infinity; in this case the structure of the set of travelling-waves has been investigated and it was proved that it contains global subcontinua in appropriate (weighted) Sobolev spaces.

Here we study the finite energy travelling-waves for (4.17)-(4.18) in dimension $N \geq 2$. In view of (4.19), by *finite energy travelling-wave* we mean a couple of functions $(\psi, \varphi) \in L^1_{loc}(\mathbf{R}^N) \times L^1_{loc}(\mathbf{R}^N)$ which satisfy (4.20)-(4.21) in $\mathcal{D}'(\mathbf{R}^N)$, the boundary conditions $|\psi| \longrightarrow 1$, $\varphi \longrightarrow 0$ as $|x| \longrightarrow \infty$ and such that $\nabla\psi$, $\nabla\varphi$, $\varphi \in L^2(\mathbf{R}^N)$, $(|\psi|^2 - 1)^2 + \frac{2}{\varepsilon^2}|\psi|^2|\varphi|^2 \in L^1(\mathbf{R}^N)$. As before, we denote $\psi_1 = \mathrm{Re}(\psi)$, $\psi_2 = \mathrm{Im}(\psi)$, $\varphi_1 = \mathrm{Re}(\varphi)$, $\varphi_2 = \mathrm{Im}(\varphi)$. We have:

Proposition 4.6 *Let $c \in \mathbf{R}$ and let (ψ, φ) be a finite energy travelling wave for (4.17)-(4.18). Then:*

i) The function ψ is bounded and C^∞ and $\varphi, \nabla\psi \in W^{k,p}(\mathbf{R}^N)$ for any $k \in \mathbf{N}$ and $p \geq 2$.

ii) There exist $R_* \geq 0$ and a real-valued function θ such that $\psi = |\psi| e^{i\theta}$ on $\mathbf{R}^N \setminus B(0, R_*)$ and $\nabla\theta \in W^{k,p}(\mathbf{R}^N \setminus B(0,R_*))$ for any $k \in \mathbf{N}$ and $p \geq 2$.

iii) Let $\chi \in C^\infty(\mathbf{R}^N)$ be a cut-off function such that $\chi = 0$ on $B(0, 2R_*)$ and $\chi = 1$ on $\mathbf{R}^N \setminus B(0, 3R_*)$. We have $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta) \in L^1(\mathbf{R}^N)$ and the following Pohozaev-type identities hold:

$$(4.22) \qquad \begin{aligned} &\int_{\mathbf{R}^N} -\left|\frac{\partial \psi}{\partial x_1}\right|^2 - \frac{1}{\varepsilon^2 q^2}\left|\frac{\partial \varphi}{\partial x_1}\right|^2 + \sum_{j=2}^N \Big(\left|\frac{\partial \psi}{\partial x_j}\right|^2 + \frac{1}{\varepsilon^2 q^2}\left|\frac{\partial \varphi}{\partial x_j}\right|^2\Big)\, dx \\ &+ \int_{\mathbf{R}^N} \frac{1}{2\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\varphi|^2\, dx = 0, \end{aligned}$$

and for any $k \in \{2, \ldots, N\}$,

$$(4.23) \qquad \begin{aligned} &\int_{\mathbf{R}^N} -\left|\frac{\partial \psi}{\partial x_k}\right|^2 - \frac{1}{\varepsilon^2 q^2}\left|\frac{\partial \varphi}{\partial x_k}\right|^2 + \sum_{j=1,\, j\neq k}^N \Big(\left|\frac{\partial \psi}{\partial x_j}\right|^2 + \frac{1}{\varepsilon^2 q^2}\left|\frac{\partial \varphi}{\partial x_j}\right|^2\Big)\, dx \\ &+ \int_{\mathbf{R}^N} \frac{1}{2\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\varphi|^2\, dx \\ &- c\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta)\, dx - \frac{2c\delta}{\varepsilon^2 q^2}\int_{\mathbf{R}^N} \varphi_1 \frac{\partial \varphi_2}{\partial x_1}\, dx = 0. \end{aligned}$$

Proof. Putting $F(x,s) = -\frac{1}{\varepsilon^2}(s + \frac{1}{\varepsilon^2}|\varphi(x)|^2 - 1)$, equation (4.20) is a particular case of (1.7). Clearly, in this case we have $r_0 = 1$.

It is obvious that F satisfies the assumptions **(H1a)** and **(H1b)** in Section 2. Clearly, $F(x,s) \leq -\frac{1}{\varepsilon^2}(s-1) \leq -\frac{1}{2\varepsilon^2}s$ for any $s \geq 2$ and $x \in \mathbf{R}^N$, hence F satisfies **(H2)** for $r_* = 2$. Moreover, $\int_{r_0^2}^{r_*} F(x,\tau)\, d\tau = -\frac{1}{\varepsilon^2}(\frac{1}{2} + \frac{1}{\varepsilon^2}|\varphi(x)|^2)$ is a locally integrable function of x. We have $|\psi|^4 \leq 2(|\psi|^2 - 1)^2 + 2$ and $(|\psi|^2-1)^2 \in L^1(\mathbf{R})$ because (ψ, φ) has finite energy, hence $\psi \in L^4_{loc}(\mathbf{R}^N)$. We also have $|\,|\varphi|^2\psi| \leq \frac{1}{2}(|\varphi|^2 + |\varphi|^2|\psi|^2)$ and $|\varphi|^2,\ |\varphi|^2|\psi|^2 \in L^1(\mathbf{R})$. It is then clear that $F(\cdot, |\psi|^2)\psi = -\frac{1}{\varepsilon^2}|\psi|^2\psi - \frac{1}{\varepsilon^4}|\varphi|^2\psi + \frac{1}{\varepsilon^2}\psi$ belongs to $L^1_{loc}(\mathbf{R}^N)$. Hence we may use Proposition 2.2 i) and we infer that $\psi \in L^\infty(\mathbf{R}^N)$.

By hypothesis we have $\varphi \in L^2(\mathbf{R}^N)$ and $\nabla\varphi \in L^2(\mathbf{R}^N)$, that is $\varphi \in W^{1,2}(\mathbf{R}^N)$. Assume that $\varphi \in W^{1,p}(\mathbf{R}^N)$ for some $p \in (1,\infty)$. Since ψ is bounded, by (4.21) we find $\Delta\varphi \in L^p(\mathbf{R}^N)$, and we infer that $\varphi \in W^{2,p}(\mathbf{R}^N)$. If $p < N$, by the Sobolev embedding we have $\varphi \in L^{p^*}(\mathbf{R}^N)$ and $\nabla\varphi \in L^{p^*}(\mathbf{R}^N)$ (where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$), hence $\varphi \in W^{1,p^*}(\mathbf{R}^N)$. Repeating the above argument if necessary, after a finite number of steps we find $\varphi \in W^{2,q}(\mathbf{R}^N)$ for some $q \geq N$ and the Sobolev embedding implies $\varphi \in L^r(\mathbf{R}^N)$ and $\nabla\varphi \in L^r(\mathbf{R}^N)$ for any $r \in [q,\infty)$. Using (4.21) again, we conclude that $\Delta\varphi \in L^r(\mathbf{R}^N)$, hence $\varphi \in W^{2,r}(\mathbf{R}^N)$ for any $r \in [2,\infty)$.

It follows that $\varphi \in C^1(\mathbf{R}^N)$, which implies $F \in C^1(\mathbf{R}^N)$ (and consequently F satisfies **(H1c)**). By Proposition 2.2 ii) we get $\psi \in W^{3,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1,\infty)$. In particular, $\psi \in C^2(\mathbf{R}^N)$.

We have $F(x,1) = -\frac{1}{\varepsilon^4}|\varphi(x)|^2$ and F clearly satisfies assumption **(H3)**. It is obvious that $\partial_{N+1}F(x,s) = -\frac{1}{\varepsilon^2}$ and $\partial^2_{N+1}F(x,s) = 0$ on $\mathbf{R}^N \times \mathbf{R}_+$, therefore F satisfies **(H4)** and **(H5)**. Thus we may use Proposition 2.5 i) and we infer that $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$.

The rest of the proof is a very easy induction. For $k \in \mathbf{N}^*$, assume that $\nabla\psi \in W^{k,p}(\mathbf{R}^N)$ and $\varphi \in W^{k+1,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$. Consider $\alpha \in \mathbf{N}^N$ such that $|\alpha| = k$. Differentiating

(4.20) and (4.21) we obtain

$$\Delta(\partial^\alpha \psi) = ic\partial^\alpha \tfrac{\partial \psi}{\partial x_1} + \tfrac{1}{\varepsilon^2}\partial^\alpha\Big((|\psi|^2 + \tfrac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi\Big), \qquad \text{respectively}$$

$$\Delta(\partial^\alpha \psi) = ic\delta\partial^\alpha \tfrac{\partial \psi}{\partial x_1} + \tfrac{1}{\varepsilon^2}\partial^\alpha\Big(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi\Big).$$

We infer that $\Delta(\partial^\alpha \psi)$, $\Delta(\partial^\alpha \varphi) \in L^p(\mathbf{R}^N)$ for any $p \in [2, \infty)$. By hypothesis we have $\partial^\alpha \psi$, $\partial^\alpha \varphi \in L^p(\mathbf{R}^N)$, therefore $\partial^\alpha \psi$, $\partial^\alpha \varphi \in W^{2,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. Since this is true for any α with $|\alpha| = k$, we have $\nabla \psi \in W^{k+1,p}(\mathbf{R}^N)$ and $\varphi \in W^{k+2,p}(\mathbf{R}^N)$. We conclude that $\nabla \psi$ and φ belong to $W^{k,p}(\mathbf{R}^N)$ for any $k \in \mathbf{N}$ and $p \in [2, \infty)$.

ii) is an immediate corollary of Proposition 2.5 ii).

iii) It follows directly from Theorem 3.1 i) that $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta) \in L^1(\mathbf{R}^N)$. The proof of (4.22) and (4.23) is similar to that of (4.3) and (4.4) (multiply (4.20) by $x_j \eta_n \frac{\partial \psi}{\partial x_j}$ and (4.21) by $\frac{1}{\varepsilon^2 q^2} x_j \eta_n \frac{\partial \varphi}{\partial x_j}$, where $\eta_n(x) = \eta(\frac{x}{n})$ is a cut-off function, add the resulting equalities, integrate by parts and pass to the limit as $n \longrightarrow \infty$). We omit the details. □

We have the following result concerning the non-existence of supersonic travelling-waves for (4.17)-(4.18):

Theorem 4.7 *Let $N \geq 2$ and let (ψ, φ) be a finite energy travelling-wave for the system (4.17)-(4.18), moving with velocity c. Assume that:*

- *either $c^2 > \frac{2}{\varepsilon^2}$,*
- *or $N = 2$ and $c^2 = \frac{2}{\varepsilon^2}$,*
- *or $N \geq 3$ and $c^2 = \frac{2}{\varepsilon^2}$ and $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbf{R}^N)$.*

Then $\varphi = 0$ and ψ is constant on $\mathbf{R}^N$.

Proof. Let θ, χ be as in Proposition 4.6 and let $F(x, s) = -\frac{1}{\varepsilon^2}(s + \frac{1}{\varepsilon^2}|\varphi(x)|^2 - 1)$. We have already seen that F satisfies assumptions **(H1)-(H5)** and it follows that identity (3.1) holds. Taking into account the particular form of F, this identity can be written as

$$\int_{\mathbf{R}^N} |\nabla \psi|^2 + \frac{1}{\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\varphi|^2|\psi|^2 \, dx$$
$$= c(1 - \frac{2}{\varepsilon^2 c^2}) \int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta) \, dx. \tag{4.24}$$

We take the scalar product of (4.21) by φ, then we integrate the resulting equality to get

$$\int_{\mathbf{R}^N} |\nabla \varphi|^2 \, dx + \frac{q^2}{\varepsilon^2} \int_{\mathbf{R}^N} |\varphi|^2 |\psi|^2 \, dx - k^2 \int_{\mathbf{R}^N} |\varphi|^2 \, dx - 2c\delta \int_{\mathbf{R}^N} \varphi_1 \frac{\partial \varphi_2}{\partial x_1} \, dx = 0. \tag{4.25}$$

Summing up the identities (4.23) for $k = 2, 3, \ldots, N$, we find

$$\int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_1}\right|^2 + \frac{1}{\varepsilon^2 q^2}\left|\frac{\partial \varphi}{\partial x_1}\right|^2 + \frac{N-3}{N-1} \sum_{j=2}^N \Big(\left|\frac{\partial \psi}{\partial x_j}\right|^2 + \frac{1}{\varepsilon^2 q^2}\left|\frac{\partial \varphi}{\partial x_j}\right|^2\Big) \, dx$$
$$+ \int_{\mathbf{R}^N} \frac{1}{2\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\varphi|^2 \, dx \tag{4.26}$$
$$-c \int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta) \, dx - \frac{2c\delta}{\varepsilon^2 q^2} \int_{\mathbf{R}^N} \varphi_1 \frac{\partial \varphi_2}{\partial x_1} \, dx = 0.$$

Next we combine the equalities (4.24)-(4.26) in order to eliminate the terms $\int_{\mathbf{R}^N} \varphi_1 \frac{\partial \varphi_2}{\partial x_1}\, dx$ and $\int_{\mathbf{R}^N} \psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\theta)\, dx$. We find:

$$\begin{aligned}
&\frac{2}{\varepsilon^2 c^2} \int_{\mathbf{R}^N} \left| \frac{\partial \psi}{\partial x_1} \right|^2 dx + \Big(1 - \Big(1 - \frac{2}{\varepsilon^2 c^2}\Big) \frac{N-3}{N-1}\Big) \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left| \frac{\partial \psi}{\partial x_j} \right|^2 dx \\
&+ \frac{2}{(N-1)\varepsilon^2 q^2} \Big(1 - \frac{2}{\varepsilon^2 c^2}\Big) \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left| \frac{\partial \varphi}{\partial x_j} \right|^2 dx \qquad (4.27) \\
&+ \frac{1}{2\varepsilon^2} \Big(1 + \frac{2}{\varepsilon^2 c^2}\Big) \int_{\mathbf{R}^N} (|\psi|^2 - 1)^2\, dx + \frac{1}{\varepsilon^4} \int_{\mathbf{R}^N} |\varphi|^2 |\psi|^2\, dx = 0.
\end{aligned}$$

Obviously, all integrals in (4.27) are nonnegative. If $c^2 \geq \frac{2}{\varepsilon^2}$, all coefficients are also nonnegative, therefore each term in (4.27) must be zero. In particular, $\int_{\mathbf{R}^N} \left| \frac{\partial \psi}{\partial x_k} \right|^2 dx = 0$ for any $k \in \{1, \ldots, N\}$, which implies $\nabla \psi = 0$ on $\mathbf{R}^N$, i.e. ψ is constant. Since $\int_{\mathbf{R}^N} (|\psi|^2 - 1)^2\, dx = 0$, necessarily $|\psi| = 1$. We have also $0 = \int_{\mathbf{R}^N} |\varphi|^2 |\psi|^2\, dx = \int_{\mathbf{R}^N} |\varphi|^2\, dx$, hence $\varphi = 0$ on $\mathbf{R}^N$. $\square$

5 The one-dimensional case

Since most of the proofs in the preceding section are not valid in space dimension $N = 1$ (in particular, we do not have identities analogous to (4.4) and (4.23)), we treat separately the one-dimensional case. It turns out that some integrations can be performed explicitly and some of the results are stronger than in higher dimensions.

Let $G : [0, \infty) \longrightarrow \mathbf{R}$ be a function satisfying the following assumption:

- **(A)** $G \in C([0, \infty))$ and there exists $r_0 > 0$ such that $G(r_0^2) = 0$.
  Moreover, $G \in C^1([r_0^2 - \eta, r_0^2 + \eta])$ for some $\eta > 0$ and $G'(r_0^2) = -L < 0$.

We consider the Schrödinger equation

$$(5.1) \qquad i \frac{\partial \Psi}{\partial t} + \Psi_{xx} + G(|\Psi|^2)\Psi = 0 \qquad \text{in } \mathbf{R},$$

together with the "boundary condition" $|\Psi| \longrightarrow r_0$ as $x \longrightarrow \pm\infty$. We have seen in the Introduction that the sound velocity at infinity associated to (5.1) and to the considered boundary condition is $v_s = r_0\sqrt{2L}$. As usually, a travelling-wave moving with velocity c is a solution of the form $\Psi(x, t) = \psi(x - ct)$. It must satisfy

$$(5.2) \qquad -ic\psi' + \psi'' + G(|\psi|^2)\psi = 0 \qquad \text{in } \mathbf{R}, \qquad |\psi(x)| \longrightarrow r_0 \qquad \text{as } x \longrightarrow \pm\infty.$$

We have the following result concerning supersonic and sonic travelling-waves:

Theorem 5.1 *Let $\psi \in L^1_{loc}(\mathbf{R})$ be a solution of (5.2) in $\mathcal{D}'(\mathbf{R})$ such that $G(|\psi|^2)\psi \in L^1_{loc}(\mathbf{R})$. Assume that G satisfies* **(A)** *and*

i) either $c^2 > v_s^2$, or

ii) $c^2 = v_s^2$ and, denoting $V(s) = \int_s^{r_0^2} G(\tau)\, d\tau$ and $W(s) = v_s^2 s^2 - 4(s + r_0^2)V(s + r_0^2)$, there exists $\varepsilon > 0$ such that one of he following conditions is verified:

a) $W(s) > 0$ *on* $(-\varepsilon, 0) \cup (0, \varepsilon)$*;*
b) $W(s) > 0$ *on* $(-\varepsilon, 0)$ *and* $W(s) < 0$ *on* $(0, \infty)$*;*
c) $W(s) > 0$ *on* $(0, \varepsilon)$ *and* $W(s) < 0$ *on* $[-r_0^2, 0)$*.*

Then either ψ *is constant, or* $\psi(x) = r_0 e^{i(cx+\theta_0)}$*, where* θ_0 *is a real constant.*

Remark. Theorem 5.1 gives *all* supersonic and sonic travelling-waves for equation (5.1), no matter whether their energy is finite or not (and we see that finite energy travelling-waves must be constant).

It is easy to see that W is C^2 near 0 and $W(0) = W'(0) = W''(0) = 0$. Condition ii) a) is satisfied, for instance, if G is C^3 near r_0^2 (this clearly implies that W is C^4 near 0) and $W'''(0) = 0$, $W^{(iv)}(0) > 0$, or equivalently $r_0^2 G''(r_0^2) = 3L$ and $4G''(r_0^2) + r_0^2 G'''(r_0^2) > 0$. The condition $W(s) > 0$ on $(-\varepsilon, 0)$ in ii) b), respectively $W(s) > 0$ on $(0, \varepsilon)$ in ii) c), is satisfied if G is C^3 near r_0^2 and $W'''(0) < 0$ (respectively $W'''(0) > 0$); however, in these cases only an information on the behavior of G in a neighborhood of r_0^2 is not sufficient to get the conclusion of Theorem 5.1.

Proof of Theorem 5.1. Let $\varphi(x) = e^{-\frac{icx}{2}}\psi(x)$. Then $\varphi \in L^1_{loc}(\mathbf{R})$ and it is easy to see that

$$\varphi'' + \Big(G(|\varphi|^2) + \frac{c^2}{4}\Big)\varphi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}). \tag{5.3}$$

From (5.3) we get $\varphi'' \in L^1_{loc}(\mathbf{R})$. This implies that φ' is a continuous function on $\mathbf{R}$ (see, e.g., Lemma VIII.2 p. 123 in [8]). Thus $\varphi \in C^1(\mathbf{R})$. Since $|\varphi| \longrightarrow r_0$ as $x \longrightarrow \pm\infty$, we infer that φ is bounded on $\mathbf{R}$. Coming back to (5.3) we see that φ'' is continuous and bounded on $\mathbf{R}$. In particular $\varphi \in C^2(\mathbf{R})$ and this implies $\psi \in C^2(\mathbf{R})$.

Denoting $\psi_1 = \mathrm{Re}(\psi)$, $\psi_2 = \mathrm{Im}(\psi)$, equation (5.2) is equivalent to the system

$$c\psi_2' + \psi_1'' + G(|\psi|^2)\psi_1 = 0, \tag{5.4}$$

$$-c\psi_1' + \psi_2'' + G(|\psi|^2)\psi_2 = 0 \qquad \text{in } \mathbf{R}. \tag{5.5}$$

We multiply (5.4) by $2\psi_1'$ and (5.5) by $2\psi_2'$, then we add the resulting equalities to get $[(\psi_1')^2 + (\psi_2')^2]' - (V(|\psi|^2))' = 0$. Hence there exists $k_1 \in \mathbf{R}$ such that

$$|\psi'|^2(x) - V(|\psi|^2)(x) = k_1 \qquad \text{for any } x \in \mathbf{R}. \tag{5.6}$$

Multiplying (5.4) by ψ_2 and (5.5) by $-\psi_1$, then summing up the corresponding equations we obtain $\frac{c}{2}(|\psi|^2 - r_0^2)' - (\psi_1\psi_2' - \psi_2\psi_1')' = 0$. Consequently there is some $k_2 \in \mathbf{R}$ such that

$$\frac{c}{2}(|\psi|^2 - r_0^2) - (\psi_1\psi_2' - \psi_2\psi_1') = k_2 \qquad \text{in } \mathbf{R}. \tag{5.7}$$

Next we multiply (5.4) by $2\psi_1$ and (5.5) by $2\psi_2$, then we add the resulting equalities to find

$$2c(\psi_1\psi_2' - \psi_2\psi_1') + (|\psi|^2 - r_0^2)'' - 2|\psi'|^2 + 2G(|\psi|^2)|\psi|^2 = 0. \tag{5.8}$$

Taking into account (5.6) and (5.7), equation (5.8) can be written as

$$(|\psi|^2 - r_0^2)'' + c^2(|\psi|^2 - r_0^2) - 2V(|\psi|^2) + 2G(|\psi|^2)|\psi|^2 = 2k_1 + 2ck_2. \tag{5.9}$$

Denote $v(x) = |\psi|^2(x) - r_0^2$. Then v is real-valued, C^2 and tends to zero as $x \longrightarrow \pm\infty$, hence there exists a sequence $x_n \longrightarrow \infty$ such that $v''(x_n) \longrightarrow 0$. Writing (5.9) for x_n, then passing to the limit as $n \longrightarrow \infty$ we see that necessariy $k_1 + ck_2 = 0$ and v satisfies the equation

$$v'' + c^2 v - 2V(v + r_0^2) + 2(v + r_0^2)G(v + r_0^2) = 0 \qquad \text{in } \mathbf{R}. \tag{5.10}$$

Next we multiply (5.10) by $2v'$, then we integrate the resulting equation and we obtain $(v')^2 + c^2v^2 - 4(v + r_0^2)V(v + r_0^2) = k_3$ in $\mathbf{R}$, where k_3 is a constant. It is clear that there exists a sequence $y_n \longrightarrow \infty$ such that $v'(y_n) \longrightarrow 0$, consequently $k_3 = 0$ and we have

$$(v')^2(x) + c^2v^2(x) - 4(v + r_0^2)V(v + r_0^2)(x) = 0 \qquad \text{for any } x \in \mathbf{R}. \tag{5.11}$$

Our aim is to prove that, under the assumptions of Theorem 5.1, we have $v = 0$ on $\mathbf{R}$.

Suppose first that $c^2 > v_s^2 = 2Lr_0^2$. Since G satisfies **(A)**, it follows that $V \in C^2([r_0^2 - \eta, r_0^2 + \eta])$ and we have by Taylor's formula

$$V(r_0^2 + s) = V(r_0^2) + sV'(r_0^2) + \frac{1}{2}s^2V''(r_0^2) + s^2h(s) = \frac{1}{2}Ls^2 + s^2h(s) \qquad \text{for } s \in [-\eta, \eta],$$

where $h(s) \longrightarrow 0$ as $s \longrightarrow 0$. Take $\varepsilon_1 \in (0, \eta]$ such that $c^2 - v_s^2 - 2Ls - 4(s + r_0^2)h(s) > 0$ for any $s \in [-\varepsilon_1, \varepsilon_1]$. Suppose that $v(x_0) \in [-\varepsilon_1, 0) \cup (0, \varepsilon_1]$ for some $x_0 \in \mathbf{R}$. By (5.11) we obtain

$$0 = (v')^2(x_0) + v^2(x_0)[c^2 - v_s^2 - 2Lv(x_0) - 4(v(x_0) + r_0^2)h(v(x_0))] > 0,$$

a contradiction. Consequently we cannot have $v(x) \in [-\varepsilon_1, 0) \cup (0, \varepsilon_1]$. Since v is continuous and $v(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$, we infer that necessarily $v(x) = 0$ for any $x \in \mathbf{R}$.

Next assume that $c^2 = v_s^2$. Equation (5.11) can be written as

$$(v')^2(x) + W(v(x)) = 0 \qquad \text{on } \mathbf{R}. \tag{5.12}$$

If assumption ii) a) is verified, we cannot have $v(x) \in (-\varepsilon, 0) \cup (0, \varepsilon)$ and we infer, as above, that $v = 0$ on $\mathbf{R}$. In case ii) b), we cannot have $v(x) \in (-\varepsilon, 0)$ and we infer that $v(x) \geq 0$ for any $x \in \mathbf{R}$. Since $v(x) \longrightarrow 0$ as $x \longrightarrow \infty$, there is some x_0 such that v achieves a nonnegative maximum at x_0. Then $v'(x_0) = 0$ and from (5.12) we get $W(v(x_0)) = 0$. But $W(s) < 0$ for $s > 0$ by ii) b), hence $v(x_0) = 0$ and consequently $v = 0$ on $\mathbf{R}$. Similarly we have $v = 0$ in the case ii) c) (note that $v = |\psi|^2 - r_0^2 \geq -r_0^2$ and it suffices to know that $W < 0$ on $[-r_0^2, 0)$).

Thus we have always $v = 0$, that is $|\psi|^2 = r_0^2$ on $\mathbf{R}$. Consequently there exists a lifting $\theta \in C^2(\mathbf{R}, \mathbf{R})$ such that $\psi(x) = r_0e^{i\theta(x)}$ for any $x \in \mathbf{R}$. It is clear that $\psi_1\psi_2' - \psi_2\psi_1' = |\psi|^2\theta' = r_0^2\theta'$ (see (3.3)). On the other hand we have $\psi_1\psi_2' - \psi_2\psi_1' = -k_2$ by (5.7), hence $\theta' = -\frac{k_2}{r_0^2}$ is constant, therefore $\theta(x) = -\frac{k_2}{r_0^2}x + \theta_0$, where θ_0 is a real constant. Since $\psi = r_0e^{i(-\frac{k_2}{r_0^2}x + \theta_0)}$ satisfies equation (5.2), we find $-c\frac{k_2}{r_0^2} - \left(\frac{k_2}{r_0^2}\right)^2 = 0$, thus either $\frac{k_2}{r_0^2} = 0$ or $\frac{k_2}{r_0^2} = -c$. Finally we have either $\psi(x) = e^{i\theta_0}$ or $\psi(x) = e^{i(cx+\theta_0)}$ and the proof is complete. □

Example 5.2 In the case of the Gross-Pitaevskii equation we have $G(s) = 1 - s$ and we obtain $W(s) = -2s^3$ (see Example 4.4). In the case of the cubic-quintic nonlinearity we have $G(s) = -\alpha_5(s - r_1^2)(s - r_0^2)$, where $\alpha_5 > 0$, $0 < r_1 < r_0$ (see Example 4.5) and a simple computation gives $W(s) = -2\alpha_5 s^3(\frac{4}{3}r_0^2 - r_1^2 + \frac{1}{3}s)$. Therefore both the Gross-Pitaevskii and the cubic-quintic nonlinearities satisfy assumption ii) b) and Theorem 5.1 gives all sonic and supersonic travelling-waves for these equations.

Remark. The proof of Theorem 5.1 provides a method to find subsonic travelling-waves for (5.1). With the above notation, it follows from (5.11) that on any interval where $v' \neq 0$ we have $v'(x) = \pm\sqrt{4(v + r_0^2)V(v + r_0^2)(x) - c^2v^2(x)}$. In many interesting applications this equation can be integrated and we obtain explicitly $v = |\psi|^2 - r_0^2$. Then it is not hard to find (up to a constant) the corresponding phase θ.

Remark. Assume that $N = 1$ and let (ψ, φ) be a finite-energy travelling-wave for the system (4.17)-(4.18). It follows from the proof of Proposition 4.6 that ψ and φ are C^∞ functions and $\psi', \varphi \in W^{k,p}(\mathbf{R})$ for any $k \in \mathbf{N}$ and $p \geq 2$. If $c^2 \geq \frac{2}{\varepsilon^2}$ (recall that $\frac{\sqrt{2}}{\varepsilon}$ is the sound velocity at infinity associated to (3.21)-(3.22)) and if there is a lifting $\psi(x) = v(x)e^{i\alpha(x)}$, $\varphi(x) = u(x)e^{i\beta(x)}$, where v, u, α, β are real-valued functions of class C^2, Proposition 3.1 p. 1545 in [26] implies that $v = 1$, α is constant and $\varphi = 0$ on $\mathbf{R}$.

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Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity

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Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity

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Dedicated to Jean-Claude Saut,
who gave me water to cross the desert.

Abstract

For a large class of nonlinear Schrödinger equations with nonzero conditions at infinity and for any speed c less than the sound velocity, we prove the existence of finite energy traveling waves moving with speed c in any space dimension $N \geq 3$. Our results are valid as well for the Gross-Pitaevskii equation and for NLS with cubic-quintic nonlinearity.

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1 Introduction

We consider the nonlinear Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} + \Delta\Phi + F(|\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N, \tag{1.1}$$

where $\Phi : \mathbf{R}^N \longrightarrow \mathbf{C}$ satisfies the "boundary condition" $|\Phi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, $r_0 > 0$ and F is a real-valued function on $\mathbf{R}_+$ satisfying $F(r_0^2) = 0$.

Equations of the form (1.1), with the considered non-zero conditions at infinity, arise in a large variety of physical problems such as superconductivity, superfluidity in Helium II, phase transitions and Bose-Einstein condensate ([2], [3], [4], [12], [20], [22], [23], [24], [25]). In nonlinear optics, they appear in the context of dark solitons ([27], [28]). Two important particular cases of (1.1) have been extensively studied both by physicists and by mathematicians: the Gross-Pitaevskii equation (where $F(s) = 1 - s$) and the so-called "cubic-quintic" Schrödinger equation (where $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, α_1, α_3, α_5 are positive and F has two positive roots).

The boundary condition $|\Phi| \longrightarrow r_0 > 0$ at infinity makes the structure of solutions of (1.1) much more complicated than in the usual case of zero boundary conditions (when the associated dynamics is essentially governed by dispersion and scattering).

Using the Madelung transformation $\Phi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$ (which is well-defined whenever $\Phi \neq 0$), equation (1.1) is equivalent to a system of Euler's equations for a compressible inviscid fluid of density ρ and velocity $2\nabla\theta$. In this context it has been shown that, if F is C^1 near r_0^2 and $F'(r_0^2) < 0$, the sound velocity at infinity associated to (1.1) is $v_s = r_0\sqrt{-2F'(r_0^2)}$ (see the introduction of [33]).

Equation (1.1) is Hamiltonian: denoting $V(s) = \int_s^{r_0^2} F(\tau)\,d\tau$, it is easy to see that, at least formally, the "energy"

$$(1.2) \qquad E(\Phi) = \int_{\mathbf{R}^N} |\nabla \Phi|^2\,dx + \int_{\mathbf{R}^N} V(|\Phi|^2)\,dx$$

is a conserved quantity.

In a series of papers (see, e.g., [2], [3], [20], [24], [25]), particular attention has been paid to a special class of solutions of (1.1), namely the traveling waves. These are solutions of the form $\Phi(x,t) = \psi(x - cty)$, where $y \in S^{N-1}$ is the direction of propagation and $c \in \mathbf{R}^*$ is the speed of the traveling wave. We say that ψ has finite energy if $\nabla\psi \in L^2(\mathbf{R}^N)$ and $V(|\psi|^2) \in L^1(\mathbf{R}^N)$). These solutions are supposed to play an important role in the dynamics of (1.1). In view of formal computations and numerical experiments, a list of conjectures, often referred to as *the Roberts programme,* has been formulated about the existence, the stability and the qualitative properties of traveling waves. The first of these conjectures asserts that finite energy traveling waves of speed c exist if and only if $|c| < v_s$.

Let ψ be a finite energy traveling-wave of (1.1) moving with speed c. Without loss of generality we may assume that $y = (1, 0, \dots, 0)$. If $N \geq 3$, it follows that $\psi - z_0 \in L^{2^*}(\mathbf{R}^N)$ for some constant $z_0 \in \mathbf{C}$, where $2^* = \frac{2N}{N-2}$ (see, e.g., Lemma 7 and Remark 4.2 pp. 774-775 in [17]). Since $|\psi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, necessarily $|z_0| = r_0$. If Φ is a solution of (1.1) and $\alpha \in \mathbf{R}$, then $e^{i\alpha}\Phi$ is also a solution; hence we may assume that $z_0 = r_0$, thus $\psi - r_0 \in L^{2^*}(\mathbf{R}^N)$. Denoting $u = r_0 - \psi$, we see that u satisfies the equation

$$(1.3) \qquad ic\frac{\partial u}{\partial x_1} - \Delta u + F(|r_0 - u|^2)(r_0 - u) = 0 \qquad \text{in } \mathbf{R}^N.$$

It is obvious that a function u satisfies (1.3) for some velocity c if and only if $u(-x_1, x')$ satisfies (1.3) with c replaced by $-c$. Hence it suffices to consider the case $c > 0$. This assumption will be made throughout the paper.

In space dimension $N = 1$, in many interesting applications equation (1.3) can be integrated explicitly and one obtains traveling waves for all subsonic speeds. The nonexistence of such solutions for supersonic speeds has also been proved under general conditions (cf. Theorem 5.1, p. 1099 in [33]).

Despite of many attempts, a rigorous proof of the existence of traveling waves in higher dimensions has been a long lasting problem. In the particular case of the Gross-Pitaevskii (GP) equation, this problem was considered in a series of papers. In space dimension $N = 2$, the existence of traveling waves has been proved in [7] for all speeds in some interval $(0, \varepsilon)$, where ε is small. In space dimension $N \geq 3$, the existence has been proved in [6] for a sequence of speeds $c_n \longrightarrow 0$ by using constrained minimization; a similar result has been established in [11] for all sufficiently small speeds by using a mountain-pass argument. In a recent paper [5], the existence of traveling waves for (GP) has been proved in space dimension $N = 2$ and $N = 3$ for any speed in a set $A \subset (0, v_s)$. If $N = 2$, A contains points arbitrarily close to 0 and to v_s (although it is not clear that $A = (0, v_s)$), while in dimension $N = 3$ we have $A \subset (0, v_0)$, where $v_0 < v_s$ and 0, v_0 are limit points of A. The traveling waves are obtained in [5] by minimizing the energy at fixed momentum (see the next section for the definition of the momentum) and the propagation speed is the Lagrange multiplier associated to minimizers. In the case of cubic-quintic type nonlinearities, it has been proved in [31] that traveling waves exist for any sufficiently small speed if $N \geq 4$. To our knowledge, even for specific nonlinearities there are no existence results in the literature that cover the whole range $(0, v_s)$ of possible speeds.

The nonexistence of traveling waves for supersonic speeds ($c > v_s$) has been proved in [21] in the case of the Gross-Pitaevskii equation, respectively in [33] for a large class of nonlinearities.

The aim of this paper is to prove the existence of finite energy traveling waves of (1.1) in space dimension $N \geq 3$, under general conditions on the nonlinearity F and for any speed $c \in (-v_s, v_s)$.

We will consider the following set of assumptions:

A1. The function F is continuous on $[0, \infty)$, C^1 in a neighborhood of r_0^2, $F(r_0^2) = 0$ and $F'(r_0^2) < 0$.

A2. There exist $C > 0$ and $p_0 < \frac{2}{N-2}$ such that $|F(s)| \leq C(1 + s^{p_0})$ for any $s \geq 0$.

A3. There exist C, $\alpha_0 > 0$ and $r_* > r_0$ such that $F(s) \leq -Cs^{\alpha_0}$ for any $s \geq r_*$.

If (A1) is satisfied, we denote $V(s) = \displaystyle\int_s^{r_0^2} F(\tau)\,d\tau$ and $a = \sqrt{-\frac{1}{2}F'(r_0^2)}$. Then the sound velocity at infinity associated to (1.1) is $v_s = 2ar_0$ and using Taylor's formula for s in a neighborhood of r_0^2 we have

$$V(s) = \frac{1}{2}V''(r_0^2)(s - r_0^2)^2 + (s - r_0^2)^2\varepsilon(s - r_0^2) = a^2(s - r_0^2)^2 + (s - r_0^2)^2\varepsilon(s - r_0^2), \tag{1.4}$$

where $\varepsilon(t) \longrightarrow 0$ as $t \longrightarrow 0$. Hence for $|\psi|$ close to r_0, $V(|\psi|^2)$ can be approximated by $a^2(|\psi|^2 - r_0^2)^2$.

We fix an odd function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(s) = s$ for $s \in [0, 2r_0]$, $0 \leq \varphi' \leq 1$ on $\mathbf{R}$ and $\varphi(s) = 3r_0$ for $s \geq 4r_0$. We denote $W(s) = V(s) - V(\varphi^2(\sqrt{s}))$, so that $W(s) = 0$ for $s \in [0, 4r_0^2]$. If assumptions (A1) and (A2) are satisfied, it is not hard to see that there exist C_1, C_2, $C_3 > 0$ such that

$$\begin{array}{l} |V(s)| \leq C_1(s - r_0^2)^2 \quad \text{for any } s \leq 9r_0^2; \\ \text{in particular, } |V(\varphi^2(\tau))| \leq C_1(\varphi^2(\tau) - r_0^2)^2 \text{ for any } \tau; \end{array} \tag{1.5}$$

$$|V(b) - V(a)| \leq C_2|b - a|\max(a^{p_0}, b^{p_0}) \qquad \text{for any } a,\, b \geq 2r_0^2; \tag{1.6}$$

$$|W(b^2) - W(a^2)| \leq C_3|b - a|\left(a^{2p_0+1}\mathbb{1}_{\{a>2r_0\}} + b^{2p_0+1}\mathbb{1}_{\{b>2r_0\}}\right) \quad \text{for any } a, b \geq 0. \tag{1.7}$$

Given $u \in H^1_{loc}(\mathbf{R}^N)$ and Ω an open set in $\mathbf{R}^N$, the modified Ginzburg-Landau energy of u in Ω is defined by

$$E^\Omega_{GL}(u) = \int_\Omega |\nabla u|^2\,dx + a^2\int_\Omega \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\,dx. \tag{1.8}$$

We simply write $E_{GL}(u)$ instead of $E^{\mathbf{R}^N}_{GL}(u)$. The modified Ginzburg-Landau energy will play a central role in our analysis. We consider the function space

$$\begin{array}{rcl} \mathcal{X} & = & \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N) \mid \varphi^2(|r_0 - u|) - r_0^2 \in L^2(\mathbf{R}^N)\} \\ & = & \{u \in \dot{H}^1(\mathbf{R}^N) \mid u \in L^{2^*}(\mathbf{R}^N),\ E_{GL}(u) < \infty\}, \end{array} \tag{1.9}$$

where $\mathcal{D}^{1,2}(\mathbf{R}^N)$ is the completion of C_c^∞ for the norm $||v|| = ||\nabla v||_{L^2}$. If $N \geq 3$ and (A1), (A2) are satisfied, it is not hard to see that a function u has finite energy if and only if $u \in \mathcal{X}$ (see Lemma 4.1 below). Note that for $N = 3$, $\mathcal{X}$ is *not* a vector space. However, in any space dimension we have $H^1(\mathbf{R}^N) \subset \mathcal{X}$. If $u \in \mathcal{X}$, it is easy to see that for any $w \in H^1(\mathbf{R}^N)$ with compact support we have $u + w \in \mathcal{X}$. For $N = 3, 4$ it can be proved that $u \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ belongs to $\mathcal{X}$ if and only if $|r_0 - u|^2 - r_0^2 \in L^2(\mathbf{R}^N)$, and consequently $\mathcal{X}$ coincides with the space F_{r_0} introduced by P. Gérard in [17], section 4. It has been proved in [17] that the Cauchy problem for the Gross-Pitaevskii equation is globally well-posed in $\mathcal{X}$ in dimension $N = 3$, respectively it is globally well-posed for small initial data if $N = 4$.

Our main results can be summarized as follows:

Theorem 1.1 *Assume that $N \geq 3$, $0 < c < v_s$, (A1) and one of the conditions (A2) or (A3) are satisfied. Then equation (1.3) admits a nontrivial solution $u \in \mathcal{X}$. Moreover, $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$ and, after a translation, u is axially symmetric with respect to Ox_1.*

At least formally, solutions of (1.3) are critical points of the functional

$$E_c(u) = \int_{\mathbf{R}^N} |\nabla u|^2 \, dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx,$$

where Q is the momentum with respect to the x_1-direction (the functional Q will be defined in the next section). If the assumptins (A1) and (A2) above are satisfied, it can be proved (see Proposition 4.1 p. 1091-1092 in [33]) that any traveling wave $u \in \mathcal{X}$ of (1.1) must satisfy a Pohozaev-type identity $P_c(u) = 0$, where

$$P_c(u) = \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_1} \right|^2 + \frac{N-3}{N-1} \sum_{k=2}^{N} \left| \frac{\partial u}{\partial x_k} \right|^2 dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx.$$

We will prove the existence of traveling waves by showing that the problem of minimizing E_c in the set $\{u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0\}$ admits solutions. Then we show that any minimizer satisfies (1.3) if $N \geq 4$, respectively any minimizer satisfies (1.3) after a scaling in the last two variables if $N = 3$.

In space dimension $N = 2$, the situation is different: if (A1) is true and (A2) holds for some $p_0 < \infty$, any solution $u \in \mathcal{X}$ of (1.3) still satisfies the identity $P_c(u) = 0$, but it can be proved that there are no minimizers of E_c subject to the constraint $P_c = 0$ (in fact, we have $\inf\{E_c(u) \mid u \in \mathcal{X}, u \neq 0, P_c(u) = 0\} = 0$). However, using a different aproach it is still possible to show the existence of traveling waves in the case $N = 2$, at least for a set of speeds that contains elements arbitrarily close to zero and to v_s (and this will be done in a forthcoming paper). Although some of the results in sections 2−4 are still valid in space dimension $N = 2$ (with straightforward modifications in proofs), for simplicity we assume throughout that $N \geq 3$.

It is easy to see that it suffices to prove Theorem 1.1 only in the case where (A1) and (A2) are satisfied. Indeed, suppose that Theorem 1.1 holds if (A1) and (A2) are true. Assume that (A1) and (A3) are satisfied. Let C, r_*, α_0 be as in (A3). There exist $\beta \in (0, \frac{2}{N-1})$, $\tilde{r} > r_*$, and $C_1 > 0$ such that

$$Cs^{2\alpha_0} - \frac{v_s^2}{4} \geq C_1(s - \tilde{r})^{2\beta} \qquad \text{for any } s \geq \tilde{r}.$$

Let $\tilde{F}$ be a function with the following properties: $F = \tilde{F}$ on $[0, 4\tilde{r}^2]$, $\tilde{F}(s) = -C_2 s^\beta$ for s sufficiently large, and $\tilde{F}(s^2) + \frac{v_s^2}{4} \leq -C_3(s - \tilde{r})^{2\beta}$ for any $s \geq \tilde{r}$, where C_2, C_3 are some positive constants. Then $\tilde{F}$ satisfies (A1), (A2), (A3) and from Theorem 1.1 it follows that equation (1.3) with $\tilde{F}$ instead of F has nontrivial solutions $u \in \mathcal{X}$. From the proof of Proposition 2.2 (i) p. 1079-1080 in [33] it follows that any such solution satisfies $|r_0 - u|^2 \leq 2\tilde{r}^2$, and consequently $F(|r_0 - u|^2) = \tilde{F}(|r_0 - u|^2)$. Thus u satisfies (1.3). Of course, if (A1) and (A3) are satisfied but (A2) does not hold, we do not claim that the solutions of (1.3) obtained as above are still minimizers of E_c subject to the constraint $P_c = 0$ (in fact, only assumptions (A1) and (A3) do not imply that E_c and P_c are well-defined on $\mathcal{X}$ and that the minimization problem makes sense).

In particular, for $F(s) = 1 - s$ the conditions (A1) and (A3) are satisfied and it follows that the Gross-Pitaevskii equation admits traveling waves of finite energy in any space dimension

$N \geq 3$ and for any speed $c \in (0, v_s)$ (although (A2) is not true for $N > 3$: the (GP) equation is critical if $N = 4$, and supercritical if $N \geq 5$). A similar result holds for the cubic-quintic NLS.

We have to mention that, according to the properties of F, for $c = 0$ equation (1.3) may or not have finite energy solutions. For instance, it is an easy consequence of the Pohozaev identities that all finite energy stationary solutions of the Gross-Pitaevskii equation are constant. On the contrary, for nonlinearities of cubic-quintic type the existence of finite energy stationary solutions has been proved in [13] under fairly general assumptions on F. In the case $c = 0$, our proofs imply that E_0 has a minimizer in the set $\{u \in \mathcal{X} \mid u \neq 0, P_0(u) = 0\}$ whenever this set is not empty. Then it is not hard to prove that minimizers satisfy (1.3) for $c = 0$ (modulo a scale change if $N = 3$). However, for simplicity we assume throughout (unless the contrary is explicitly mentioned) that $0 < c < v_s$.

This paper is organized as follows. In the next section we give a convenient definition of the momentum and we study the properties of this functional.

In section 3 we introduce a regularization procedure for functions in $\mathcal{X}$ which will be a key tool for all the variational machinery developed later.

In section 4 we describe the variational framework. In particular, we prove that the set $\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0,\ P_c(u) = 0\}$ is not empty and we have $\inf\{E_c(u) \mid u \in \mathcal{C}\} > 0$.

In section 5 we consider the case $N \geq 4$ and we prove that the functional E_c has minimizers in $\mathcal{C}$ and these minimizers are solutions of (1.3). To show the existence of minimizers we use the concentration-compactness principle and the regularization procedure developed in section 3. Then we use the Pohozaev identities to control the Lagrange multiplier associated to the minimization problem.

Although the results in space dimension $N = 3$ are similar to those in higher dimensions (with one exception: not all minimizers of E_c in $\mathcal{C}$ are solutions of (1.3), as one can easily see by scaling), it turns out that the proofs are quite different. We treat the case $N = 3$ in section 6.

Finally, we prove that traveling waves found by minimization in sections 5 and 6 are axially symmetric (as one would expect from physical considerations, see [24]).

Throughout the paper, $\mathcal{L}^N$ is the Lebesgue measure on $\mathbf{R}^N$. For $x = (x_1, \dots, x_N) \in \mathbf{R}^N$, we denote $x' = (x_2, \dots, x_N) \in \mathbf{R}^{N-1}$. We write $\langle z_1, z_2 \rangle$ for the scalar product of two complex numbers z_1, z_2. Given a function f defined on $\mathbf{R}^N$ and λ, $\sigma > 0$, we denote by

$$f_{\lambda,\sigma} = f\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right) \tag{1.10}$$

the dilations of f. The behavior of functions and of functionals with respect to dilations in $\mathbf{R}^N$ will be very important. For $1 \leq p < N$, we denote by p^* the Sobolev exponent associated to p, that is $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

2 The momentum

A good definition of the momentum is essential in any attempt to find solutions of (1.3) by using a variational approach. Roughly speaking, the momentum (with respect to the x_1-direction) should be a functional with derivative $2iu_{x_1}$. Various definitions have been given in the literature (see [7], [5], [6], [31]), any of them having its advantages and its inconvenients. Unfortunately, none of them is valid for all functions in $\mathcal{X}$. We propose a new and more general definition in this section.

It is clear that for functions $u \in H^1(\mathbf{R}^N)$, the momentum should be given by

$$Q_1(u) = \int_{\mathbf{R}^N} \langle iu_{x_1}, u \rangle \, dx, \tag{2.1}$$

and this is indeed a nice functional on $H^1(\mathbf{R}^N)$. The problem is that there are functions $u \in \mathcal{X} \setminus H^1(\mathbf{R}^N)$ such that $\langle iu_{x_1}, u\rangle \notin L^1(\mathbf{R}^N)$.

If $u \in \mathcal{X}$ is such that $r_0 - u$ admits a lifting $r_0 - u = \rho e^{i\theta}$, a formal computation gives

$$\int_{\mathbf{R}^N} \langle iu_{x_1}, u\rangle \, dx = -\int_{\mathbf{R}^N} \rho^2 \theta_{x_1} \, dx = -\int_{\mathbf{R}^N} (\rho^2 - r_0^2)\theta_{x_1} \, dx. \tag{2.2}$$

It is not hard to see that if $u \in \mathcal{X}$ is as above, then $(\rho^2 - r_0^2)\theta_{x_1} \in L^1(\mathbf{R}^N)$. However, there are many "interesting" functions $u \in \mathcal{X}$ such that $r_0 - u$ does not admit a lifting.

Our aim is to define the momentum on $\mathcal{X}$ in such a way that it agrees with (2.1) for functions in $H^1(\mathbf{R}^N)$ and with (2.2) when a lifting as above exists.

Lemma 2.1 *Let $u \in \mathcal{X}$ be such that $m \le |r_0 - u(x)| \le 2r_0$ a.e. on $\mathbf{R}^N$, where $m > 0$. There exist two real-valued functions ρ, θ such that $\rho - r_0 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $r_0 - u = \rho e^{i\theta}$ a.e. on $\mathbf{R}^N$ and*

$$\langle iu_{x_1}, u\rangle = -r_0 \frac{\partial}{\partial x_1}(\mathrm{Im}(u) + r_0\theta) - (\rho^2 - r_0^2)\frac{\partial \theta}{\partial x_1} \qquad a.e. \text{ on } \mathbf{R}^N. \tag{2.3}$$

Moreover, we have $\displaystyle\int_{\mathbf{R}^N} \left|(\rho^2 - r_0^2)\theta_{x_1}\right| dx \le \frac{1}{2am} E_{GL}(u).$

Proof. Since $r_0 - u \in H^1_{loc}(\mathbf{R}^N)$, the fact that there exist $\rho, \theta \in H^1_{loc}(\mathbf{R}^N)$ such that $r_0 - u = \rho e^{i\theta}$ a.e. is standard and follows from Theorem 3 p. 38 in [9]. We have

$$\left|\frac{\partial u}{\partial x_j}\right|^2 = \left|\frac{\partial \rho}{\partial x_j}\right|^2 + \rho^2 \left|\frac{\partial \theta}{\partial x_j}\right|^2 \qquad \text{a.e. on } \mathbf{R}^N \text{ for } j = 1, \dots, N. \tag{2.4}$$

Since $\rho = |r_0 - u| \ge m$ a.e., it follows that $\nabla\rho, \nabla\theta \in L^2(\mathbf{R}^N)$. If $N \ge 3$, we infer that there exist $\rho_0, \theta_0 \in \mathbf{R}$ such that $\rho - \rho_0$ and $\theta - \theta_0$ belong to $L^{2^*}(\mathbf{R}^N)$. Then it is not hard to see that $\rho_0 = r_0$ and $\theta_0 = 2k_0\pi$, where $k_0 \in \mathbf{Z}$. Replacing θ by $\theta - 2k_0\pi$, we have $\rho - r_0, \theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. Since $\rho \le 2r_0$ a.e., we have $\rho^2 - r_0^2 = \varphi(|r_0 - u|^2) - r_0^2 \in L^2(\mathbf{R}^N)$ because $u \in \mathcal{X}$. Clearly $|\rho - r_0| = \frac{|\rho^2 - r_0^2|}{\rho + r_0} \le \frac{1}{r_0}|\rho^2 - r_0^2|$, hence $\rho - r_0 \in L^2(\mathbf{R}^N)$.

A straightforward computation gives

$$\langle iu_{x_1}, u\rangle = \langle iu_{x_1}, r_0\rangle - \rho^2\theta_{x_1} = -r_0\frac{\partial}{\partial x_1}(\mathrm{Im}(u) + r_0\theta) - (\rho^2 - r_0^2)\frac{\partial\theta}{\partial x_1}.$$

By (2.4) we have $\left|\frac{\partial\theta}{\partial x_j}\right| \le \frac{1}{\rho}\left|\frac{\partial u}{\partial x_j}\right| \le \frac{1}{m}\left|\frac{\partial u}{\partial x_j}\right|$ and the Cauchy-Schwarz inequality gives

$$\int_{\mathbf{R}^N} \left|(\rho^2 - r_0^2)\theta_{x_1}\right| dx \le ||\rho^2 - r_0^2||_{L^2} ||\theta_{x_1}||_{L^2} \le \frac{1}{m}||\rho^2 - r_0^2||_{L^2}||u_{x_1}||_{L^2} \le \frac{1}{2am}E_{GL}(u).$$

□

Lemma 2.2 *Let $\chi \in C_c^\infty(\mathbf{C}, \mathbf{R})$ be a function such that $\chi = 1$ on $B(0, \frac{r_0}{4})$, $0 \le \chi \le 1$ and $\mathrm{supp}(\chi) \subset B(0, \frac{r_0}{2})$. For an arbitrary $u \in \mathcal{X}$, denote $u_1 = \chi(u)u$ and $u_2 = (1 - \chi(u))u$. Then $u_1 \in \mathcal{X}$, $u_2 \in H^1(\mathbf{R}^N)$ and the following estimates hold:*

$$|\nabla u_i| \le C|\nabla u| \qquad a.e. \text{ on } \mathbf{R}^N, i = 1, 2, \text{ wehere } C \text{ depends only on } \chi, \tag{2.5}$$

$$||u_2||_{L^2(\mathbf{R}^N)} \le C_1 ||\nabla u||_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}} \quad and \quad ||(1 - \chi^2(u))u||_{L^2(\mathbf{R}^N)} \le C_1||\nabla u||_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}}, \tag{2.6}$$

$$(2.7)\qquad \int_{\mathbf{R}^N} \left(\varphi^2(|r_0-u_1|)-r_0^2\right)^2\,dx \le \int_{\mathbf{R}^N} \left(\varphi^2(|r_0-u|)-r_0^2\right)^2\,dx + C_2||\nabla u||_{L^2(\mathbf{R}^N)}^{2^*},$$

$$(2.8)\qquad \int_{\mathbf{R}^N} \left(\varphi^2(|r_0-u_2|)-r_0^2\right)^2\,dx \le C_2||\nabla u||_{L^2(\mathbf{R}^N)}^{2^*}.$$

Let $r_0-u_1=\rho e^{i\theta}$ be the lifting of r_0-u_1, as given by Lemma 2.1. Then we have

$$(2.9)\qquad \langle iu_{x_1},u\rangle = (1-\chi^2(u))\langle iu_{x_1},u\rangle - (\rho^2-r_0^2)\frac{\partial\theta}{\partial x_1} - r_0\frac{\partial}{\partial x_1}(\mathrm{Im}(u_1)+r_0\theta)$$

a.e. on $\mathbf{R}^N$.

Proof. Since $|u_i|\le|u|$, we have $u_i\in L^{2^*}(\mathbf{R}^N)$, $i=1,2$. It is standard to prove that $u_i\in H^1_{loc}(\mathbf{R}^N)$ (see, e.g., Lemma C1 p. 66 in [9]) and we have

$$(2.10)\qquad \frac{\partial u_1}{\partial x_j} = \left(\partial_1\chi(u)\frac{\partial(\mathrm{Re}(u))}{\partial x_j} + \partial_2\chi(u)\frac{\partial(\mathrm{Im}(u))}{\partial x_j}\right)u + \chi(u)\frac{\partial u}{\partial x_j}.$$

A similar formula holds for u_2. Since the functions $z\longmapsto \partial_i\chi(z)z$, $i=1,2$, are bounded on $\mathbf{C}$, (2.5) follows immediately from (2.10).

Using the Sobolev embedding we have

$$||u_2||_{L^2}^2 \le \int_{\mathbf{R}^N}|u|^2\mathbb{1}_{\{|u|>\frac{r_0}{4}\}}(x)\,dx \le \left(\frac{4}{r_0}\right)^{2^*-2}\int_{\mathbf{R}^N}|u|^{2^*}\mathbb{1}_{\{|u|>\frac{r_0}{4}\}}(x)\,dx \le C_1||\nabla u||_{L^2}^{2^*}.$$

This gives the first estimate in (2.6); the second one is similar.

For $|u|\le\frac{r_0}{4}$ we have $u_1(x)=u(x)$, hence

$$\int_{\{|u|\le\frac{r_0}{4}\}}\left(\varphi^2(|r_0-u_1|)-r_0^2\right)^2\,dx = \int_{\{|u|\le\frac{r_0}{4}\}}\left(\varphi^2(|r_0-u|)-r_0^2\right)^2\,dx.$$

There exists $C'>0$ such that $\left(\varphi^2(|r_0-z|)-r_0^2\right)^2\le C'|z|^2$ if $|z|\ge\frac{r_0}{4}$. Proceeding as in the proof of (2.6) we have for $i=1,2$

$$\int_{\{|u|>\frac{r_0}{4}\}}\left(\varphi^2(|r_0-u_i|)-r_0^2\right)^2\,dx \le C'\int_{\{|u|>\frac{r_0}{4}\}}|u_i|^2\,dx \le C_2||\nabla u||_{L^2}^{2^*}.$$

This clearly implies (2.7) and (2.8).

Since $\partial_1\chi(u)\frac{\partial(\mathrm{Re}(u))}{\partial x_j}+\partial_2\chi(u)\frac{\partial(\mathrm{Im}(u))}{\partial x_j}\in\mathbf{R}$, using (2.10) we see that $\langle i\frac{\partial u_1}{\partial x_1},u_1\rangle=\chi^2(u)\langle iu_{x_1},u\rangle$ a.e. on $\mathbf{R}$. Then (2.9) follows from Lemma 2.1. □

We consider the space $\mathcal{Y}=\{\partial_{x_1}\phi\mid\phi\in\mathcal{D}^{1,2}(\mathbf{R}^N)\}$. It is clear that $\phi_1,\phi_2\in\mathcal{D}^{1,2}(\mathbf{R}^N)$ and $\partial_{x_1}\phi_1=\partial_{x_1}\phi_2$ imply $\phi_1=\phi_2$. Defining

$$||\partial_{x_1}\phi||_{\mathcal{Y}} = ||\phi||_{\mathcal{D}^{1,2}} = ||\nabla\phi||_{L^2(\mathbf{R}^N)},$$

it is easy to see that $||\cdot||_{\mathcal{Y}}$ is a norm on $\mathcal{Y}$ and $(\mathcal{Y},||\cdot||_{\mathcal{Y}})$ is a Banach space. The following holds.

Lemma 2.3 *For any $v\in L^1(\mathbf{R}^N)\cap\mathcal{Y}$ we have $\int_{\mathbf{R}^N}v(x)\,dx=0$.*

Proof. Let $\phi \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ be such that $v = \partial_{x_1}\phi$. Then $\phi \in \mathcal{S}'(\mathbf{R}^N)$ and $|\xi|\widehat{\phi} \in L^2(\mathbf{R}^N)$. Hence $\widehat{\phi} \in L^1_{loc}(\mathbf{R}^N \setminus \{0\})$. On the other hand we have $v = \partial_{x_1}\phi \in L^1 \cap L^2(\mathbf{R}^N)$ by hypothesis, hence $\widehat{v} = i\xi_1\widehat{\phi} \in L^2 \cap C^0_b(\mathbf{R}^N)$.

We prove that $\widehat{v}(0) = 0$. We argue by contradiction and assume that $\widehat{v}(0) \neq 0$. By continuity, there exists $m > 0$ and $\varepsilon > 0$ such that $|\widehat{v}(\xi)| \geq m$ for $|\xi| \leq \varepsilon$. For $j = 2, \dots N$ we get

$$|i\xi_j\widehat{\phi}(\xi)| \geq \frac{|\xi_j|}{|\xi_1|}|\widehat{v}(\xi)| \geq m\frac{|\xi_j|}{|\xi_1|} \qquad \text{for a.e. } \xi \in B(0,\varepsilon).$$

But this contradicts the fact that $i\xi_j\widehat{\phi} \in L^2(\mathbf{R}^N)$. Thus necessarily $\widehat{v}(0) = 0$ and this is exactly the conclusion of Lemma 2.3. □

It is obvious that $L_1(v) = \int_{\mathbf{R}^N} v(x)\,dx$ and $L_2(w) = 0$ are continuous linear forms on $L^1(\mathbf{R}^N)$ and on $\mathcal{Y}$, respectively. Moreover, by Lemma 2.3 we have $L_1 = L_2$ on $L^1(\mathbf{R}^N) \cap \mathcal{Y}$. Putting

$$L(v+w) = L_1(v) + L_2(w) = \int_{\mathbf{R}^N} v(x)\,dx \qquad \text{for } v \in L^1(\mathbf{R}^N) \text{ and } w \in \mathcal{Y}, \tag{2.11}$$

we see that L is well-defined and is a continuous linear form on $L^1(\mathbf{R}^N) + \mathcal{Y}$.

It follows from (2.9) and Lemmas 2.1 and 2.2 that for any $u \in \mathcal{X}$ we have $\langle iu_{x_1}, u\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$. This enables us to give the following

Definition 2.4 *Given $u \in \mathcal{X}$, the momentum of u (with respect to the x_1-direction) is*

$$Q(u) = L(\langle iu_{x_1}, u\rangle).$$

If $u \in \mathcal{X}$ and $\chi, u_1, u_2, \rho, \theta$ are as in Lemma 2.2, from (2.9) we get

$$Q(u) = \int_{\mathbf{R}^N} (1-\chi^2(u))\langle iu_{x_1}, u\rangle - (\rho^2 - r_0^2)\theta_{x_1}\,dx. \tag{2.12}$$

It is easy to check that the right-hand side of (2.12) does not depend on the choice of the cut-off function χ, provided that χ is as in Lemma 2.2.

It follows directly from (2.12) that the functional Q has a nice behavior with respect to dilations in $\mathbf{R}^N$: for any $u \in \mathcal{X}$ and λ, $\sigma > 0$ we have

$$Q(u_{\lambda,\sigma}) = \sigma^{N-1}Q(u). \tag{2.13}$$

The next lemma will enable us to perform "integrations by parts".

Lemma 2.5 *For any $u \in \mathcal{X}$ and $w \in H^1(\mathbf{R}^N)$ we have $\langle iu_{x_1}, w\rangle \in L^1(\mathbf{R}^N)$, $\langle iu, w_{x_1}\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$ and*

$$L(\langle iu_{x_1}, w\rangle + \langle iu, w_{x_1}\rangle) = 0. \tag{2.14}$$

Proof.

Since $w, u_{x_1} \in L^2(\mathbf{R}^N)$, the Cauchy-Schwarz inequality implies $\langle iu_{x_1}, w\rangle \in L^1(\mathbf{R}^N)$.

Let χ, u_1, u_2 be as in Lemma 2.2 and denote $w_1 = \chi(w)w$, $w_2 = (1-\chi(w))w$. Then $u = u_1 + u_2$, $w = w_1 + w_2$ and it follows from Lemma 2.2 that $u_1 \in \mathcal{X} \cap L^\infty(\mathbf{R}^N)$ and u_2, w_1, $w_2 \in H^1(\mathbf{R}^N)$.

As above we have $\langle i\frac{\partial u_2}{\partial x_1}, w\rangle$, $\langle iu_2, \frac{\partial w}{\partial x_1}\rangle \in L^1(\mathbf{R}^N)$ by the Cauchy-Schwarz inequality. The standard integration by parts formula for functions in $H^1(\mathbf{R}^N)$ (see, e.g., [8], p. 197) gives

$$\int_{\mathbf{R}^N} \langle i\frac{\partial u_2}{\partial x_1}, w\rangle + \langle iu_2, \frac{\partial w}{\partial x_1}\rangle \, dx = 0. \tag{2.15}$$

Since $u_1 \in \mathcal{D}^{1,2} \cap L^\infty(\mathbf{R}^N)$ and $w_1 \in H^1 \cap L^\infty(\mathbf{R}^N)$, it is standard to prove that $\langle iu_1, w_1\rangle \in \mathcal{D}^{1,2} \cap L^\infty(\mathbf{R}^N)$ and

$$\langle i\frac{\partial u_1}{\partial x_1}, w_1\rangle + \langle iu_1, \frac{\partial w_1}{\partial x_1}\rangle = \frac{\partial}{\partial x_1}\langle iu_1, w_1\rangle \qquad \text{a.e. on } \mathbf{R}^N. \tag{2.16}$$

Let $A_w = \{x \in \mathbf{R}^N \mid |w(x)| \geq \frac{r_0}{4}\}$. We have $\left(\frac{r_0}{4}\right)^2 \mathcal{L}^N(A_w) \leq \int_{A_w} |w|^2 \, dx \leq ||w||_{L^2}^2$, and consequently A_w has finite measure. It is clear that $w_2 = 0$ and $\nabla w_2 = 0$ a.e. on $\mathbf{R}^N \setminus A_w$. Since $w_2 \in L^{2^*}(\mathbf{R}^N)$ and $\nabla w_2 \in L^2(\mathbf{R}^N)$, we infer that $w_2 \in L^1 \cap L^{2^*}(\mathbf{R}^N)$ and $\nabla w_2 \in L^1 \cap L^2(\mathbf{R}^N)$. Together with the fact that $u_1 \in L^{2^*} \cap L^\infty(\mathbf{R}^N)$ and $\nabla u_1 \in L^2(\mathbf{R}^N)$, this gives $\langle iu_1, w_2\rangle \in L^1 \cap L^{2^*}(\mathbf{R}^N)$ and

$$\langle i\frac{\partial u_1}{\partial x_j}, w_2\rangle \in L^1 \cap L^{\frac{N}{N-1}}(\mathbf{R}^N), \qquad \langle iu_1, \frac{\partial w_2}{\partial x_j}\rangle \in L^1 \cap L^2(\mathbf{R}^N) \qquad \text{for } j = 1, \ldots, N.$$

It is easy to see that $\frac{\partial}{\partial x_j}\langle iu_1, w_2\rangle = \langle i\frac{\partial u_1}{\partial x_j}, w_2\rangle + \langle iu_1, \frac{\partial w_2}{\partial x_j}\rangle$ in $\mathcal{D}'(\mathbf{R}^N)$. From the above we infer that $\langle iu_1, w_2\rangle \in W^{1,1}(\mathbf{R}^N)$. It is obvious that $\int_{\mathbf{R}^N} \frac{\partial \psi}{\partial x_j} \, dx = 0$ for any $\psi \in W^{1,1}(\mathbf{R}^N)$ (indeed, let $(\psi_n)_{n\geq 1} \subset C_c^\infty(\mathbf{R}^N)$ be a sequence such that $\psi_n \longrightarrow \psi$ in $W^{1,1}(\mathbf{R}^N)$ as $n \longrightarrow \infty$; then $\int_{\mathbf{R}^N} \frac{\partial \psi_n}{\partial x_j} \, dx = 0$ for each n and $\int_{\mathbf{R}^N} \frac{\partial \psi_n}{\partial x_j} \, dx \longrightarrow \int_{\mathbf{R}^N} \frac{\partial \psi}{\partial x_j} \, dx$ as $n \longrightarrow \infty$). Thus we have $\langle i\frac{\partial u_1}{\partial x_1}, w_2\rangle$, $\langle iu_1, \frac{\partial w_2}{\partial x_1}\rangle \in L^1(\mathbf{R}^N)$ and

$$\int_{\mathbf{R}^N} \langle i\frac{\partial u_1}{\partial x_1}, w_2\rangle + \langle iu_1, \frac{\partial w_2}{\partial x_1}\rangle \, dx = \int_{\mathbf{R}^N} \frac{\partial}{\partial x_1}\langle iu_1, w_2\rangle \, dx = 0. \tag{2.17}$$

Now (2.14) follows from (2.15), (2.16), (2.17) and Lemma 2.5 is proved. □

Corollary 2.6 *Let u, $v \in \mathcal{X}$ be such that $u - v \in L^2(\mathbf{R}^N)$. Then*

$$|Q(u) - Q(v)| \leq ||u - v||_{L^2(\mathbf{R}^N)} \left(\left|\left|\frac{\partial u}{\partial x_1}\right|\right|_{L^2(\mathbf{R}^N)} + \left|\left|\frac{\partial v}{\partial x_1}\right|\right|_{L^2(\mathbf{R}^N)} \right) \tag{2.18}$$

Proof. It is clear that $w = u - v \in H^1(\mathbf{R}^N)$ and using (2.14) we get

$$\begin{aligned} Q(u) - Q(v) &= L(\langle i(u - v)_{x_1}, u\rangle + \langle iv_{x_1}, u - v\rangle) \\ &= L(\langle iu_{x_1}, u - v\rangle + \langle iv_{x_1}, u - v\rangle) \\ &= \int_{\mathbf{R}^N} \langle iu_{x_1} + iv_{x_1}, u - v\rangle \, dx. \end{aligned} \tag{2.19}$$

Then (2.18) follows from (2.19) and the Cauchy-Schwarz inequality. □

The next result will be useful to estimate the contribution to the momentum of a domain where the modified Ginzburg-Landau energy is small.

Lemma 2.7 *Let $M > 0$ and let Ω be an open subset of $\mathbf{R}^N$. Assume that $u \in \mathcal{X}$ satisfies $E_{GL}(u) \leq M$ and let χ, ρ, θ be as in Lemma 2.2. Then we have*

$$\int_\Omega \left|(1 - \chi^2(u))\langle iu_{x_1}, u\rangle - (\rho^2 - r_0^2)\theta_{x_1}\right| dx \leq C(M^{\frac{1}{2}} + M^{\frac{2^*}{4}}) \left(E_{GL}^\Omega(u)\right)^{\frac{1}{2}}. \tag{2.20}$$

Proof. Using (2.6) and the Cauchy-Schwarz inequality we get

$$\begin{aligned}\int_\Omega \left|(1-\chi^2(u))\langle iu_{x_1}, u\rangle\right| dx &\leq ||u_{x_1}||_{L^2(\Omega)}||(1-\chi^2(u))u||_{L^2(\Omega)} \\ &\leq C_1||u_{x_1}||_{L^2(\Omega)}||\nabla u||^{\frac{2^*}{2}}_{L^2(\mathbf{R}^N)}.\end{aligned} \tag{2.21}$$

We have $|u_1| \leq \frac{r_0}{2}$, hence $|r_0 - u_1| \leq \frac{3r_0}{2}$ and $\varphi(|r_0 - u_1|) = |r_0 - u_1| = \rho$. Then (2.7) gives

$$||\rho^2 - r_0^2||_{L^2(\mathbf{R}^N)} \leq C'(E_{GL}(u) + E_{GL}(u)^{\frac{2^*}{2}}) \leq C'(M + M^{\frac{2^*}{2}}). \tag{2.22}$$

From (2.4) and (2.5) we have $\left|\frac{\partial\theta}{\partial x_j}\right| \leq \frac{1}{\rho}\left|\frac{\partial u_1}{\partial x_j}\right| \leq C''\left|\frac{\partial u}{\partial x_j}\right|$ a.e. on $\mathbf{R}^N$. Therefore

$$\begin{aligned}&\int_\Omega \left|(\rho^2 - r_0^2)\theta_{x_1}\right| dx \leq ||\rho^2 - r_0^2||_{L^2(\Omega)}||\theta_{x_1}||_{L^2(\Omega)} \\ &\leq C''||\rho^2 - r_0^2||_{L^2(\mathbf{R}^N)}||u_{x_1}||_{L^2(\Omega)} \leq C'''\left(M + M^{\frac{2^*}{2}}\right)^{\frac{1}{2}}\left(E^\Omega_{GL}(u)\right)^{\frac{1}{2}}.\end{aligned} \tag{2.23}$$

Then (2.20) follows from (2.21) and (2.23). □

3 A regularization procedure

Given a function $u \in \mathcal{X}$ and a region $\Omega \subset \mathbf{R}^N$ such that $E^\Omega_{GL}(u)$ is small, we would like to get a fine estimate of the contribution of Ω to the momentum of u. To do this, we will use a kind of "regularization" procedure for arbitrary functions in $\mathcal{X}$. A similar device has been introduced in [1] to get rid of small-scale topological defects of functions; variants of it have been used for various purposes in [7], [6], [5].

Throughout this section, Ω is an open set in $\mathbf{R}^N$. We do not assume Ω bounded, nor connected. If $\partial\Omega \neq \emptyset$, we assume that $\partial\Omega$ is C^2. Let φ be as in the introduction. Let $u \in \mathcal{X}$ and let $h > 0$. We consider the functional

$$G^u_{h,\Omega}(v) = E^\Omega_{GL}(v) + \frac{1}{h^2}\int_\Omega \varphi\left(\frac{|v-u|^2}{32r_0}\right) dx.$$

Note that $G^u_{h,\Omega}(v)$ may equal ∞ for some $v \in \mathcal{X}$; however, $G^u_{h,\Omega}(v)$ is finite whenever $v \in \mathcal{X}$ and $v - u \in L^2(\Omega)$. We denote $H^1_0(\Omega) = \{u \in H^1(\mathbf{R}^N) \mid u = 0 \text{ on } \mathbf{R}^N \setminus \Omega\}$ and

$$H^1_u(\Omega) = \{v \in \mathcal{X} \mid v - u \in H^1_0(\Omega)\}.$$

The next lemma gives the properties of functions that minimize $G^u_{h,\Omega}$ in the space $H^1_u(\Omega)$.

Lemma 3.1 *i) The functional $G^u_{h,\Omega}$ has a minimizer in $H^1_u(\Omega)$.*

ii) Let v_h be a minimizer of $G^u_{h,\Omega}$ in $H^1_u(\Omega)$. There exist constants C_1, C_2, $C_3 > 0$, depending only on N, a and r_0 such that v_h satisfies:

$$E^\Omega_{GL}(v_h) \leq E^\Omega_{GL}(u); \tag{3.1}$$

$$||v_h - u||^2_{L^2(\Omega)} \leq 32r_0h^2E^\Omega_{GL}(u) + C_1\left(E^\Omega_{GL}(u)\right)^{1+\frac{2}{N}}h^{\frac{4}{N}}; \tag{3.2}$$

$$\int_\Omega \left|\left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - v_h|) - r_0^2\right)^2\right| dx \leq C_2hE^\Omega_{GL}(u); \tag{3.3}$$

$$(3.4)\qquad |Q(u)-Q(v_h)| \le C_3 \left(h^2 + \left(E_{GL}^{\Omega}(u)\right)^{\frac{2}{N}} h^{\frac{4}{N}} \right)^{\frac{1}{2}} E_{GL}^{\Omega}(u).$$

iii) For $z \in \mathbf{C}$, denote $H(z) = \left(\varphi^2(|z-r_0|) - r_0^2\right) \varphi(|z-r_0|)\varphi'(|z-r_0|)\frac{z-r_0}{|z-r_0|}$ if $z \ne r_0$ and $H(r_0) = 0$. Then any minimizer v_h of $G_{h,\Omega}^u$ in $H_u^1(\Omega)$ satisfies the equation

$$(3.5)\qquad -\Delta v_h + 2a^2 H(v_h) + \frac{1}{32r_0h^2}\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u) = 0 \qquad in\ \mathcal{D}'(\Omega).$$

Moreover, for any $\omega \subset\subset \Omega$ we have $v_h \in W^{2,p}(\omega)$ for $p \in [1,\infty)$; thus, in particular, $v_h \in C^{1,\alpha}(\omega)$ for $\alpha \in [0,1)$.

iv) For any $h > 0$, $\delta > 0$ and $R > 0$ there exists a constant $K = K(a, r_0, N, h, \delta, R) > 0$ such that for any $u \in \mathcal{X}$ with $E_{GL}^{\Omega}(u) \le K$ and for any minimizer v_h of $G_{h,\Omega}^u$ in $H_u^1(\Omega)$ we have

$$(3.6)\qquad r_0 - \delta < |r_0 - v_h(x)| < r_0 + \delta \qquad whenever\ x \in \Omega\ and\ dist(x, \partial\Omega) \ge 4R.$$

Proof. i) It is obviuos that $u \in H_u^1(\Omega)$. Let $(v_n)_{n\ge1}$ be a minimizing sequence for $G_{h,\Omega}^u$ in $H_u^1(\Omega)$. We may assume that $G_{h,\Omega}^u(v_n) \le G_{h,\Omega}^u(u) = E_{GL}^{\Omega}(u)$ and this implies $\int_\Omega |\nabla v_n|^2\,dx \le E_{GL}^{\Omega}(u)$. It is clear that

$$(3.7)\qquad \int_{\Omega\cap\{|v_n-u|\le 8r_0\}} |v_n-u|^2\,dx \le 32r_0 \int_\Omega \varphi\left(\frac{|v_n-u|^2}{32r_0}\right)dx \le 32r_0h^2E_{GL}^{\Omega}(u).$$

Since $v_n - u \in H_0^1(\Omega) \subset H^1(\mathbf{R}^N)$, by the Sobolev embedding we have $||v_n-u||_{L^{2^*}(\mathbf{R}^N)} \le C_S||\nabla v_n - \nabla u||_{L^2(\mathbf{R}^N)}$, where C_S depends only on N. Therefore

$$(3.8)\qquad \begin{aligned} &\int_{\{|v_n-u|>8r_0\}} |v_n-u|^2\,dx \le (8r_0)^{2-2^*} \int_{\{|v_n-u|>8r_0\}} |v_n-u|^{2^*}\,dx \\ &\le (8r_0)^{2-2^*}\,||v_n-u||_{L^{2^*}(\mathbf{R}^N)}^{2^*} \le C'||\nabla v_n - \nabla u||_{L^2(\mathbf{R}^N)}^{2^*} \le C\left(E_{GL}^{\Omega}(u)\right)^{\frac{2^*}{2}}. \end{aligned}$$

It follows from (3.7) and (3.8) that $||v_n-u||_{L^2(\Omega)}$ is bounded, hence $v_n - u$ is bounded in $H_0^1(\Omega)$. We infer that there exists a sequence (still denoted $(v_n)_{n\ge1}$) and there is $w \in H_0^1(\Omega)$ such that $v_n - u \rightharpoonup w$ weakly in $H_0^1(\Omega)$, $v_n - u \longrightarrow w$ a.e. and $v_n - u \longrightarrow w$ in $L_{loc}^p(\Omega)$ for $1 \le p < 2^*$. Let $v = u + w$. Then $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^2(\mathbf{R}^N)$ and this implies

$$\int_\Omega |\nabla v|^2\,dx \le \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2\,dx.$$

Using the a.e. convergence and Fatou's Lemma we infer that

$$\int_\Omega \left(\varphi^2(|r_0-v|)-r_0^2\right)^2 dx \le \liminf_{n\to\infty} \int_\Omega \left(\varphi^2(|r_0-v_n|)-r_0^2\right)^2 dx \qquad \text{and}$$

$$\int_\Omega \varphi\left(\frac{|v-u|^2}{32r_0}\right)dx \le \liminf_{n\to\infty} \int_\Omega \varphi\left(\frac{|v_n-u|^2}{32r_0}\right)dx.$$

Therefore $G_{h,\Omega}^u(v) \le \liminf_{n\to\infty} G_{h,\Omega}^u(v_n)$ and consequently v is a minimizer of $G_{h,\Omega}^u$ in $H_u^1(\Omega)$.

ii) Since $u \in H_u^1(\Omega)$, we have $E_{GL}^{\Omega}(v_h) \le G_{h,\Omega}^u(v_h) \le E_{GL}^{\Omega}(u)$; hence (3.1) holds. It is clear that $\varphi\left(\frac{|v_h-u|^2}{32r_0}\right) \ge 2r_0$ if $|v_h-u| \ge 8r_0$, thus

$$2r_0\mathcal{L}^N(\{|v_h-u| \ge 8r_0\}) \le \int_{\mathbf{R}^N} \varphi\left(\frac{|v_h-u|^2}{32r_0}\right)dx \le h^2G_{h,\Omega}^u(v_h) \le h^2E_{GL}^{\Omega}(u).$$

Using Hölder's inequality, the above estimate and the Sobolev inequality we get

$$
\begin{aligned}
&\int_{\{|v_h-u|\geq 8r_0\}} |v_h-u|^2\,dx \\
&\leq ||v_h-u||^2_{L^{2^*}(\{|v_h-u|\geq 8r_0\})}\left(\mathcal{L}^N(\{|v_h-u|\geq 8r_0\})\right)^{1-\frac{2}{2^*}} \\
&\leq ||v_h-u||^2_{L^{2^*}(\mathbf{R}^N)}\left(\mathcal{L}^N(\{|v_h-u|\geq 8r_0\})\right)^{1-\frac{2}{2^*}} \\
&\leq C_S||\nabla v_h-\nabla u||^2_{L^2(\mathbf{R}^N)}\left(\frac{h^2}{2r_0}E^{\Omega}_{GL}(u)\right)^{1-\frac{2}{2^*}} \leq C_1 h^{\frac{4}{N}}\left(E^{\Omega}_{GL}(u)\right)^{1+\frac{2}{N}}.
\end{aligned}
\tag{3.9}
$$

It is clear that (3.7) holds with v_h instead of v_n and then (3.2) follows from (3.7) and (3.9).

We claim that

$$
\Big|\varphi(|r_0-z|)-\varphi(|r_0-\zeta|)\Big| \leq \left[32r_0\varphi\left(\frac{|z-\zeta|^2}{32r_0}\right)\right]^{\frac{1}{2}} \qquad \text{for any } z,\ \zeta\in\mathbf{C}. \tag{3.10}
$$

Indeed, if $|z-r_0|\leq 4r_0$ and $|\zeta-r_0|\leq 4r_0$, then $|z-\zeta|\leq 8r_0$, $\varphi\left(\frac{|z-\zeta|^2}{32r_0}\right)=\frac{|z-\zeta|^2}{32r_0}$ and $\Big|\varphi(|r_0-z|)-\varphi(|r_0-\zeta|)\Big|\leq\Big||r_0-z|-|r_0-\zeta|\Big|\leq|z-\zeta|$, hence (3.10) holds.

If $|z-r_0|\leq 4r_0$ and $|\zeta-r_0|>4r_0$, there exists $t\in[0,1)$ such that $w=(1-t)z+t\zeta$ satisfies $|r_0-w|=4r_0$ and

$$
\begin{aligned}
&\Big|\varphi(|r_0-z|)-\varphi(|r_0-\zeta|)\Big| = \Big|\varphi(|r_0-z|)-\varphi(|r_0-w|)\Big| \\
&\leq\left[32r_0\varphi\left(\tfrac{|z-w|^2}{32r_0}\right)\right]^{\frac{1}{2}}\leq\left[32r_0\varphi\left(\tfrac{|z-\zeta|^2}{32r_0}\right)\right]^{\frac{1}{2}}.
\end{aligned}
$$

We argue similarly if $|z-r_0|>4r_0$ and $|\zeta-r_0|\leq 4r_0$. Finally, in the case $|z-r_0|>4r_0$ and $|\zeta-r_0|>4r_0$ we have $\varphi(|r_0-z|)=\varphi(|r_0-\zeta|)=3r_0$ and (3.10) trivially holds.

It is obvious that

$$
\begin{aligned}
&\Big|\left(\varphi^2(|r_0-u|)-r_0^2\right)^2-\left(\varphi^2(|r_0-v_h|)-r_0^2\right)^2\Big| \\
&\leq 6r_0\Big|\varphi(|r_0-u|)-\varphi(|r_0-v_h|)\Big|\cdot\Big|\varphi^2(|r_0-u|)+\varphi^2(|r_0-v_h|)-2r_0^2\Big|.
\end{aligned}
\tag{3.11}
$$

Using (3.11), the Cauchy-Schwarz inequality and (3.10) we get

$$
\begin{aligned}
&\int_\Omega\Big|\left(\varphi^2(|r_0-u|)-r_0^2\right)^2-\left(\varphi^2(|r_0-v_h|)-r_0^2\right)^2\Big|\,dx \\
&\leq 6r_0\left(\int_\Omega\Big|\varphi(|r_0-u|)-\varphi(|r_0-v_h|)\Big|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega\Big|\varphi^2(|r_0-u|)+\varphi^2(|r_0-v_h|)-2r_0^2\Big|^2dx\right)^{\frac{1}{2}} \\
&\leq 6r_0\left(\int_\Omega 32r_0\varphi\left(\tfrac{|v_h-u|^2}{32r_0}\right)dx\right)^{\frac{1}{2}}\left(2\int_\Omega\left(\varphi^2(|r_0-u|)-r_0^2\right)^2+\left(\varphi^2(|r_0-v_h|)-r_0^2\right)^2dx\right)^{\frac{1}{2}} \\
&\leq 48r_0^{\frac{3}{2}}\left(h^2G^u_{h,\Omega}(v_h)\right)^{\frac{1}{2}}\left(\tfrac{1}{a^2}E^{\Omega}_{GL}(u)+\tfrac{1}{a^2}E^{\Omega}_{GL}(v_h)\right)^{\frac{1}{2}}\leq\tfrac{48\sqrt{2}}{a}r_0^{\frac{3}{2}}hE^{\Omega}_{GL}(u)
\end{aligned}
$$

and (3.3) is proved. Finally, (3.4) follows directly from (3.1), (3.2) and Corollary 2.6.

iii) The proof of (3.5) is standard. For any $\psi\in C_c^\infty(\Omega)$ we have $v+\psi\in H^1_u(\Omega)$ and the function $t\longmapsto G^u_{h,\Omega}(v+t\psi)$ achieves its minumum at $t=0$. Hence $\frac{d}{dt}\Big|_{t=0}\left(G^u_{h,\Omega}(v+t\psi)\right)=0$ for any $\psi\in C_c^\infty(\Omega)$ and this is precisely (3.5).

For any $z \in \mathbf{C}$ we have

$$|H(z)| \leq 3r_0|\varphi^2(|z-r_0|) - r_0^2| \leq 24r_0^3. \tag{3.12}$$

Since $v_h \in \mathcal{X}$, we have $\varphi^2(|r_0 - v_h|) - r_0^2 \in L^2(\mathbf{R}^N)$ and (3.12) gives $H(v_h) \in L^2 \cap L^\infty(\mathbf{R}^N)$. We also have $\left|\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right| \leq |v_h-u|$ and $\left|\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right| \leq \sup\limits_{s\geq 0}\varphi'\left(\frac{s^2}{32r_0}\right)s < \infty$. Since $v_h - u \in L^2(\mathbf{R}^N)$, we get $\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u) \in L^2 \cap L^\infty(\mathbf{R}^N)$. Using (3.5) we infer that $\Delta v_h \in L^2 \cap L^\infty(\Omega)$. Then (iii) follows from standard elliptic estimates (see, e.g., Theorem 9.11 p. 235 in [19]) and a straightforward bootstrap argument.

iv) Using (3.12) we get

$$\int_\Omega |H(v_h)|^2\,dx \leq 9r_0^2\int_\Omega \left(\varphi^2(|r_0-v_h|) - r_0^2\right)^2\,dx \leq \frac{9r_0^2}{a^2}E_{GL}^\Omega(v_h) \leq \frac{9r_0^2}{a^2}E_{GL}^\Omega(u),$$

hence $||H(v_h)||_{L^2(\Omega)} \leq C'\left(E_{GL}^\Omega(u)\right)^{\frac{1}{2}}$. By interpolation we find for any $p \in [2,\infty]$,

$$||H(v_h)||_{L^p(\Omega)} \leq ||H(v_h)||_{L^\infty(\Omega)}^{\frac{p-2}{p}}||H(v_h)||_{L^2(\Omega)}^{\frac{2}{p}} \leq C\left(E_{GL}^\Omega(u)\right)^{\frac{1}{p}}. \tag{3.13}$$

There exist m_1, $m_2 > 0$ such that $\left|\varphi'\left(\frac{s^2}{32r_0}\right)s\right|^2 \leq m_1\varphi\left(\frac{s^2}{32r_0}\right)$ and $\left|\varphi'\left(\frac{s^2}{32r_0}\right)s\right| \leq m_2$ for any $s \geq 0$. Then we have

$$\int_\Omega\left|\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|^2dx \leq m_1\int_\Omega\varphi\left(\frac{|v_h-u|^2}{32r_0}\right)dx \leq m_1h^2E_{GL}^\Omega(u),$$

thus $\left|\left|\varphi'\left(\frac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|\right|_{L^2(\Omega)} \leq h\left(m_1E_{GL}^\Omega(u)\right)^{\frac{1}{2}}$. By interpolation we get

$$\begin{aligned}&\left|\left|\varphi'\left(\tfrac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|\right|_{L^p(\Omega)}\\ &\leq \left|\left|\varphi'\left(\tfrac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|\right|_{L^\infty(\Omega)}^{\frac{p-2}{p}}\left|\left|\varphi'\left(\tfrac{|v_h-u|^2}{32r_0}\right)(v_h-u)\right|\right|_{L^2(\Omega)}^{\frac{2}{p}}\\ &\leq Ch^{\frac{2}{p}}\left(E_{GL}^\Omega(u)\right)^{\frac{1}{p}}\end{aligned} \tag{3.14}$$

for any $p \in [2,\infty]$. From (3.5), (3.13) and (3.14) we obtain

$$||\Delta v_h||_{L^p(\Omega)} \leq C(1+h^{\frac{2}{p}-2})\left(E_{GL}^\Omega(u)\right)^{\frac{1}{p}} \qquad \text{for any } p \geq 2. \tag{3.15}$$

For a measurable set $\omega \subset \mathbf{R}^N$ with $\mathcal{L}^N(\omega) < \infty$ and for any $f \in L^1(\omega)$, we denote by $m(f,\omega) = \frac{1}{\mathcal{L}^N(\omega)}\int_\omega f(x)\,dx$ the mean value of f on ω.

Let x_0 be such that $B(x_0, 4R) \subset \Omega$. Using the Poincaré inequality and (3.1) we have

$$||v_h - m(v_h, B(x_0,4R))||_{L^2(B(x_0,4R))} \leq C_PR||\nabla v_h||_{L^2(B(x_0,4R))} \leq C_PR\left(E_{GL}^\Omega(u)\right)^{\frac{1}{2}}. \tag{3.16}$$

We claim that there exist $k \in \mathbf{N}$, depeding only on N, and $C_* = C_*(a, r_0, N, h, R)$ such that

$$||v_h - m(v_h, B(x_0,4R))||_{W^{2,N}(B(x_0,\frac{R}{2^{k-2}}))} \leq C_*\left(\left(E_{GL}^\Omega(u)\right)^{\frac{1}{2}} + \left(E_{GL}^\Omega(u)\right)^{\frac{1}{N}}\right). \tag{3.17}$$

It is well-known (see Theorem 9.11 p. 235 in [19]) that for $p \in (1,\infty)$ there exists $C = C(N,r,p) > 0$ such that for any $w \in W^{2,p}(B(a,2r))$ we have

$$||w||_{W^{2,p}(B(a,r))} \leq C\left(||w||_{L^p(B(a,2r))} + ||\Delta w||_{L^p(B(a,2r))}\right). \tag{3.18}$$

From (3.15), (3.16) and (3.18) we infer that

$$(3.19) \qquad ||v_h - m(v_h, B(x_0, 4R))||_{W^{2,2}(B(x_0,2R))} \le C(a, r_0, N, h, R) \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}.$$

If $\frac{1}{2} - \frac{2}{N} \le \frac{1}{N}$, from (3.19) and the Sobolev embedding we find

$$(3.20) \qquad ||v_h - m(v_h, B(x_0, 4R))||_{L^N(B(x_0,2R))} \le C(a, r_0, N, h, R) \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}.$$

Then using (3.15) (for $p = N$), (3.20) and (3.18) we infer that (3.17) holds for $k = 2$.

If $\frac{1}{2} - \frac{2}{N} > \frac{1}{N}$, (3.19) and the Sobolev embedding imply

$$(3.21) \qquad ||v_h - m(v_h, B(x_0, 4R))||_{L^{p_1}(B(x_0,2R))} \le C(a, r_0, N, h, R) \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}},$$

where $\frac{1}{p_1} = \frac{1}{2} - \frac{2}{N}$. Then (3.21), (3.15) and (3.18) give

$$(3.22) \quad ||v_h - m(v_h, B(x_0, 4R))||_{W^{2,p_1}(B(x_0,R))} \le C(a, r_0, N, h, R) \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}} \right).$$

If $\frac{1}{p_1} - \frac{2}{N} \le \frac{1}{N}$, using (3.22), the Sobolev embedding, (3.15) and (3.18) we get

$$||v_h - m(v_h, B(x_0, 4R))||_{W^{2,N}(B(x_0,\frac{R}{2}))} \le C(a, r_0, N, h, R) \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}} \right);$$

otherwise we repeat the process. After a finite number of steps we find $k \in \mathbf{N}$ such that (3.17) holds.

We will use the following variant of the Gagliardo-Nirenberg inequality:

$$(3.23) \qquad ||w - m(w, B(a, r))||_{L^p(B(a,r))} \le C(p, q, N, r) ||w||_{L^q(B(a,2r))}^{\frac{q}{p}} ||\nabla w||_{L^N(B(a,2r))}^{1-\frac{q}{p}}$$

for any $w \in W^{1,N}(B(a, 2r))$, where $1 \le q \le p < \infty$ (see, e.g., [26] p. 78).

Using (3.23) with $w = \nabla v_h$ and (3.17) we find

$$(3.24) \qquad \begin{aligned} &||\nabla v_h - m(\nabla v_h, B(x_0, \tfrac{R}{2^{k-1}}))||_{L^p(B(x_0,\frac{R}{2^{k-1}}))} \\ &\le C ||\nabla v_h||_{L^2(B(x_0,\frac{R}{2^{k-2}}))}^{\frac{2}{p}} ||\nabla^2 v_h||_{L^N(B(x_0,\frac{R}{2^{k-2}}))}^{1-\frac{2}{p}} \\ &\le C \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{p}} \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}} \right)^{1-\frac{2}{p}} \end{aligned}$$

for any $p \in [2, \infty)$, where the constants depend only on a, r_0, N, p, h, R.

Using the Cauchy-Schwarz inequality and (3.1) we have

$$\left| m(\nabla v_h, B(x_0, \frac{R}{2^{k-1}})) \right| \le \mathcal{L}^N(B(x_0, \frac{R}{2^{k-1}}))^{-\frac{1}{2}} ||\nabla v_h||_{L^2(B(x_0,\frac{R}{2^{k-1}}))} \le C \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}$$

and we infer that for any $p \in [1, \infty]$ we have the estimate

$$(3.25) \qquad \begin{aligned} &||m(\nabla v_h, B(x_0, \tfrac{R}{2^{k-1}}))||_{L^p(B(x_0,\frac{R}{2^{k-1}}))} \\ &\le \left| m(\nabla v_h, B(x_0, \tfrac{R}{2^{k-1}})) \right| \left(\mathcal{L}^N(B(x_0, \tfrac{R}{2^{k-1}})) \right)^{\frac{1}{p}} \le C(N, p, R) \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}}. \end{aligned}$$

From (3.24) and (3.25) we obtain for any $p \in [2, \infty)$,

$$(3.26) \qquad ||\nabla v_h||_{L^p(B(x_0,\frac{R}{2^{k-1}}))} \le C(a, r_0, N, p, h, R) \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{p}+\frac{1}{N}(1-\frac{2}{p})} \right).$$

We will use the Morrey inequality which asserts that, for any $w \in C^0 \cap W^{1,p}(B(x_0, r))$ with $p > N$ we have

$$|w(x) - w(y)| \le C(p, N)|x - y|^{1-\frac{N}{p}} ||\nabla w||_{L^p(B(x_0,r))} \qquad \text{for any } x, y \in B(x_0, r)) \tag{3.27}$$

(see, e.g., the proof of Theorem IX.12 p. 166 in [8]). Using (3.26) and the Morrey's inequality (3.27) for $p = 2N$ we get

$$|v_h(x) - v_h(y)| \le C(a, r_0, N, h, R)|x - y|^{\frac{1}{2}} \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}(1+\frac{1}{2^*})} \right) \tag{3.28}$$

for any $x, y \in B(x_0, \frac{R}{2^{k-1}})$).

Let $\delta > 0$ and assume that there exists $x_0 \in \Omega$ such that $|\,|v_h(x_0) - r_0| - r_0| \ge \delta$ and $B(x_0, 4R) \subset \Omega$. Since $\big|\,|\,|v_h(x) - r_0| - r_0| - |\,|v_h(y) - r_0| - r_0|\,\big| \le |v_h(x) - v_h(y)|$, from (3.28) we infer that

$$|\,|v_h(x) - r_0| - r_0| \ge \frac{\delta}{2} \qquad \text{for any } x \in B(x_0, r_\delta),$$

where

$$r_\delta = \min \left(\frac{R}{2^{k-1}}, \left(\frac{\delta}{2C(a, r_0, N, h, R)} \right)^2 \left(\left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{2}} + \left(E_{GL}^{\Omega}(u)\right)^{\frac{1}{N}(1+\frac{1}{2^*})} \right)^{-2} \right). \tag{3.29}$$

Let

$$\eta(s) = \inf\{(\varphi^2(\tau) - r_0^2)^2 \mid \tau \in (-\infty, r_0 - s] \cup [r_0 + s, \infty)\}. \tag{3.30}$$

It is clear that η is nondecreasing and positive on $(0, \infty)$. We have:

$$\begin{aligned} &E_{GL}^{\Omega}(u) \ge E_{GL}^{\Omega}(v_h) \ge a^2 \int_{B(x_0,r_\delta)} \left(\varphi^2(|r_0 - v_h|) - r_0^2\right)^2 dx \\ &\ge a^2 \int_{B(x_0,r_\delta)} \eta(\tfrac{\delta}{2})\, dx = \mathcal{L}^N(B(0,1))a^2\eta(\tfrac{\delta}{2})r_\delta^N, \end{aligned} \tag{3.31}$$

where r_δ is given by (3.29). It is obvious that there exists a constant $K > 0$, depending only on a, r_0, N, h, R, δ such that (3.31) cannot hold for $E_{GL}^{\Omega}(u) \le K$. We infer that $|\,|v_h(x_0) - r_0| - r_0| < \delta$ if $B(x_0, 4R) \subset \Omega$ and $E_{GL}^{\Omega}(u) \le K$. This completes the proof of Lemma 3.1. □

Lemma 3.2 *Let $(u_n)_{n\ge1} \subset \mathcal{X}$ be a sequence of functions satisfying:*

a) $E_{GL}(u_n)$ is bounded and

b) $\lim_{n\to\infty} \left(\sup_{y\in\mathbf{R}^N} E_{GL}^{B(y,1)}(u_n) \right) = 0.$

There exists a sequence $h_n \longrightarrow 0$ such that for any minimizer v_n of $G_{h_n,\mathbf{R}^N}^{u_n}$ in $H_{u_n}^1(\mathbf{R}^N)$ we have $||\,|v_n - r_0| - r_0||_{L^\infty(\mathbf{R}^N)} \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. Let $M = \sup_{n\ge1} E_{GL}(u_n)$. For $n \ge 1$ and $x \in \mathbf{R}^N$ we denote

$$m_n(x) = m(u_n, B(x, 1)) = \frac{1}{\mathcal{L}^N(B(0,1))} \int_{B(x,1)} u_n(y)\, dy.$$

By the Poincaré inequality, there exists $C_0 > 0$ such that

$$\int_{B(x,1)} |u_n(y) - m_n(x)|^2\, dy \ \le C_0 \int_{B(x,1)} ||\nabla u_n(y)||^2\, dy.$$

From (b) it follows that

$$\sup_{x\in\mathbf{R}^N} ||u_n - m_n(x)||_{L^2(B(x,1))} \longrightarrow 0 \qquad \text{as } x \longrightarrow \infty. \tag{3.32}$$

Let H be as in Lemma 3.1 (iii). From (3.12) and (b) we get

$$\sup_{x\in\mathbf{R}^N} ||H(u_n)||^2_{L^2(B(x,1))} \le \sup_{x\in\mathbf{R}^N} 9r_0^2 \int_{B(x,1)} \left(\varphi^2(|r_0 - u_n(y)|) - r_0^2\right)^2 dy \longrightarrow 0 \tag{3.33}$$

as $n \longrightarrow \infty$. It is obvious that H is Lipschitz on $\mathbf{C}$. Using (3.32) we find

$$\sup_{x\in\mathbf{R}^N} ||H(u_n) - H(m_n(x))||_{L^2(B(x,1))} \le C_1 \sup_{x\in\mathbf{R}^N} ||u_n - m_n(x)||_{L^2(B(x,1))} \longrightarrow 0 \tag{3.34}$$

as $n \longrightarrow \infty$. From (3.33) and (3.34) we infer that $\sup_{x\in\mathbf{R}^N} ||H(m_n(x))||_{L^2(B(x,1))} \longrightarrow 0$ as $n \longrightarrow \infty$. Since $||H(m_n(x))||_{L^2(B(x,1))} = \mathcal{L}^N(B(0,1)|H(m_n(x))|$, we have proved that

$$\lim_{n\to\infty} \sup_{x\in\mathbf{R}^N} |H(m_n(x))| = 0. \tag{3.35}$$

Let

$$h_n = \max\left(\left(\sup_{x\in\mathbf{R}^N} ||u_n - m_n(x)||_{L^2(B(x,1))} \right)^{\frac{1}{N+2}}, \left(\sup_{x\in\mathbf{R}^N} |H(m_n(x))| \right)^{\frac{1}{N}} \right). \tag{3.36}$$

From (3.32) and (3.35) it follows that $h_n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus we may assume that $0 < h_n < 1$ for any n (if $h_n = 0$, we see that u_n is constant a.e. and there is nothing to prove). Let v_n be a minimizer of $G^{u_n}_{h_n,\mathbf{R}^N}$ (such minimizers exist by Lemma 3.1 (i)). It follows from Lemma 3.1 (iii) that v_n satisfies (3.5). We will prove that there exist $R_N > 0$ and $C > 0$, independent on n, such that

$$||\Delta v_n||_{L^N(B(x,R_N))} \le C \qquad \text{for any } x \in \mathbf{R}^N \text{ and } n \in \mathbf{N}^*. \tag{3.37}$$

Clearly, it suffices to prove (3.37) for $x = 0$. We denote $m_n = m_n(0)$ and $\tilde\varphi(s) = \varphi(\frac{s}{32r_0})$. Then (3.5) can be written as

$$-\Delta v_n + \frac{1}{h_n^2}\tilde\varphi'(|v_n - m_n|^2)(v_n - m_n) = f_n, \tag{3.38}$$

where

$$\begin{aligned} f_n &= -2a^2\left(H(v_n) - H(m_n)\right) - 2a^2 H(m_n) \\ &\quad + \tfrac{1}{h_n^2}\left(\tilde\varphi'(|v_n - m_n|^2)(v_n - m_n) - \tilde\varphi'(|v_n - u_n|^2)(v_n - u_n)\right). \end{aligned} \tag{3.39}$$

In view of Lemma 3.1 (iii), equality (3.38) holds in $L^p_{loc}(\mathbf{R}^N)$ (and not only in $\mathcal{D}'(\mathbf{R}^N)$).

The function $z \longmapsto \tilde\varphi'(|z|^2)z$ belongs to $C_c^\infty(\mathbf{C})$ and consequently it is Lipschitz. Using (3.36), we see that there exists $C_2 > 0$ such that

$$\begin{aligned} &||\tilde\varphi'(|v_n - m_n|^2)(v_n - m_n) - \tilde\varphi'(|v_n - u_n|^2)(v_n - u_n)||_{L^2(B(0,1))} \\ &\le C_2||u_n - m_n||_{L^2(B(0,1))} \le C_2 h_n^{N+2}. \end{aligned} \tag{3.40}$$

By (3.36) we have also $||H(m_n)||_{L^2(B(0,1))} = \left(\mathcal{L}^N(B(0,1)\right)^{\frac{1}{2}}|H(m_n)| \le \left(\mathcal{L}^N(B(0,1)\right)^{\frac{1}{2}} h_n^N$. From this estimate, (3.39), (3.40) and the fact that H is Lipschitz we get

$$||f_n||_{L^2(B(0,R))} \le C_3||v_n - m_n||_{L^2(B(0,R))} + C_4 h_n^N \qquad \text{for any } R \in (0,1]. \tag{3.41}$$

Let $\chi \in C_c^\infty(\mathbf{R}^N, \mathbf{R})$. Taking the scalar product (in $\mathbf{C}$) of (3.38) by $\chi(x)(v_n(x) - m_n)$ and integrating by parts we find

$$\begin{aligned}(3.42)\qquad &\int_{\mathbf{R}^N} \chi |\nabla v_n|^2 \, dx + \frac{1}{h_n^2}\int_{\mathbf{R}^N} \chi \tilde{\varphi}'(|v_n - m_n|^2)|v_n - m_n|^2 \, dx \\ &= \frac{1}{2}\int_{\mathbf{R}^N} (\Delta \chi)|v_n - m_n|^2 \, dx + \int_{\mathbf{R}^N} \langle f_n(x), v_n(x) - m_n\rangle \chi(x) \, dx.\end{aligned}$$

From (3.2) we have $||v_n - u_n||_{L^2(\mathbf{R}^N)} \le C_5 h_n^{\frac{2}{N}}$, thus

$$(3.43)\qquad ||v_n - m_n||_{L^2(B(0,1))} \le ||v_n - u_n||_{L^2(B(0,1))} + ||u_n - m_n||_{L^2(B(0,1))} \le K_0 h_n^{\frac{2}{N}}.$$

We prove that

$$(3.44)\qquad ||v_n - m_n||_{L^2(B(0,\frac{1}{2^{j-1}}))} \le K_j h_n^{\frac{2j}{N}} \qquad \text{for } 1 \le j \le \left[\frac{N^2}{2}\right] + 1,$$

where K_j does not depend on n. We proceed by induction. From (3.43) it follows that (3.44) is true for $j = 1$.

Assume that (3.44) holds for some $j \in \mathbf{N}^*$, $j \le \left[\frac{N^2}{2}\right]$. Let $\chi_j \in C_c^\infty(\mathbf{R}^N)$ be a real-valued function such that $0 \le \chi_j \le 1$, $\text{supp}(\chi_j) \subset B(0, \frac{1}{2^{j-1}})$ and $\chi_j = 1$ on $B(0, \frac{1}{2^j})$. Replacing χ by χ_j in (3.42), then using the Cauchy-Schwarz inequality and (3.41) we find

$$\begin{aligned}(3.45)\qquad &\int_{B(0,\frac{1}{2^j})} |\nabla v_n|^2 \, dx + \frac{1}{h_n^2}\int_{B(0,\frac{1}{2^j})} \tilde{\varphi}'(|v_n - m_n|^2)|v_n - m_n|^2 \, dx \\ &\le \tfrac{1}{2}||\Delta \chi_j||_{L^\infty(\mathbf{R}^N)}||v_n - m_n||^2_{L^2(B(0,\frac{1}{2^{j-1}}))} + ||f_n||_{L^2(B(0,\frac{1}{2^{j-1}}))}||v_n - m_n||_{L^2(B(0,\frac{1}{2^{j-1}}))} \\ &\le A_j ||v_n - m_n||^2_{L^2(B(0,\frac{1}{2^{j-1}}))} + C_4 h_n^N ||v_n - m_n||_{L^2(B(0,\frac{1}{2^{j-1}}))} \le A_j' h_n^{\frac{4j}{N}}.\end{aligned}$$

From (3.44) and (3.45) we infer that $||v_n - m_n||_{H^1 B(0,\frac{1}{2^j})} \le B_j h_n^{\frac{2j}{N}}$. Then the Sobolev embedding implies

$$(3.46)\qquad ||v_n - m_n||_{L^{2^*} B(0,\frac{1}{2^j})} \le D_j h_n^{\frac{2j}{N}}.$$

The function $z \longmapsto \tilde{\varphi}(|z|^2)$ is clearly Lipschitz on $\mathbf{C}$, thus we have

$$\begin{aligned}&\int_{B(0,1)} |\tilde{\varphi}(|v_n - u_n|^2) - \tilde{\varphi}(|v_n - m_n|^2)| \, dx \le C_6' \int_{B(0,1)} |u_n - m_n| \, dx \\ &\le C_6 ||u_n - m_n||_{L^2(B(0,1))} \le C_6 h_n^{N+2}.\end{aligned}$$

It is clear that $\displaystyle\int_{B(0,1)} \tilde{\varphi}(|v_n - u_n|^2) \, dx \le h_n^2 G^{u_n}_{h_n,\mathbf{R}^N}(v_n) \le h_n^2 E_{GL}(u_n) \le h_n^2 M$ and we obtain

$$(3.47)\qquad \int_{B(0,1)} \tilde{\varphi}(|v_n - m_n|^2) \, dx \le C_7 h_n^2.$$

If $|v_n(x) - m_n| \ge 8r_0$ we have $\tilde{\varphi}(|v_n(x) - m_n|^2) = \varphi\left(\frac{|v_n(x) - m_n|^2}{32 r_0}\right) \ge 2r_0$, hence

$$(3.48)\qquad 2r_0 \mathcal{L}^N(\{x \in B(0,1) \mid |v_n(x) - m_n| \ge 8r_0\}) \le \int_{B(0,1)} \tilde{\varphi}\left(|v_n - m_n|^2\right) \, dx \le C_7 h_n^2.$$

By Hölder's inequality, (3.46) and (3.48) we have

$$\begin{aligned}&\int_{\{|v_n-m_n|\geq 8r_0\}\cap B(0,\frac{1}{2^j})} |v_n-m_n|^2\,dx\\ &\leq ||v_n-m_n||^2_{L^{2^*}B(0,\frac{1}{2^j})}\left(\mathcal{L}^N(\{x\in B(0,1)\mid |v_n(x)-m_n|\geq 8r_0\})\right)^{1-\frac{2}{2^*}}\\ &\leq \left(D_j h_n^{\frac{2j}{N}}\right)^2\left(\frac{C_7}{2r_0}h_n^2\right)^{1-\frac{2}{2^*}}\leq E_j h_n^{\frac{4j+4}{N}}.\end{aligned}\tag{3.49}$$

From (3.45) it follows that

$$\begin{aligned}&\int_{\{|v_n-m_n|<8r_0\}\cap B(0,\frac{1}{2^j})} |v_n-m_n|^2\,dx \leq \int_{B(0,\frac{1}{2^j})}\tilde{\varphi}'(|v_n-m_n|^2)|v_n-m_n|^2\,dx\\ &\leq A'_j h_n^{2+\frac{4j}{N}}\leq A'_j h_n^{\frac{4j+4}{N}}.\end{aligned}\tag{3.50}$$

Then (3.49) and (3.50) imply that (3.44) holds for $j+1$ and the induction is complete. Thus (3.44) is established. Denoting $j_N=\left[\frac{N^2}{2}\right]+1$ and $R_N=\frac{1}{2^{j_N-1}}$, we have proved that

$$||v_n-m_n||_{L^2(B(0,R_N))}\leq K_{j_N}h_n^{\frac{2j_N}{N}}\leq K_{j_N}h_n^N.\tag{3.51}$$

It follows that

$$\begin{aligned}&\int_{B(0,R_N)}\left|\frac{1}{h_n^2}\tilde{\varphi}'(|v_n-m_n|^2)(v_n-m_n)\right|^N dx\\ &\leq \frac{1}{h_n^{2N}}\sup_{z\in\mathbf{C}}\left|\tilde{\varphi}'\left(|z|^2\right)z\right|^{N-2}\int_{B(0,R_N)}|v_n-m_n|^2\,dx\leq C_8.\end{aligned}\tag{3.52}$$

Arguing as in (3.40) and using (3.36) we get

$$\begin{aligned}&||\tilde{\varphi}'(|v_n-m_n|^2)(v_n-m_n)-\tilde{\varphi}'(|v_n-u_n|^2)(v_n-u_n)||^N_{L^N(B(0,1))}\\ &\leq C_9\sup_{z\in\mathbf{C}}\left|\tilde{\varphi}'\left(|z|^2\right)z\right|^{N-2}||u_n-m_n||^2_{L^2(B(0,1)}\leq C_{10}h_n^{2N+4}.\end{aligned}\tag{3.53}$$

From (3.39), (3.53) and the fact that H is bounded on $\mathbf{C}$ it follows that $||f_n||_{L^N(B(0,R_N))}\leq C_{11}$, where C_{11} does not depend on n. Using this estimate, (3.52) and (3.38), we infer that (3.37) holds.

Since any ball of radius 1 can be covered by a finite number of balls of radius R_N, it follows that there exists $C>0$ such that

$$||\Delta v_n||_{L^N(B(x,1))}\leq C\qquad\text{for any } x\in\mathbf{R}^N \text{ and } n\in\mathbf{N}^*.\tag{3.54}$$

We will use (3.18) and (3.54) to prove that there exist $\tilde{R}_N\in(0,1]$ and $C>0$ such that

$$||v_n-m_n(x)||_{W^{2,N}(B(x,\tilde{R}_N))}\leq C\qquad\text{for any } x\in\mathbf{R}^N \text{ and } n\in\mathbf{N}^*.\tag{3.55}$$

As previously, it suffices to prove (3.55) for $x_0=0$. From (3.54) and Hölder's inequality it follows that for $1\leq p\leq N$ we have

$$||\Delta v_n||_{L^p(B(x,1))}\leq\left(\mathcal{L}^N(B(0,1))\right)^{1-\frac{p}{N}}||\Delta v_n||^{\frac{p}{N}}_{L^N(B(x,1))}\leq C(p).\tag{3.56}$$

Using (3.43), (3.54) and (3.18) we obtain

$$||v_n-m_n(0)||_{W^{2,2}(B(x,\frac{1}{2}))}\leq C.\tag{3.57}$$

If $\frac{1}{2} - \frac{2}{N} \leq \frac{1}{N}$, (3.57) and the Sobolev embedding give

$$||v_n - m_n(0)||_{L^N(B(x,\frac{1}{2}))} \leq C,$$

and this estimate together with (3.54) and (3.18) imply that (3.55) holds for $\tilde{R}_N = \frac{1}{4}$.

If $\frac{1}{2} - \frac{2}{N} > \frac{1}{N}$, from (3.57) and the Sobolev embedding we find $||v_n - m_n(0)||_{L^{p_1}(B(x,\frac{1}{2}))} \leq C$, where $\frac{1}{p_1} = \frac{1}{2} - \frac{2}{N}$. This estimate, (3.56) and (3.18) imply $||v_n - m_n(0)||_{W^{2,p_1}(B(x,\frac{1}{4}))} \leq C$. If $\frac{1}{p_1} - \frac{2}{N} \leq \frac{1}{N}$, from the Sobolev embedding we obtain $||v_n - m_n(0)||_{L^N(B(x,\frac{1}{4}))} \leq C$, and then using (3.54) and (3.18) we infer that (3.55) holds for $\tilde{R}_N = \frac{1}{8}$. Otherwise we repeat the above argument. After a finite number of steps we see that (3.55) holds.

Next we proceed as in the proof of Lemma 3.1 (iv). By (3.23) and (3.55) we have for $p \in [2, \infty)$ and any $x_0 \in \mathbf{R}^N$,

$$\begin{aligned}
&||\nabla v_n - m(\nabla v_n, B(x_0, \tfrac{1}{2}\tilde{R}_N))||_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))} \\
&\leq C||\nabla v_n||^{\frac{2}{p}}_{L^2(B(x_0,\tilde{R}_N))} ||\nabla^2 v_n||^{1-\frac{2}{p}}_{L^N(B(x_0,\tilde{R}_N))} \leq C_1(p).
\end{aligned} \tag{3.58}$$

Arguing as in (3.25) we see that $||m(\nabla v_n, B(x_0, \frac{1}{2}\tilde{R}_N))||_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))}$ is bounded independently on n and hence

$$||\nabla v_n||_{L^p(B(x_0,\frac{1}{2}\tilde{R}_N))} \leq C_2(p) \qquad \text{for any } n \in \mathbf{N}^* \text{ and } x_0 \in \mathbf{R}^N.$$

Using this estimate for $p = 2N$ together with the Morrey inequality (3.27), we see that there exists $C_* > 0$ such that for any $x, y \in \mathbf{R}^N$ with $|x - y| \leq \frac{\tilde{R}_N}{2}$ and any $n \in \mathbf{N}^*$ we have

$$|v_n(x) - v_n(y)| \leq C_*|x - y|^{\frac{1}{2}}. \tag{3.59}$$

Let $\delta_n = ||\,|v_n - r_0| - r_0||_{L^\infty(\mathbf{R}^N)}$ and choose $x_n \in \mathbf{R}^N$ such that $|\,|v_n(x_n) - r_0| - r_0| \geq \frac{\delta_n}{2}$. From (3.59) it follows that $|\,|v_n(x) - r_0| - r_0| \geq \frac{\delta_n}{4}$ for any $x \in B(x_n, r_n)$, where

$$r_n = \min\left(\frac{\tilde{R}_N}{2}, \left(\frac{\delta_n}{4C_*}\right)^2\right).$$

Then we have

$$\begin{aligned}
&\int_{B(x_n,1)} \left(\varphi^2(|r_0 - v_n(y)|) - r_0^2\right)^2 dy \geq \int_{B(x_n,r_n)} \left(\varphi^2(|r_0 - v_n(y)|) - r_0^2\right)^2 dy \\
&\geq \int_{B(x_n,r_n)} \eta\left(\tfrac{\delta_n}{4}\right) dy = \mathcal{L}^N(B(0,1)\eta\left(\tfrac{\delta_n}{4}\right) r_n^N,
\end{aligned} \tag{3.60}$$

where η is as in (3.30).

On the other hand, the function $z \longmapsto \left(\varphi^2(|r_0 - z|) - r_0^2\right)^2$ is Lipschitz on $\mathbf{C}$. Using this fact, the Cauchy-Schwarz inequality, (3.2) and assumption (a) we get

$$\begin{aligned}
&\int_{B(x,1)} \left| \left(\varphi^2(|r_0 - v_n(y)|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - u_n(y)|) - r_0^2\right)^2 \right| dy \\
&\leq C \int_{B(x,1)} |v_n(y) - u_n(y)|\, dy \leq C'||v_n - u_n||_{L^2(B(x,1))} \leq C'||v_n - u_n||_{L^2(\mathbf{R}^N)} \leq C'' h_n^{\frac{2}{N}}.
\end{aligned}$$

Then using assumption (b) we infer that

$$\sup_{x \in \mathbf{R}^N} \int_{B(x,1)} \left(\varphi^2(|r_0 - v_n(y)|) - r_0^2\right)^2 dy \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.61}$$

From (3.60) and (3.61) we get $\lim_{n\to\infty} \eta\left(\frac{\delta_n}{4}\right) r_n^N = 0$ and this clearly implies $\lim_{n\to\infty} \delta_n = 0$. Lemma 3.2 is thus proved. □

The next result is based on Lemma 3.1 and will be very useful in the next sections to prove the "concentration" of minimizing sequences. For $0 < R_1 < R_2$ we denote $\Omega_{R_1,R_2} = B(0,R_2) \setminus \overline{B}(0,R_1)$.

Lemma 3.3 *Let $A > A_3 > A_2 > 1$. There exist $\varepsilon_0 = \varepsilon_0(a,r_0,N,A,A_2,A_3) > 0$ and $C_i = C_i(a,r_0,N,A,A_2,A_3) > 0$ such that for any $R \geq 1$, $\varepsilon \in (0,\varepsilon_0)$ and $u \in \mathcal{X}$ verifying $E_{GL}^{\Omega_{AR,R}}(u) \leq \varepsilon$, there exist two functions u_1, $u_2 \in \mathcal{X}$ and a constant $\theta_0 \in [0,2\pi)$ satisfying the following properties:*

i) $\mathrm{supp}(u_1) \subset B(0,A_2R)$ *and* $r_0 - u_1 = e^{-i\theta_0}(r_0 - u)$ *on* $B(0,R)$,

ii) $u_2 = u$ *on* $\mathbf{R}^N \setminus B(0,AR)$ *and* $r_0 - u_2 = r_0 e^{i\theta_0} =$ *constant on* $B(0,A_3R)$,

iii) $\displaystyle\int_{\mathbf{R}^N} \left| \left|\frac{\partial u}{\partial x_j}\right|^2 - \left|\frac{\partial u_1}{\partial x_j}\right|^2 - \left|\frac{\partial u_2}{\partial x_j}\right|^2 \right| dx \leq C_1\varepsilon$ *for* $j = 1,\dots,N$,

iv) $\displaystyle\int_{\mathbf{R}^N} \left| \left(\varphi^2(|r_0-u|) - r_0^2\right)^2 - \left(\varphi^2(|r_0-u_1|) - r_0^2\right)^2 - \left(\varphi^2(|r_0-u_2|) - r_0^2\right)^2 \right| dx \leq C_2\varepsilon$,

v) $|Q(u) - Q(u_1) - Q(u_2)| \leq C_3\varepsilon$,

vi) If assumptions (A1) and (A2) in the introduction hold, then

$$\int_{\mathbf{R}^N} \left|V(|r_0-u|^2) - V(|r_0-u_1|^2) - V(|r_0-u_2|^2)\right| dx \leq C_4\varepsilon + C_5\sqrt{\varepsilon}\left(E_{GL}(u)\right)^{\frac{2^*-1}{2}}.$$

Proof. Fix $k > 0$, A_1 and A_4 such that $1 + 4k < A_1 < A_2 < A_3 < A_4 < A - 4k$. Let $h = 1$ and $\delta = \frac{r_0}{2}$. We will prove that Lemma 3.3 holds for $\varepsilon_0 = K(a,r_0,N,h=1,\delta=\frac{r_0}{2},k)$, where $K(a,r_0,N,h,\delta,R)$ is as in Lemma 3.1 (iv).

Consider $\eta_1, \eta_2 \in C^\infty(\mathbf{R})$ satisfying the following properties:

$$\begin{array}{lll} \eta_1 = 1 \text{ on } (-\infty, A_1], & \eta_1 = 0 \text{ on } [A_2,\infty), & \eta_1 \text{ is nonincreasing,} \\ \eta_2 = 0 \text{ on } (-\infty, A_3], & \eta_2 = 1 \text{ on } [A_4,\infty), & \eta_2 \text{ is nondecreasing.} \end{array}$$

Let $\varepsilon < \varepsilon_0$ and let $u \in \mathcal{X}$ be such that $E_{GL}^{\Omega_{R,AR}}(u) \leq \varepsilon$. Let v_1 be a minimizer of $G^u_{1,\Omega_{R,AR}}$ in the space $H^1_u(\Omega_{R,AR})$. The existence of v_1 is guaranteed by Lemma 3.1. We also know that $v_1 \in W^{2,p}_{loc}(\Omega_{R,AR})$ for any $p \in [1,\infty)$. Moreover, since $E_{GL}^{\Omega_{R,AR}}(u) \leq K(a,r_0,N,1,\frac{r_0}{2},k)$, Lemma 3.1 (iv) implies that

$$\frac{r_0}{2} < |r_0 - v_1(x)| < \frac{3r_0}{2} \qquad \text{if } R + 4k \leq |x| \leq AR - 4k. \tag{3.62}$$

Since $N \geq 3$, Ω_{A_1R,A_4R} is simply connected and it follows directly from Theorem 3 p. 38 in [9] that there exist two real-valued functions ρ, $\theta \in W^{2,p}(\Omega_{A_1R,A_4R})$, $1 \leq p < \infty$, such that

$$r_0 - v_1(x) = \rho(x)e^{i\theta(x)} \qquad \text{on } \Omega_{A_1R,A_4R}. \tag{3.63}$$

For $j = 1,\dots,N$ we have

$$\frac{\partial v_1}{\partial x_j} = \left(-\frac{\partial \rho}{\partial x_j} - i\rho\frac{\partial \theta}{\partial x_j}\right)e^{i\theta} \quad \text{and} \quad \left|\frac{\partial v_1}{\partial x_j}\right|^2 = \left|\frac{\partial \rho}{\partial x_j}\right|^2 + \rho^2\left|\frac{\partial \theta}{\partial x_j}\right|^2 \quad \text{a.e. on } \Omega_{A_1R,A_4R}. \tag{3.64}$$

Thus we get the following estimates:

$$\int_{\Omega_{A_1R,\,A_4R}} |\nabla\rho|^2\,dx \leq \int_{\Omega_{A_1R,\,A_4R}} |\nabla v_1|^2\,dx \leq \varepsilon, \tag{3.65}$$

$$a^2 \int_{\Omega_{A_1R,\,A_4R}} \left(\rho^2 - r_0^2\right)^2 \, dx \le E_{GL}^{\Omega_{A_1R,\,A_4R}}(v_1) \le \varepsilon, \tag{3.66}$$

$$\int_{\Omega_{A_1R,\,A_4R}} |\nabla\theta|^2 \, dx \le \frac{4}{r_0^2} \int_{\Omega_{A_1R,\,A_4R}} |\nabla v_1|^2 \, dx \le \frac{4}{r_0^2}\varepsilon. \tag{3.67}$$

The Poincaré inequality and a scaling argument imply that

$$\int_{\Omega_{A_1R,\,A_4R}} |f - m(f, \Omega_{A_1R,\,A_4R})|^2 \, dx \le C(N, A_1, A_4) R^2 \int_{\Omega_{A_1R,\,A_4R}} |\nabla f|^2 \, dx \tag{3.68}$$

for any $f \in H^1(\Omega_{A_1R,\,A_4R})$, where $C(N, A_1, A_4)$ does not depend on R. Let $\theta_0 = m(\theta, \Omega_{A_1R,\,A_4R})$. We may assume that $\theta_0 \in [0, 2\pi)$ (otherwise we replace θ by $\theta - 2\pi\left[\frac{\theta}{2\pi}\right]$). Using (3.67) and (3.68) we get

$$\int_{\Omega_{A_1R,\,A_4R}} |\theta - \theta_0|^2 \, dx \le C(r_0, N, A_1, A_4) R^2 \int_{\Omega_{A_1R,\,A_4R}} |\nabla v_1|^2 \, dx \le C(r_0, N, A_1, A_4) R^2 \varepsilon. \tag{3.69}$$

We define $\tilde{u}_1$ and u_2 by

$$r_0 - \tilde{u}_1(x) = \begin{cases} r_0 - u(x) & \text{if } x \in \overline{B}(0,R), \\ r_0 - v_1(x) & \text{if } x \in B(0, A_1R) \setminus \overline{B}(0,R), \\ \left(r_0 + \eta_1(\frac{|x|}{R})(\rho(x) - r_0)\right) e^{i\left(\theta_0 + \eta_1(\frac{|x|}{R})(\theta(x) - \theta_0)\right)} & \\ \qquad\qquad \text{if } x \in B(0, A_4R) \setminus B(0, A_1R), & \\ r_0 e^{i\theta_0} & \text{if } x \in \mathbf{R}^N \setminus B(0, A_4R), \end{cases} \tag{3.70}$$

$$r_0 - u_2(x) = \begin{cases} r_0 e^{i\theta_0} & \text{if } x \in \overline{B}(0, A_1R), \\ \left(r_0 + \eta_2(\frac{|x|}{R})(\rho(x) - r_0)\right) e^{i\left(\theta_0 + \eta_2(\frac{|x|}{R})(\theta(x) - \theta_0)\right)} & \\ \qquad\qquad \text{if } x \in B(0, A_4R) \setminus \overline{B}(0, A_1R), & \\ r_0 - v_1(x) & \text{if } x \in B(0, AR) \setminus B(0, A_4R), \\ r_0 - u(x) & \text{if } x \in \mathbf{R}^N \setminus B(0, AR), \end{cases} \tag{3.71}$$

then we define u_1 in such a way that $r_0 - u_1 = e^{-i\theta_0}(r_0 - \tilde{u}_1)$. Since $u \in \mathcal{X}$ and $u - v_1 \in H_0^1(\Omega_{R,\,AR})$, it is clear that $u_1 \in H^1(\mathbf{R}^N)$, $u_2 \in \mathcal{X}$ and (i), (ii) hold.

Since $\rho + r_0 \ge \frac{3}{2} r_0$ on $\Omega_{A_1R,\,A_4R}$, from (3.66) we get

$$||\rho - r_0||^2_{L^2(\Omega_{A_1R,\,A_4R})} \le \frac{4}{9 r_0^2 a^2}\varepsilon. \tag{3.72}$$

Obviously, $\nabla\left(r_0 + \eta_i(\frac{|x|}{R})(\rho(x) - r_0)\right) = \frac{1}{R}\eta_i'(\frac{|x|}{R})(\rho(x) - r_0)\frac{x}{|x|} + \eta_i(\frac{|x|}{R})\nabla\rho$ and using (3.65), (3.72) and the fact that $R \ge 1$ we get

$$\begin{aligned} & ||\nabla\left(r_0 + \eta_i(\tfrac{|x|}{R})(\rho(x) - r_0)\right)||_{L^2(\Omega_{A_1R,\,A_4R})} \\ & \le \tfrac{1}{R}\sup|\eta_i'| \cdot ||\rho - r_0||_{L^2(\Omega_{A_1R,\,A_4R})} + ||\eta_i(\tfrac{|\cdot|}{R})\nabla\rho||_{L^2(\Omega_{A_1R,\,A_4R})} \le C\sqrt{\varepsilon}. \end{aligned} \tag{3.73}$$

Similarly, using (3.67) and (3.69) we find

$$\begin{aligned} & ||\nabla\left(\theta_0 + \eta_i(\tfrac{|x|}{R})(\theta(x) - \theta_0)\right)||_{L^2(\Omega_{A_1R,\,A_4R})} \\ & \le \tfrac{1}{R}\sup|\eta_i'| \cdot ||\theta - \theta_0||_{L^2(\Omega_{A_1R,\,A_4R})} + ||\eta_i(\tfrac{|\cdot|}{R})\nabla\theta||_{L^2(\Omega_{A_1R,\,A_4R})} \le C\sqrt{\varepsilon}. \end{aligned} \tag{3.74}$$

From (3.73), (3.74) and the definition of u_1, u_2 it follows that $||\nabla u_i||_{L^2(\Omega_{A_1 R,\, A_4 R})} \le C\sqrt{\varepsilon}$, $i = 1, 2$. Therefore

$$\int_{\mathbf{R}^N} \left| \left|\frac{\partial u}{\partial x_j}\right|^2 - \left|\frac{\partial u_1}{\partial x_j}\right|^2 - \left|\frac{\partial u_2}{\partial x_j}\right|^2 \right| dx = \int_{\Omega_{R,AR}} \left| \left|\frac{\partial u}{\partial x_j}\right|^2 - \left|\frac{\partial u_1}{\partial x_j}\right|^2 - \left|\frac{\partial u_2}{\partial x_j}\right|^2 \right| dx$$

$$\le \int_{\Omega_{R,A_1R}\cup\Omega_{A_4R,AR}} \left|\frac{\partial u}{\partial x_j}\right|^2 + \left|\frac{\partial v_1}{\partial x_j}\right|^2 dx + \int_{\Omega_{A_1R,A_4R}} \left|\frac{\partial u}{\partial x_j}\right|^2 + \left|\frac{\partial u_1}{\partial x_j}\right|^2 + \left|\frac{\partial u_2}{\partial x_j}\right|^2 dx \le C_1\varepsilon$$

and (iii) is proved. On Ω_{A_1R,A_4R} we have $\rho \in [\frac{r_0}{2}, \frac{3r_0}{2}]$, hence $\varphi\left(r_0 + \eta_i(\frac{|x|}{R})(\rho(x) - r_0)\right) = r_0 + \eta_i(\frac{|x|}{R})(\rho(x) - r_0)$ and

$$\begin{aligned}(3.75)\qquad &\left(\varphi^2\left(r_0 + \eta_i(\tfrac{|x|}{R})(\rho(x) - r_0)\right) - r_0^2\right)^2 = (\rho - r_0)^2\eta_i^2(\tfrac{|x|}{R})\left(2r_0 + \eta_i(\tfrac{|x|}{R})(\rho - r_0)\right)^2 \\ &\le \left(\tfrac{5}{2}r_0\right)^2 (\rho - r_0)^2.\end{aligned}$$

From (3.70)–(3.72) and (3.75) it follows that $||\varphi^2(|r_0 - u_i|) - r_0^2||_{L^2(\Omega_{A_1R,\, A_4R})} \le C\sqrt{\varepsilon}$. As above, we get

$$\int_{\mathbf{R}^N} \left| \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - u_1|) - r_0^2\right)^2 - \left(\varphi^2(|r_0 - u_2|) - r_0^2\right)^2 \right|$$

$$\le \int_{\Omega_{R,A_1R}\cup\Omega_{A_4R,AR}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 + \left(\varphi^2(|r_0 - v_1|) - r_0^2\right)^2 dx$$

$$+ \int_{\Omega_{A_1R,A_4R}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 + \left(\varphi^2(|r_0 - u_1|) - r_0^2\right)^2 + \left(\varphi^2(|r_0 - u_2|) - r_0^2\right)^2 dx \le C_2\varepsilon.$$

This proves (iv).

Next we prove (v). Since $\langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle$ has compact support, a simple computation gives

$$(3.76)\quad Q(u_1) = L(\langle i\frac{\partial u_1}{\partial x_1}, u_1\rangle) = L(\langle ie^{-i\theta_0}\frac{\partial \tilde{u}_1}{\partial x_1}, r_0 - e^{-i\theta_0}r_0 + e^{-i\theta_0}\tilde{u}_1\rangle) = \int_{\mathbf{R}^N} \langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle\, dx.$$

From the definition of $\tilde{u}_1$ and u_2 and the fact that $u = v_1$ on $\mathbf{R}^N \setminus \Omega_{R,AR}$ we get $\langle i\frac{\partial v_1}{\partial x_1}, v_1\rangle - \langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle - \langle i\frac{\partial u_2}{\partial x_1}, u_2\rangle = 0$ a.e. on $\mathbf{R}^N \setminus \Omega_{A_1R,\, A_4R}$. Using this identity, Definition 2.4, (3.76), then (2.3) and (3.70), (3.71) we obtain

$$\begin{aligned}&Q(v_1) - Q(u_1) - Q(u_2) = \int_{\Omega_{A_1R,\, A_4R}} \langle i\frac{\partial v_1}{\partial x_1}, v_1\rangle - \langle i\frac{\partial \tilde{u}_1}{\partial x_1}, \tilde{u}_1\rangle - \langle i\frac{\partial u_2}{\partial x_1}, u_2\rangle\, dx \\ &= \int_{\Omega_{A_1R,\, A_4R}} \langle i\frac{\partial v_1}{\partial x_1} - \frac{\partial \tilde{u}_1}{\partial x_1} - \frac{\partial u_2}{\partial x_1}, r_0\rangle\, dx - \int_{\Omega_{A_1R,\, A_4R}} (\rho^2 - r_0^2)\frac{\partial \theta}{\partial x_1}\, dx \\ (3.77)\qquad &+ \int_{\Omega_{A_1R,\, A_4R}} \sum_{i=1}^{2} \left(\left(r_0 + \eta_i(\frac{|x|}{R})(\rho - r_0)\right)^2 - r_0^2\right) \frac{\partial}{\partial x_1}\left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta - \theta_0)\right) dx \\ &- \int_{\Omega_{A_1R,\, A_4R}} r_0^2 \left(\frac{\partial \theta}{\partial x_1} - \sum_{i=1}^{2} \frac{\partial}{\partial x_1}\left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta(x) - \theta_0)\right)\right) dx.\end{aligned}$$

The functions $v_1 - \tilde{u}_1 - u_2$ and $\theta^* = \theta - \sum_{i=1}^2 \left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta(x) - \theta_0)\right)$ belong to $C^1(\Omega_{R,AR})$ and $v_1 - \tilde{u}_1 - u_2 = r_0(e^{i\theta_0} - 1) = const.$, $\theta^* = -\theta_0 = const.$ on $\Omega_{R,AR} \setminus \Omega_{A_1R,A_4R}$. Therefore

$$\int_{\Omega_{A_1R,\, A_4R}} \langle i \frac{\partial}{\partial x_1}(v_1 - \tilde{u}_1 - u_2), r_0 \rangle \, dx = 0 \qquad \text{and} \qquad \int_{\Omega_{A_1R,\, A_4R}} \frac{\partial \theta^*}{\partial x_1} \, dx = 0. \tag{3.78}$$

Using (3.66), (3.67) and the Cauchy-Schwarz inequality we have

$$\Big| \int_{\Omega_{A_1R,\, A_4R}} (\rho^2 - r_0^2) \frac{\partial \theta}{\partial x_1} \, dx \Big| \leq C\varepsilon. \tag{3.79}$$

Similarly, from (3.72), (3.74), (3.75) and the Cauchy-Schwarz inequality we get

$$\Big| \int_{\Omega_{A_1R,\, A_4R}} \left(\left(r_0 + \eta_i(\frac{|x|}{R})(\rho - r_0) \right)^2 - r_0^2 \right) \frac{\partial}{\partial x_1} \left(\theta_0 + \eta_i(\frac{|x|}{R})(\theta - \theta_0) \right) dx \Big| \leq C\varepsilon. \tag{3.80}$$

From (3.77)–(3.80) we obtain $|Q(v_1) - Q(u_1) - Q(u_2)| \leq C\varepsilon$ and (3.4) gives $|Q(u) - Q(v_1)| \leq CE_{GL}^{\Omega_{R,AR}}(u) \leq C\varepsilon$. These estimates clearly imply (v).

It remains to prove (vi). Assume that (A1) and (A2) are satisfied and let W be as in the introduction. Using (1.5) and (1.7), then Hölder's inequality we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^N} \Big| V(|r_0 - u|^2) - V(|r_0 - v_1|^2) \Big| \, dx \\
&\leq \int_{\Omega_{R,\, AR}} \Big| V(\varphi^2(|r_0 - u|)) - V(\varphi^2(|r_0 - v_1|)) \Big| + \Big| W(|r_0 - u|^2) - W(|r_0 - v_1|^2) \Big| dx \\
&\leq C \int_{\Omega_{R,\, AR}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 + \left(\varphi^2(|r_0 - v_1|) - r_0^2\right)^2 \, dx \\
&\quad + C \int_{\Omega_{R,\, AR}} \Big| |r_0 - u| - |r_0 - v_1| \Big| \left(|r_0 - u|^{2p_0+1} \mathbb{1}_{\{|r_0 - u| > 2r_0\}} \right. \\
&\qquad\qquad \left. + |r_0 - v_1|^{2p_0+1} \mathbb{1}_{\{|r_0 - v_1| > 2r_0\}} \right) dx \\
&\leq C'\varepsilon + C' \int_{\Omega_{R,\, AR}} |u - v_1| \left(|r_0 - u|^{2^*-1} \mathbb{1}_{\{|r_0 - u| > 2r_0\}} + |r_0 - v_1|^{2^*-1} \mathbb{1}_{\{|r_0 - v_1| > 2r_0\}} \right) dx \\
&\leq C'\varepsilon + C' ||u - v_1||_{L^{2^*}(\Omega_{R,\, AR})} \left(|| \, |r_0 - u| \mathbb{1}_{\{|r_0 - u| > 2r_0\}} ||_{L^{2^*}(\Omega_{R,\, AR})}^{2^*-1} \right. \\
&\qquad\qquad \left. + || \, |r_0 - v_1| \mathbb{1}_{\{|r_0 - v_1| > 2r_0\}} ||_{L^{2^*}(\Omega_{R,\, AR})}^{2^*-1} \right).
\end{aligned} \tag{3.81}$$

From the Sobolev embedding we have

$$\begin{aligned}
&||u - v_1||_{L^{2^*}(\mathbf{R}^N)} \leq C_S ||\nabla(u - v_1)||_{L^2(\mathbf{R}^N)} \\
&\leq C_S (||\nabla u||_{L^2(\Omega_{R,\, AR})} + ||\nabla v_1||_{L^2(\Omega_{R,\, AR})}) \leq 2C_S \sqrt{\varepsilon}.
\end{aligned} \tag{3.82}$$

It is clear that $|r_0 - u| > 2r_0$ implies $|u| > r_0$ and $|r_0 - u| < 2|u|$, hence

$$\begin{aligned}
&|| \, |r_0 - u| \mathbb{1}_{\{|r_0 - u| > 2r_0\}} ||_{L^{2^*}(\Omega_{R,\, AR})} \\
&\leq 2||u||_{L^{2^*}(\mathbf{R}^N)} \leq 2C_S ||\nabla u||_{L^2(\mathbf{R}^N)} \leq 2C_S \left(E_{GL}(u)\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.83}$$

Obviously, a similar estimate holds for v_1. Combining (3.81), (3.82) and (3.83) we find

$$\int_{\Omega_{R,\,AR}} \Big| V(|r_0-u|^2) - V(|r_0-v_1|^2) \Big| \, dx \le C'\varepsilon + C''\sqrt{\varepsilon}\,(E_{GL}(u))^{\frac{2^*-1}{2}}. \tag{3.84}$$

From (3.70) and (3.71) it follows that $V(|r_0-v_1|^2) - V(|r_0-u_1|^2) - V(|r_0-u_2|^2) = 0$ on $\mathbf{R}^N \setminus \Omega_{A_1R,\,A_4R}$ and $|r_0-v_1|$, $|r_0-u_1|$, $|r_0-u_2| \in \left[\frac{r_0}{2}, \frac{3r_0}{2}\right]$ on $\Omega_{A_1R,\,A_4R}$. Then using (1.5), (3.66), (3.75) and (3.72) we get

$$\int_{\Omega_{A_1R,\,A_4R}} |V(|r_0-v_1|^2)| \, dx \le C \int_{\Omega_{A_1R,\,A_4R}} (\rho^2 - r_0^2)^2 \, dx \le C\varepsilon, \qquad \text{respectively} \tag{3.85}$$

$$\int_{\Omega_{A_1R,\,A_4R}} |V(|r_0-u_i|^2)| \, dx \le C \int_{\Omega_{A_1R,\,A_4R}} \left(\left(r_0 + \eta_i(\tfrac{|x|}{R})(\rho - r_0) \right)^2 - r_0^2 \right)^2 dx \le C\varepsilon. \tag{3.86}$$

Therefore

$$\begin{aligned} &\int_{\mathbf{R}^N} \Big| V(|r_0-v_1|^2) - V(|r_0-u_1|^2) - V(|r_0-u_2|^2) \Big| \, dx \\ &\le \int_{\Omega_{A_1R,\,A_4R}} |V(|r_0-v_1|^2)| + |V(|r_0-u_1|^2)| + |V(|r_0-u_2|^2)| \, dx \le C\varepsilon. \end{aligned} \tag{3.87}$$

Then (iv) follows from (3.84) and (3.87) and Lemma 3.3 is proved. □

4 Variational formulation

We assume throughout that assumptions (A1) and (A2) in the introduction are satisfied. We introduce the following functionals:

$$E_c(u) = \int_{\mathbf{R}^N} |\nabla u|^2 \, dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0-u|^2) \, dx,$$

$$A(u) = \int_{\mathbf{R}^N} \sum_{j=2}^{N} \left| \frac{\partial u}{\partial x_j} \right|^2 dx,$$

$$B_c(u) = \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_1} \right|^2 dx + cQ(u) + \int_{\mathbf{R}^N} V(|r_0-u|^2) \, dx,$$

$$P_c(u) = \tfrac{N-3}{N-1} A(u) + B_c(u).$$

It is clear that $E_c(u) = A(u) + B_c(u) = \frac{2}{N-1} A(u) + P_c(u)$. Let

$$\mathcal{C} = \{ u \in \mathcal{X} \mid u \neq 0, P_c(u) = 0 \}.$$

The aim of this section is to study the properties of the above functionals. In particular, we will prove that $\mathcal{C} \neq \emptyset$ and $\inf\{E_c(u) \mid u \in \mathcal{C}\} > 0$. This will be done in a sequence of lemmas. In the next sections we show that E_c admits a minimizer in $\mathcal{C}$ and this minimizer is a solution of (1.3).

We begin by proving that the above functionals are well-defined on $\mathcal{X}$. Since we have already seen in section 2 that Q is well-defined on $\mathcal{X}$, all we have to do is to prove that $V(|r_0-u|^2) \in L^1(\mathbf{R}^N)$ for any $u \in \mathcal{X}$. This will be done in the next lemma.

Lemma 4.1 *For any $u \in \mathcal{X}$ we have $V(|r_0 - u|^2) \in L^1(\mathbf{R}^N)$. Moreover, for any $\delta > 0$ there exist $C_1(\delta)$, $C_2(\delta) > 0$ such that for any $u \in \mathcal{X}$ we have*

$$
\begin{aligned}
&(1-\delta)a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 \, dx - C_1(\delta)||\nabla u||_{L^2(\mathbf{R}^N)}^{2^*} \\
&\leq \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx \qquad (4.1) \\
&\leq (1+\delta)a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 \, dx + C_2(\delta)||\nabla u||_{L^2(\mathbf{R}^N)}^{2^*}.
\end{aligned}
$$

Proof. Fix $\delta > 0$. Using (1.4) we see that there exists $\beta = \beta(\delta) \in (0, r_0]$ such that

$$(1-\delta)a^2(s - r_0^2)^2 \leq V(s) \leq (1+\delta)a^2(s - r_0^2)^2 \qquad \text{for any } s \in ((r_0 - \beta)^2, (r_0 + \beta)^2). \tag{4.2}$$

Let $u \in \mathcal{X}$. If $|u(x)| < \beta$ we have $|r_0 - u(x)|^2 \in ((r_0 - \beta)^2, (r_0 + \beta)^2)$ and it follows from (4.2) that $V(|r_0 - u|^2)\mathbb{1}_{\{|u|<\beta\}} \in L^1(\mathbf{R}^N)$ and

$$
\begin{aligned}
&(1-\delta)a^2 \int_{\{|u|<\beta\}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 \, dx \leq \int_{\{|u|<\beta\}} V(|r_0 - u|^2) \, dx \qquad (4.3) \\
&\leq (1+\delta)a^2 \int_{\{|u|<\beta\}} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 \, dx.
\end{aligned}
$$

Assumption (A2) implies that there exists $C_1'(\delta) > 0$ such that

$$\left|V(|r_0 - z|^2) - (1-\delta)a^2(\varphi^2(|r_0 - z|) - r_0^2)^2\right| \leq C_1'(\delta)|z|^{2p_0+2} \leq C_1''(\delta)|z|^{2^*}$$

for any $z \in \mathbf{C}$ satisfying $|z| \geq \beta$. Using the Sobolev embedding we obtain

$$
\begin{aligned}
&\int_{\{|u|\geq\beta\}} \left|V(|r_0 - u|^2) - (1-\delta)a^2(\varphi^2(|r_0 - u|) - r_0^2)^2\right| dx \qquad (4.4) \\
&\leq C_1''(\delta) \int_{\{|u|\geq\beta\}} |u|^{2^*} \, dx \leq C_1''(\delta) \int_{\mathbf{R}^N} |u|^{2^*} \, dx \leq C_1(\delta)||\nabla u||_{L^2(\mathbf{R}^N)}^{2^*}.
\end{aligned}
$$

Consequently $V(|r_0 - u|^2)\mathbb{1}_{\{|u|\geq\beta\}} \in L^1(\mathbf{R}^N)$ and it follows from (4.3) and (4.4) that the first inequality in (4.1) holds; the proof of the second inequality is similar. □

Lemma 4.2 *Let $\delta \in (0, r_0)$ and let $u \in \mathcal{X}$ be such that $r_0 - \delta \leq |r_0 - u| \leq r_0 + \delta$ a.e. on $\mathbf{R}^N$. Then*

$$|Q(u)| \leq \frac{1}{2a(r_0 - \delta)} E_{GL}(u).$$

Proof. From Lemma 2.1 we know that there are two real-valued functions ρ, θ such that $\rho - r_0 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ and $r_0 - u = \rho e^{i\theta}$ a.e. on $\mathbf{R}^N$. Moreover, from (2.3) and Definition 2.4 we infer that

$$Q(u) = -\int_{\mathbf{R}^N} (\rho^2 - r_0^2)\theta_{x_1} \, dx.$$

Using the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
&2a(r_0 - \delta)|Q(u)| \leq 2a(r_0 - \delta)||\theta_{x_1}||_{L^2(\mathbf{R}^N)}||\rho^2 - r_0^2||_{L^2(\mathbf{R}^N)} \\
&\leq (r_0 - \delta)^2 \int_{\mathbf{R}^N} |\theta_{x_1}|^2 \, dx + a^2 \int_{\mathbf{R}^N} \left(\rho^2 - r_0^2\right)^2 \, dx \\
&\leq \int_{\mathbf{R}^N} \rho^2 |\nabla\theta|^2 + a^2 \left(\rho^2 - r_0^2\right)^2 \, dx \leq E_{GL}(u).
\end{aligned}
$$

□

Lemma 4.3 *Assume that $0 \le c < v_s$ and let $\varepsilon \in (0, 1 - \frac{c}{v_s})$. There exists a constant $K_1 = K_1(F, N, c, \varepsilon) > 0$ such that for any $u \in \mathcal{X}$ satisfying $E_{GL}(u) < K_1$ we have*

$$\int_{\mathbf{R}^N} |\nabla u|^2 \, dx + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx - c|Q(u)| \ge \varepsilon E_{GL}(u).$$

Proof. Fix ε_1 such that $\varepsilon < \varepsilon_1 < 1 - \frac{c}{v_s}$. Then fix $\delta_1 \in (0, \varepsilon_1 - \varepsilon)$. By Lemma 4.1, there exists $C_1(\delta_1) > 0$ such that for any $u \in \mathcal{X}$ we have

$$\int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx \ge (1 - \delta_1) a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2 \right)^2 \, dx - C_1(\delta_1) \left(E_{GL}(u) \right)^{\frac{2^*}{2}} . \tag{4.5}$$

Using (3.4) we see that there exists $A > 0$ such that for any $w \in \mathcal{X}$ with $E_{GL}(w) \le 1$, for any $h \in (0, 1]$ and for any minimizer v_h of $G^w_{h,\mathbf{R}^N}$ in $H^1_w(\mathbf{R}^N)$ we have

$$|Q(w) - Q(v_h)| \le A h^{\frac{2}{N}} E_{GL}(w). \tag{4.6}$$

Choose $h \in (0, 1]$ such that $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} > \varepsilon$ (this choice is possible because $\varepsilon_1 - \delta_1 - \varepsilon > 0$). Then fix $\delta > 0$ such that $\frac{c}{2a(r_0 - \delta)} < 1 - \varepsilon_1$ (such δ exist because $\varepsilon_1 < 1 - \frac{c}{v_s} = 1 - \frac{c}{2ar_0}$).

Let $K = K(a, r_0, N, h, \delta, 1)$ be as in Lemma 3.1 (iv).

Consider $u \in \mathcal{X}$ such that $E_{GL}(u) \le \min(K, 1)$. Let v_h be a minimizer of $G^u_{h,\mathbf{R}^N}$ in $H^1_u(\mathbf{R}^N)$. The existence of v_h follows from Lemma 3.1 (i). By Lemma 3.1 (iv) we have $r_0 - \delta < |r_0 - v_h| < r_0 + \delta$ a.e. on $\mathbf{R}^N$ and then Lemma 4.2 implies

$$c|Q(v_h)| \le \frac{c}{2a(r_0 - \delta)} E_{GL}(v_h) \le (1 - \varepsilon_1) E_{GL}(v_h) \le (1 - \varepsilon_1) E_{GL}(u). \tag{4.7}$$

We have:

$$\begin{aligned}
&\int_{\mathbf{R}^N} |\nabla u|^2 \, dx + \int_{\mathbf{R}^N} V(|r_0 - u|^2) \, dx - c|Q(u)| \\
&\ge (1 - \delta_1) E_{GL}(u) - C_1(\delta_1) \left(E_{GL}(u) \right)^{\frac{2^*}{2}} - c|Q(u)| \qquad \text{by (4.5)} \\
&\ge (1 - \delta_1) E_{GL}(u) - C_1(\delta_1) \left(E_{GL}(u) \right)^{\frac{2^*}{2}} - c|Q(u) - Q(v_h)| - c|Q(v_h)| \\
&\ge (1 - \delta_1) E_{GL}(u) - C_1(\delta_1) \left(E_{GL}(u) \right)^{\frac{2^*}{2}} - cAh^{\frac{2}{N}} E_{GL}(u) - (1 - \varepsilon_1) E_{GL}(u) \\
&\qquad\qquad \text{by (4.6) and (4.7)} \\
&= \left(\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} - C_1(\delta_1) \left(E_{GL}(u) \right)^{\frac{2^*}{2} - 1} \right) E_{GL}(u).
\end{aligned} \tag{4.8}$$

Note that (4.8) holds for any $u \in \mathcal{X}$ with $E_{GL}(u) \le \min(K, 1)$. Since $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} > \varepsilon$, it is obvious that $\varepsilon_1 - \delta_1 - cAh^{\frac{2}{N}} - C_1(\delta_1) \left(E_{GL}(u) \right)^{\frac{2^*}{2} - 1} > \varepsilon$ if $E_{GL}(u)$ is sufficiently small and the conclusion of Lemma 4.3 follows. $\square$

An obvious consequence of Lemma 4.3 is that $E_c(u) > 0$ if $u \in \mathcal{X} \setminus \{0\}$ and $E_{GL}(u)$ is sufficiently small. An easy corollary of the next lemma is that there are functions $v \in \mathcal{X}$ such that $E_c(v) < 0$.

Lemma 4.4 *Let $N \ge 2$. Let $D = \{(R, \varepsilon) \in \mathbf{R}^2 \mid R > 0, \ 0 < \varepsilon < \frac{R}{2}\}$. There exists a continuous map from D to $H^1(\mathbf{R}^N)$, $(R, \varepsilon) \longmapsto v^{R,\varepsilon}$ such that $v^{R,\varepsilon} \in C_c(\mathbf{R}^N)$ for any $(R, \varepsilon) \in D$ and the following estimates hold:*

i) $\displaystyle\int_{\mathbf{R}^N} |\nabla v^{R,\varepsilon}|^2 \, dx \le C_1 R^{N-2} + C_2 R^{N-2} \ln \frac{R}{\varepsilon}$, *ii)* $\left|\int_{\mathbf{R}^N} V(|r_0 - v^{R,\varepsilon}|^2)\, dx\right| \le C_3\varepsilon^2 R^{N-2}$,

iii) $\left|\int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - v^{R,\varepsilon}|) - r_0^2\right)^2\, dx\right| \le C_4\varepsilon^2 R^{N-2}$,

iv) $-2\pi r_0^2\omega_{N-1}R^{N-1} \le Q(v^{R,\varepsilon}) \le -2\pi r_0^2\omega_{N-1}(R-2\varepsilon)^{N-1}$,

where the constants $C_1 - C_4$ *depend only on* N *and* $\omega_{N-1} = \mathcal{L}^{N-1}(B_{\mathbf{R}^{N-1}}(0,1))$.

Proof. Let $A > 0$ and

$$T_{A,R} = \{x \in \mathbf{R}^N \mid 0 \le |x'| \le R,\ \ -\tfrac{A(R-|x'|)}{R} < x_1 < \tfrac{A(R-|x'|)}{R}\}.$$

We define $\theta^{A,R} : \mathbf{R}^N \longrightarrow \mathbf{R}$ in the following way: if $|x'| \ge R$ we put $\theta^{A,R}(x) = 0$ and if $|x'| < R$ we define

$$\theta^{A,R}(x) = \begin{cases} 0 & \text{if } x_1 \le -\frac{A(R-|x'|)}{R}, \\ \frac{\pi R}{A(R-|x'|)}x_1 + \pi & \text{if } x \in T_{A,R}, \\ 2\pi & \text{if } x_1 \ge \frac{A(R-|x'|)}{R}. \end{cases} \tag{4.9}$$

It is easy to see that $x \longmapsto e^{i\theta^{A,R}(x)}$ is continuous on $\mathbf{R}^N \setminus \{x \mid x_1 = 0,\ |x'| = R\}$ and equals 1 on $\mathbf{R}^N \setminus T_{A,R}$.

Fix $\psi \in C^\infty(\mathbf{R})$ such that $\psi = 0$ on $(-\infty, 1]$, $\psi = 1$ on $[2,\infty)$ and $0 \le \psi' \le 2$. Let

$$\psi^{R,\varepsilon}(x) = \psi\Big(\frac{1}{\varepsilon}\sqrt{x_1^2 + (|x'| - R)^2}\Big) \quad \text{and} \quad w_{A,R,\varepsilon}(x) = r_0\left(1 - \psi^{R,\varepsilon}(x)e^{i\theta^{A,R}(x)}\right). \tag{4.10}$$

It is obvious that $w_{A,R,\varepsilon} \in C_c(\mathbf{R}^N)$ (in fact, $w_{A,R,\varepsilon}$ is C^∞ on $\mathbf{R}^N \setminus B$, where $B = \partial T_{A,R} \cup \{(x_1, 0, \ldots, 0) \mid x_1 \in [-A, A]\}$). On $\mathbf{R}^N \setminus B$ we have

$$\frac{\partial\theta^{A,R}}{\partial x_1} = \begin{cases} \frac{\pi R}{A(R-|x'|)} & \text{if } x \in T_{A,R}, \\ 0 \quad \text{otherwise}, \end{cases} \qquad \frac{\partial\theta^{A,R}}{\partial x_j} = \begin{cases} \frac{\pi R x_1}{A(R-|x'|)^2}\frac{x_j}{|x'|} & \text{if } x \in T_{A,R}, \\ 0 \quad \text{otherwise}, \end{cases} \tag{4.11}$$

$$\frac{\partial\psi^{R,\varepsilon}}{\partial x_1}(x) = \frac{1}{\varepsilon}\psi'\left(\frac{\sqrt{x_1^2 + (|x'| - R)^2}}{\varepsilon}\right)\frac{x_1}{\sqrt{x_1^2 + (|x'| - R)^2}}, \tag{4.12}$$

$$\frac{\partial\psi^{R,\varepsilon}}{\partial x_j}(x) = \frac{1}{\varepsilon}\psi'\left(\frac{\sqrt{x_1^2 + (|x'| - R)^2}}{\varepsilon}\right)\frac{|x'| - R}{\sqrt{x_1^2 + (|x'| - R)^2}}\frac{x_j}{|x'|} \quad \text{for } j \ge 2. \tag{4.13}$$

Then a simple computation gives $\langle i\frac{\partial w_{A,R,\varepsilon}}{\partial x_1}, w_{A,R,\varepsilon}\rangle = -r_0^2(\psi^{R,\varepsilon})^2\frac{\partial\theta^{A,R}}{\partial x_1} + r_0^2\frac{\partial}{\partial x_1}\left(\psi^{R,\varepsilon}\sin(\theta^{A,R})\right)$ on $\mathbf{R}^N \setminus B$. Thus we have

$$Q(w_{A,R,\varepsilon}) = -r_0^2\int_{\mathbf{R}^N}(\psi^{R,\varepsilon})^2\frac{\partial\theta^{A,R}}{\partial x_1}\, dx.$$

It is obvious that

$$\int_{-\infty}^{\infty}\frac{\partial\theta^{A,R}}{\partial x_1}\, dx_1 = 0 \quad \text{if } |x'| > R \quad \text{and} \quad \int_{-\infty}^{\infty}\frac{\partial\theta^{A,R}}{\partial x_1}\, dx_1 = 2\pi \quad \text{if } 0 < |x'| < R. \tag{4.14}$$

Since $\frac{\partial\theta^{A,R}}{\partial x_1} \ge 0$ a.e. on $\mathbf{R}^N$ and $0 \le \psi^{R,\varepsilon} \le 1$, we get

$$\int_{\{|R-|x'||\ge 2\varepsilon\}}\frac{\partial\theta^{A,R}}{\partial x_1}\, dx \le \int_{\mathbf{R}^N}\left(\psi^{R,\varepsilon}\right)^2\frac{\partial\theta^{A,R}}{\partial x_1}\, dx_1 \le \int_{\mathbf{R}^N}\frac{\partial\theta^{A,R}}{\partial x_1}\, dx_1,$$

and using Fubini's theorem and (4.14) we obtain that $w_{A,R,\varepsilon}$ satisfies (iv).

Using cylindrical coordinates (x_1, r, ζ) in $\mathbf{R}^N$, where $r = |x'|$ and $\zeta = \frac{x'}{|x'|} \in S^{N-2}$, we get

$$\int_{\mathbf{R}^N} V(|r_0 - w_{A,R,\varepsilon}|^2)\,dx = |S^{N-2}| \int_{-\infty}^{\infty}\int_0^{\infty} V\!\left(r_0^2 \psi^2\left(\frac{\sqrt{x_1^2+(r-R)^2}}{\varepsilon}\right)\right) r^{N-2}\,dr\,dx_1. \tag{4.15}$$

Next we use polar coordinates in the (x_1, r) plane, that is we write $x_1 = \tau\cos\alpha$, $r = R + \tau\sin\alpha$ (thus $\tau = \sqrt{x_1^2 + (R-r)^2}$). Since $V(r_0^2\psi^2(s)) = 0$ for $s \geq 2$, we get

(4.16)

$$\int_{-\infty}^{\infty}\int_0^{\infty} V\left(r_0^2\psi^2\left(\frac{\sqrt{x_1^2+(r-R)^2}}{\varepsilon}\right)\right) r^{N-2}\,dr\,dx_1 = \int_0^{2\varepsilon}\int_0^{2\pi} V(r_0^2\psi^2(\tfrac{\tau}{\varepsilon}))(R+\tau\sin\alpha)^{N-2}\,d\alpha\,\tau\,d\tau$$

$$= \varepsilon^2 \int_0^2\int_0^{2\pi} V(r_0^2\psi^2(s))(R+\varepsilon s\sin\alpha)^{N-2}\,d\alpha\,s\,ds.$$

It is obvious that $\left|\int_0^{2\pi}(R+\varepsilon s\sin\alpha)^{N-2}d\alpha\right| \leq 2\pi(R+2\varepsilon)^{N-2}$ for any $s \in [0,2]$, and then using (4.15) and (4.16) we infer that $w_{A,R,\varepsilon}$ satisfies (ii). The proof of (iii) is similar.

It is clear that on $\mathbf{R}^N \setminus B$ we have

$$|\nabla w_{A,R,\varepsilon}| = r_0^2|\nabla\psi^{R,\varepsilon}|^2 + r_0^2|\psi^{R,\varepsilon}|^2|\nabla\theta^{A,R}|^2. \tag{4.17}$$

From (4.12) and (4.13) we see that $|\nabla\psi^{R,\varepsilon}(x)|^2 = \frac{1}{\varepsilon^2}\left|\psi'\left(\frac{\sqrt{x_1^2+(|x'|-R)^2}}{\varepsilon}\right)\right|^2$. Proceeding as above and using cylindrical coordinates (x_1, r, ζ) in $\mathbf{R}^N$, then passing to polar coordinates $x_1 = \tau\cos\alpha$, $r = R + \tau\sin\alpha$, we obtain

$$\int_{\mathbf{R}^N}\left|\psi'\left(\frac{\sqrt{x_1^2+(|x'|-R)^2}}{\varepsilon}\right)\right|^2 dx \leq 2\pi|S^{N-2}|\varepsilon^2(R+2\varepsilon)^{N-2}\int_0^2 s|\psi'(s)|^2\,ds. \tag{4.18}$$

It is easily seen from (4.11) that $|\nabla\theta^{A,R}(x)|^2 = \frac{\pi^2R^2}{A^2(R-|x'|)^2}\left(1 + \frac{x_1^2}{(R-|x'|)^2}\right)$ if $x \in T_{A,R}$, $|x'| \neq 0$, and $\nabla\theta^{A,R}(x) = 0$ a.e. on $\mathbf{R}^N \setminus \overline{T}_{A,R}$. Moreover, if $(x_1, x') \in T_{A,R}$ and $|x'| \geq R - \frac{R\varepsilon}{\sqrt{A^2+R^2}}$, we have $\psi^{R,\varepsilon}(x_1, x') = 0$. Therefore

$$\int_{\mathbf{R}^N}|\psi^{R,\varepsilon}|^2|\nabla\theta^{A,R}|^2\,dx \leq \int_{T_{A,R}\cap\{|x'|<R-\frac{R\varepsilon}{\sqrt{A^2+R^2}}\}}|\nabla\theta^{A,R}|^2\,dx$$

$$= \int_{\{|x'|<R-\frac{R\varepsilon}{\sqrt{A^2+R^2}}\}}\int_{-\frac{A(R-|x'|)}{R}}^{\frac{A(R-|x'|)}{R}}|\nabla\theta^{A,R}|^2\,dx_1\,dx'$$

$$= \int_{\{|x'|<R-\frac{R\varepsilon}{\sqrt{A^2+R^2}}\}}\frac{2\pi^2R}{A(R-|x'|)} + \frac{2\pi^2}{3}\frac{A}{R}\frac{1}{R-|x'|}\,dx' \tag{4.19}$$

$$= 2\pi^2\left(\frac{R}{A} + \frac{3A}{R}\right)|S^{N-2}|\int_0^{R-\frac{R\varepsilon}{\sqrt{A^2+R^2}}}\frac{r^{N-2}}{R-r}\,dr$$

$$= 2\pi^2\left(\frac{R}{A} + \frac{3A}{R}\right)|S^{N-2}|R^{N-2}\left(\sum_{k=1}^{N-2}\frac{1}{k}\left(1 - \frac{\varepsilon}{\sqrt{A^2+R^2}}\right)^k + \ln\left(\frac{\sqrt{A^2+R^2}}{\varepsilon}\right)\right).$$

Now it suffices to take $v^{R,\varepsilon} = w_{R,R,\varepsilon}$. From (4.17), (4.18) and (4.19) it follows that $v^{R,\varepsilon}$ satisfies (i). It is not hard to see that the mapping $(R, \varepsilon) \longmapsto v^{R,\varepsilon}$ is continuous from D to $H^1(\mathbf{R}^N)$ and Lemma 4.4 is proved. $\square$

Lemma 4.5 *For any $k > 0$, the functional Q is bounded on the set*

$$\{u \in \mathcal{X} \mid E_{GL}(u) \leq k\}.$$

Proof. Let $c \in (0, v_s)$ and let $\varepsilon \in (0, 1-\frac{c}{v_s})$. From Lemmas 4.1 and 4.3 it follows that there exist two positive constants $C_2(\frac{\varepsilon}{2})$ and K_1 such that for any $u \in \mathcal{X}$ satisfying $E_{GL}(u) < K_1$ we have

$$(1 + \tfrac{\varepsilon}{2})E_{GL}(u) + C_2(\tfrac{\varepsilon}{2})\,(E_{GL}(u))^{\frac{2^*}{2}} - c|Q(u)|$$

$$\geq \int_{\mathbf{R}^N} |\nabla u|^2\,dx + \int_{\mathbf{R}^N} V(|r_0 - u|^2)\,dx - c|Q(u)| \geq \varepsilon E_{GL}(u).$$

This inequality implies that there exists $K_2 \leq K_1$ such that for any $u \in \mathcal{X}$ satisfying $E_{GL}(u) \leq K_2$ we have

$$c|Q(u)| \leq E_{GL}(u). \tag{4.20}$$

Hence Lemma 4.5 is proved if $k \leq K_2$.

Now let $u \in \mathcal{X}$ be such that $E_{GL}(u) > K_2$. Using the notation (1.10), it is clear that for $\sigma > 0$ we have $Q(u_{\sigma,\sigma}) = \sigma^{N-1}Q(u)$ (see (2.14) and

$$E_{GL}(u_{\sigma,\sigma}) = \sigma^{N-2}\int_{\mathbf{R}^N} |\nabla u|^2\,dx + \sigma^N a^2 \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\,dx.$$

Let $\sigma_0 = \left(\frac{K_2}{E_{GL}(u)}\right)^{\frac{1}{N-2}}$. Then $\sigma_0 \in (0,1)$ and we have $E_{GL}(u_{\sigma_0,\sigma_0}) \leq \sigma_0^{N-2}E_{GL}(u) = K_2$. Using (4.20) we infer that $c|Q(u_{\sigma_0,\sigma_0})| \leq E_{GL}(u_{\sigma_0,\sigma_0})$, and this implies $c\sigma_0^{N-1}|Q(u)| \leq \sigma_0^{N-2}E_{GL}(u)$, or equivalently

$$|Q(u)| \leq \frac{1}{c\sigma_0}E_{GL}(u) = \frac{1}{c}K_2^{-\frac{1}{N-2}}\,(E_{GL}(u))^{\frac{N-1}{N-2}}. \tag{4.21}$$

Since (4.21) holds for any $u \in \mathcal{X}$ with $E_{GL}(u) > K_2$, Lemma 4.5 is proved. $\square$

From Lemma 4.1 and Lemma 4.5 it follows that for any $k > 0$, the functional E_c is bounded on the set $\{u \in \mathcal{X} \mid E_{GL}(u) = k\}$. For $k > 0$ we define

$$E_{c,\,min}(k) = \inf\{E_c(u) \mid u \in \mathcal{X},\ E_{GL}(u) = k\}.$$

Clearly, the function $E_{c,\,min}$ is bounded on any bounded interval in $\mathbf{R}$. The next result will be important for our variational argument.

Lemma 4.6 *Assume that $N \geq 3$ and $0 < c < v_s$. The function $E_{c,\,min}$ has the following properties:*

i) There exists $k_0 > 0$ such that $E_{c,\,min}(k) > 0$ for any $k \in (0, k_0)$.

ii) We have $\lim_{k\to\infty} E_{c,\,min}(k) = -\infty$.

iii) For any $k > 0$ we have $E_{c,\,min}(k) < k$.

Proof. (i) is an easy consequence of Lemma 4.3.

(ii) It is obvious that $H^1(\mathbf{R}^N) \subset \mathcal{X}$ and the functionals E_{GL}, E_c and Q are continuous on $H^1(\mathbf{R}^N)$. For $\varepsilon = 1$ and $R > 2$, consider the functions $v^{R,1}$ constructed in Lemma 4.4. Clearly, $R \longmapsto v^{R,1}$ is a continuous curve in $H^1(\mathbf{R}^N)$. Lemma 4.4 implies $E_c(v^{R,1}) \longrightarrow -\infty$ as $R \longrightarrow \infty$. From Lemma 4.5 we infer that $E_{GL}(v^{R,1}) \longrightarrow \infty$ as $R \longrightarrow \infty$ and then it is not hard to see that (ii) holds.

(iii) Fix $k > 0$. Let $v^{R,1}$ be as above and let $u = v^{R,1}$ for some R sufficiently large, so that

$$E_{GL}(u) > k, \quad Q(u) < 0 \quad \text{ and } E_c(u) < 0.$$

In particular, we have

$$E_c(u) - E_{GL}(u) = cQ(u) + \int_{\mathbf{R}^N} V(|r_0 - u|^2) - a^2 \left(\varphi^2(|r_0 - u|^2) - r_0^2\right)^2 \, dx < 0.$$

It is obvious that $E_{GL}(u_{\sigma,\sigma}) \longrightarrow 0$ as $\sigma \longrightarrow 0$, hence there exists $\sigma_0 \in (0,1)$ such that $E_{GL}(u_{\sigma_0,\sigma_0}) = k$. We have

$$\begin{aligned} & E_c(u_{\sigma_0,\sigma_0}) - E_{GL}(u_{\sigma_0,\sigma_0}) \\ &= \sigma_0^{N-1} cQ(u) + \sigma_0^N \int_{\mathbf{R}^N} V(|r_0 - u|^2) - a^2 \left(\varphi^2(|r_0 - u|^2) - r_0^2\right)^2 \, dx \\ &= (\sigma_0^{N-1} - \sigma_0^N) cQ(u) + \sigma_0^N (E_c(u) - E_{GL}(u)) < 0. \end{aligned}$$

Thus $E_c(u_{\sigma_0,\sigma_0}) < E_{GL}(u_{\sigma_0,\sigma_0})$. Since $E_{GL}(u_{\sigma_0,\sigma_0}) = k$, we have necessarily $E_{c,\,min}(k) \leq E_c(u_{\sigma_0,\sigma_0}) < k$. □

From Lemma 4.6 (i) and (ii) it follows that

$$0 < S_c := \sup\{E_{c,min}(k) \mid k > 0\} < \infty. \tag{4.22}$$

Lemma 4.7 *The set $\mathcal{C} = \{u \in \mathcal{X} \mid u \neq 0,\ P_c(u) = 0\}$ is not empty and we have*

$$T_c := \inf\{E_c(u) \mid u \in \mathcal{C}\} \geq S_c > 0.$$

Proof. Let $u \in \mathcal{X} \setminus \{0\}$ be such that $E_c(w) < 0$ (we have seen in the proof of Lemma 4.6 that such functions exist). It is obvious that $A(w) > 0$ and $\displaystyle\int_{\mathbf{R}^N} \left|\frac{\partial w}{\partial x_1}\right|^2 dx > 0$; therefore $B_c(w) = E_c(w) - A(w) < 0$ and $P_c(w) = E_c(w) - \frac{2}{N-1} A(w) < 0$. Clearly,

$$P_c(w_{\sigma,1}) = \frac{1}{\sigma} \int_{\mathbf{R}^N} \left|\frac{\partial w}{\partial x_1}\right|^2 dx + \frac{N-3}{N-1} \sigma A(w) + cQ(w) + \sigma \int_{\mathbf{R}^3} V(|r_0 - w|^2) \, dx. \tag{4.23}$$

Since $P_c(w_{1,1}) = P_c(w) < 0$ and $\lim\limits_{\sigma \to 0} P_c(w_{\sigma,1}) = \infty$, there exists $\sigma_0 \in (0,1)$ such that $P_c(w_{\sigma_0,1}) = 0$, that is $w_{\sigma_0,1} \in \mathcal{C}$. Thus $\mathcal{C} \neq \emptyset$.

To prove the second part of Lemma 4.7, consider first the case $N \geq 4$. Let $u \in \mathcal{C}$. It is clear that $A(u) > 0$, $B_c(u) = -\frac{N-3}{N-1} A(u) < 0$ and for any $\sigma > 0$ we have $E_c(u_{1,\sigma}) = A(u_{1,\sigma}) + B_c(u_{1,\sigma}) = \sigma^{N-3} A(u) + \sigma^{N-1} B_c(u)$, hence

$$\frac{d}{d\sigma}(E_c(u_{1,\sigma})) = (N-3)\sigma^{N-4} A(u) + (N-1)\sigma^{N-2} B_c(u)$$

is positive on $(0,1)$ and negative on $(1,\infty)$. Consequently the function $\sigma \longmapsto E_c(u_{1,\sigma})$ achieves its maximum at $\sigma = 1$.

On the other hand, we have

$$E_{GL}(u_{1,\sigma}) = \sigma^{N-3}A(u) + \sigma^{N-1}\left(\int_{\mathbf{R}^N}\left|\frac{\partial u}{\partial x_1}\right|^2 + a^2\left(\varphi^2(|r_0-u|)-r_0^2\right)^2\,dx\right).$$

It is easy to see that the mapping $\sigma \longmapsto E_{GL}(u_{1,\sigma})$ is strictly increasing and one-to-one from $(0,\infty)$ to $(0,\infty)$. Hence for any $k>0$, there is a unique $\sigma(k,u)>0$ such that $E_{GL}(u_{1,\sigma(k,u)}) = k$. Then we have

$$E_{c,min}(k) \le E_c(u_{1,\sigma(k,u)}) \le E_c(u_{1,1}) = E_c(u).$$

Since this is true for any $k>0$ and any $u \in \mathcal{C}$, the conclusion follows.

Next we consider the case $N=3$. Let $u \in \mathcal{C}$. We have $P_c(u) = B_c(u) = 0$ and $E_c(u) = A(u) > 0$. For $\sigma > 0$ we get

$$E_c(u_{1,\sigma}) = A(u) + \sigma^2 B_c(u) = A(u) \qquad \text{and}$$
$$E_{GL}(u_{1,\sigma}) = A(u) + \sigma^2\left(\int_{\mathbf{R}^3}\left|\frac{\partial u}{\partial x_1}\right|^2 + a^2\left(\varphi^2(|r_0-u|)-r_0^2\right)^2\,dx\right).$$

Clearly, $\sigma \longmapsto E_{GL}(u_{1,\sigma})$ is increasing on $(0,\infty)$ and is one-to-one from $(0,\infty)$ to $(A(u),\infty)$.

Let $\varepsilon>0$. Let $k_\varepsilon>0$ be such that $E_{c,min}(k_\varepsilon) > S_c - \varepsilon$. If $A(u) \ge k_\varepsilon$, from Lemma 4.6 (iii) we have

$$E_c(u) = A(u) \ge k_\varepsilon > E_{c,min}(k_\varepsilon) > S_c - \varepsilon.$$

If $A(u) < k_\varepsilon$, there exists $\sigma(k_\varepsilon,u)>0$ such that $E_{GL}(u_{1,\sigma(k_\varepsilon,u)}) = k_\varepsilon$. Then we get

$$E_c(u) = A(u) = E_c(u_{1,\sigma(k_\varepsilon,u)}) \ge E_{c,min}(k_\varepsilon) > S_c - \varepsilon.$$

So far we have proved that for any $u \in \mathcal{C}$ and any $\varepsilon>0$ we have $E_c(u) > S_c - \varepsilon$. The conclusion follows letting $\varepsilon \longrightarrow 0$, then taking the infimum for $u \in \mathcal{C}$. □

In Lemma 4.7, we do not know whether $T_c = S_c$.

Lemma 4.8 *Let T_c be as in Lemma 4.7. The following assertions hold.*

i) For any $u \in \mathcal{X}$ with $P_c(u) < 0$ we have $A(u) > \frac{N-1}{2}T_c$.

ii) Let $(u_n)_{n\ge1} \subset \mathcal{X}$ be a sequence such that $(E_{GL}(u_n))_{n\ge1}$ is bounded and $\lim_{n\to\infty} P_c(u_n) = \mu < 0$. Then $\liminf_{n\to\infty} A(u_n) > \frac{N-1}{2}T_c$.

Proof. i) Since $P_c(u)<0$, it is clear that $u \ne 0$ and $\int_{\mathbf{R}^N}\left|\frac{\partial u}{\partial x_1}\right|^2 dx > 0$. As in the proof of Lemma 4.7, we have $P_c(u_{1,1}) = P_c(u) < 0$ and (4.23) implies that $\lim_{\sigma\to0} P_c(u_{\sigma,1}) = \infty$, hence there exists $\sigma_0 \in (0,1)$ such that $P_c(u_{\sigma_0,1}) = 0$. From Lemma 4.7 we get $E_c(u_{\sigma_0,1}) \ge T_c$ and this implies $E_c(u_{\sigma_0,1}) - P_c(u_{\sigma_0,1}) \ge T_c$, that is $\frac{2}{N-1}A(u_{\sigma_0,1}) \ge T_c$. From the last inequality we find

$$A(u) \ge \frac{N-1}{2}\frac{1}{\sigma_0}T_c > \frac{N-1}{2}T_c. \tag{4.24}$$

ii) For n sufficiently large (so that $P_c(u_n)<0$) we have $u_n \ne 0$ and $\int_{\mathbf{R}^N}\left|\frac{\partial u_n}{\partial x_1}\right|^2 dx > 0$. As in the proof of part (i), using (4.23) we see that for each n sufficiently big there exists $\sigma_n \in (0,1)$ such that

$$P_c((u_n)_{\sigma_n,1}) = 0 \tag{4.25}$$

and we infer that $A(u_n) \geq \frac{N-1}{2}\frac{1}{\sigma_n}T_c$. We claim that

$$\limsup_{n\to\infty} \sigma_n < 1. \tag{4.26}$$

Notice that if (4.26) holds, we have $\liminf_{n\to\infty} A(u_n) \geq \frac{N-1}{2}\frac{1}{\limsup_{n\to\infty}\sigma_n}T_c > \frac{N-1}{2}T_c$ and Lemma 4.8 is proved.

To prove (4.26) we argue by contradition and assume that there is a subsequence $(\sigma_{n_k})_{k\geq 1}$ such that $\sigma_{n_k} \longrightarrow 1$ as $k \longrightarrow \infty$. Since $(E_{GL}(u_n))_{n\geq 1}$ is bounded, using Lemmas 4.1 and 4.5 we infer that $\left(\int_{\mathbf{R}^N} \left|\frac{\partial u_n}{\partial x_1}\right|^2 dx\right)_{n\geq 1}$, $\left(\int_{\mathbf{R}^N} V(|r_0 - u_n|^2)\, dx\right)_{n\geq 1}$, $(A(u_n))_{n\geq 1}$, and $(Q(u_n))_{n\geq 1}$ are bounded. Consequently there is a subsequence $(n_{k_\ell})_{\ell\geq 1}$ and there are α_1, α_2, β, $\gamma \in \mathbf{R}$ such that

$$\int_{\mathbf{R}^N} \left|\frac{\partial u_{n_{k_\ell}}}{\partial x_1}\right|^2 dx \longrightarrow \alpha_1, \qquad \int_{\mathbf{R}^N} V(|r_0 - u_{n_{k_\ell}}|^2)\, dx \longrightarrow \gamma$$

$$A(u_{n_{k_\ell}}) \longrightarrow \alpha_2, \qquad Q(u_{n_{k_\ell}}) \longrightarrow \beta \quad \text{as } \ell \longrightarrow \infty.$$

Writing (4.25) and (4.23) (with $(u_{n_{k_\ell}})_{\sigma_{n_{k_\ell}},1}$ instead of $(u_n)_{\sigma_n,1}$ and $w_{\sigma,1}$, respectively) then passing to the limit as $\ell \longrightarrow \infty$ and using the fact that $\sigma_{n_k} \longrightarrow 1$ we find $\alpha_1 + \frac{N-3}{N-1}\alpha_2 + c\beta + \gamma = 0$. On the other hand we have $\lim_{\ell\to\infty} P_c(u_{n_{k_\ell}}) = \mu < 0$ and this gives $\alpha_1 + \frac{N-3}{N-1}\alpha_2 + c\beta + \gamma = \mu < 0$. This contradiction proves that (4.26) holds and the proof of Lemma 4.8 is complete. □

5 The case $N \geq 4$

Throughout this section we assume that $N \geq 4$, $0 < c < v_s$ and the assumptions (A1) and (A2) are satisfied. Most of the results below do *not* hold for $c > v_s$. Some of them may not hold for $c = 0$ and some particular nonlinearities F.

Lemma 5.1 *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that $(E_c(u_n))_{n\geq 1}$ is bounded and $P_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Then $(E_{GL}(u_n))_{n\geq 1}$ is bounded.

Proof. We have $\frac{2}{N-1}A(u_n) = E_c(u_n) - P_c(u_n)$, hence $(A(u_n))_{n\geq 1}$ is bounded. It remains to prove that $\int_{\mathbf{R}^N} \left|\frac{\partial u_n}{\partial x_1}\right| + a^2\left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx$ is bounded. We argue by contradiction and we assume that there is a subsequence, still denoted $(u_n)_{n\geq 1}$, such that

$$\int_{\mathbf{R}^N} \left|\frac{\partial u_n}{\partial x_1}\right| + a^2\left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty. \tag{5.1}$$

Fix $k_0 > 0$ such that $E_{c,min}(k_0) > 0$. Arguing as in the proof of Lemma 4.7, it is easy to see that there exists a sequence $(\sigma_n)_{n\geq 1}$ such that

$$E_{GL}((u_n)_{1,\sigma_n}) = \sigma_n^{N-3}A(u_n) + \sigma_n^{N-1}\int_{\mathbf{R}^N} \left|\frac{\partial u_n}{\partial x_1}\right| + a^2\left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx = k_0. \tag{5.2}$$

From (5.1) and (5.2) we have $\sigma_n \longrightarrow 0$ as $n \longrightarrow \infty$. Since $B_c(u_n) = -\frac{N-3}{N-1}A(u_n) + P_c(u_n)$, it is clear that $(B_c(u_n))_{n\geq 1}$ is bounded and we obtain

$$E_c((u_n)_{1,\sigma_n}) = \sigma_n^{N-3}A(u_n) + \sigma_n^{N-1}B_c(u_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

But this contradicts the fact that $E_{c,min}(k_0) > 0$ and Lemma 5.1 is proved. □

Lemma 5.2 *Let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence satisfying the following properties:*
a) There exist C_1, $C_2 > 0$ such that $C_1 \leq E_{GL}(u_n)$ and $A(u_n) \leq C_2$ for any $n \geq 1$.
b) $P_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.
Then $\liminf\limits_{n\to\infty} E_c(u_n) \geq T_c$, where T_c is as in Lemma 4.7.

Note that in Lemma 5.2 the assumption $E_{GL}(u_n) \geq C_1 > 0$ is necessary. To see this, consider a sequence $(u_n)_{n\geq 1} \subset H^1(\mathbf{R}^N)$ such that $u_n \neq 0$ and $u_n \longrightarrow 0$ as $n \longrightarrow \infty$. It is clear that $P_c(u_n) \longrightarrow 0$ and $E_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. First we prove that

$$C_3 := \liminf_{n\to\infty} A(u_n) > 0. \tag{5.3}$$

To see this, fix $k_0 > 0$ such that $E_{c,min}(k_0) > 0$. Exactly as in the proof of Lemma 4.7, it is easy to see that for each n there exists a unique $\sigma_n > 0$ such that (5.2) holds. Since $k_0 = E_{GL}((u_n)_{1,\sigma_n}) \geq \min(\sigma_n^{N-3}, \sigma_n^{N-1}) E_{GL}((u_n)) \geq \min(\sigma_n^{N-3}, \sigma_n^{N-1}) C_1$, it follows that $(\sigma_n)_{n\geq 1}$ is bounded. On the other hand, we have $E_c((u_n)_{1,\sigma_n}) = \sigma_n^{N-3} A(u_n) + \sigma_n^{N-1} B_c(u_n) \geq E_{c,min}(k_0) > 0$, that is

$$\sigma_n^{N-3} A(u_n) + \sigma_n^{N-1} \left(P_c(u_n) - \frac{N-3}{N-1} A(u_n) \right) \geq E_{c,min}(k_0) > 0. \tag{5.4}$$

If there is a subsequence $(u_{n_k})_{k\geq 1}$ such that $A(u_{n_k}) \longrightarrow 0$, putting u_{n_k} in (5.4) and letting $k \longrightarrow \infty$ we would get $0 \geq E_{c,min}(k_0) > 0$, a contradiction. Thus (5.3) is proved.

We have $B_c(u_n) = P_c(u_n) - \frac{N-3}{N-1} A(u_n)$ and using (b) and (5.3) we obtain

$$\limsup_{n\to\infty} B_c(u_n) \leq -\frac{N-3}{N-1} C_3 < 0. \tag{5.5}$$

Clearly, for any $\sigma > 0$ we have

$$P_c((u_n)_{1,\sigma}) = \sigma^{N-3} \frac{N-3}{N-1} A(u_n) + \sigma^{N-1} B_c(u_n) = \sigma^{N-3} \left(\frac{N-3}{N-1} A(u_n) + \sigma^2 B_c(u_n) \right).$$

For n sufficiently big (so that $B_c(u_n) < 0$), let $\tilde{\sigma}_n = \left(\frac{\frac{N-3}{N-1} A(u_n)}{-B_c(u_n)} \right)^{\frac{1}{2}}$. Then $P_c((u_n)_{1,\tilde{\sigma}_n}) = 0$, or equivalently $(u_n)_{1,\tilde{\sigma}_n} \in \mathcal{C}$. From Lemma 4.7 we obtain $E_c((u_n)_{1,\tilde{\sigma}_n}) = \tilde{\sigma}_n^{N-3} \frac{N-3}{N-1} A(u_n) + \tilde{\sigma}_n^{N-1} B_c(u_n) \geq T_c$, that is

$$E_c(u_n) + \left(\tilde{\sigma}_n^{N-3} - 1 \right) A(u_n) + \left(\tilde{\sigma}_n^{N-1} - 1 \right) \left(P_c(u_n) - \frac{N-3}{N-1} A(u_n) \right) \geq T_c. \tag{5.6}$$

Clearly, $\tilde{\sigma}_n$ can be written as $\tilde{\sigma}_n = \left(\frac{P_c(u_n)}{-B_c(u_n)} + 1 \right)^{\frac{1}{2}}$ and using (b) and (5.5) it follows that $\lim\limits_{n\to\infty} \tilde{\sigma}_n = 1$. Then passing to the limit as $n \longrightarrow \infty$ in (5.6) and using the fact that $(A(u_n))_{n\geq 1}$ and $(P_c(u_n))_{n\geq 1}$ are bounded, we obtain $\liminf\limits_{n\to\infty} E_c(u_n) \geq T_c$. □

We can now state the main result of this section.

Theorem 5.3 *Let $(u_n)_{n\geq 1} \subset \mathcal{X} \setminus \{0\}$ be a sequence such that*

$$P_c(u_n) \longrightarrow 0 \qquad \text{and} \qquad E_c(u_n) \longrightarrow T_c \qquad \text{as } n \longrightarrow \infty.$$

There exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^N$ and $u \in \mathcal{C}$ such that

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad \text{and} \quad \varphi^2(|r_0 - u_{n_k}(\cdot + x_k)|) - r_0^2 \longrightarrow \varphi^2(|r_0 - u|) - r_0^2 \quad \text{in } L^2(\mathbf{R}^N).$$

Moreover, we have $E_c(u) = T_c$, that is u minimizes E_c in $\mathcal{C}$.

Proof. From Lemma 5.1 we know that $E_{GL}(u_n)$ is bounded. We have $\frac{2}{N-1}A(u_n) = E_c(u_n) - P_c(u_n) \longrightarrow T_c$ as $n \longrightarrow \infty$. Therefore

$$\lim_{n\to\infty} A(u_n) = \frac{N-1}{2}T_c \quad \text{and} \quad \liminf_{n\to\infty} E_{GL}(u_n) \geq \lim_{n\to\infty} A(u_n) = \frac{N-1}{2}T_c. \tag{5.7}$$

Passing to a subsequence if necessary, we may assume that there exists $\alpha_0 \geq \frac{N-1}{2}T_c$ such that

$$E_{GL}(u_n) \longrightarrow \alpha_0 \qquad \text{as } n \longrightarrow \infty. \tag{5.8}$$

We will use the concentration-compactness principle ([30]). We denote by $q_n(t)$ the concentration function of $E_{GL}(u_n)$, that is

$$q_n(t) = \sup_{y\in\mathbf{R}^N} \int_{B(y,t)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \, dx. \tag{5.9}$$

As in [30], it follows that there exists a subsequence of $((u_n, q_n))_{n\geq1}$, still denoted $((u_n, q_n))_{n\geq1}$, there exists a nondecreasing function $q : [0, \infty) \longrightarrow \mathbf{R}$ and there is $\alpha \in [0, \alpha_0]$ such that

$$q_n(t) \longrightarrow q(t) \text{ a.e on } [0,\infty) \text{ as } n \longrightarrow \infty \qquad \text{and} \qquad q(t) \longrightarrow \alpha \text{ as } t \longrightarrow \infty. \tag{5.10}$$

We claim that

$$\text{there is a nondecreasing sequnce } t_n \longrightarrow \infty \text{ such that } \lim_{n\to\infty} q_n(t_n) = \alpha. \tag{5.11}$$

To prove the claim, fix an increasing sequence $x_k \longrightarrow \infty$ such that $q_n(x_k) \longrightarrow q(x_k)$ as $n \longrightarrow \infty$ for any k. Then there exists $n_k \in \mathbf{N}$ such that $|q_n(x_k) - q(x_k)| < \frac{1}{k}$ for any $n \geq n_k$; clearly, we may assume that $n_k < n_{k+1}$ for all k. If $n_k \leq n < n_{k+1}$, put $t_n = x_k$. Then for $n_k \leq n < n_{k+1}$ we have

$$|q_n(t_n) - \alpha| = |q_n(x_k) - \alpha| \leq |q_n(x_k) - q(x_k)| + |q(x_k) - \alpha| \leq \frac{1}{k} + |q(x_k) - \alpha| \longrightarrow 0$$

as $k \longrightarrow \infty$ and (5.11) is proved.

Next we claim that

$$q_n(t_n) - q_n\left(\frac{t_n}{2}\right) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{5.12}$$

To see this, fix $\varepsilon > 0$. Take $y > 0$ such that $q(y) > \alpha - \frac{\varepsilon}{4}$ and $q_n(y) \longrightarrow q(y)$ as $n \longrightarrow \infty$. There is some $\tilde{n} \geq 1$ such that $q_n(y) > \alpha - \frac{\varepsilon}{2}$ for $n \geq \tilde{n}$. Then we can find $n_* \geq \tilde{n}$ such that $t_n > 2y$ for $n \geq n_*$, and consequently we have $q_n(\frac{t_n}{2}) \geq q_n(y) > \alpha - \frac{\varepsilon}{2}$. Therefore $\limsup_{n\to\infty} \left(q_n(t_n) - q_n(\frac{t_n}{2})\right) = \lim_{n\to\infty} q_n(t_n) - \liminf_{n\to\infty} q_n(\frac{t_n}{2}) < \varepsilon$. Since ε was arbitrary, (5.12) follows.

Our aim is to show that $\alpha = \alpha_0$ in (5.10). It follows from the next lemma that $\alpha > 0$.

Lemma 5.4 *Let $(u_n)_{n\geq1} \subset \mathcal{X}$ be a sequence satisfying*

a) $M_1 \leq E_{GL}(u_n) \leq M_2$ for some positive constants M_1, M_2.

b) $\lim_{n\to\infty} P_c(u_n) = 0$.

There exists $k > 0$ such that $\sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \geq k$ for all sufficiently large n.

Proof. We argue by contradiction and we suppose that the conclusion is false. Then there exists a subsequence (still denoted $(u_n)_{n\geq 1}$) such that

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \, dx = 0. \tag{5.13}$$

We will prove that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \left| V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx = 0. \tag{5.14}$$

Fix $\varepsilon > 0$. Assumptions (A1) and (A2) imply that there exists $\delta(\varepsilon) > 0$ such that

$$\left|V(|r_0 - z|^2) - a^2 \left(\varphi^2(|r_0 - z|) - r_0^2\right)^2 \right| \leq \varepsilon a^2 \left(\varphi^2(|r_0 - z|) - r_0^2\right)^2 \tag{5.15}$$

for any $z \in \mathbf{C}$ satisfying $\big| |r_0 - z| - r_0 \big| \leq \delta(\varepsilon)$ (see (4.2)). Therefore

$$\begin{aligned} &\int_{\{|\,|r_0-u_n|-r_0|\leq\delta(\varepsilon)\}} \left| V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx \\ &\leq \varepsilon a^2 \int_{\{|\,|r_0-u_n|-r_0|\leq\delta(\varepsilon)\}} \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \, dx \leq \varepsilon M_2. \end{aligned} \tag{5.16}$$

Assumption (A2) implies that there exists $C(\varepsilon) > 0$ such that

$$\left| V(|r_0 - z|^2) - a^2 \left(\varphi^2(|r_0 - z|) - r_0^2\right)^2 \right| \leq C(\varepsilon) \big|\, |r_0 - z| - r_0 \big|^{2p_0+2} \tag{5.17}$$

for any $z \in \mathbf{C}$ verifying $\big| |r_0 - z| - r_0 \big| \geq \delta(\varepsilon)$.

Let $w_n = \big|\, |r_0 - u_n| - r_0 \big|$. It is clear that $|w_n| \leq |u_n|$. Using the inequality $|\nabla |v|\,| \leq |\nabla v|$ a.e. for $v \in H^1_{loc}(\mathbf{R}^N)$, we infer that $w_n \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ and

$$\int_{\mathbf{R}^N} |\nabla w_n|^2 \, dx \leq M_2 \qquad \text{for any } n. \tag{5.18}$$

Using (5.17), Hölder's inequality, the Sobolev embedding and (5.18) we find

$$\begin{aligned} &\int_{\{|r_0-u_n|-r_0|>\delta(\varepsilon)\}} \left| V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx \\ &\leq C(\varepsilon) \int_{\{w_n>\delta(\varepsilon)\}} |w_n|^{2p_0+2} \, dx \\ &\leq C(\varepsilon) \left(\int_{\{w_n>\delta(\varepsilon)\}} |w_n|^{2^*} \, dx \right)^{\frac{2p_0+2}{2^*}} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\})\right)^{1-\frac{2p_0+2}{2^*}} \\ &\leq C(\varepsilon) C_S^{2p_0+2} ||\nabla w_n||^{2p_0+2}_{L^2(\mathbf{R}^N)} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\})\right)^{1-\frac{2p_0+2}{2^*}} \\ &\leq C(\varepsilon) C_S^{2p_0+2} M_2^{p_0+1} \left(\mathcal{L}^N(\{w_n > \delta(\varepsilon)\})\right)^{1-\frac{2p_0+2}{2^*}}. \end{aligned} \tag{5.19}$$

We claim that for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathcal{L}^N(\{w_n > \delta(\varepsilon)\}) = 0. \tag{5.20}$$

To prove the claim, we argue by contradiction and assume that there exist $\varepsilon_0 > 0$, a subsequence $(w_{n_k})_k \geq 1$ and $\gamma > 0$ such that $\mathcal{L}^N\left(\{w_{n_k} > \delta(\varepsilon_0)\}\right) \geq \gamma > 0$ for any $k \geq 1$. Since $||\nabla w_n||_{L^2(\mathbf{R}^N)}$ is bounded, using Lieb's lemma (see Lemma 6 p. 447 in [29] or Lemma 2.2 p. 101 in [10]), we infer that there exists $\beta > 0$ and $y_k \in \mathbf{R}^N$ such that $\mathcal{L}^N\left(\{w_{n_k} > \frac{\delta(\varepsilon_0)}{2}\} \cap B(y_k, 1)\right) \geq \beta$. Let η be as in (3.30). Then $w_{n_k}(x) \geq \frac{\delta(\varepsilon_0)}{2}$ implies $\left(\varphi^2(|r_0 - u_{n_k}(x)|) - r_0^2\right)^2 \geq \eta\left(\frac{\delta(\varepsilon_0)}{2}\right) > 0$. Therefore

$$\int_{B(y_k,1)} \left(\varphi^2(|r_0 - u_{n_k}(x)|) - r_0^2\right)^2 dx \geq \eta\left(\frac{\delta(\varepsilon_0)}{2}\right)\beta > 0$$

for any $k \geq 1$, and this clearly contradicts (5.13). Thus we have proved that (5.20) holds.

From (5.16), (5.19) and (5.20) it follows that

$$\int_{\mathbf{R}^N} \left| V(|r_0 - u_n|^2) - a^2\left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx \leq 2\varepsilon M_2$$

for all sufficiently large n. Thus (5.14) is proved.

From Lemma 5.2 we know that $\liminf_{n\to\infty} E_c(u_n) \geq T_c$. Combined with (b), this implies $\liminf_{n\to\infty} \frac{2}{N-1}A(u_n) \geq T_c$. Let $\sigma_0 = \sqrt{\frac{2(N-1)}{N-3}}$ and let $\tilde{u}_n = (u_n)_{1,\sigma_0}$. It is obvious that

$$\liminf_{n\to\infty} A(\tilde{u}_n) = \sigma_0^{N-3} \liminf_{n\to\infty} A(u_n) \geq \frac{N-1}{2}\sigma_0^{N-3} T_c. \tag{5.21}$$

Using assumption (a), (5.13) and (5.14) it is easy to see that

(5.22) there exist $\tilde{M}_1$, $\tilde{M}_2 > 0$ such that $\tilde{M}_1 \leq E_{GL}(\tilde{u}_n) \leq \tilde{M}_2$ for any n,

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |\nabla \tilde{u}_n|^2 + a^2\left(\varphi^2(|r_0 - \tilde{u}_n|) - r_0^2\right)^2 dx = 0 \quad \text{and} \tag{5.23}$$

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \left| V(|r_0 - \tilde{u}_n|^2) - a^2\left(\varphi^2(|r_0 - \tilde{u}_n|) - r_0^2\right)^2 \right| dx = 0. \tag{5.24}$$

It is clear that $P_c(u_n) = \frac{N-3}{N-1}\sigma_0^{3-N}A(\tilde{u}_n) + \sigma_0^{1-N}B_c(\tilde{u}_n)$ and then assumption (b) implies

$$\lim_{n\to\infty}\left(\frac{N-3}{N-1}\sigma_0^2 A(\tilde{u}_n) + B_c(\tilde{u}_n)\right) = \lim_{n\to\infty}\left(A(\tilde{u}_n) + E_c(\tilde{u}_n)\right) = 0. \tag{5.25}$$

Using (5.22), (5.23) and Lemma 3.2 we infer that there exists a sequence $h_n \longrightarrow 0$ and for each n there exists a minimizer v_n of $G^{\tilde{u}_n}_{h_n,\mathbf{R}^N}$ in $H^1_{\tilde{u}_n}(\mathbf{R}^N)$ such that $\delta_n := ||\,|v_n - r_0| - r_0||_{L^\infty(\mathbf{R}^N)} \longrightarrow 0$ as $n \longrightarrow \infty$. Then using Lemma 4.2 and the fact that $|c| < v_s = 2ar_0$ we obtain

$$E_{GL}(v_n) + cQ(v_n) \geq 0 \qquad \text{for all sufficiently large } n. \tag{5.26}$$

From (5.22) and (3.4) we obtain

$$|Q(\tilde{u}_n) - Q(v_n)| \leq \left(h_n^2 + h_n^{\frac{4}{N}}\tilde{M}_2^{\frac{2}{N}}\right)^{\frac{1}{2}}\tilde{M}_2 \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{5.27}$$

Since $E_{GL}(v_n) \le E_{GL}(\tilde{u}_n)$, it is clear that

$$\begin{aligned} E_c(\tilde{u}_n) &= E_{GL}(\tilde{u}_n) + cQ(\tilde{u}_n) + \int_{\mathbf{R}^N} V(|r_0 - \tilde{u}_n|^2) - a^2\left(\varphi^2(|r_0 - \tilde{u}_n|) - r_0^2\right)^2 dx \\ &\ge E_{GL}(v_n) + cQ(v_n) + c(Q(\tilde{u}_n) - Q(v_n)) \\ &\qquad - \int_{\mathbf{R}^N} \left| V(|r_0 - \tilde{u}_n|^2) - a^2\left(\varphi^2(|r_0 - \tilde{u}_n|) - r_0^2\right)^2 \right| dx \end{aligned}$$

Using the last inequality and (5.24), (5.26), (5.27) we infer that $\liminf_{n\to\infty} E_c(\tilde{u}_n) \ge 0$. Combined with (5.25), this gives $\limsup_{n\to\infty} A(\tilde{u}_n) \le 0$, which clearly contradicts (5.21). This completes the proof of Lemma 5.4. $\square$

Next we prove that we cannot have $\alpha \in (0, \alpha_0)$. To do this we argue again by contradiction and we assume that $0 < \alpha < \alpha_0$. Let t_n be as in (5.11) and let $R_n = \frac{t_n}{2}$. For each $n \ge 1$, fix $y_n \in \mathbf{R}^N$ such that $E_{GL}^{B(y_n,R_n)}(u_n) \ge q_n(R_n) - \frac{1}{n}$. Using (5.12), we have

$$\begin{aligned} \varepsilon_n &:= \int_{B(y_n,2R_n)\setminus B(y_n,R_n)} |\nabla u_n|^2 + a^2\left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \\ &\le q_n(2R_n) - \left(q_n(R_n) - \tfrac{1}{n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{5.28}$$

After a translation, we may assume that $y_n = 0$. Using Lemma 3.3 with $A = 2$, $R = R_n$, $\varepsilon = \varepsilon_n$, we infer that for all n sufficiently large there exist two functions $u_{n,1}$, $u_{n,2}$ having the properties (i)-(vi) in Lemma 3.3.

From Lemma 3.3 (iii) and (iv) we get $|E_{GL}(u_n) - E_{GL}(u_{n,1}) - E_{GL}(u_{n,2})| \le C\varepsilon_n$, while Lemma 3.3 (i) and (ii) implies $E_{GL}(u_{n,1}) \ge E_{GL}^{B(0,R_n)}(u_n) > q_n(R_n) - \frac{1}{n}$, respectively $E_{GL}(u_{n,2}) \ge E_{GL}^{\mathbf{R}^N\setminus B(0,2R_n)}(u_n) \ge E_{GL}(u_n) - q_n(2R_n)$. Taking into account (5.11), (5.12) and (5.28), we infer that

$$E_{GL}(u_{n,1}) \longrightarrow \alpha \qquad \text{and} \qquad E_{GL}(u_{n,2}) \longrightarrow \alpha_0 - \alpha \qquad \text{as } n \longrightarrow \infty. \tag{5.29}$$

By (5.28) and Lemma 3.3 (iii)−(vi) we obtain

$$|A(u_n) - A(u_{n,1}) - A(u_{n,2})| \longrightarrow 0, \tag{5.30}$$

$$|E_c(u_n) - E_c(u_{n,1}) - E_c(u_{n,2})| \longrightarrow 0, \qquad \text{and} \tag{5.31}$$

$$|P_c(u_n) - P_c(u_{n,1}) - P_c(u_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{5.32}$$

From (5.32) and the fact that $P_c(u_n) \longrightarrow 0$ we infer that $P_c(u_{n,1}) + P_c(u_{n,2}) \longrightarrow 0$ as $n \longrightarrow \infty$. Moreover, Lemmas 4.1 and 4.5 imply that the sequences $(P_c(u_{n,i}))_{n\ge 1}$ and $(E_c(u_{n,i}))_{n\ge 1}$ are bounded, $i = 1, 2$. Passing again to a subsequence (still denoted $(u_n)_{n\ge 1}$), we may assume that $\lim_{n\to\infty} P_c(u_{n,1}) = p_1$ and $\lim_{n\to\infty} P_c(u_{n,2}) = p_2$ where p_1, $p_2 \in \mathbf{R}$ and $p_1 + p_2 = 0$. There are only two possibilities: either $p_1 = p_2 = 0$, or one element of $\{p_1, p_2\}$ is negative.

If $p_1 = p_2 = 0$, then (5.29) and Lemma 5.2 imply that $\liminf_{n\to\infty} E_c(u_{n,i}) \ge T_c$, $i = 1, 2$. Using (5.31), we obtain $\liminf_{n\to\infty} E_c(u_n) \ge 2T_c$ and this clearly contradicts the assumption $E_c(u_n) \longrightarrow T_c$ in Theorem 5.3.

If $p_i < 0$, it follows from (5.29) and Lemma 4.8 (ii) that $\liminf_{n\to\infty} A(u_{n,i}) > \frac{N-1}{2} T_c$. Using (5.30) and the fact that $A \ge 0$, we obtain $\liminf_{n\to\infty} A(u_n) > \frac{N-1}{2} T_c$, which is in contradiction with (5.7).

We conclude that we cannot have $\alpha \in (0, \alpha_0)$.

So far we have proved that $\lim_{t\to\infty} q(t) = \alpha_0$. Proceeding as in [30], it follows that for each $n \geq 1$ there exists $x_n \in \mathbf{R}^N$ such that for any $\varepsilon > 0$ there is $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ satisfying

$$E_{GL}^{B(x_n,R_\varepsilon)}(u_n) > \alpha_0 - \varepsilon \qquad \text{for any } n \geq n_\varepsilon. \tag{5.33}$$

Let $\tilde{u}_n = u_n(\cdot + x_n)$, so that $\tilde{u}_n$ satisfies (5.33) with $B(0, R_\varepsilon)$ instead of $B(x_n, R_\varepsilon)$. Let $\chi \in C_c^\infty(\mathbf{C}, \mathbf{R})$ be as in Lemma 2.2 and denote $\tilde{u}_{n,1} = \chi(\tilde{u}_n)\tilde{u}_n$, $\tilde{u}_{n,1} = (1 - \chi(\tilde{u}_n))\tilde{u}_n$. Since $E_{GL}(\tilde{u}_n) = E_{GL}(u_n)$ is bounded, we infer from Lemma 2.2 that $(\tilde{u}_{n,1})_{n\geq1}$ is bounded in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, $(\tilde{u}_{n,2})_{n\geq1}$ is bounded in $H^1(\mathbf{R}^N)$ and $(E_{GL}(\tilde{u}_{n,i}))_{n\geq1}$ is bounded, $i = 1, 2$.

Using Lemma 2.1 we may write $r_0 - \tilde{u}_{n,1} = \rho_n e^{i\theta_n}$, where $\frac{1}{2}r_0 \leq \rho_n \leq \frac{3}{2}r_0$ and $\theta_n \in \mathcal{D}^{1,2}(\mathbf{R}^N)$. From (2.4) and (2.7) we find that $(\rho_n - r_0)_{n\geq1}$ is bounded in $H^1(\mathbf{R}^N)$ and $(\theta_n)_{n\geq1}$ is bounded in $\mathcal{D}^{1,2}(\mathbf{R}^N)$.

We infer that there exists a subsequence $(n_k)_{k\geq1}$ and there are functions $u_1 \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $u_2 \in H^1(\mathbf{R}^N)$, $\theta \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, $\rho \in r_0 + H^1(\mathbf{R}^N)$ such that

$$\tilde{u}_{n_k,1} \rightharpoonup u_1 \qquad \text{and} \qquad \theta_{n_k} \rightharpoonup \theta \qquad \text{weakly in } \mathcal{D}^{1,2}(\mathbf{R}^N),$$

$$\tilde{u}_{n_k,2} \rightharpoonup u_2 \qquad \text{and} \qquad \rho_{n_k} - r_0 \rightharpoonup \rho - r_0 \qquad \text{weakly in } H^1(\mathbf{R}^N),$$

$$\tilde{u}_{n_k,1} \longrightarrow u_1, \qquad \tilde{u}_{n_k,2} \longrightarrow u_2, \qquad \theta_{n_k} \longrightarrow \theta, \qquad \rho_{n_k} - r_0 \longrightarrow \rho - r_0$$

strongly in $L^p(K)$, $1 \leq p < 2^*$ for any compact set $K \subset \mathbf{R}^N$ and almost everywhere on $\mathbf{R}^N$. Since $\tilde{u}_{n_k,1} = r_0 - \rho_{n_k} e^{i\theta_{n_k}} \longrightarrow r_0 - \rho e^{i\theta}$ a.e., we have $r_0 - u_1 = \rho e^{i\theta}$ a.e. on $\mathbf{R}^N$.

Denoting $u = u_1 + u_2$, we see that $\tilde{u}_{n_k} \rightharpoonup u$ weakly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, $\tilde{u}_{n_k} \longrightarrow u$ a.e. on $\mathbf{R}^N$ and strongly in $L^p(K)$, $1 \leq p < 2^*$ for any compact set $K \subset \mathbf{R}^N$.

Since $E_{GL}(\tilde{u}_n)$ is bounded, it is clear that $\left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)_{k\geq1}$ is bounded in $L^2(\mathbf{R}^N)$ and converges a.e. on $\mathbf{R}^N$ to $\varphi^2(|r_0 - u|) - r_0^2$. From Lemma 4.8 p. 11 in [26] it follows that

$$\left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right) \rightharpoonup \varphi^2(|r_0 - u|) - r_0^2 \quad \text{weakly in } L^2(\mathbf{R}^N). \tag{5.34}$$

The weak convergence $\tilde{u}_{n_k} \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ implies

$$\int_{\mathbf{R}^N} \left|\frac{\partial u}{\partial x_j}\right|^2 dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^N} \left|\frac{\partial \tilde{u}_{n_k}}{\partial x_j}\right|^2 dx < \infty \qquad \text{for } j = 1, \ldots, N. \tag{5.35}$$

Using the a.e. convergence and Fatou's lemma we obtain

$$\int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2 dx \tag{5.36}$$

From (5.35) and(5.36) it follows that $u \in \mathcal{X}$ and $E_{GL}(u) \leq \liminf_{k\to\infty} E_{GL}(\tilde{u}_{n_k})$.

We will prove that

$$\lim_{k\to\infty} \int_{\mathbf{R}^N} V(|r_0 - \tilde{u}_{n_k}|^2)\, dx = \int_{\mathbf{R}^N} V(|r_0 - u|^2)\, dx \tag{5.37}$$

and

$$\lim_{k\to\infty} Q(\tilde{u}_{n_k}) = Q(u). \tag{5.38}$$

Fix $\varepsilon > 0$. Let R_ε be as in (5.33). Since $E_{GL}(\tilde{u}_{n_k}) \longrightarrow \alpha_0$ as $k \longrightarrow \infty$, it follows from (5.33) that there exists $k_\varepsilon \geq 1$ such that

$$E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(\tilde{u}_{n_k}) < 2\varepsilon \qquad \text{for any } k \geq k_\varepsilon. \tag{5.39}$$

As in (5.35)−(5.36), the weak convergence $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^N \setminus B(0,R_\varepsilon))$ implies

$$\int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |\nabla u|^2\,dx \le \liminf_{k\to\infty} \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |\nabla \tilde{u}_{n_k}|^2\,dx,$$

while the fact that $\tilde{u}_{n_k} \longrightarrow u$ a.e. on $\mathbf{R}^N$ and Fatou's lemma imply

$$\int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\,dx \le \liminf_{k\to\infty} \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2\,dx.$$

Therefore

$$E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(u) \le \liminf_{k\to\infty} E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(\tilde{u}_{n_k}) \le 2\varepsilon. \tag{5.40}$$

Let $v \in \mathcal{X}$ be a function satisfying $E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(v) \le 2\varepsilon$. As in the introduction, we write $V(s) = V(\varphi^2(\sqrt{s})) + W(s)$. Using (1.5) we find

$$\begin{aligned}\int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |V(\varphi^2(|r_0 - v|))|\,dx &\le C_1 \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|r_0 - v|) - r_0^2\right)^2\,dx \\ &\le \tfrac{C_1}{a^2} E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(v) \le \tfrac{2C_1}{a^2}\varepsilon.\end{aligned} \tag{5.41}$$

It is clear that $W(|r_0 - v(x)|^2) = 0$ if $|r_0 - v(x)| \le 2r_0$. On the other hand, $|r_0 - v(x)| > 2r_0$ implies $\left(\varphi^2(|r_0 - v(x)|) - r_0^2\right)^2 > 9r_0^4$, consequently

$$9r_0^4 \mathcal{L}^N(\{x \in \mathbf{R}^N \setminus B(0,R_\varepsilon) \mid |r_0 - v(x)| > 2r_0\}) \le \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|r_0 - v|) - r_0^2\right)^2\,dx \le \frac{2\varepsilon}{a^2}.$$

Using (1.7), Hölder's inequality, the above estimate and the Sobolev embedding we find

$$\begin{aligned}\int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |W(|r_0 - v|^2)|\,dx &\le C \int_{(\mathbf{R}^N \setminus B(0,R_\varepsilon)) \cap \{|r_0 - v| > 2r_0\}} |v|^{2p_0+2}\,dx \\ &\le C \left(\int_{\mathbf{R}^N} |v|^{2^*}\,dx \right)^{\frac{2p_0+2}{2^*}} \left(\mathcal{L}^N(\{x \in \mathbf{R}^N \setminus B(0,R_\varepsilon) \mid |r_0 - v(x)| > 2r_0\}) \right)^{1 - \frac{2p_0+2}{2^*}} \\ &\le C' ||\nabla v||_{L^2(\mathbf{R}^N)}^{2p_0+2} \varepsilon^{1 - \frac{2p_0+2}{2^*}} \le C' \left(E_{GL}(v)\right)^{p_0+1} \varepsilon^{1 - \frac{2p_0+2}{2^*}}.\end{aligned} \tag{5.42}$$

It is obvious that u and $\tilde{u}_{n_k}$ (with $k \ge k_\varepsilon$) satisfy (5.41) and (5.42). If $M > 0$ is such that $E_{GL}(u_n) \le M$ for any n, from (5.41) and (5.42) we infer that

$$\begin{aligned}&\int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |V(|r_0 - \tilde{u}_{n_k}|^2) - V(|r_0 - u|^2)|\,dx \\ &\le \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |V(|r_0 - \tilde{u}_{n_k}|^2)| + |V(|r_0 - u|^2)|\,dx \le C\varepsilon + CM^{p_0+1}\varepsilon^{1 - \frac{2p_0+2}{2^*}}.\end{aligned} \tag{5.43}$$

Since $z \longmapsto V(|r_0 - z|^2)$ is C^1, $|V(|r_0 - z|^2)| \le C(1 + |z|^{2p_0+2})$ and $\tilde{u}_{n_k} \longrightarrow u$ in $L^{2p_0+2}(B(0,R_\varepsilon))$ and almost everywhere, it follows that $V(|r_0 - \tilde{u}_{n_k}|^2) \longrightarrow V(|r_0 - u|^2)$ in $L^1(B(0,R_\varepsilon))$ (see, e.g., Theorem A2 p. 133 in [36]). Hence

$$\int_{B(0,R_\varepsilon)} |V(|r_0 - \tilde{u}_{n_k}|^2) - V(|r_0 - u|^2)|\,dx \le \varepsilon \qquad \text{if } k \text{ is sufficiently large.} \tag{5.44}$$

Since $\varepsilon > 0$ is arbitrary, (5.37) follows from (5.43) and (5.44).

From (2.6) we obtain $||(1-\chi^2(u_n))u_n||_{L^2(\mathbf{R}^N)} \le C||\nabla u_n||_{L^2(\mathbf{R}^N)}^{\frac{2^*}{2}} \le C\left(E_{GL}(u_n)\right)^{\frac{2^*}{4}}$. Using the Cauchy-Schwarz inequality and (5.39) we get

$$
\begin{aligned}
&\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \Big|(1-\chi^2(\tilde{u}_{n_k})\langle i\frac{\partial \tilde{u}_{n_k}}{\partial x_1}, \tilde{u}_{n_k}\rangle\Big|dx \\
&\le ||(1-\chi^2(u_n))u_n||_{L^2(\mathbf{R}^N)}||\tfrac{\partial \tilde{u}_{n_k}}{\partial x_1}||_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \le CM^{\frac{2^*}{4}}\sqrt{\varepsilon} \quad \text{for any } k \ge k_\varepsilon.
\end{aligned} \tag{5.45}
$$

From (2.7) we infer that

$$
||\rho_n^2 - r_0^2||_{L^2(\mathbf{R}^N)} \le C\left(E_{GL}(u_n) + ||\nabla u_n||^{2^*}_{L^2(\mathbf{R}^N)}\right)^{\frac{1}{2}} \le C\left(M + M^{\frac{2^*}{2}}\right)^{\frac{1}{2}}.
$$

Using (2.4) and (2.5) we obtain $\left|\frac{\partial \theta_n}{\partial x_1}\right| \le \frac{2}{r_0}\left|\frac{\partial(\chi(\tilde{u}_n)\tilde{u}_n)}{\partial x_1}\right| \le C\left|\frac{\partial \tilde{u}_n}{\partial x_1}\right|$ a.e. on $\mathbf{R}^N$ and then (5.39) implies $||\frac{\partial \theta_{n_k}}{\partial x_1}||_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \le C\sqrt{\varepsilon}$ for any $k \ge k_\varepsilon$. Using again the Cauchy-Schwarz inequality we find

$$
\begin{aligned}
&\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \Big|\left(\rho_{n_k}^2 - r_0^2\right)\frac{\partial \theta_{n_k}}{\partial x_1}\Big|dx \le ||\rho_{n_k}^2 - r_0^2||_{L^2(\mathbf{R}^N)}\Big|\Big|\frac{\partial \theta_{n_k}}{\partial x_1}\Big|\Big|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \\
&\le C\left(M + M^{\frac{2^*}{2}}\right)^{\frac{1}{2}}\sqrt{\varepsilon} \quad \text{for any } k \ge k_\varepsilon.
\end{aligned} \tag{5.46}
$$

It is obvious that the estimates (5.45) and (5.46) also hold with u instead of $\tilde{u}_{n_k}$.

Using the fact that $\tilde{u}_{n_k} \longrightarrow u$ and $\rho_{n_k} - r_0 \longrightarrow \rho - r_0$ in $L^2(B(0,R_\varepsilon))$ and a.e. and the dominated convergence theorem we infer that

$$
(1-\chi^2(\tilde{u}_{n_k}))\tilde{u}_{n_k} \longrightarrow (1-\chi^2(u))u \qquad \text{and } \rho_{n_k}^2 - r_0^2 \longrightarrow \rho^2 - r_0^2 \qquad \text{in } L^2(B(0,R_\varepsilon)).
$$

This information and the fact that $\frac{\partial \tilde{u}_{n_k}}{\partial x_1} \rightharpoonup \frac{\partial u}{\partial x_1}$ and $\frac{\partial \theta_{n_k}}{\partial x_1} \rightharpoonup \frac{\partial \theta}{\partial x_1}$ weakly in $L^2(B(0,R_\varepsilon))$ imply

$$
\int_{B(0,R_\varepsilon)} \langle i\frac{\partial \tilde{u}_{n_k}}{\partial x_1}, (1-\chi^2(\tilde{u}_{n_k}))\tilde{u}_{n_k}\rangle\, dx \longrightarrow \int_{B(0,R_\varepsilon)} \langle i\frac{\partial u}{\partial x_1}, (1-\chi^2(u))u\rangle\, dx \quad \text{and} \tag{5.47}
$$

$$
\int_{B(0,R_\varepsilon)} \left(\rho_{n_k}^2 - r_0^2\right)\frac{\partial \theta_{n_k}}{\partial x_1}\, dx \longrightarrow \int_{B(0,R_\varepsilon)} \left(\rho^2 - r_0^2\right)\frac{\partial \theta}{\partial x_1}\, dx. \tag{5.48}
$$

Using (5.45)$-$(5.48) and the representation formula (2.12) we infer that there is some $k_1(\varepsilon) \ge k_\varepsilon$ such that for any $k \ge k_1(\varepsilon)$ we have

$$
|Q(\tilde{u}_{n_k}) - Q(u)| \le C\left(M^{\frac{1}{2}} + M^{\frac{2^*}{4}}\right)\sqrt{\varepsilon},
$$

where C does not depend on $k \ge k_1(\varepsilon)$ and ε. Since $\varepsilon > 0$ is arbitrary, (5.38) is proved.

It is obvious that

$$
\begin{aligned}
&-cQ(\tilde{u}_{n_k}) - \int_{\mathbf{R}^N} V(|r_0 - \tilde{u}_{n_k}|^2)\, dx \\
&= \frac{N-3}{N-1}A(\tilde{u}_{n_k}) + \int_{\mathbf{R}^N}\Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\Big|^2 dx - P_c(\tilde{u}_{n_k}) \ge \frac{N-3}{N-1}A(\tilde{u}_{n_k}) - P_c(\tilde{u}_{n_k}).
\end{aligned}
$$

Passing to the limit as $k \longrightarrow \infty$ in this inequality and using (5.37), (5.38) and the fact that $A(u_n) \longrightarrow \frac{N-1}{2}T_c$, $P_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ we find

$$-cQ(u) - \int_{\mathbf{R}^N} V(|r_0 - u|^2)\, dx \geq \frac{N-3}{2}T_c > 0. \tag{5.49}$$

In particular, (5.49) implies that $u \neq 0$.

From (5.35) we get

$$A(u) \leq \liminf_{k\to\infty} A(\tilde{u}_{n_k}) = \frac{N-1}{2}T_c. \tag{5.50}$$

Using (5.35), (5.37) and (5.38) we find

$$P_c(u) \leq \liminf_{k\to\infty} P_c(\tilde{u}_{n_k}) = 0. \tag{5.51}$$

If $P_c(u) < 0$, from Lemma 4.8 (i) we get $A(u) > \frac{N-1}{2}T_c$, contradicting (5.50). Thus necessarily $P_c(u) = 0$, that is $u \in \mathcal{C}$. Since $A(v) \geq \frac{N-1}{2}T_c$ for any $v \in \mathcal{C}$, we infer from (5.50) that $A(u) = \frac{N-1}{2}T_c$, therefore $E_c(u) = T_c$ and u is a minimizer of E_c in $\mathcal{C}$.

It follows from the above that

$$A(u) = \frac{N-1}{2}T_c = \lim_{k\to\infty} A(\tilde{u}_{n_k}). \tag{5.52}$$

Since $P_c(u) = 0$, $\lim\limits_{k\to\infty} P_c(\tilde{u}_{n_k}) = 0$ and (5.37), (5.38) and (5.52) hold, it is obvious that

$$\int_{\mathbf{R}^N} \left|\frac{\partial u}{\partial x_1}\right|^2 dx = \lim_{k\to\infty} \int_{\mathbf{R}^N} \left|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\right|^2 dx. \tag{5.53}$$

Now (5.52) and (5.53) imply $\lim\limits_{k\to\infty} ||\nabla \tilde{u}_{n_k}||^2_{L^2(\mathbf{R}^N)} = ||\nabla u||^2_{L^2(\mathbf{R}^N)}$. Since $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ weakly in $L^2(\mathbf{R}^N)$, we infer that $\nabla \tilde{u}_{n_k} \longrightarrow \nabla u$ strongly in $L^2(\mathbf{R}^N)$, that is $\tilde{u}_{n_k} \longrightarrow u$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$.

Proceeding as in the proof of (5.37) we show that

$$\lim_{k\to\infty} \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2 dx = \int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2 dx. \tag{5.54}$$

Together with the weak convergence $\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \rightharpoonup \varphi^2(|r_0 - u|) - r_0^2$ in $L^2(\mathbf{R}^N)$ (see (5.34)), this implies $\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \longrightarrow \varphi^2(|r_0 - u|) - r_0^2$ strongly in $L^2(\mathbf{R}^N)$. The proof of Theorem 5.3 is complete. □

In order to prove that the minimizers provided by Theorem 5.3 solve equation (1.3), we need the following regularity result.

Lemma 5.5 *Let $N \geq 3$. Assume that the conditions (A1) and (A2) in the Introduction hold and that $u \in \mathcal{X}$ satisfies (1.3) in $\mathcal{D}'(\mathbf{R}^N)$. Then $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1,\infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for $p \in [2,\infty)$, $u \in C^{1,\alpha}(\mathbf{R}^N)$ for $\alpha \in [0,1)$ and $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.*

Proof. First we prove that for any $R > 0$ and $p \in [2,\infty)$ there exists $C(R,p) > 0$ (depending on u, but not on $x \in \mathbf{R}^N$) such that

$$\|u\|_{W^{2,p}(B(x,R))} \leq C(R,p) \qquad \text{for any } x \in \mathbf{R}^N. \tag{5.55}$$

We write $u = u_1 + u_2$, where u_1 and u_2 are as in Lemma 2.2. Then $|u_1| \leq \frac{r_0}{2}$, $\nabla u_1 \in L^2(\mathbf{R}^N)$ and $u_2 \in H^1(\mathbf{R}^N)$, hence for any $R > 0$ there exists $C(R) > 0$ such that

$$\|u\|_{H^1(B(x,R))} \leq C(R) \qquad \text{for any } x \in \mathbf{R}^N. \tag{5.56}$$

Let $\phi(x) = e^{-\frac{icx_1}{2}}(r_0 - u(x))$. It is easy to see that ϕ satisfies

$$\Delta\phi + \left(F(|\phi|^2) + \frac{c^2}{4}\right)\phi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{5.57}$$

Moreover, (5.56) holds for ϕ instead of u. From (5.56), (5.57), (3.18) and a standard bootstrap argument we infer that ϕ satisfies (5.55). (Note that assumption (A2) is needed for this bootstrap argument.) It is then clear that (5.55) also holds for u.

From (5.55), the Sobolev embeddings and Morrey's inequality (3.27) we find that u and ∇u are continuous and bounded on $\mathbf{R}^N$ and $u \in C^{1,\alpha}(\mathbf{R}^N)$ for $\alpha \in [0, 1)$. In particular, u is Lipschitz; since $u \in L^{2^*}(\mathbf{R}^N)$, we have necessarily $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

The boundedness of u implies that there is some $C > 0$ such that $\left|F(|r_0 - u|^2)(r_0 - u)\right| \leq C\left|\varphi^2(|r_0 - u|) - r_0^2\right|$ on $\mathbf{R}^N$. Therefore $F(|r_0 - u|^2)(r_0 - u) \in L^2 \cap L^\infty(\mathbf{R}^N)$. Since $\nabla u \in L^2(\mathbf{R}^N)$, from (1.3) we find $\Delta u \in L^2(\mathbf{R}^N)$. It is well known that $\Delta u \in L^p(\mathbf{R}^N)$ with $1 < p < \infty$ implies $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbf{R}^N)$ for any i, j (see, e.g., Theorem 3 p. 96 in [34]). Thus we get $\nabla u \in W^{1,2}(\mathbf{R}^N)$. Then the Sobolev embedding implies $\nabla u \in L^p(\mathbf{R}^N)$ for $p \in [2, 2^*]$. Repeating the previous argument, after an easy induction we find $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$. ☐

Proposition 5.6 *Assume that the conditions (A1) and (A2) in the introduction are satisfied. Let $u \in \mathcal{C}$ be a minimizer of E_c in $\mathcal{C}$. Then $u \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1, \infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^N)$ for $p \in [2, \infty)$ and u is a solution of (1.3).*

Proof. It is standard to prove that for any $R > 0$, $J_u(v) = \int_{\mathbf{R}^N} V(|r_0 - u - v|^2)\,dx$ is a C^1 functional on $H_0^1(B(0, R))$ and $J_u'(v).w = 2\int_{\mathbf{R}^N} F(|r_0 - u - v|^2)\langle r_0 - u - v, w\rangle\,dx$ (see, e.g., Lemma 17.1 p. 64 in [26] or the appendix A in [36]). It follows easily that for any $R > 0$, the functionals $\tilde{P}_c(v) = P_c(u + v)$ and $\tilde{E}_c(v) = E_c(u + v)$ are C^1 on $H_0^1(B(0, R))$. We divide the proof of Proposition 5.6 into several steps.

Step 1. There exists a function $w \in C_c^1(\mathbf{R}^N)$ such that $\tilde{P}_c'(0).w \neq 0$.

To prove this, we argue by contradiction and we assume that the above statement is false. Then u satisfies

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{N-3}{N-1}\left(\sum_{k=2}^{N}\frac{\partial^2 u}{\partial x_k^2}\right) + icu_{x_1} + F(|r_0 - u|^2)(r_0 - u) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{5.58}$$

Let $\sigma = \sqrt{\frac{N-1}{N-3}}$. It is not hard to see that $u_{1,\sigma}$ satisfies (1.3) in $\mathcal{D}'(\mathbf{R}^N)$. Hence the conclusion of Lemma 5.5 holds for $u_{1,\sigma}$ (and thus for u). This regularity is enough to prove that u satisfies the Pohozaev identity

$$\int_{\mathbf{R}^N}\left|\frac{\partial u_{1,\sigma}}{\partial x_1}\right|^2 dx + \frac{N-3}{N-1}\int_{\mathbf{R}^N}\sum_{k=2}^{N}\left|\frac{\partial u_{1,\sigma}}{\partial x_k}\right|^2 dx + cQ(u_{1,\sigma}) + \int_{\mathbf{R}^N} V(|r_0 - u_{1,\sigma}|^2)\,dx = 0. \tag{5.59}$$

To prove (5.59), we multiply (1.3) by $\sum_{k=2}^N \tilde{\chi}(\frac{x}{n})\frac{\partial u_{1,\sigma}}{\partial x_k}$, where $\tilde{\chi} \in C_c^\infty(\mathbf{R}^N)$ is a cut-off function such that $\tilde{\chi} = 1$ on $B(0, 1)$ and $\operatorname{supp}(\tilde{\chi}) \subset B(0, 2)$, we integrate by parts, then we let $n \longrightarrow \infty$; see the proof of Proposition 4.1 and equation (4.13) in [33] for details.

Since $\sigma = \sqrt{\frac{N-1}{N-3}}$, (5.59) is equivalent to $\left(\frac{N-3}{N-1}\right)^2 A(u) + B_c(u) = 0$. On the other hand we have $P_c(u) = \frac{N-3}{N-1}A(u) + B_c(u) = 0$ and we infer that $A(u) = 0$. But this contradicts the fact that $A(u) = T_c > 0$ and the proof of step 1 is complete.

Step 2. Existence of a Lagrange multiplier.

Let w be as above and let $v \in H^1(\mathbf{R}^N)$ be a function with compact support such that $\tilde{P}_c'(0).v = 0$. For $s, t \in \mathbf{R}$, put $\Phi(t,s) = P_c(u + tv + sw) = \tilde{P}_c(tv + sw)$, so that $\Phi(0,0) = 0$, $\frac{\partial \Phi}{\partial t}(0,0) = \tilde{P}_c'(0).v = 0$ and $\frac{\partial \Phi}{\partial s}(0,0) = \tilde{P}_c'(0).w \neq 0$. The implicit function theorem implies that there exist $\delta > 0$ and a C^1 function $\eta : (-\delta, \delta) \longrightarrow \mathbf{R}$ such that $\eta(0) = 0$, $\eta'(0) = 0$ and $P_c(u + tv + \eta(t)w) = P_c(u) = 0$ for $t \in (-\delta, \delta)$. Since u is a minimizer of A in $\mathcal{C}$, the function $t \longmapsto A(u + tv + \eta(t)w)$ achieves a minimum at $t = 0$. Differentiating at $t = 0$ we get $A'(u).v = 0$.

Hence $A'(u).v = 0$ for any $v \in H^1(\mathbf{R}^N)$ with compact support satisfying $\tilde{P}_c'(0).v = 0$. Taking $\alpha = \frac{A'(u).w}{\tilde{P}_c'(0).w}$ (where w is as in step 1), we see that

$$(5.60) \qquad A'(u).v = \alpha P_c'(u).v \quad \text{for any } v \in H^1(\mathbf{R}^N) \text{ with compact support.}$$

Step 3. We have $\alpha < 0$.

To see this, we argue by contradition. Suppose that $\alpha > 0$. Let w be as in step 1. We may assume that $P_c'(u).w > 0$. From (5.60) we obtain $A'(u).w > 0$. Since $A'(u).w = \lim\limits_{t \to 0} \frac{A(u+tw) - A(u)}{t}$ and $P_c'(u).w = \lim\limits_{t \to 0} \frac{P_c(u+tw) - P_c(u)}{t}$, we see that for $t < 0$, t sufficiently close to 0 we have $u + tw \neq 0$, $P_c(u + tw) < P_c(u) = 0$ and $A(u + tw) < A(u) = \frac{N-1}{2} T_c$. But this contradicts Lemma 4.8 (i). Therefore $\alpha \leq 0$.

Assume that $\alpha = 0$. Then (5.60) implies

$$(5.61) \qquad \int_{\mathbf{R}^N} \sum_{k=2}^{N} \langle \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \rangle \, dx = 0 \quad \text{for any } v \in H^1(\mathbf{R}^N) \text{ with compact support.}$$

Let $\tilde{\chi} \in C_c^\infty(\mathbf{R}^N)$ be such that $\chi = 1$ on $B(0,1)$ and $\text{supp}(\tilde{\chi}) \subset B(0,2)$. Put $v_n(x) = \chi(\frac{x}{n}) u(x)$, so that $\nabla v_n(x) = \frac{1}{n} \nabla \tilde{\chi}(\frac{x}{n}) u + \tilde{\chi}(\frac{x}{n}) \nabla u$. It is easy to see that $\tilde{\chi}(\frac{\cdot}{n}) \nabla u \longrightarrow \nabla u$ in $L^2(\mathbf{R}^N)$ and $\frac{1}{n} \nabla \tilde{\chi}(\frac{\cdot}{n}) u \rightharpoonup 0$ weakly in $L^2(\mathbf{R}^N)$. Replacing v by v_n in (5.61) and passing to the limit as $n \longrightarrow \infty$ we get $A(u) = 0$, which contradicts the fact that $A(u) = \frac{N-1}{2} T_c$. Hence we cannot have $\alpha = 0$. Thus necessarily $\alpha < 0$.

Step 4. Conclusion.

Since $\alpha < 0$, it follows from (5.60) that u satisfies

$$(5.62) \qquad -\frac{\partial^2 u}{\partial x_1^2} - \left(\frac{N-3}{N-1} - \frac{1}{\alpha} \right) \sum_{k=2}^{N} \frac{\partial^2 u}{\partial x_k^2} + icu_{x_1} + F(|r_0 - u|^2)(r_0 - u) = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

Let $\sigma_0 = \left(\frac{N-3}{N-1} - \frac{1}{\alpha} \right)^{-\frac{1}{2}}$. It is easy to see that u_{1,σ_0} satisfies (1.3) in $\mathcal{D}'(\mathbf{R}^N)$. Therefore the conclusion of Lemma 5.5 holds for u_{1,σ_0} (and consequently for u). Then Proposition 4.1 in [33] implies that u_{1,σ_0} satisfies the Pohozaev identity $\frac{N-3}{N-1} A(u_{1,\sigma_0}) + B_c(u_{1,\sigma_0}) = 0$, or equivalently $\frac{N-3}{N-1} \sigma_0^{N-3} A(u) + \sigma_0^{N-1} B_c(u) = 0$, which implies

$$\frac{N-3}{N-1} \left(\frac{N-3}{N-1} - \frac{1}{\alpha} \right) A(u) + B_c(u) = 0.$$

On the other hand we have $P_c(u) = \frac{N-3}{N-1} A(u) + B_c(u) = 0$. Since $A(u) > 0$, we get $\frac{N-3}{N-1} - \frac{1}{\alpha} = 1$. Then coming back to (5.62) we see that u satisfies (1.3). □

6 The case $N = 3$

This section is devoted to the proof of Theorem 1.1 in space dimension $N = 3$. We only indicate the differences with respect to the case $N \geq 4$. Clearly, if $N = 3$ we have $P_c = B_c$. For $v \in \mathcal{X}$ we denote

$$D(v) = \int_{\mathbf{R}^3} \left| \frac{\partial v}{\partial x_1} \right|^2 dx + a^2 \int_{\mathbf{R}^3} \left(\varphi^2(|r_0 - v|) - r_0^2\right)^2 dx.$$

For any $v \in \mathcal{X}$ and $\sigma > 0$ we have

$$(6.1) \qquad A(v_{1,\sigma}) = A(v), \qquad B_c(v_{1,\sigma}) = \sigma^2 B_c(v) \qquad \text{and} \qquad D(v_{1,\sigma}) = \sigma^2 D(v).$$

If $N = 3$ we cannot have a result similar to Lemma 5.1. To see this consider $u \in \mathcal{C}$, so that $B_c(u) = 0$. Using (6.1) we see that $u_{1,\sigma} \in \mathcal{C}$ for any $\sigma > 0$ and we have $E_c(u_{1,\sigma}) = A(u) + \sigma^2 B_c(u) = A(u)$, while $E_{GL}(u_{1,\sigma}) = A(u) + \sigma^2 D(u) \longrightarrow \infty$ as $\sigma \longrightarrow \infty$.

However, for any $u \in \mathcal{C}$ there exists $\sigma > 0$ such that $D(u_{1,\sigma}) = 1$ (and obviously $u_{1,\sigma} \in \mathcal{C}$, $E_c(u_{1,\sigma}) = E_c(u)$). Since $\mathcal{C} \neq \emptyset$ and $T_c = \inf\{E_c(u) \mid u \in \mathcal{C}\}$, we see that there exists a sequence $(u_n)_{n\geq 1} \subset \mathcal{C}$ such that

$$(6.2) \qquad D(u_n) = 1 \qquad \text{and} \qquad E_c(u_n) = A(u_n) \longrightarrow T_c \quad \text{as } n \longrightarrow \infty.$$

In particular, (6.2) implies $E_{GL}(u_n) \longrightarrow T_c + 1$ as $n \longrightarrow \infty$.

The following result is the equivalent of Lemma 5.2 in the case $N = 3$.

Lemma 6.1 *Let $N = 3$ and let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence satisfying*
a) There exists $C > 0$ such that $D(u_n) \geq C$ for any n, and
b) $B_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.
Then $\liminf_{n\to\infty} E_c(u_n) = \liminf_{n\to\infty} A(u_n) \geq S_c$, where S_c is given by (4.22).

Proof. It suffices to prove that for any $k > 0$ we have

$$(6.3) \qquad \liminf_{n\to\infty} A(u_n) \geq E_{c,min}(k).$$

Fix $k > 0$. Let $n \geq 1$. If $A(u_n) \geq k$, by Lemma 4.6 (iii) we have $A(u_n) \geq k > E_{c,min}(k)$. If $A(u_n) < k$, since $E_{GL}((u_n)_{1,\sigma}) = A(u_n) + \sigma^2 D(u_n)$ we see that there exists $\sigma_n > 0$ such that $E_{GL}((u_n)_{1,\sigma_n}) = k$. Obviously, we have $\sigma_n^2 D(u_n) < k$, hence $\sigma_n^2 \leq \frac{k}{C}$ by (a). It is clear that $E_c((u_n)_{1,\sigma_n}) = A(u_n) + \sigma_n^2 B_c(u_n) \geq E_{c,min}(k)$, therefore $A(u_n) \geq E_{c,min}(k) - \sigma_n^2 |B_c(u_n)| \geq E_{c,min}(k) - \frac{k}{C}|B_c(u_n)|$. Passing to the limit as $n \longrightarrow \infty$ we obtain (6.3). Since $k > 0$ is arbitrary, Lemma 6.1 is proved. □

Let

$$\begin{aligned}\Lambda_c \; = \; \{\lambda \in \mathbf{R} \mid & \text{ there exists a sequence } (u_n)_{n\geq 1} \subset \mathcal{X} \text{ such that} \\ & D(u_n) \geq 1, \ B_c(u_n) \longrightarrow 0 \text{ and } A(u_n) \longrightarrow \lambda \text{ as } n \longrightarrow \infty\}.\end{aligned}$$

Using a scaling argument, we see that

$$\begin{aligned}\Lambda_c \; = \; \{\lambda \in \mathbf{R} \mid & \text{ there exist a sequence } (u_n)_{n\geq 1} \subset \mathcal{X} \text{ and } C > 0 \text{ such that} \\ & D(u_n) \geq C, \ B_c(u_n) \longrightarrow 0 \text{ and } A(u_n) \longrightarrow \lambda \text{ as } n \longrightarrow \infty\}.\end{aligned}$$

Let $\lambda_c = \inf \Lambda_c$. From (6.2) it follows that $T_c \in \Lambda_c$. It is standard to prove that Λ_c is closed in $\mathbf{R}$, hence $\lambda_c \in \Lambda_c$. From Lemma 6.1 we obtain

$$(6.4) \qquad S_c \leq \lambda_c \leq T_c.$$

The main result of this section is as follows.

Theorem 6.2 *Let $N = 3$ and let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that*

$$(6.5) \qquad D(u_n) \longrightarrow 1, \quad B_c(u_n) \longrightarrow 0 \quad \text{and} \quad A(u_n) \longrightarrow \lambda_c \quad \text{as } n \longrightarrow \infty.$$

There exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^3$ and $u \in \mathcal{C}$ such that

$$\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u \quad \text{and} \quad |r_0 - u_{n_k}(\cdot + x_k)|^2 - r_0^2 \longrightarrow |r_0 - u|^2 - r_0^2 \quad \text{in } L^2(\mathbf{R}^3).$$

Moreover, we have $E_c(u) = A(u) = T_c = \lambda_c$ and u minimizes E_c in $\mathcal{C}$.

Proof. By (6.5) we have $E_{GL}(u_n) = A(u_n) + D(u_n) \longrightarrow \lambda_c + 1$ as $n \longrightarrow \infty$. Let $q_n(t)$ be the concentration function of $E_{GL}(u_n)$, as in (5.9). Proceeding as in the proof of Theorem 5.3, we infer that there exist a subsequence of $(u_n, q_n)_{n\geq 1}$, still denoted $(u_n, q_n)_{n\geq 1}$, a nondecreasing function $q : [0, \infty) \longrightarrow [0, \infty)$ and $\alpha \in [0, \lambda_c + 1]$ such that (5.10) holds. We see also that there exists a sequence $t_n \longrightarrow \infty$ satisfying (5.11) and (5.12).

Clearly, our aim is to prove that $\alpha = \lambda_c + 1$. The next result implies that $\alpha > 0$.

Lemma 6.3 *Assume that $N = 3$, $0 \leq c < v_s$ and let $(u_n)_{n\geq 1} \subset \mathcal{X}$ be a sequence such that $D(u_n) \longrightarrow 1$, $B_c(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sup\limits_{n\geq 1} E_{GL}(u_n) = M < \infty$.*

There exists $k > 0$ such that $\sup\limits_{y\in\mathbf{R}^3} \int_{B(y,1)} |\nabla u_n|^2 + a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \geq k$ for all sufficiently large n.

Proof. We argue by contradiction and assume that the conclusion of Lemma 6.3 is false. Then there exists a subsequence, still denoted $(u_n)_{n\geq 1}$, such that

$$(6.6) \qquad \sup_{y\in\mathbf{R}^3} E_{GL}^{B(y,1)}(u_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

Exactly as in Lemma 5.4 we prove that (5.14) holds, that is

$$(6.7) \qquad \lim_{n\to\infty} \int_{\mathbf{R}^3} \left| V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 \right| dx = 0.$$

Using (6.7) and the assumptions of Lemma 6.3 we find

$$(6.8) \quad cQ(u_n) = B_c(u_n) - D(u_n) - \int_{\mathbf{R}^3} V(|r_0 - u_n|^2) - a^2 \left(\varphi^2(|r_0 - u_n|) - r_0^2\right)^2 dx \longrightarrow -1$$

as $n \longrightarrow \infty$. If $c = 0$, (6.8) gives a contradiction and Lemma 6.3 is proved. From now on we assume that $0 < c < v_s$.

Fix $c_1 \in (c, v_s)$, then fix $\sigma > 0$ such that

$$(6.9) \qquad \sigma^2 > \frac{Mc}{c_1 - c}.$$

A simple change of variables shows that $\tilde{M} := \sup\limits_{n\geq 1} E_{GL}((u_n)_{1,\sigma}) < \infty$ and (6.7) holds with $(u_n)_{1,\sigma}$ instead of u_n. It is easy to see that $((u_n)_{1,\sigma})_{n\geq 1}$ also satisfies (6.6). Using Lemma 3.2 we infer that there exists a sequence $h_n \longrightarrow 0$ and for each n there exists a minimizer v_n of $G_{h_n,\mathbf{R}^3}^{(u_n)_{1,\sigma}}$ in $H^1_{(u_n)_{1,\sigma}}(\mathbf{R}^3)$ such that

$$(6.10) \qquad \| \, |v_n - r_0| - r_0 \|_{L^\infty(\mathbf{R}^3)} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

From (3.4) we obtain

$$|Q((u_n)_{1,\sigma}) - Q(v_n)| \leq \left(h_n^2 + h_n^{\frac{4}{3}} \tilde{M}^{\frac{2}{3}}\right)^{\frac{1}{2}} \tilde{M} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{6.11}$$

Using (6.10), the fact that $0 < c_1 < 2ar_0$ and Lemma 4.2 we infer that for all sufficiently large n we have

$$E_{GL}(v_n) + c_1 Q(v_n) \geq 0. \tag{6.12}$$

Since $E_{GL}(v_n) \leq E_{GL}((u_n)_{1,\sigma})$, for large n we have

$$\begin{aligned}
0 &\leq E_{GL}(v_n) + c_1 Q(v_n) \\
&\leq E_{GL}((u_n)_{1,\sigma}) + c_1 Q((u_n)_{1,\sigma}) + c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| \\
&= A(u_n) + B_c((u_n)_{1,\sigma}) + (c_1 - c) Q((u_n)_{1,\sigma}) + c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| \\
&\quad + \int_{\mathbf{R}^3} a^2 \left(\varphi^2(|r_0 - (u_n)_{1,\sigma}|) - r_0^2\right)^2 - V(|r_0 - (u_n)_{1,\sigma}|^2)\, dx \\
&= A(u_n) + \sigma^2 B_c(u_n) + \sigma^2 (c_1 - c) Q(u_n) + a_n \\
&\leq M + \sigma^2 B_c(u_n) + \sigma^2 (c_1 - c) Q(u_n) + a_n,
\end{aligned} \tag{6.13}$$

where

$$a_n = c_1 |Q((u_n)_{1,\sigma}) - Q(v_n)| + \int_{\mathbf{R}^3} a^2 \left(\varphi^2(|r_0 - (u_n)_{1,\sigma}|) - r_0^2\right)^2 - V(|r_0 - (u_n)_{1,\sigma}|^2)\, dx.$$

From (6.7) and (6.11) we infer that $\lim_{n\to\infty} a_n = 0$. Then passing to the limit as $n \longrightarrow \infty$ in (6.13), using (6.8) and the fact that $\lim_{n\to\infty} B_c(u_n) = 0$ we find $0 \leq M - \sigma^2 \frac{c_1 - c}{c}$. The last inequality clearly contradicts the choice of σ in (6.9). This contradiction shows that (6.6) cannot hold and Lemma 6.3 is proved. □

Next we show that we cannot have $\alpha \in (0, \lambda_c + 1)$. We argue again by contradiction and we assume that $\alpha \in (0, \lambda_c + 1)$. Proceeding exactly as in the proof of Theorem 5.3 and using Lemma 3.3, we infer that for each n sufficiently large there exist two functions $u_{n,1}$, $u_{n,2}$ having the following properties:

$$E_{GL}(u_{n,1}) \longrightarrow \alpha, \qquad E_{GL}(u_{n,1}) \longrightarrow \lambda_c + 1 - \alpha, \tag{6.14}$$

$$|A(u_n) - A(u_{n,1}) - A(u_{n,2})| \longrightarrow 0, \tag{6.15}$$

$$|B_c(u_n) - B_c(u_{n,1}) - B_c(u_{n,2})| \longrightarrow 0, \tag{6.16}$$

$$|D(u_n) - D(u_{n,1}) - D(u_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{6.17}$$

Since $(E_{GL}(u_{n,i}))_{n\geq 1}$ are bounded, from Lemmas 4.1 and 4.5 we see that $B_c(u_{n,i}))_{n\geq 1}$ are bounded. Moreover, by (6.16) we have $\lim_{n\to\infty} (B_c(u_{n,1}) + B_c(u_{n,2})) = \lim_{n\to\infty} B_c(u_n) = 0$. Similarly, $(D(u_{n,i}))_{n\geq 1}$ are bounded and $\lim_{n\to\infty} (D(u_{n,1}) + D(u_{n,2})) = \lim_{n\to\infty} D(u_n) = 1$. Passing again to a subsequence (still denoted $(u_n)_n \geq 1$), we may assume that

$$\lim_{n\to\infty} B_c(u_{n,1}) = b_1, \qquad \lim_{n\to\infty} B_c(u_{n,2}) = b_2, \qquad \text{where } b_i \in \mathbf{R},\ b_1 + b_2 = 0, \tag{6.18}$$

$$\lim_{n\to\infty} D(u_{n,1}) = d_1, \qquad \lim_{n\to\infty} D(u_{n,2}) = d_2, \qquad \text{where } d_i \geq 0,\ d_1 + d_2 = 1. \tag{6.19}$$

From (6.18) it follows that either $b_1 = b_2 = 0$, or one of b_1 or b_2 is negative.

Case 1. If $b_1 = b_2 = 0$, we distinguish two subcases:

Subcase 1a. We have $d_1 > 0$ and $d_2 > 0$. Let $\sigma_i = \frac{2}{\sqrt{d_i}}$, $i = 1, 2$. Then $D((u_{n,i})_{1,\sigma_i}) = \sigma_i^2 D(u_{n,i}) \longrightarrow 4$ and $B_c((u_{n,i})_{1,\sigma_i}) = \sigma_i^2 B_c(u_{n,i}) \longrightarrow 0$ as $n \longrightarrow \infty$. From (6.1) and the definition of λ_c it follows that $\liminf_{n\to\infty} A(u_{n,i}) = \liminf_{n\to\infty} A((u_{n,i})_{1,\sigma_i}) \geq \lambda_c$, $i = 1, 2$. Then (6.15) implies

$$\liminf_{n\to\infty} A(u_n) \geq \liminf_{n\to\infty} A(u_{n,1}) + \liminf_{n\to\infty} A(u_{n,2}) \geq 2\lambda_c$$

an this is a contradiction because by (6.5) we have $\lim_{n\to\infty} A(u_n) = \lambda_c$.

Subcase 1b. One of d_i's is zero, say $d_1 = 0$. Then necessarily $d_2 = 1$, that is $\lim_{n\to\infty} D(u_{n,2}) = 1$. Since $E_{GL}(u_{n,2}) = A(u_{n,2}) + D(u_{n,2}) \longrightarrow 1 + \lambda_c - \alpha$ as $n \longrightarrow \infty$, we infer that $\lim_{n\to\infty} A(u_{n,2}) = \lambda_c - \alpha$. Hence $D(u_{n,2}) \longrightarrow 1$, $B_c(u_{n,2}) \longrightarrow 0$ and $A(u_{n,2}) \longrightarrow \lambda_c - \alpha$ as $n \longrightarrow \infty$, which implies $\lambda_c - \alpha \in \Lambda_c$. Since $\alpha > 0$, this contradicts the definition of λ_c.

Case 2. One of b_i's is negative, say $b_1 < 0$. From Lemma 4.8 (ii) we get $\liminf_{n\to\infty} A(u_{n,1}) > T_c \geq \lambda_c$ and then using (6.15) we find $\liminf_{n\to\infty} A(u_n) > \lambda_c$, in contradiction with (6.5).

Consequently in all cases we get a contradiction and this proves that we cannot have $\alpha \in (0, \lambda_c + 1)$.

Up to now we have proved that $\lim_{t\to\infty} q(t) = \lambda_c + 1$, that is "concentration" occurs.

Proceeding as in the case $N \geq 4$, we see that there exist a subsequence $(u_{n_k})_{k\geq 1}$, a sequence of points $(x_k)_{k\geq 1} \subset \mathbf{R}^3$ and $u \in \mathcal{X}$ such that, denoting $\tilde{u}_{n_k}(x) = u_{n_k}(x + x_k)$, we have:

$$\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u \text{ and } \varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \rightharpoonup \varphi^2(|r_0 - u|) - r_0^2 \text{ weakly in } L^2(\mathbf{R}^3), \tag{6.20}$$

$$\tilde{u}_{n_k} \longrightarrow u \quad \text{in } L^p_{loc}(\mathbf{R}^3) \text{ for } 1 \leq p < 6 \text{ and a.e. on } \mathbf{R}^3, \tag{6.21}$$

$$\int_{\mathbf{R}^3} V(|r_0 - \tilde{u}_{n_k}|^2)\, dx \longrightarrow \int_{\mathbf{R}^3} V(|r_0 - u|^2)\, dx, \tag{6.22}$$

$$\int_{\mathbf{R}^3} \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2\, dx \longrightarrow \int_{\mathbf{R}^3} \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\, dx, \tag{6.23}$$

$$Q(\tilde{u}_{n_k}) \longrightarrow Q(u) \qquad \text{as } k \longrightarrow \infty. \tag{6.24}$$

Passing to the limit as $k \longrightarrow \infty$ in the identity

$$\int_{\mathbf{R}^3} V(|r_0 - \tilde{u}_{n_k}|^2) - a^2 \left(\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2\right)^2\, dx + cQ(\tilde{u}_{n_k}) = B_c(\tilde{u}_{n_k}) - D(\tilde{u}_{n_k}),$$

using (6.22)–(6.24) and the fact that $B_c(\tilde{u}_{n_k}) \longrightarrow 0$, $D(\tilde{u}_{n_k}) \longrightarrow 1$ we get

$$\int_{\mathbf{R}^3} V(|r_0 - u|^2) - a^2 \left(\varphi^2(|r_0 - u|) - r_0^2\right)^2\, dx + cQ(u) = -1.$$

Thus $u \neq 0$.

From the weak convergence $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^3)$ we get

$$\int_{\mathbf{R}^3} \Big|\frac{\partial u}{\partial x_j}\Big|^2 dx \leq \liminf_{k\to\infty} \int_{\mathbf{R}^3} \Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_j}\Big|^2 dx \qquad \text{for } j = 1, \dots, N. \tag{6.25}$$

In particular, we have

$$A(u) \leq \lim_{k\to\infty} A(\tilde{u}_{n_k}) = \lambda_c. \tag{6.26}$$

From (6.22), (6.24) and (6.25) we obtain

$$B_c(u) \leq \lim_{k\to\infty} B_c(\tilde{u}_{n_k}) = 0. \tag{6.27}$$

Since $u \neq 0$, (6.27) and Lemma 4.8 (i) imply $A(u) \geq T_c$. Then using (6.26) and the fact that $\lambda_c \leq T_c$, we infer that necessarily

$$A(u) = T_c = \lambda_c = \lim_{k\to\infty} A(\tilde{u}_{n_k}). \tag{6.28}$$

The fact that $B_c(\tilde{u}_{n_k}) \longrightarrow 0$, (6.22) and (6.24) imply that $\left(\int_{\mathbf{R}^3} \Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\Big|^2 dx \right)_{k\geq 1}$ converges. If $\int_{\mathbf{R}^3} \Big|\frac{\partial u}{\partial x_1}\Big|^2 dx < \lim_{k\to\infty} \int_{\mathbf{R}^3} \Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\Big|^2 dx$, we get $B_c(u) < \lim_{k\to\infty} B_c(\tilde{u}_{n_k}) = 0$ in (6.27) and then Lemma 4.8 (i) implies $A(u) > T_c$, a contradiction. Taking (6.25) into account, we see that necessarily

$$\int_{\mathbf{R}^3} \Big|\frac{\partial u}{\partial x_1}\Big|^2 dx = \lim_{k\to\infty} \int_{\mathbf{R}^3} \Big|\frac{\partial \tilde{u}_{n_k}}{\partial x_1}\Big|^2 dx \qquad \text{and} \qquad B_c(u) = 0. \tag{6.29}$$

Thus we have proved that $u \in \mathcal{C}$ and $||\nabla u||_{L^2(\mathbf{R}^3)} = \lim_{k\to\infty} ||\nabla \tilde{u}_{n_k}||_{L^2(\mathbf{R}^3)}$. Combined with the weak convergence $\nabla \tilde{u}_{n_k} \rightharpoonup \nabla u$ in $L^2(\mathbf{R}^3)$, this implies the strong convergence $\nabla \tilde{u}_{n_k} \longrightarrow \nabla u$ in $L^2(\mathbf{R}^3)$. Then using the Sobolev embedding we find $\tilde{u}_{n_k} \longrightarrow u$ in $L^6(\mathbf{R}^3)$.

From the second part of (6.20) and (6.23) it follows that

$$\varphi^2(|r_0 - \tilde{u}_{n_k}|) - r_0^2 \longrightarrow \varphi^2(|r_0 - u|) - r_0^2 \qquad \text{in } L^2(\mathbf{R}^3). \tag{6.30}$$

Let $G(z) = |r_0 - z|^2 - \varphi^2(|r_0 - z|)$. It is obvious that $G \in C^\infty(\mathbf{C}, \mathbf{R})$ and $|G(z)| \leq C|r_0 - z|^2 \mathbb{1}_{\{|r_0 - z| > 2r_0\}} \leq C'|z|^2 \mathbb{1}_{\{|z|>r_0\}} \leq C''|z|^3 \mathbb{1}_{\{|z|>r_0\}}$. Since $\tilde{u}_{n_k} \longrightarrow u$ in $L^6(\mathbf{R}^3)$, it is easy to see that $G(\tilde{u}_{n_k}) \longrightarrow G(u)$ in $L^2(\mathbf{R}^3)$ (see Theorem A4 p. 134 in [36]). Together with (6.30), this gives $|r_0 - \tilde{u}_{n_k}|^2 - r_0^2 \longrightarrow |r_0 - u|^2 - r_0^2$ in $L^2(\mathbf{R}^3)$ and the proof of Theorem 6.2 is complete. □

To prove that any minimizer provided by Theorem 6.2 satisfies an Euler-Lagrange equation, we will need the next lemma. It is clear that for any $v \in \mathcal{X}$ and any $R > 0$, the functional $\tilde{B}_c^v(w) := B_c(v + w)$ is C^1 on $H_0^1(B(0, R))$. We denote by $(\tilde{B}_c^v)'(0).w = \lim_{t\to 0} \frac{B_c(v+tw) - B_c(v)}{t}$ its derivative at the origin.

Lemma 6.4 *Assume that $N \geq 3$ and the conditions (A1) and (A2) are satisfied. Let $v \in \mathcal{X}$ be such that $(\tilde{B}_c^v)'(0).w = 0$ for any $w \in C_c^1(\mathbf{R}^N)$. Then $v = 0$ almost everywhere on $\mathbf{R}^N$.*

Proof. We denote by v^* be the precise representative of v, that is $v^*(x) = \lim_{r\to 0} m(v, B(x, r))$ if this limit exists, and 0 otherwise. Since $v \in L^1_{loc}(\mathbf{R}^N)$, it is well-known that $v = v^*$ almost everywhere on $\mathbf{R}^N$ (see, e.g., Corollary 1 p. 44 in [14]). Throughout the proof of Lemma 6.4 we replace v by v^*. We proceed in three steps.

Step 1. There exists a set $S \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(S) = 0$ and for any $x' \in \mathbf{R}^{N-1} \setminus S$ the function $v_{x'} := v(\cdot, x')$ belongs to $C^2(\mathbf{R})$ and solves the differential equation

$$(6.31) \qquad -(v_{x'})''(s) + ic(v_{x'})'(s) + F(|r_0 - v_{x'}(s)|^2)(r_0 - v_{x'}(s)) = 0 \qquad \text{for any } s \in \mathbf{R}.$$

Moreover, we have $|v_{x'}(s)| \longrightarrow 0$ as $s \longrightarrow \pm\infty$ and $v_{x'}$ satisfies the following properties:

$$(6.32) \quad v_{x'} \in L^{2^*}(\mathbf{R}), \qquad \varphi^2(|r_0 - v_{x'}|) - r_0^2 \in L^2(\mathbf{R}) \quad \text{and} \quad (v_{x'})' = \frac{\partial v}{\partial x_1}(\cdot, x') \in L^2(\mathbf{R}),$$

$$(6.33) \qquad F(|r_0 - v_{x'}|^2)(r_0 - v_{x'}) \in L^2(\mathbf{R}) + L^{\frac{2^*}{2p_0+1}}(\mathbf{R}).$$

It is easy to see that $F(|r_0 - v|^2)(r_0 - v) \in L^2(\mathbf{R}^N) + L^{\frac{2^*}{2p_0+1}}(\mathbf{R}^N)$. Since $v \in H^1_{loc}(\mathbf{R}^3)$, using Theorem 2 p. 164 in [14] and Fubini's Theorem, respectively, we see that there exists a set $\tilde{S} \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(\tilde{S}) = 0$ and for any $x' \in \mathbf{R}^{N-1} \setminus \tilde{S}$ the function $v_{x'}$ is absolutely continuous, $v_{x'} \in H^1_{loc}(\mathbf{R})$ and (6.32)$-$(6.33) hold.

Given $\phi \in C^1_c(\mathbf{R})$, we denote $\Lambda_\phi(x_1, x') = \langle \frac{\partial v}{\partial x_1}(x_1, x'), \phi'(x_1)\rangle + c\langle i\frac{\partial v}{\partial x_1}(x_1, x'), \phi(x_1)\rangle + \langle F(|r_0 - v|^2)(r_0 - v)(x_1, x'), \phi(x_1)\rangle$. From (6.32) and (6.33) it follows that $\Lambda_\phi(\cdot, x') \in L^1(\mathbf{R})$ for $x' \in \mathbf{R}^{N-1} \setminus \tilde{S}$. For such x' we define $\lambda_\phi(x') = \displaystyle\int_{\mathbf{R}} \Lambda_\phi(x_1, x')dx_1$, then we extend the function λ_ϕ in an arbitrary way to $\mathbf{R}^{N-1}$. Let $\psi \in C^1_c(\mathbf{R}^{N-1})$. It is obvious that the function $(x_1, x') \longmapsto \Lambda_\phi(x_1, x')\psi(x')$ belongs to $L^1(\mathbf{R}^N)$ and using Fubini's Theorem we get $\displaystyle\int_{\mathbf{R}^N} \Lambda_\phi(x_1, x')\psi(x')\,dx = \int_{\mathbf{R}^{N-1}} \lambda_\phi(x')\psi(x')\,dx'$. On the other hand, using the assumption of Lemma 6.4 we obtain $2\displaystyle\int_{\mathbf{R}^N} \Lambda_\phi(x_1, x')\psi(x')\,dx = \left(\tilde{B}^v_c\right)'(0).(\phi(x_1)\psi(x')) = 0$. Hence we have $\displaystyle\int_{\mathbf{R}^{N-1}} \lambda_\phi(x')\psi(x')\,dx' = 0$ for any $\psi \in C^1_c(\mathbf{R}^{N-1})$ and this implies that there exists a set $S_\phi \subset \mathbf{R}^{N-1} \setminus \tilde{S}$ such that $\mathcal{L}^{N-1}(S_\phi) = 0$ and $\lambda_\phi = 0$ on $\mathbf{R}^{N-1} \setminus (\tilde{S} \cup S_\phi)$.

Denote $q_0 = \frac{2^*}{2p_0+1} \in (1, \infty)$. There exists a coutable set $\{\phi_n \in C^1_c(\mathbf{R}) \mid n \in \mathbf{N}\}$ which is dense in $H^1(\mathbf{R}) \cap L^{q_0'}(\mathbf{R})$. For each n consider the set $S_{\phi_n} \subset \mathbf{R}^{N-1}$ as above. Let $S = \tilde{S} \cup \displaystyle\bigcup_{n \in \mathbf{N}} S_{\phi_n}$. It is clear that $\mathcal{L}^{N-1}(S) = 0$.

Let $x' \in \mathbf{R}^{N-1} \setminus S$. Fix $\phi \in C^1_c(\mathbf{R})$. There is a sequence $(\phi_{n_k})_{k \geq 1}$ such that $\phi_{n_k} \longrightarrow \phi$ in $H^1(\mathbf{R})$ and in $L^{q_0'}(\mathbf{R})$. Then $\lambda_{\phi_{n_k}}(x') = 0$ for each k and (6.32)$-$(6.33) imply that $\lambda_{\phi_{n_k}}(x') \longrightarrow \lambda_\phi(x')$. Consequently $\lambda_\phi(x') = 0$ for any $\phi \in C^1_c(\mathbf{R})$ and this implies that $v_{x'}$ satisfies the equation (6.31) in $\mathcal{D}'(\mathbf{R})$. Using (6.31) we infer that $(v_{x'})''$ (the *weak* second derivative of $v_{x'}$) belongs to $L^1_{loc}(\mathbf{R})$ and then it follows that $(v_{x'})'$ is continuous on $\mathbf{R}$ (see, e.g., Lemma VIII.2 p. 123 in [8]). In particular, we have $v_{x'} \in C^1(\mathbf{R})$. Coming back to (6.31) we see that $(v_{x'})''$ is continuous, hence $v_{x'} \in C^2(\mathbf{R})$ and (6.31) holds at each point of $\mathbf{R}$. Finally, we have $|v_{x'}(s_2) - v_{x'}(s_1)| \leq |s_2 - s_1|^{\frac{1}{2}} ||\,(v_{x'})'\,||_{L^2}$; this estimate and the fact that $v_{x'} \in L^{2^*}(\mathbf{R})$ imply that $v_{x'}(s) \longrightarrow 0$ as $s \longrightarrow \pm\infty$.

Step 2. There exist two positive constants k_1, k_2 (depending only on F and c) such that for any $x' \in \mathbf{R}^{N-1} \setminus S$ we have either $v_{x'} = 0$ on $\mathbf{R}$ or there exists an interval $I_{x'} \subset \mathbf{R}$ with $\mathcal{L}^1(I_{x'}) \geq k_1$ and $|\,|r_0 - v_{x'}| - r_0| \geq k_2$ on $I_{x'}$.

To see this, fix $x' \in \mathbf{R}^{N-1} \setminus S$ and denote $g = |r_0 - v_{x'}|^2 - r_0^2$. Then $g \in C^2(\mathbf{R}, \mathbf{R})$ and g tends to zero at $\pm\infty$. Proceeding exactly as in [33], p. 1100-1101 we integrate (6.31) and we see that g satisfies

$$(6.34) \qquad (g')^2(s) + c^2 g^2(s) - 4(g(s) + r_0^2)V(g(s) + r_0^2) = 0 \qquad \text{in } \mathbf{R}.$$

Using (1.4) we have $c^2t^2 - 4(t+r_0^2)V(t+r_0^2) = t^2(c^2 - v_s^2 + \varepsilon_1(t))$, where $\varepsilon_1(t) \longrightarrow 0$ as $t \longrightarrow 0$. In particular, there exists $k_0 > 0$ such that

$$c^2t^2 - 4(t+r_0^2)V(t+r_0^2) < 0 \qquad \text{for } t \in [-2k_0, 0) \cup (0, 2k_0]. \tag{6.35}$$

If $g = 0$ on $\mathbf{R}$ then $|r_0 - v_{x'}| = r_0$ and consequently there exists a lifting $r_0 - v_{x'}(s) = r_0 e^{i\theta(s)}$ with $\theta \in C^2(\mathbf{R}, \mathbf{R})$. Using equation (6.31) and proceeding as in [33] p. 1101 we see that either $r_0 - v_{x'}(s) = r_0 e^{i\theta_0}$ or $r_0 - v_{x'}(s) = r_0 e^{ics+\theta_0}$, where $\theta_0 \in \mathbf{R}$ is a constant. Since $v_{x'} \in L^{2^*}(\mathbf{R})$, we must have $v_{x'} = 0$.

If $g \not\equiv 0$, the function g achieves a negative minimum or a positive maximum at some $s_0 \in \mathbf{R}$. Then $g'(s_0) = 0$ and using (6.34) and (6.35) we infer that $|g(s_0)| > 2k_0$. Let $s_2 = \inf\{s < s_0 \mid |g(s)| \geq 2k_0\}$, $s_1 = \sup\{s < s_2 \mid g(s) \leq k_0\}$, so that $s_1 < s_2$, $|g(s_1)| = k_0$, $|g(s_2)| = 2k_0$ and $k_0 \leq |g(s)| \leq 2k_0$ for $s \in [s_1, s_2]$. Denote $M = \sup\{4(t+r_0^2)V(t+r_0^2) - c^2t^2 \mid t \in [-2k_0, 2k_0]\}$. From (6.34) we obtain $|g'(s)| \leq \sqrt{M}$ if $g(s) \in [-2k_0, 2k_0]$ and we infer that

$$k_0 = |g(s_2)| - |g(s_1)| \leq \left| \int_{s_1}^{s_2} g'(s)\, ds \right| \leq \sqrt{M}(s_2 - s_1),$$

hence $s_2 - s_1 \geq \frac{k_0}{\sqrt{M}}$. Obviously, there exists $k_2 > 0$ such that $|\,|r_0 - z|^2 - r_0^2| \geq k_0$ implies $|\,|r_0 - z| - r_0| \geq k_2$. Taking $k_1 = \frac{k_0}{\sqrt{M}}$ and $I_{x'} = [s_1, s_2]$, the proof of step 2 is complete.

Step 3. Conclusion.

Let $K = \{x' \in \mathbf{R}^{N-1} \setminus S \mid v_{x'} \not\equiv 0\}$. It is standard to prove that K is $\mathcal{L}^{N-1}-$measurable. The conclusion of Lemma 6.4 follows if we prove that $\mathcal{L}^{N-1}(K) = 0$. We argue by contradiction and we assume that $\mathcal{L}^{N-1}(K) > 0$.

If $x' \in K$, it follows from step 2 that there exists an interval $I_{x'}$ of length at least k_1 such that $\left(\varphi^2(|r_0 - v_{x'}|) - r_0^2\right)^2 \geq \eta(k_2)$ on $I_{x'}$, where η is as in (3.30). This implies $\displaystyle\int_{\mathbf{R}} \left(\varphi^2(|r_0 - v(x_1, x')|) - r_0^2\right)^2 dx_1 \geq k_1 \eta(k_2)$ and using Fubini's theorem we get

$$\begin{aligned}&\int_{\mathbf{R}^N} \left(\varphi^2(|r_0 - v(x)|) - r_0^2\right)^2 dx = \int_K \left(\int_{\mathbf{R}} \left(\varphi^2(|r_0 - v(x_1, x')|) - r_0^2\right)^2 dx_1 \right) dx' \\ &\geq k_1 \eta(k_2) \mathcal{L}^{N-1}(K).\end{aligned}$$

Since $v \in \mathcal{X}$, we infer that $\mathcal{L}^{N-1}(K)$ is finite.

It is obvious that there exist $x_1' \in K$ and $x_2' \in \mathbf{R}^{N-1} \setminus (K \cup S)$ arbitrarily close to each other. Then $|v_{x_1'}| \geq k_2$ on an interval $I_{x_1'}$ of length k_1, while $v_{x_2'} \equiv 0$. If we knew that v is uniformly continuous, this would lead to a contradiction. However, the equation (6.31) satisfied by v involves only derivatives with respect to x_1 and does not imply any regularity properties of v with respect to the transverse variables (note that if v is a solution of (6.31), then $v(x_1 + \delta(x'), x')$ is also a solution, even if δ is discontinuous). For instance, for the Gross-Pitaevskii nonlinearity $F(s) = 1 - s$ it is possible to construct bounded, C^∞ functions v such that $v \in L^{2^*}(\mathbf{R}^N)$, (6.31) is satisfied for a.e. x', and the set K constructed as above is a nontrivial ball in $\mathbf{R}^{N-1}$ (of course, these functions do not tend uniformly to zero at infinity, are not uniformly continuous and their gradient is not in $L^2(\mathbf{R}^N)$).

We use that fact that one transverse derivative of v (for instance, $\frac{\partial v}{\partial x_2}$) is in $L^2(\mathbf{R}^N)$ to get a contradiction.

For $x' = (x_2, x_3, \ldots, x_N) \in \mathbf{R}^{N-1}$, we denote $x'' = (x_3, \ldots, x_N)$. Since $v \in H^1_{loc}(\mathbf{R}^N)$, from Theorem 2 p. 164 in [14] it follows that there exists $J \subset \mathbf{R}^{N-1}$ such that $\mathcal{L}^{N-1}(J) = 0$ and $u(x_1, \cdot, x'') \in H^1_{loc}(\mathbf{R}^N)$ for any $(x_1, x'') \in \mathbf{R}^{N-1} \setminus J$. Given $x'' \in \mathbf{R}^{N-2}$, we denote

$$\begin{aligned}K_{x''} &= \{x_2 \in \mathbf{R} \mid (x_2, x'') \in K\}, \\ S_{x''} &= \{x_2 \in \mathbf{R} \mid (x_2, x'') \in S\}, \\ J_{x''} &= \{x_1 \in \mathbf{R} \mid (x_1, x'') \in J\}.\end{aligned}$$

Fubini's Theorem implies that the sets $K_{x''}$, $S_{x''}$, $J_{x''}$ are $\mathcal{L}^1-$measurable, $\mathcal{L}^1(K_{x''}) < \infty$ and $\mathcal{L}^1(S_{x''}) = \mathcal{L}^1(J_{x''}) = 0$ for $\mathcal{L}^{N-2}-$a.e. $x'' \in \mathbf{R}^{N-2}$. Let

$$\begin{aligned} G \;=\; & \{x'' \in \mathbf{R}^{N-2} \mid K_{x''}, S_{x''}, J_{x''} \text{ are } \mathcal{L}^1 \text{ measurable}, \\ & \mathcal{L}^1(S_{x''}) = \mathcal{L}^1(J_{x''}) = 0 \text{ and } 0 < \mathcal{L}^1(K_{x''}) < \infty\}. \end{aligned} \tag{6.36}$$

Clearly, G is $\mathcal{L}^{N-2}-$measurable and $\displaystyle\int_G \mathcal{L}^1(K_{x''})\,dx'' = \mathcal{L}^{N-1}(K) > 0$, thus $\mathcal{L}^{N-2}(G) > 0$. We claim that

$$\int_{\mathbf{R}^2} \Big|\frac{\partial v}{\partial x_2}(x_1, x_2, x'')\Big|^2 dx_1\,dx_2 = \infty \qquad \text{for any } x'' \in G. \tag{6.37}$$

Indeed, let $x'' \in G$. Fix $\varepsilon > 0$. Using (6.36) we infer that there exist s_1, $s_2 \in \mathbf{R}$ such that $(s_1, x'') \in \mathbf{R}^{N-1} \setminus (K \cup S)$, $(s_2, x'') \in K$ and $|s_2 - s_1| < \varepsilon$. Then $v(t, s_1, x'') = 0$ for any $t \in \mathbf{R}$. From step 2 it follows that there exists an interval I with $\mathcal{L}^1(I) \geq k_1$ such that $|v(t, s_2, x'')| \geq |\,|r_0 - v(t, s_2, x'')| - r_0| \geq k_2$ for $t \in I$. Assume $s_1 < s_2$. If $t \in I \setminus J_{x''}$ we have $v(t, \cdot, x'') \in H^1_{loc}(\mathbf{R})$, hence

$$\begin{aligned} k_2 &\leq |v(t, s_2, x'') - v(t, s_1, x'')| = \Big|\int_{s_1}^{s_2} \frac{\partial v}{\partial x_2}(t, \tau, x'')\,d\tau\Big| \\ &\leq (s_2 - s_1)^{\frac12} \left(\int_{s_1}^{s_2} \Big|\frac{\partial v}{\partial x_2}(t, \tau, x'')\Big|^2 d\tau\right)^{\frac12}. \end{aligned}$$

Clearly, this implies $\displaystyle\int_{s_1}^{s_2} \Big|\frac{\partial v}{\partial x_2}(t, \tau, x'')\Big|^2 d\tau \geq \frac{k_2^2}{\varepsilon}$. Consequently

$$\int_{\mathbf{R}^2} \Big|\frac{\partial v}{\partial x_2}(x_1, x_2, x'')\Big|^2 dx_1\,dx_2 \geq \int_I \int_{s_1}^{s_2} \Big|\frac{\partial v}{\partial x_2}(t, \tau, x'')\Big|^2 d\tau\,dt \geq \frac{k_1 k_2^2}{\varepsilon}.$$

Since the last inequality holds for any $\varepsilon > 0$, (6.37) is proved. Using (6.37), the fact that $\mathcal{L}^{N-2}(G) > 0$ and Fubini's Theorem we get $\displaystyle\int_{\mathbf{R}^N} \Big|\frac{\partial v}{\partial x_2}\Big|^2 dx = \infty$, contradicting the fact that $v \in \mathcal{X}$. Thus necessarily $\mathcal{L}^{N-1}(K) = 0$ and the proof of Lemma 6.4 is complete. □

Proposition 6.5 *Assume that $N = 3$ and the conditions (A1) and (A2) are satisfied. Let $u \in \mathcal{C}$ be a minimizer of E_c in $\mathcal{C}$. Then $u \in W^{2,p}_{loc}(\mathbf{R}^3)$ for any $p \in [1, \infty)$, $\nabla u \in W^{1,p}(\mathbf{R}^3)$ for $p \in [2, \infty)$ and there exists $\sigma > 0$ such that $u_{1,\sigma}$ is a solution of (1.3).*

Proof. The proof is very similar to the proof of Proposition 5.6. It is clear that $A(u) = E_c(u) = T_c$ and u is a minimizer of A in $\mathcal{C}$. For any $R > 0$, the functionals $\tilde{B}^u_c$ and $\tilde{A}(v) := A(u + v)$ are C^1 on $H^1_0(B(0, R))$. We proceed in four steps.

Step 1. There exists $w \in C^1_c(\mathbf{R}^3)$ such that $(\tilde{B}^u_c)'(0).w \neq 0$. This follows from Lemma 6.4.

Step 2. There exists a Lagrange multiplier $\alpha \in \mathbf{R}$ such that

$$\tilde{A}'(0).v = \alpha(\tilde{B}^u_c)'(0).v \qquad \text{for any } v \in H^1(\mathbf{R}^3),\ v \text{ with compact support}. \tag{6.38}$$

Step 3. We have $\alpha < 0$.

The proof of steps 2 and 3 is the same as the proof of steps 2 and 3 in Proposition 5.6.

Step 4. Conclusion.

Let $\beta = -\frac{1}{\alpha}$. Then (6.38) implies that u satisfies

$$-\frac{\partial^2 u}{\partial x_1^2} - \beta\left(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) + icu_{x_1} + F(|r_0 - u|^2)(r_0 - u) = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3).$$

For $\sigma^2 = \frac{1}{\beta}$ we see that $u_{1,\sigma}$ satisfies (1.3). It is clear that $u_{1,\sigma} \in \mathcal{C}$ and $u_{1,\sigma}$ minimizes A (respectively E_c) in $\mathcal{C}$. Finally, the regularity of $u_{1,\sigma}$ (thus the regularity of u) follows from Lemma 5.5. □

7 Further properties of traveling waves

By Propositions 5.6 and 6.5 we already know that the solutions of (1.3) found there are in $W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1,\infty)$ and in $C^2(\mathbf{R}^N)$. In general, a straightforward boot-strap argument shows that the finite energy traveling waves of (1.1) have the best regularity allowed by the nonlinearity F. For instance, if $F \in C^k([0,\infty))$ for some $k \in \mathbf{N}^*$, it can be proved that all finite energy solutions of (1.3) are in $W^{k+2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1,\infty)$ (see, for instance, Proposition 2.2 (ii) in [33]). If F is analytic, it can be proved that finite energy traveling waves are also analytic. In the case of the Gross-Pitaevskii equation, this has been done in [5].

A lower bound $K(c,N)$ on the energy of traveling waves of speed $c < v_s$ for the Gross-Pitaevskii equation has been found in [35]. The constant $K(c,N)$ is known explicitly and we have $K(c,N) \longrightarrow 0$ as $c \longrightarrow v_s$. In the case of general nonlinearities, we know that *any* finite energy traveling wave u of speed c satisfies the Pohozaev identity $P_c(u) = 0$, that is $u \in \mathcal{C}$. Then it follows from Lemma 4.7 that $A(u) \geq \frac{N-1}{2} T_c > 0$.

Our next result concerns the symmetry of those solutions of (1.3) that minimize E_c in $\mathcal{C}$.

Proposition 7.1 *Assume that $N \geq 3$ and the conditions (A1), (A2) in the introduction hold. Let $u \in \mathcal{C}$ be a minimizer of E_c in $\mathcal{C}$. Then, after a translation in the variables $(x_2, \dots, x_N)$, u is axially symmetric with respect to Ox_1.*

Proof. Let T_c be as in Lemma 4.7. We know that any minimizer u of E_c in $\mathcal{C}$ satisfies $A(u) = \frac{N-1}{2} T_c > 0$. Using Lemma 4.8 (i), it is easy to prove that a function $u \in \mathcal{X}$ is a minimizer of E_c in $\mathcal{C}$ if and only if

$$(7.1) \qquad u \text{ minimizes the functional } -P_c \text{ in the set } \{v \in \mathcal{X} \mid A(v) = \frac{N-1}{2} T_c\}.$$

The minimization problem (7.1) is of the type studied in [32]. All we have to do is to verify that the assumptions made in [32] are satisfied, then to apply the general theory developed there.

Let Π be an affine hyperplane in $\mathbf{R}^N$ parallel to Ox_1. We denote by s_Π the symmetry of $\mathbf{R}^N$ with respect to Π and by Π^+, Π^- the two half-spaces determined by Π. Given a function $v \in \mathcal{X}$, we denote

$$v_{\Pi^+}(x) = \begin{cases} v(x) & \text{if } x \in \Pi^+ \cup \Pi, \\ v(s_\Pi(x)) & \text{if } x \in \Pi^-, \end{cases} \qquad \text{and} \qquad v_{\Pi^-}(x) = \begin{cases} v(x) & \text{if } x \in \Pi^- \cup \Pi, \\ v(s_\Pi(x)) & \text{if } x \in \Pi^+. \end{cases}$$

It is easy to see that $v_{\Pi^+}, v_{\Pi^-} \in \mathcal{X}$. Moreover, for any $v \in \mathcal{X}$ we have

$$A(v_{\Pi^+}) + A(v_{\Pi^-}) = 2A(v) \quad \text{and} \quad P_c(v_{\Pi^+}) + P_c(v_{\Pi^-}) = 2P_c(v).$$

This implies that assumption $(\mathbf{A1}_c)$ in [32] is satisfied.

By Propositions 5.6 and 6.5 and Lemma 5.5 we know that any minimizer of (7.1) is C^1 on $\mathbf{R}^N$, hence assumption $(\mathbf{A2}_c)$ in [32] holds. Then the axial symmetry of solutions of (7.1) follows directly from Theorem 2' in [32]. □

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Global branches of travelling waves to a Gross-Pitaevskii-Schrödinger system in one dimension

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Global branches of travelling-waves to a Gross-Pitaevskii-Schrödinger system in one dimension

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Abstract

We are interested in the existence of travelling-wave solutions to a system which modelizes the motion of an uncharged impurity in a Bose condensate. We prove that in space dimension one, there exist travelling-waves moving with velocity c if and only if c is less than the sound velocity at infinity. In this case we investigate the structure of the set of travelling-waves and we show that it contains global subcontinua in appropriate Sobolev spaces.
Key words. Gross-Pitaevskii system, travelling-waves, global bifurcation.
AMS subject classifications. 35Q55, 35Q51, 37K50, 35P05, 35J10

1 Introduction.

This paper is devoted to the study of a special kind of solutions of a system describing the motion of an uncharged impurity in a Bose condensate. In dimensionless variables, the system reads

$$(1.1)\qquad \left\{ \begin{array}{rcl} 2i\dfrac{\partial \psi}{\partial t} & = & -\Delta\psi + \frac{1}{\varepsilon^2}(|\psi|^2 + \frac{1}{\varepsilon^2}|\varphi|^2 - 1)\psi \\ 2i\delta\dfrac{\partial \varphi}{\partial t} & = & -\Delta\varphi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi. \end{array} \right.$$

Here ψ and φ are the wavefunctions for bosons, respectively for the impurity, $\delta = \frac{\mu}{M}$ where μ is the mass of impurity and M is the boson mass (δ is supposed to be small), $q^2 = \frac{l}{2d}$, l being the boson-impurity scattering length and d the boson diameter, k is a dimensionless measure for the single-particle impurity energy and ε is a dimensionless constant ($\varepsilon = (\frac{a\mu}{lM})^{\frac{1}{5}}$, where a is the "healing length"; in applications, $\varepsilon \cong 0.2$). Assuming that we are in a frame in which the condensate is at rest at infinity, the solutions must satisfy the "boundary conditions"

$$(1.2)\qquad |\psi| \longrightarrow 1, \quad \varphi \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty.$$

This system (originally introduced by Clark and Gross) was studied by J. Grant and P. H. Roberts (see [5]). Using formal asymptotic expansions and numerical experiments, they computed the effective radius and the induced mass of the uncharged impurity.

We consider here the system (1.1) in a one dimensional space and we look for solitary waves, that is for solutions of the form

$$\psi(x,t) = \tilde{\psi}(x-ct), \quad \varphi(x,t) = \tilde{\varphi}(x-ct). \tag{1.3}$$

This kind of solutions corresponds to the case where the only disturbance present in the condensate is that caused by the uniform motion of the impurity with velocity c. In view of the boundary conditions, we seek for solutions of the form

$$\tilde{\psi}(x) = (1+\tilde{r}(x))e^{i\psi_0(x)}, \quad \tilde{\varphi}(x) = \tilde{u}(x)e^{i\varphi_0(x)} \tag{1.4}$$

with $\tilde{r}(x) \longrightarrow 0$, $\tilde{u}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. By an easy computation we find that the real functions ψ_0, φ_0, $\tilde{r}$, $\tilde{u}$ must satisfy

$$\psi_0' = c(1 - \frac{1}{(1+\tilde{r})^2}), \tag{1.5}$$

$$\varphi_0' = c\delta, \tag{1.6}$$

$$\tilde{r}'' = c^2\Big(\frac{1}{(1+\tilde{r})^3} - (1+\tilde{r})\Big) + \frac{1}{\varepsilon^2}\Big((1+\tilde{r})^3 - (1+\tilde{r}) + \frac{1}{\varepsilon^2}(1+\tilde{r})\tilde{u}^2\Big), \tag{1.7}$$

$$\tilde{u}'' = \Big(\frac{q^2}{\varepsilon^2}(1+\tilde{r})^2 - c^2\delta^2 - k^2\Big)\tilde{u}. \tag{1.8}$$

From (1.6) we see that necessarily $\varphi_0(x) = c\delta x + C$. Note that the system is invariant under the transform $(\psi, \varphi) \longmapsto (e^{i\alpha}\psi, e^{i\beta}\varphi)$, so the integration constants in (1.5) and (1.6) are not important. Thus all we have to do is to solve the system (1.7)-(1.8). Thereafter it will be easy to find the corresponding phases from (1.5)-(1.6) and (1.4) will give a solitary-wave solution of (1.1).

After the scale change $\tilde{u}(x) = \frac{1}{\varepsilon}u(\frac{x}{\varepsilon})$, $\tilde{r}(x) = r(\frac{x}{\varepsilon})$, we find that the functions r and u satisfy

$$r'' = (1+r)^3 - (1+r) - c^2\varepsilon^2\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)u^2, \tag{1.9}$$

$$u'' = (q^2(1+r)^2 - \lambda)u, \tag{1.10}$$

where

$$\lambda = \varepsilon^2(c^2\delta^2 + k^2). \tag{1.11}$$

The equation $r'' = (1+r)^3 - (1+r) - \frac{v^2}{4}\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)U$, where U is a positive Borel measure, was studied in [7]. In the case $U \equiv 0$, it has been shown that this equation has only the trivial solution $r \equiv 0$ if $|v| \geq \sqrt{2}$; for $0 < |v| < \sqrt{2}$, it also admits the solution

$$r_v(x) = -1 + \sqrt{\frac{v^2}{2} + (1 - \frac{v^2}{2}) \tanh^2(\frac{\sqrt{2-v^2}}{2} x)}. \tag{1.12}$$

Moreover, any other nontrivial solution is of the form $r_v(\cdot - x_0)$ for some $x_0 \in \mathbf{R}$. Equation (1.10) is linear in u ; more precisely, u must be an eigenvector of the linear operator $-\frac{d^2}{dx^2} + q^2(1+r)^2$ corresponding to the eigenvalue $\lambda = \varepsilon^2(c^2\delta^2 + k^2)$.

It is now clear that except for translations, the only solutions of (1.9)-(1.10) of the form $(r, 0)$ are $(0,0)$ and $(r_{2c\varepsilon}, 0)$ (the latter one exists only for $|c\varepsilon| < \frac{1}{\sqrt{2}}$). We call these solutions the *trivial solutions* of (1.9)-(1.10). We will prove that there exist non-trivial solutions of (1.9)-(1.10) in a neighbourhood of $(r_{2c\varepsilon}, 0)$ (for suitable values of the parameter λ) and we will study the global structure of the set of non-trivial solutions.

It has been shown (see e.g. [7] and references therein) that using the Madelung's transform $\psi = \sqrt{\rho} e^{i\psi_0}$, the first equation in (1.1) can be put into a hydrodynamical form (i.e. it is equivalent to a system of Euler equations for a compressible inviscid fluid of density ρ and velocity $\nabla\psi_0$). In this context, $\frac{1}{\varepsilon\sqrt{2}}$ represents the sound velocity at infinity. It will be proved at the beginning of section 3 that (1.1) does not possess non-constant travelling-vaves moving with velocity $|c| \geq \frac{1}{\varepsilon\sqrt{2}}$. Hence we will assume throughout that $|c| < \frac{1}{\varepsilon\sqrt{2}}$.

Observe that the system (1.9)-(1.10) has a good variational formulation : its solutions are critical points of the "energy" functional. Indeed, since $1 + \tilde{r} = |\tilde{\psi}| \geq 0$, it is clear that we must have $\tilde{r} \geq -1$. Therefore we will seek for solutions r of (1.9) with $r > -1$. Let $V = \{r \in H^1(\mathbf{R}) \mid \inf_{x\in\mathbf{R}} r(x) > -1\}$. It is obvious that V is open in $H^1(\mathbf{R})$ because $H^1(\mathbf{R}) \subset C_b^0(\mathbf{R})$ by the Sobolev embedding. A pair $(r,u) \in V \times H^1(\mathbf{R})$ satisfy (1.9)-(1.10) if and only if (r,u) is a critical point of the C^∞ functional $E : V \times H^1(\mathbf{R}) \longrightarrow \mathbf{R}$,

$$\begin{aligned} E(r,u) &= \int_{\mathbf{R}} |r'|^2 dx + \frac{1}{2}\int_{\mathbf{R}} \Big((1+r)^2 - 1\Big)^2 \Big(1 - \frac{2c^2\varepsilon^2}{(1+r)^2}\Big) dx \\ &\quad + \int_{\mathbf{R}} u^2(1+r)^2 dx + \frac{1}{q^2}\int_{\mathbf{R}} |u'|^2 dx - \frac{\lambda}{q^2}\int_{\mathbf{R}} u^2 dx. \end{aligned} \tag{1.13}$$

However, $E(r,\cdot)$ is quadratic in u for any fixed r and it would be very difficult to find critical points of E by using a classical topological argument.

In this paper we use bifurcation theory to show the existence of nontrivial solitary waves for the system (1.1). Note that this system (or equivalently (1.9)-(1.10)) is invariant by translations. To avoid the degeneracy of the linearized system due to this invariace, we work on symmetric function spaces. Consequently, the travelling-waves that we obtain will also present a symmetry. To be more precise, we will use the spaces

$$\mathbf{H} = H^2_{rad}(\mathbf{R}) = \{u \in H^2(\mathbf{R}) \mid u(x) = u(-x), \ \ \forall x \in \mathbf{R}\} \quad \text{and}$$

$$\mathbf{L} = L^2_{rad}(\mathbf{R}) = \{u \in L^2(\mathbf{R}) \mid u(x) = u(-x), \ \ a.e. \ x \in \mathbf{R}\}.$$

Clearly $\mathbf{H} \cap V$ is an open set of $\mathbf{H}$. We define $S : (\mathbf{H} \cap V) \times \mathbf{H} \longrightarrow \mathbf{L}$, $T : \mathbf{R} \times \mathbf{H} \times \mathbf{H} \longrightarrow \mathbf{L}$,

$$S(r,u) = -r'' + (1+r)^3 - (1+r) - c^2\varepsilon^2\Big(1 + r - \frac{1}{(1+r)^3}\Big) + (1+r)u^2, \tag{1.14}$$

$$T(\lambda, r, u) = -u'' + (q^2(1+r)^2 - \lambda)u. \tag{1.15}$$

It is obvious that S and T are well defined and of class C^∞ (recall that $\mathbf{H} \subset C_b^1(\mathbf{R})$ and $\mathbf{H}$ is an algebra). Clearly r and u satisfy the system (1.9)-(1.10) if and only if $S(r, u) = 0$ and $T(\lambda, r, u) = 0$.

In the next section, we will study the structure of the set of nontrivial solutions in a neighbourhood of the trivial ones. It follows easily from the Implicit Function Theorem that there are no nontrivial solutions of (1.9)-(1.10) in a neighbourhood of $(\lambda, 0, 0)$ for $\lambda < q^2$ (see the proof of Theorem 3.8). It is well-known that we may have nontrivial solutions arbitrarily close to $(\lambda, r_{2c\varepsilon}, 0)$ if and only if the differential $d_{(r,u)}(S,T)(\lambda, r_{2c\varepsilon}, 0)$ is not invertible. For $\lambda < q^2$, we will see that $d_{(r,u)}(S,T)(\lambda, r_{2c\varepsilon}, 0)$ is not invertible if and only if λ is an eigenvalue of the particular Schrödinger operator given by (1.10). In this case we show that all the nontrivial solutions in a neighbourhood of $(\lambda, r_{2c\varepsilon}, 0)$ form a smooth curve in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$.

It is natural to ask how long such a branch of solutions exists. Recently, there were obtained general global bifurcation results for C^1 Fredholm mappings of index 0 which apply to a broad class of elliptic equations in $\mathbf{R}^N$ (see, e.g., [9], [10]). Using the ideas and techniques developed in [11] it can be proved that for any fixed $\lambda < q^2$, the mapping $(S, T(\lambda, \cdot, \cdot)) : (\mathbf{H} \cap V) \times \mathbf{H} \longrightarrow \mathbf{L} \times \mathbf{L}$ is Fredholm of index 0. By a general global bifurcation theorem (a variant of Theorem 6.1 in [9]) one can prove that either the branch of nontrivial solutions of (1.9)-(1.10) starting from a bifurcation point $(\lambda, r_{2c\varepsilon}, 0)$ is noncompact in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ or it meets $[q^2, \infty) \times \mathbf{H} \times \mathbf{H}$ (note that $[q^2, \infty)$ is the essential spectrum of the linear Schrödinger operator appearing in (1.10)).

To obtain further information (such as unboundedness) about the branches of nontrivial solutions, a key ingredient would be the properness of the operator (S, T), at least on closed bounded sets. Unfortunately, it is easy to see that the operator (S, T) is not proper on closed bounded sets. Indeed, it suffices to take $r_n = r_{2c\varepsilon}(\cdot - n) + r_{2c\varepsilon}(\cdot + n)$ and to observe that $(S,T)(\lambda, r_n, 0) \longrightarrow (0, 0)$ as $n \longrightarrow \infty$, the sequence (r_n) is bounded in $\mathbf{H}$ but has no convergent subsequence.

In order to obtain a more precise description of the branches of nontrivial solutions and to avoid troubles due to the lack of properness, we choose a different approach : we reformulate the problem and we work on a weighted Sobolev space (which is a subspace of $\mathbf{H}$). In section 3, we use a variant of the Global Bifurcation Theorem of Rabinowitz ([12]) to obtain global branches of solutions of (1.9)-(1.10) in that space. Note that the use of a slowly increasing weight (for example, $(1 + x^2)^s$ for $s > 0$) is sufficient to eliminate the lack of properness and to obtain global branches of travelling-waves. It is worth to note that for $\lambda < q^2$, any nontrivial travelling-wave which is in $\mathbf{H}$ also belongs to the weighted space which is used (i.e., there is no loss of solutions). We show that there exists exactly one branch of nontrivial solutions bifurcating from the curve $(\lambda, r_{2c\varepsilon}, 0)$ if $q \le \frac{1}{\sqrt{2 \ln 2}}$. The number of these branches is increasing with q and tends to infinity as $q \longrightarrow \infty$. We will prove that any of these branches is either unbounded (in the weighted space) or λ tends to q^2 along it. On the other hand, we prove that there are no nontrivial solutions of (1.9)-(1.10) for $\lambda > q^2$.

2 Local curves of solutions

In order to prove a local existence result of nontrivial solitary waves for the system (1.1), we have to study the properties of the linear operator $A = -\frac{d^2}{dx^2} + q^2(1 + r_{2c\varepsilon})^2$, which can be written as $A = -\frac{d^2}{dx^2} + q^2 r_{2c\varepsilon}(2 + r_{2c\varepsilon}) + q^2$. Since $-1 < r(x) < 0$ for any $x \in \mathbf{R}$, the function $r_{2c\varepsilon}(2 + r_{2c\varepsilon})$ is everywhere negative (and even). Actually, in a slightly more general framework, we will study the operator $L = -\frac{d^2}{dx^2} + V(x)$ for a negative potential V, the properties of A being then deduced from those of L by a shift. For any $\lambda \le 0$, we also consider the Cauchy problem

$$\text{(2.1)} \qquad \begin{cases} -u''(x) + V(x)u(x) = \lambda u(x), \\ u(0) = 1, \quad u'(0) = 0. \end{cases}$$

If V is continuous and even (i.e., $V(x) = V(-x)$), it is clear that problem (2.1) has an unique *global* solution which is also even. We denote by u_λ this solution and by $n(\lambda)$ the number of zeroes of u_λ in $(0, \infty)$.

Proposition 2.1 *Let $V \in L^2 \cap L^\infty(\mathbf{R}^N)$, $V \not\equiv 0$ be continuous, less than or equal to zero, even, and satisfy $\lim_{x \longrightarrow \pm\infty} V(x) = 0$. The operator $L = -\frac{d^2}{dx^2} + V(x) : \mathbf{H} \longrightarrow \mathbf{L}$ is self-adjoint and has the following properties :*

i) $\sigma_{ess}(L) = [0, \infty)$.

ii) L has at least one negative eigenvalue.

iii) Any eigenvalue of L is simple.

iv) For any $\lambda < 0$ and $\varepsilon > 0$, there exists $C > 0$ such that

$$\text{(2.2)} \qquad |u_\lambda^{(m)}(x)| \le C e^{\sqrt{-\lambda+\varepsilon}|x|}, \qquad m = 0, 1, 2.$$

If $\lambda < 0$ is an eigenvalue and $0 < \varepsilon < -\lambda$, there exist $C_1, C_2, M > 0$ such that

$$\text{(2.3)} \qquad C_1 e^{-\sqrt{-\lambda+\varepsilon}|x|} \le |u_\lambda^{(m)}(x)| \le C_2 e^{-\sqrt{-\lambda-\varepsilon}|x|} \qquad \text{on} \quad [M, \infty), \qquad m = 0, 1, 2.$$

v) For any $\lambda \le 0$, the number of eigenvalues of L in $(-\infty, \lambda)$ is exactly $n(\lambda)$, the number of zeroes of u_λ in $(0, \infty)$.

vi) If $\int_0^\infty x|V(x)|dx < \infty$, then L has at most $1 + \int_0^\infty x|V(x)|dx$ negative eigenvalues.

Proof. i) The operator $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbf{R})$ (with domain $H^2(\mathbf{R})$) is self-adjoint, so it is easy to see that L is self-adjoint. Multiplication by V is a relatively compact perturbation of $-\Delta$ and it follows from a classical theorem of Weyl that $\sigma_{ess}(L) = \sigma_{ess}(-\Delta) = [0, \infty)$.

ii) It suffices to show that there exists $u \in \mathbf{H}$ such that $\langle Lu, u \rangle_{\mathbf{L}} < 0$ and it will follow from the Min-Max Principle (see [13], Theorem XIII.1, p.76) that L has negative eigenvalues. Consider an even function $u \in C_0^\infty$ such that $u \equiv 1$ on $[-1, 1]$ and u is non-increasing on $[0, \infty)$. Let $u_n(x) = u(\frac{x}{n})$. Then

$$\langle Lu_n, u_n \rangle_{\mathbf{L}} = \frac{1}{n} \int_{\mathbf{R}} |u'(x)|^2 dx + \int_{\mathbf{R}} |u(\frac{x}{n})|^2 V(x) dx \longrightarrow \int_{\mathbf{R}} V(x) dx < 0$$

as $n \longrightarrow \infty$, so $\langle Lu_n, u_n \rangle_{\mathbf{L}} < 0$ for n sufficiently large.

iii) Clearly, λ is an eigenvalue of L if and only if $u_\lambda \in \mathbf{H}$. If this is the case, it is obvious that $Ker(L-\lambda) = Span\{u_\lambda\}$. Since L is self-adjoint, we have $Ker(L-\lambda) \cap Im(L-\lambda) = \{0\}$, so for any $n \in \mathbf{N}^*$ we have $Ker(L-\lambda)^n = Ker(L-\lambda) = Span\{u_\lambda\}$, that is λ is a simple eigenvalue.

iv) By (2.1), u_λ and u'_λ cannot vanish simultaneously, so u_λ must change sign any time it vanishes and u_λ has only isolated zeroes. There exists $d > 0$ such that $V(x) - \lambda > -\frac{\lambda}{2} > 0$ on $[d, \infty)$ because $V(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Two situations may occur :

1°. There exists $x_0 > d$ such that $u_\lambda(x_0)$ and $u'_\lambda(x_0)$ have the same sign, say, they are positive. Then $u''_\lambda = (V(x) - \lambda)u_\lambda$, so u''_λ will remain positive after x_0 as long as $u_\lambda > 0$, which implies that u'_λ is increasing, hence it remains positive as long as $u_\lambda > 0$. Consequently, u_λ is increasing after x_0 as long as it remains positive, which implies that u_λ is positive and increasing on $[x_0, \infty)$. Since $u'_\lambda(x) \geq u'_\lambda(x_0) > 0$ for any $x > x_0$, we have necessarily $\lim_{x\to\infty} u_\lambda(x) = \infty$. By (2.1) we find that $\lim_{x\to\infty} u''_\lambda(x) = \infty$, so we have also $\lim_{x\to\infty} u'_\lambda(x) = \infty$. Let $f(x) = (u'_\lambda(x))^2$ and $g(x) = u^2_\lambda(x)$. Clearly, $f(x) \longrightarrow \infty$, $g(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ and

$$\frac{f'(x)}{g'(x)} = \frac{u''_\lambda(x)}{u_\lambda(x)} = V(x) - \lambda \longrightarrow -\lambda \qquad \text{as } x \longrightarrow \infty.$$

L'Hôspital's rule implies that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = -\lambda$, which gives $\lim_{x\to\infty} \frac{u'_\lambda(x)}{u_\lambda(x)} = \sqrt{-\lambda}$. Thus for any $\epsilon > 0$, there exists $x_\epsilon > 0$ such that

$$\sqrt{-\lambda-\epsilon} < \frac{u'_\lambda(x)}{u_\lambda(x)} < \sqrt{-\lambda+\epsilon} \qquad \text{on } [x_\epsilon, \infty). \tag{2.4}$$

Integrating (2.4) from x_ϵ to x we get for any $x > x_\epsilon$,

$$\sqrt{-\lambda-\epsilon}(x - x_\epsilon) < \ln u_\lambda(x) - \ln u_\lambda(x_\epsilon) < \sqrt{-\lambda+\epsilon}(x - x_\epsilon),$$

that is

$$u_\lambda(x_\epsilon)e^{\sqrt{-\lambda-\epsilon}(x-x_\epsilon)} < u_\lambda(x) < u_\lambda(x_\epsilon)e^{\sqrt{-\lambda+\epsilon}(x-x_\epsilon)} \qquad \text{for any } x > x_\epsilon. \tag{2.5}$$

Note that the above situation always occurs if u_λ has a zero in (d, ∞). Indeed, if $u_\lambda(x_0) = 0$, then necessarily $u_\lambda(x)$ and $u'_\lambda(x)$ have opposite signs for $x < x_0$ and x close to x_0 (because if u_λ and u'_λ have the same sign at some $x_1 \in (d, x_0)$, we have just seen that u_λ cannot vanish in after x_1). But u_λ changes sign at x_0 and $u'_\lambda(x_0) \neq 0$, hence u_λ and u'_λ have the same sign just after x_0.

2°. The functions u_λ and u'_λ have opposite sign in (d, ∞). Replacing u_λ by $-u_\lambda$ if necessary, we may suppose that $u_\lambda > 0$ and $u'_\lambda < 0$ in (d, ∞) (observe that u'_λ cannot vanish because it also changes sign at any zero and we would be in case 1°). So u_λ is decreasing and positive on (d, ∞). Let $l = \lim_{x\to\infty} u_\lambda(x)$. Clearly, $l \geq 0$. If $l > 0$, then $u''_\lambda(x) \longrightarrow -\lambda l > 0$ as $x \longrightarrow \infty$ by (2.1), which implies $u'_\lambda(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, a contradiction. Thus necessarily $l = 0$. Also, u'_λ is increasing on (d, ∞) (because $u''_\lambda(x) = (V(x) - \lambda)u_\lambda(x) > 0$) and negative, so it also has a limit at infinity. Since u_λ converges (to zero) at infinity, we must have $\lim_{x\to\infty} u'_\lambda(x) = 0$. Now we may apply l'Hôspital's rule to get

$$\lim_{x\to\infty} \frac{(u'_\lambda(x))^2}{u^2_\lambda(x)} = \lim_{x\to\infty} \frac{u''_\lambda(x)}{u_\lambda(x)} = \lim_{x\to\infty} (V(x) - \lambda) = -\lambda.$$

Thus $\frac{u'_\lambda(x)}{u_\lambda(x)} \longrightarrow -\sqrt{-\lambda}$ as $x \longrightarrow \infty$ because u_λ and u'_λ have opposite sign at infinity. Given $\epsilon > 0$, there exists $M > d$ such that

$$-\sqrt{-\lambda+\epsilon} < \frac{u'_\lambda(x)}{u_\lambda(x)} < -\sqrt{-\lambda-\epsilon} \qquad \text{on } [M,\infty). \tag{2.6}$$

Integrating (2.6) on $[M, x]$ we obtain, as in case 1°,

$$u_\lambda(M)e^{-\sqrt{-\lambda+\epsilon}(x-M)} < u_\lambda(x) < u_\lambda(M)e^{-\sqrt{-\lambda-\epsilon}(x-M)} \qquad \text{for any } x > M. \tag{2.7}$$

Finally, (2.2) and (2.3) follow from (2.5), respectivey (2.7) and the fact that $\lim\limits_{x\to\infty} \frac{u''_\lambda(x)}{u_\lambda(x)} = -\lambda$, $\lim\limits_{x\to\infty} \frac{u'_\lambda(x)}{u_\lambda(x)} = \pm\sqrt{-\lambda}$. It is obvious that λ is an eigenvalue of L if and only if $u_\lambda \in \mathbf{H}$, i.e. if and only if we are in case 2°. Therefore assertion iv) is proved.

Note also that u_λ has only a finite number of zeroes. Indeed, it follows from the above arguments that u_λ has at most one zero in (d,∞) and we know that any zero is isolated, so there are only finitely many zeroes in $[0, d]$.

The proofs of v) and vi) are rather classical and are similar to the proofs of Theorems XIII.8 and XIII.9 p. 90-94 in [13]. The bound on the number of eigenvalues given by vi) is due to Bargmann (see [13] and references therein). □

Corollary 2.2 *The linear operator $A = -\frac{d^2}{dx^2} + q^2(1+r_{2c\varepsilon})^2$ (considered on $\mathbf{L}$ with domain $D(A) = \mathbf{H}$) is self-adjoint and has the following properties :*

i) $A \geq 2c^2\varepsilon^2q^2$ and $\sigma_{ess}(A) = [q^2,\infty)$.

ii) A has at least one eigenvalue in $[2c^2\varepsilon^2q^2, q^2)$.

iii) Any eigenvalue of A is simple. If $\mu < q^2$ is an eigenvalue and u_μ is a corresponding eigenvector, then for any $\epsilon > 0$, there exist $C_1, C_2, M > 0$ such that

$$C_1 e^{-\sqrt{q^2-\mu+\epsilon}|x|} \leq |u_\mu^{(m)}(x)| \leq C_2 e^{-\sqrt{q^2-\mu-\epsilon}|x|} \qquad \text{if } |x| \geq M, \quad m = 0, 1, 2. \tag{2.8}$$

iv) Let N_q be the number of eigenvalues of A in $[2c^2\varepsilon^2q^2, q^2)$. We have $N_q < 1+(2\ln 2)q^2$. In particular, if $q \leq \frac{1}{\sqrt{2\ln 2}}$, then A has exactly one eigenvalue less than q^2.

v) We have $N_q \longrightarrow \infty$ as $q \longrightarrow \infty$.

It can be proved that there exist $c_1, c_2, q_0 > 0$ such that $c_1 q \leq N_q \leq c_2 q$ for any $q \geq q_0$, but we will not make use of this result in what follows.

Proof. Recall that $r_{2c\varepsilon}$ is given by (1.12). We have $A = -\frac{d^2}{dx^2} + q^2V(x) + q^2$, where the function V given by $V(x) = (1+r_{2c\varepsilon}(x))^2 - 1 = (1-2c^2\varepsilon^2)\Big(-1+\tanh^2(\sqrt{\frac{1-2c^2\varepsilon^2}{2}}x)\Big)$ is even, negative, tends exponentially to zero as $x \longrightarrow \pm\infty$ and $\inf\limits_{x\in\mathbf{R}} V(x) = 2c^2\varepsilon^2 - 1$. Obviously, μ is an eigenvalue of A if and only if $\mu - q^2$ is an eigenvalue of $-\frac{d^2}{dx^2} + q^2V(x)$, so i), ii) and iii) follow at once from Proposition 2.1. An easy computation gives

$$\int_0^\infty x|V(x)|dx = (1-2c^2\varepsilon^2)\int_0^\infty x\Big(1-\tanh^2(\sqrt{\frac{1-2c^2\varepsilon^2}{2}}x)\Big)dx$$
$$= 2\int_0^\infty y(1-\tanh^2 y)dy = 2\int_0^\infty y(\tanh y - 1)'dy = 2\ln 2.$$

Now iv) is a direct consequence of Proposition 2.1, vi).

v) Fix $n \in \mathbf{N}$, $n \geq 1$ and take n symmetric functions $\varphi_1, \ldots, \varphi_n \in C_0^\infty(\mathbf{R})$, $\varphi_i \not\equiv 0$, such that $supp(\varphi_i) \cap supp(\varphi_j) = \emptyset$ if $i \neq j$. Clearly,

$$\langle A\varphi_i, \varphi_i\rangle_{\mathbf{L}} - q^2\langle\varphi_i,\varphi_i\rangle_{\mathbf{L}} = \int_{\mathbf{R}} |\nabla\varphi_i|^2dx + q^2\int_{\mathbf{R}} V(x)|\varphi_i(x)|^2dx \longrightarrow -\infty \quad \text{ as } q \longrightarrow \infty$$

hence there exists $q_0 > 0$ such that for any $q \geq q_0$ and any $i = 1, \dots, n$ we have $\langle A\varphi_i, \varphi_i\rangle_{\mathbf{L}} - q^2\langle \varphi_i, \varphi_i\rangle_{\mathbf{L}} < 0$. Since the φ_i's have disjoint supports we get

$$\begin{aligned}
&\langle A\Big(\sum_{i=1}^{n}\alpha_i\varphi_i\Big), \sum_{i=1}^{n}\alpha_i\varphi_i\rangle_{\mathbf{L}} - q^2||\sum_{i=1}^{n}\alpha_i\varphi_i||^2_{\mathbf{L}} \\
&= \sum_{i=1}^{n}|\alpha_i|^2\Big(\int_{\mathbf{R}}|\nabla\varphi_i|^2dx + q^2\int_{\mathbf{R}}V(x)|\varphi_i(x)|^2dx\Big) < 0
\end{aligned}$$

Therefore we have found an n-dimensional subspace of $\mathbf{H}$, $V_n = Span\{\varphi_1, \dots, \varphi_n\}$ such that $\langle Au, u\rangle_{\mathbf{L}} - q^2||u||_{\mathbf{L}} < 0$ for any $u \in V_n$ and any $q \geq q_0$. By the Min-Max Principle (see, e.g., [13], Theorem XIII.1 p.76) it follows that for $q \geq q_0$, A has at least n eigenvalues less than q^2, that is $N_q \geq n$ if $q \geq q_0$. This proves v). □

We have the following result concerning the existence of non-trivial solitary waves:

Theorem 2.3 *Let $\lambda_* < q^2$ be an eigenvalue of A and let u_* be a corresponding eigenvector. There exists $\eta > 0$ and C^∞ functions*

$$s \longmapsto (\lambda(s), r(s), u(s)) \in \mathbf{R} \times \mathbf{H} \times (u_*^\perp \cap \mathbf{H})$$

defined on $(-\eta, \eta)$ such that $\lambda(0) = \lambda_*$, $r(0) = 0$, $u(0) = 0$ and

$$S(r_{2c\varepsilon} + sr(s), s(u_* + u(s))) = 0, \quad T(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_* + u(s))) = 0.$$

Moreover, there exists a neighbourhood U of $(\lambda_*, r_{2c\varepsilon}, 0)$ in $\mathbf{R}\times\mathbf{H}\times\mathbf{H}$ such that any solution of $S(r, u) = 0$, $T(\lambda, r, u) = 0$ in U is either of the form $(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_* + u(s)))$ or of the form $(\lambda, r_{2c\varepsilon}, 0)$.

That is, $r = r_{2c\varepsilon} + sr(s)$, $u = s(u_* + u(s))$ are nontrivial solutions of (1.9)-(1.10) for $\lambda = \lambda(s)$.

Let $g_{2c\varepsilon} : (-1, \infty) \longrightarrow \mathbf{R}$, $g_{2c\varepsilon}(x) = (1+x)^3 - (1+x) - c^2\varepsilon^2\Big(1 + x - \frac{1}{(1+x)^3}\Big)$. Then $S(r, u)$ can be written as $S(r, u) = -r'' + g_{2c\varepsilon}(r) + (1 + r)u^2$. It is easily seen that $d_rS(r_{2c\varepsilon}, 0) = -\frac{d^2}{dx^2} + g'_{2c\varepsilon}(r_{2c\varepsilon})$.

For the proof of Theorem 2.3, we need the following lemmas :

Lemma 2.4 *The linear operator $J := -\frac{d^2}{dx^2} + g'_{2c\varepsilon}(r_{2c\varepsilon}) : \mathbf{H} \longrightarrow \mathbf{L}$ has the following properties :*

i) J is self-adjoint, invertible and has the essential spectrum $\sigma_{ess}(J) = [2 - 4c^2\varepsilon^2, \infty)$.

ii) J has exactly one negative eigenvalue and any eigenvalue of J is simple.

Proof. i) The linear operator $B = -\frac{d^2}{dx^2} + g'_{2c\varepsilon}(r_{2c\varepsilon})$ with domain $D(B) = H^2(\mathbf{R})$ is self-adjoint in $L^2(\mathbf{R})$. We claim that $Ker(B) = Span\{\frac{d}{dx}r_{2c\varepsilon}\}$. Indeed, we have

$$\frac{d^2}{dx^2}r_{2c\varepsilon} = g_{2c\varepsilon}(r_{2c\varepsilon}). \tag{2.9}$$

Thus $r''_{2c\varepsilon} \in C^1(\mathbf{R})$. Differentiating (2.9) with respect to x we get $\frac{d}{dx}r_{2c\varepsilon} \in Ker(B)$.

Conversely, let $h \in Ker(B)$. Then $h'' = g'_{2c\varepsilon}(r_{2c\varepsilon})h$, so that

$$(h'r'_{2c\varepsilon})' = h''r'_{2c\varepsilon} + h'r''_{2c\varepsilon} = hg'_{2c\varepsilon}(r_{2c\varepsilon})r'_{2c\varepsilon} + h'g_{2c\varepsilon}(r_{2c\varepsilon}) = (hg_{2c\varepsilon}(r_{2c\varepsilon}))'.$$

Hence $h'r'_{2c\varepsilon} = hg_{2c\varepsilon}(r_{2c\varepsilon}) + C$ on $\mathbf{R}$. Taking the limits as $|x| \longrightarrow \infty$, we get $C = 0$, so $h'r'_{2c\varepsilon} = hg_{2c\varepsilon}(r_{2c\varepsilon}) = hr''_{2c\varepsilon}$. Since $r'_{2c\varepsilon} \neq 0$ on $(-\infty, 0)$ and on $(0, \infty)$, on each of these intervals we have $\left(\frac{h}{r'_{2c\varepsilon}}\right)' = \frac{h'r'_{2c\varepsilon}-hr''_{2c\varepsilon}}{(r'_{2c\varepsilon})^2} = 0$. Thus there exist constants C_1, C_2 such that $h(x) = C_1 r'_{2c\varepsilon}(x)$ on $(-\infty, 0)$ and $h(x) = C_2 r'_{2c\varepsilon}(x)$ on $(0,\infty)$. Consequently, $h'(x) = C_1 r''_{2c\varepsilon}(x) = C_1 g(r_{2c\varepsilon}(x))$ on $(-\infty, 0)$ and $h'(x) = C_2 r''_{2c\varepsilon}(x) = C_2 g_{2c\varepsilon}(r_{2c\varepsilon}(x))$ on $(0,\infty)$. But h' is continuous because $h \in H^2(\mathbf{R})$ and therefore $C_1 = C_2$, which proves our claim.

Since $r'_{2c\varepsilon} \notin \mathbf{H}$, it is clear that the restriction of B to $\mathbf{H}$ is one-to-one from $\mathbf{H}$ into $\mathbf{L}$. It remains to prove that $B\mathbf{H} = \mathbf{L}$. It is well-known that $Im(B) = Ker(B)^{\perp} = (r'_{2c\varepsilon})^{\perp}$ since B is self-adjoint. We have $\mathbf{L} \subset Im(B)$ because $r'_{2c\varepsilon}$ is an odd function. Let $f \in \mathbf{L}$. Clearly there exists $r \in H^2(\mathbf{R})$ such that $Br = f$. Let $\tilde{r}(x) = r(-x)$. It is easy to see that $B\tilde{r} = f$, hence there exists C such that $r - \tilde{r} = Cr'_{2c\varepsilon}$. Then $r - \frac{1}{2}Cr'_{2c\varepsilon} = \frac{1}{2}(r + \tilde{r}) \in \mathbf{H}$ and $B(r - \frac{1}{2}Cr'_{2c\varepsilon}) = f$.

Now it is clear that J, which is the restriction of B to $\mathbf{H}$, is self-adjoint in $\mathbf{L}$ and invertible. The function $g'_{2c\varepsilon}(r_{2c\varepsilon})$ tends (exponentially) to $g'_{2c\varepsilon}(0) = 2 - 4c^2\varepsilon^2$ as $x \longrightarrow \infty$. It follows from Weyl's theorem that $\sigma_{ess}(J) = \sigma_{ess}(B) = [2 - 4c^2\varepsilon^2, \infty)$. This completes the proof of i).

ii) It follows from Proposition 2.1 iii) and v) that any eigenvalue of J is simple and the number of negative eigenvalues of J is exactly the number of zeroes of u in $(0,\infty)$, where u is the solution of the Cauchy problem

$$\begin{cases} -u'' + g'_{2c\varepsilon}(r_{2c\varepsilon})u = 0 & \text{in } [0,\infty), \\ u(0) = 1, \quad u'(0) = 0. \end{cases} \tag{2.10}$$

We use the following simplified version of the well-known Sturm oscillation lemma (this is also a paticular case of Lemma 5 in [8]) :

Sturm oscillation lemma. *Let Y and Z be nontrivial solutions of the differential equation*

$$-\varphi'' + h(x)\varphi = 0$$

on some interval (μ, ν), where h is continuous on (μ, ν). If Y and Z are linearly independent and $Y(\mu) = Y(\nu) = 0$, then Z has at least one zero in (μ, ν).

From this lemma it follows at once that J has at most one negative eigenvalue. Indeed, suppose that J has at least two negative eigenvalues. Then the solution u of (2.10) has at least two zeroes in $(0,\infty)$, say, $x_1 < x_2$. But the function $r'_{2c\varepsilon}$ also satisfies the differential equation in (2.10) and obviously u and $r'_{2c\varepsilon}$ are linearly independent (because $r'_{2c\varepsilon}(0) = 0$). Using Sturm's oscillation lemma, we infer that $r'_{2c\varepsilon}$ must have a zero on (x_1, x_2), which is absurd because $r'_{2c\varepsilon}(x) > 0$ on $(0,\infty)$.

Now let us prove that J has (at least) one negative eigenvalue. We argue again by contradiction and we suppose that J has no negative eigenvalues. Then the solution u of (2.10) has no zeroes in $[0,\infty)$, consequently $u(x) > 0$ for any $x \in [0,\infty)$. Since $g'_{2c\varepsilon}(r_{2c\varepsilon}(x)) \longrightarrow 2 - 4c^2\varepsilon^2 > 0$ as $x \longrightarrow \infty$, repeating the argument used in the proof of Proposition 2.1 iv) we infer that either $u(x) \longrightarrow \infty$ or $u(x) \longrightarrow 0$ as $x \longrightarrow \infty$. In the latter case we have also

$$|u^{(m)}(x)| \le Ce^{-\sqrt{2-4c^2\varepsilon^2-\delta}|x|}, \qquad m = 0, 1, 2$$

for some constant $C > 0$, $\delta \in (0, 2 - 4c^2\varepsilon^2)$ and x sufficiently large. Consequently, $u \in \mathbf{H}$ and 0 is an eigenvalue of J. But this is excluded by i). Therefore we must have $u(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Since $u(0) = 1$, we have $u > 0$ in a neighbourhood of 0. Note that $g'_{2c\varepsilon}(r_{2c\varepsilon}(0)) = (5 + \frac{3}{2c^2\varepsilon^2})(c^2\varepsilon^2 - \frac{1}{2}) < 0$, hence $g'_{2c\varepsilon}(r_{2c\varepsilon}) < 0$ near 0. From (2.10) we get $u''(x) < 0$ for

$x>0$ and x close to 0. We have $u'(0)=0$, so there exists $\delta>0$ such that $u'(x)<0$ on $(0,\delta]$. We may choose δ so small that $u(\delta)>0$ and $r''_{2c\varepsilon}(\delta)>0$ (note that $r''_{2c\varepsilon}(0)=g_{2c\varepsilon}(r_{2c\varepsilon}(0))=\frac{(1-2c^2\varepsilon^2)^2}{2\sqrt{2}c\varepsilon}>0$). Let $\beta=\frac{u(\delta)}{r'_{2c\varepsilon}(\delta)}>0$ and let $h(x)=\beta r'_{2c\varepsilon}(x)-u(x)$. Clearly, h is a solution of the differential equation in (2.10) and $h(\delta)=0$, $h'(\delta)=\beta r''_{2c\varepsilon}(\delta)-u'(\delta)>0$. Hence $h(x)>0$ for $x>\delta$ and x close to δ. On the other hand, we have $\lim\limits_{x\longrightarrow\infty}h(x)=-\infty$, so there exists $\eta>\delta$ such that $h(\eta)=0$. Since both $r'_{2c\varepsilon}$ and h satisfy the differential equation in (2.10), by the Sturm oscillation lemma we infer that $r'_{2c\varepsilon}$ must have a zero in (δ,η), which is absurd. This finishes the proof of Lemma 2.4. □

Lemma 2.5 *We have:*
i) $Ker(T(\lambda_*,r_{2c\varepsilon},\cdot))=Span(u_*)$;
ii) $Im(T(\lambda_*,r_{2c\varepsilon},\cdot))=u_*^{\perp}\cap\mathbf{L}$.

The proof is obvious.

Proof of Theorem 2.3. Let $\tilde V=\{r\in\mathbf{H}\mid \sup\limits_{x\in\mathbf{R}}|r(x)|<1\}$ and $I=(-\sqrt{2}c\varepsilon,\sqrt{2}c\varepsilon)$. Clearly $\tilde V$ is open in $\mathbf{H}$. We define $F: I\times\mathbf{R}\times\tilde V\times(\mathbf{H}\cap u_*^{\perp})\longrightarrow\mathbf{L}\times\mathbf{L}$ by

$$F(s,\lambda,r,u)=\begin{cases}\begin{pmatrix}\dfrac{1}{s}S(r_{2c\varepsilon}+sr,s(u_*+u))\\ \\ \dfrac{1}{s}T(\lambda,r_{2c\varepsilon}+sr,s(u_*+u))\end{pmatrix} & \text{if } s\neq 0,\\ \\ \begin{pmatrix}(d_rS(r_{2c\varepsilon},0).r\\ T(\lambda,r_{2c\varepsilon},u_*+u)\end{pmatrix} & \text{if } s=0.\end{cases}$$

It is easily seen that F is C^∞ because

$$F_1(s,\lambda,r,u)=\tfrac{1}{s}\left(S(r_{2c\varepsilon}+sr,s(u_*+u))-S(r_{2c\varepsilon},0)\right)$$

$$=\tfrac{1}{s}\int_0^1\frac{d}{dt}S(r_{2c\varepsilon}+tsr,ts(u_*+u))dt$$

$$=\tfrac{1}{s}\int_0^1 d_rS(r_{2c\varepsilon}+tsr,ts(u_*+u)).sr+d_uS(r_{2c\varepsilon}+tsr,ts(u_*+u)).s(u_*+u)dt$$

$$=\int_0^1 d_rS(r_{2c\varepsilon}+tsr,ts(u_*+u)).r+d_uS(r_{2c\varepsilon}+tsr,ts(u_*+u)).(u_*+u)dt$$

and $F_2(s,\lambda,r,u)=T(\lambda,r_{2c\varepsilon}+sr,u_*+u)$.

It is also clear that $F(0,\lambda_*,0,0)=\begin{pmatrix}0\\0\end{pmatrix}$ and

$$d_{(\lambda,r,u)}F(0,\lambda_*,0,0)(\tilde\lambda,\tilde r,\tilde u)=\begin{pmatrix}0\\-\tilde\lambda u_*\end{pmatrix}+\begin{pmatrix}d_rS(r_{2c\varepsilon},0).\tilde r\\0\end{pmatrix}+\begin{pmatrix}0\\T(\lambda_*,r_{2c\varepsilon},\tilde u)\end{pmatrix}$$

In view of Lemmas 2.4 and 2.5, $d_{(\lambda,r,u)}F(0,\lambda_*,0,0)$ is invertible. By the Implicit Function Theorem, there exist $\eta>0$ and C^∞ functions defined on $(-\eta,\eta)$,

$$s\longmapsto(\lambda(s),r(s),u(s))\in\mathbf{R}\times\mathbf{H}\times(\mathbf{H}\cap u_*^{\perp})$$

such that $\lambda(0) = \lambda_*$, $r(0) = 0$, $u(0) = 0$ and $F(s, \lambda(s), u(s), r(s)) = (0, 0)$. It is obvious that for $s \neq 0$, $(\lambda(s), (r_{2c\varepsilon} + sr(s), s(u_0 + u(s))))$ satisfy the system (1.9)-(1.10). Finally, the uniqueness part in Theorem 2.3 is proved exactly in the same way as in the Bifurcation from a Simple Eigenvalue Theorem. □

Remark 2.6 Let $\lambda(s)$, $r(s)$, $u(s)$ be given by Theorem 2.3. We have $\dot\lambda(0) = 0$, $\dot u(0) = 0$ and

$$\ddot\lambda(0) = -\frac{4q^2}{||u_*||^2_{\mathbf{L}}}\langle(1 + r_{2c\varepsilon})u_*^2, J^{-1}((1 + r_{2c\varepsilon})u_*^2)\rangle_{\mathbf{L}}, \tag{2.11}$$

where the dots denote derivatives with respect to s and J is the operator in Lemma 2.4.

To see this, we differentiate with respect to s the equation $T(\lambda(s), r_{2c\varepsilon} + sr(s), u_* + u(s)) = 0$ and then we take $s = 0$ to obtain

$$-\frac{d^2}{dx^2}\dot u(0) + [q^2(1 + r_{2c\varepsilon})^2 - \lambda_*]\dot u(0) - \dot\lambda(0)u_* = 0, \tag{2.12}$$

that is $(A - \lambda_*)\dot u(0) - \dot\lambda(0)u_* = 0$. But $Im(A - \lambda_*)$ and $Ker(A - \lambda_*) = Span\{u_*\}$ are orthogonal (because A is self-adjoint), so (2.12) implies that $\dot\lambda(0) = 0$ and $\dot u(0) = 0$.

We differentiate twice with respect to s the equation $T(\lambda(s), r_{2c\varepsilon} + sr(s), u_* + u(s)) = 0$, then we take $s = 0$ to get

$$(A - \lambda_*)\ddot u(0) + 4q^2(1 + r_{2c\varepsilon})\dot r(0)u_* - \ddot\lambda(0)u_* = 0. \tag{2.13}$$

Substracting the equation $-r''_{2c\varepsilon} + g_{2c\varepsilon}(r_{2c\varepsilon}) = 0$ from the equation $S(r_{2c\varepsilon} + sr(s), s(u_* + u(s))) = 0$ and then dividing by s we get

$$-\frac{d^2}{dx^2}r(s) + \int_0^1 g'_{2c\varepsilon}(r_{2c\varepsilon} + tsr(s))\, dt \cdot r(s) + s(1 + r_{2c\varepsilon} + sr(s))(u_* + u(s))^2 = 0. \tag{2.14}$$

We differentiate (2.14) with respect to s, then we take $s = 0$ to obtain

$$-\frac{d^2}{dx^2}\dot r(0) + g'_{2c\varepsilon}(r_{2c\varepsilon})\dot r(0) + (1 + r_{2c\varepsilon})u_*^2 = 0,$$

that is $J\dot r(0) + (1 + r_{2c\varepsilon})u_*^2 = 0$, which can still be written as

$$\dot r(0) = -J^{-1}((1 + r_{2c\varepsilon})u_*^2). \tag{2.15}$$

Taking the scalar product of (2.13) with u_* we find $\ddot\lambda(0)||u_*||^2_{\mathbf{L}} = 4q^2\langle 1 + r_{2c\varepsilon})u_*^2, \dot r(0)\rangle_{\mathbf{L}}$. We replace $\dot r(0)$ from (2.15) in the last equality to obtain (2.11).

3 Global branches of solutions

Our purpose is to obtain information about the global structure of the set of nontrivial solutions of (1.9)-(1.10). We give a nonexistence result first.

Proposition 3.1 *i) The system (1.9)-(1.10) does not admit solutions* $(\lambda, r, u) \in \mathbf{R} \times V \times H^1(\mathbf{R})$ *with* $(r, u) \neq (0, 0)$ *if* $c \geq \frac{1}{\varepsilon\sqrt{2}}$.

ii) Suppose that $c < \frac{1}{\varepsilon\sqrt{2}}$ *and let* $(\lambda, r, u) \in \mathbf{R} \times V \times H^1(\mathbf{R})$ *be a nontrivial solution of the system (1.9)-(1.10). Then* $2c^2\varepsilon^2q^2 < \lambda \leq q^2$ *and* $-1 + \sqrt{2}c\varepsilon < r(x) \leq 0$ *for any* $x \in \mathbf{R}$.

Proof. Let $(\lambda, r, u) \in \mathbf{R} \times V \times H^1(\mathbf{R})$ be a solution of (1.9)-(1.10). Since $H^1(\mathbf{R}) \subset C_b(\mathbf{R})$, the equations (1.9)-(1.10) imply that r'' and u'' are continuous, hence $r, u \in C^2(\mathbf{R})$.

If $u \equiv 0$ and $c \geq \frac{1}{\varepsilon\sqrt{2}}$, the only solution of (1.9) which tends to zero at $\pm\infty$ is $r \equiv 0$ (this was proved in [7], but can be easily deduced from the arguments below). From now on we suppose that $u \not\equiv 0$. Multiplying (1.10) by u and integrating we find

$$\int_{\mathbf{R}} |u'|^2 dx + q^2 \int_{\mathbf{R}} (1+r)^2 |u|^2 dx = \lambda \int_{\mathbf{R}} |u|^2 dx. \tag{3.1}$$

Since $u \not\equiv 0$, we have necessarily $\lambda > 0$. Let
$G_{2c\varepsilon}(s) = \int_0^s g_{2c\varepsilon}(\tau) d\tau = \frac{1}{4}((1+s)^2 - 1)^2 \Big(1 - \frac{2c^2\varepsilon^2}{(1+s)^2}\Big)$. Multiplying (1.9) by r' gives

$$-\frac{1}{2}[(r')^2]' + [G_{2c\varepsilon}(r)]' + \frac{1}{2}[(1+r)^2]' u^2 = 0, \tag{3.2}$$

and multiplying (1.10) by u' leads to

$$-\frac{1}{2}[(u')^2]' + \frac{1}{2} q^2 (1+r)^2 (u^2)' - \frac{\lambda}{2}(u^2)' = 0. \tag{3.3}$$

From (3.2) and (3.3) we get

$$-\frac{1}{2}[(r')^2]' - \frac{1}{2q^2}[(u')^2]' + [G_{2c\varepsilon}(r)]' + \frac{1}{2}[(1+r)^2 u^2]' - \frac{\lambda}{2q^2}(u^2)' = 0. \tag{3.4}$$

Integrating (3.4) from $-\infty$ to x and taking into account that $r(x) \longrightarrow 0$, $r'(x) \longrightarrow 0$, $u(x) \longrightarrow 0$ and $u'(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$ we obtain

$$|r'|^2(x) + \frac{1}{q^2}|u'|^2(x) + \Big(\frac{\lambda}{q^2} - (1+r(x))^2\Big) u^2(x) = 2G_{2c\varepsilon}(r(x)) \qquad \text{for any } x \in \mathbf{R}. \tag{3.5}$$

Suppose that there exists $x_0 \in \mathbf{R}$ such that $r(x_0) < \min(-1 + \frac{\sqrt{\lambda}}{q}, -1 + \sqrt{2}c\varepsilon)$. Then $\frac{\lambda}{q^2} - (1 + r(x_0))^2 > 0$ and the left hand side of (3.5) is positive at x_0 (because $u(x_0) = u'(x_0) = 0$ and (1.10) would imply $u \equiv 0$) while $G_{2c\varepsilon}(r(x_0)) < 0$, a contradiction. Thus $r(x) \geq \min(-1 + \frac{\sqrt{\lambda}}{q}, -1 + \sqrt{2}c\varepsilon)$ for any $x \in \mathbf{R}$.

Suppose that $\lambda \leq 2c^2\varepsilon^2 q^2$ (that is, $\frac{\sqrt{\lambda}}{q} \leq \sqrt{2}c\varepsilon$). Then we have $(1 + r(x))^2 \geq \frac{\lambda}{q^2}$ for any $x \in \mathbf{R}$ and (3.1) gives

$$\int_{\mathbf{R}} |u'|^2 dx + q^2 \int_{\mathbf{R}} \Big((1+r)^2 - \frac{\lambda}{q^2}\Big) u^2 dx = 0,$$

which implies $u \equiv 0$, again a contradiction. Therefore we have $\lambda > 2c^2\varepsilon^2 q^2$ and $r(x) \geq -1 + \sqrt{2}c\varepsilon$ for any $x \in \mathbf{R}$. This is impossible if $\sqrt{2}c\varepsilon > 1$ because $r(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$.

Hence we cannot have other solutions than $(\lambda, 0, 0)$ if $\sqrt{2}c\varepsilon > 1$. From now on we suppose that $\sqrt{2}c\varepsilon \leq 1$. In this case we have $r \leq 0$ on $\mathbf{R}$ by the Maximum Principle. Indeed, the function $g_{2c\varepsilon}$ is strictly increasing and positive on $(0, \infty)$. Suppose that r achieves a positive maximum at x_0. Then $r''(x_0) \leq 0$. On the other hand, from (1.9) we infer that $r''(x_0) = g_{2c\varepsilon}(r(x_0)) + (1 + r(x_0))u^2(x_0) > 0$, which is absurd.

If $\sqrt{2}c\varepsilon = 1$ we have seen that $0 \geq r(x) \geq -1 + \sqrt{2}c\varepsilon = 0$, hence $r \equiv 0$. Then (1.10) becomes $u'' = (q^2 - \lambda)u$; together with the boundary condition $u(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$, this gives $u \equiv 0$. Thus i) is proved.

From now on we suppose throughout that $2c^2\varepsilon^2 < 1$. Clearly, if $r(x_0) = -1 + \sqrt{2}c\varepsilon$ for some $x_0 \in \mathbf{R}$, then (3.5) would imply $u(x_0) = u'(x_0) = 0$ (because $\lambda > 2c^2\varepsilon^2 q^2$), hence $u \equiv 0$ by (1.10), which is impossible. Hence $0 \geq r(x) > -1 + \sqrt{2}c\varepsilon$ for any $x \in \mathbf{R}$.

It only remains to show that we cannot have nontrivial solutions with $\lambda > q^2$. Suppose that (λ, r, u) is such a solution. First, observe that r cannot vanish because (3.5) would give a contradiction. We prove that r decays sufficiently fast at infinity. Take $0 < \epsilon < \frac{\lambda}{q^2} - 1$.There exists $M_\epsilon > 0$ such that $(1 + r(x))^2 \leq 1 + \epsilon$ on $[M_\epsilon, \infty)$ (because $r(x) \longrightarrow 0$ as $x \longrightarrow \infty$). Using (3.5), we have on $[M_\epsilon, \infty)$

$$0 \leq \Big(\frac{\lambda}{q^2} - 1 - \epsilon\Big) u^2(x) \leq 2G_{2c\varepsilon}(r(x)),$$

hence $0 \leq \left(\frac{\lambda}{q^2} - 1 - \epsilon\right) \frac{u^2(x)}{|r(x)|} \leq 2\frac{|G_{2c\varepsilon}(r(x))|}{|r(x)|}$. Passing to the limit as $x \longrightarrow \infty$ we obtain $\lim\limits_{x\to\infty} \frac{u^2(x)}{r(x)} = 0$. Dividing (1.9) by r we get

$$\frac{r''(x)}{r(x)} = \frac{g_{2c\varepsilon}(r(x))}{r(x)} + (1 + r(x))\frac{u^2(x)}{r(x)} \longrightarrow g'_{2c\varepsilon}(0) > 0 \qquad \text{as } x \longrightarrow \infty. \tag{3.6}$$

Since r'' must have at least one zero between two zeroes of r', (3.6) shows that r' has no zeroes in some neighbourhood of infinity. In that neighbourhood we have

$$\frac{(|r'(x)|^2)'}{(r^2(x))'} = \frac{r''(x)}{r(x)} \longrightarrow g'_{2c\varepsilon}(0) > 0 \qquad \text{as } x \longrightarrow \infty.$$

Since $r(x) \longrightarrow 0$ and $r'(x) \longrightarrow 0$ at infinity, we may apply l'Hôspital's rule to get $\lim\limits_{x\to\infty} \left(\frac{r'(x)}{r(x)}\right)^2 = g'_{2c\varepsilon}(0)$. We know that r and r' have constant sign in a neighbourhood of infinity and they cannot have the same sign because r tends to 0 at infinity, so necessarily $\lim\limits_{x\to\infty} \frac{r'(x)}{r(x)} = -\sqrt{g'_{2c\varepsilon}(0)}$. The argument already used in the proof of Proposition 2.1 shows that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|r(x)| \leq C_\epsilon e^{-\sqrt{g'_{2c\varepsilon}(0) - \epsilon}\, x} \qquad \text{for any } x \in [0, \infty).$$

Of course that a similar estimate is valid on $(-\infty, 0]$. In particular, $r^2 + 2r$ is a continuous, bounded function on $\mathbf{R}$ and $\lim\limits_{x\to\pm\infty} |x|(r^2(x) + 2r(x)) = 0$. Moreover, multiplication by $r^2 + 2r$ is a bounded aperator on $L^2(\mathbf{R})$, hence it is also bounded with respect to $-\frac{d^2}{dx^2}$ with relative bound zero. Consequently, by the Kato-Agmon-Simon Theorem (see, e.g., [13], Theorem XIII.58 p. 226), the operator $-\frac{d^2}{dx^2} + q^2(r^2 + 2r)$ (with domain $H^2(\mathbf{R})$ and range $L^2(\mathbf{R})$) cannot have eigenvalues embedded in the continuous spectrum $(0, \infty)$. This means exactly that the operator $-\frac{d^2}{dx^2} + q^2(1 + r)^2$ has no eigenvalues in (q^2, ∞) and contradicts the existence of a non-tivial solution (λ, r, u) with $\lambda > q^2$. □

We will use the following variant of the Global Bifurcation Theorem of Rabinowitz :

Proposition 3.2 *Let E be a real Banach space and $\Omega \subset \mathbf{R} \times E$ an open set. Suppose that $G : \Omega \longrightarrow E$ is compact on closed, bounded subsets $\omega \subset \Omega$ such that $dist(\omega, \partial\Omega) > 0$ and is of the form $G(a, u) = L(a, u) + H(a, u)$, where L and H satisfy the following assumptions :*

a) $L(a, \cdot)$ is linear, compact for any fixed a and $(a, u) \longmapsto L(a, u)$ is continuous and compact on closed, bounded subsets $\omega \subset \Omega$ such that $dist(\omega, \partial\Omega) > 0$.

b) For any closed, bounded subset $\omega \subset \Omega$ such that $dist(\omega, \partial\Omega) > 0$, there exists a function h_ω such that $h_\omega(s) \longrightarrow 0$ as $s \longrightarrow 0$ and

$$||H(a,u)|| \leq ||u|| h_\omega(||u||) \qquad \text{for any } (a,u) \in \omega.$$

c) There exists a_0 and $\epsilon > 0$ such that
- $(a_0, 0) \in \Omega$,
- *for any $a \in [a_0 - \epsilon, a_0 + \epsilon] \setminus \{a_0\}$ we have $Ker(Id - L(a,\cdot)) = \{0\}$,*
- *if $a_1 \in [a_0 - \epsilon, a_0)$ and $a_2 \in (a_0, a_0 + \epsilon]$, then*

$$ind(Id - L(a_1,\cdot),0) \neq ind(Id - L(a_2,\cdot),0).$$

Let

$$\mathcal{S} = \{(a,u) \in \Omega \mid u \neq 0 \text{ and } u = G(a,u)\}$$

be the set of nontrivial solutions of the equation $u = G(a,u)$. Then $\mathcal{S} \cup \{(a_0, 0)\}$ possesses a maximal subcontinuum (i.e. a maximal closed connected subset) $\mathcal{C}_{a_0}$ which contains $(a_0, 0)$ and has at least one of the following properties :

i) $\mathcal{C}_{a_0}$ is unbounded ;

ii) $dist(\mathcal{C}_{a_0}, \partial\Omega) = 0$;

iii) $\mathcal{C}_{a_0}$ meets $(a_1, 0)$, where $a_1 \neq a_0$ and $Ker(Id - L(a_1,\cdot)) \neq \{0\}$.

From the first assertion in c) it follows that the index $ind(Id - L(a,\cdot),0) = deg(Id - L(a,\cdot), B(0,\rho), 0)$ is well defined for any $a \in [a_0 - \epsilon, a_0 + \epsilon] \setminus \{a_0\}$. By a) and the homotopy invariance of the Leray-Schauder degree, it is a continuous function of a. So we have necessarily $Ker(Id - L(a_0,\cdot)) \neq \{0\}$ (since otherwise $ind(Id - L(a_0,\cdot),0)$ would be defined and $ind(Id - L(a,\cdot)\ 0)$ would be constant for $a \in [a_0 - \epsilon, a_0 + \epsilon]$, contradicting the last assertion in c)).

The proof of Proposition 3.2 is similar to that of Theorem 1.3, p. 490 in [12] (see also Corollary 1.12 in [12]).

Next, we give a reformulation of problem (1.9)-(1.10) suitable for the use of Proposition 3.2.

Equation (1.9) can be written as $-r'' + g_{2c\varepsilon}(r) + (1+r)u^2 = 0$, where $g_{2c\varepsilon}(x) = (1+x)^3 - (1+x) - c^2\varepsilon^2\left(1 + x - \frac{1}{(1+x)^3}\right)$. We will seek for solutions of the form $r(x) = r_{2c\varepsilon}(x) + w(x)$. Taking into account that $r_{2c\varepsilon}$ satisfies $-r''_{2c\varepsilon} + g_{2c\varepsilon}(r_{2c\varepsilon}) = 0$, equation (1.9) becomes

$$-w'' + g_{2c\varepsilon}(r_{2c\varepsilon} + w) - g_{2c\varepsilon}(r_{2c\varepsilon}) + (1 + r_{2c\varepsilon} + w)u^2 = 0. \tag{3.7}$$

Note that $g'_{2c\varepsilon}(0) = 2 - 4c^2\varepsilon^2 > 0$, thus the linear operator $-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)$ (with domain $\mathbf{H}$ and range $\mathbf{L}$) is invertible, so equation (3.7) is equivalent to

(3.8)

$$w = -\Big(-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[g_{2c\varepsilon}(r_{2c\varepsilon} + w) - g_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(r_{2c\varepsilon})w + (1 + r_{2c\varepsilon} + w)u^2]$$

$$-\Big(-\frac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[(g'_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(0))w].$$

In the same way, equation (1.10) can be written as

$$-u'' + (q^2 - \lambda)u = q^2(1 - (1 + r_{2c\varepsilon} + w)^2)u.$$

For $\lambda < q^2$, the linear operator $-\frac{d^2}{dx^2} + q^2 - \lambda$ is invertible and (1.10) becomes

$$u = -q^2\Big(-\frac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(r^2_{2c\varepsilon} + 2r_{2c\varepsilon})u] - q^2\Big(-\frac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(w^2 + 2wr_{2c\varepsilon} + 2w)u]. \tag{3.9}$$

We denote

$$H_1(w,u) = \Big(-\tfrac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[g_{2c\varepsilon}(r_{2c\varepsilon}+w) - g_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(r_{2c\varepsilon})w + (1+r_{2c\varepsilon}+w)u^2],$$

$$H_2(\lambda,w,u) = q^2\Big(-\tfrac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(w^2 + 2wr_{2c\varepsilon} + 2w)u],$$

$$A_\lambda(u) = A(\lambda,u) = q^2\Big(-\tfrac{d^2}{dx^2} + q^2 - \lambda\Big)^{-1}[(r_{2c\varepsilon}^2 + 2r_{2c\varepsilon})u],$$

$$B(w) = \Big(-\tfrac{d^2}{dx^2} + g'_{2c\varepsilon}(0)\Big)^{-1}[(g'_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(0))w].$$

It is easy to see that $A_\lambda,\ B : \mathbf{L} \longrightarrow \mathbf{H}$ are linear and continuous. Denote $V_{2c\varepsilon} = \{r \in \mathbf{H} \mid r + r_{2c\varepsilon} \in V\}$. It is obvious that $V_{2c\varepsilon}$ is open in $\mathbf{H}$. Since $\mathbf{H} \subset C_b^1(\mathbf{R})$ and $\mathbf{H}$ is an algebra, H_1 and H_2 are well-defined and continuous from $V_{2c\varepsilon} \times \mathbf{H}$ and $(-\infty, q^2) \times \mathbf{H} \times \mathbf{H}$, respectively, to $\mathbf{H}$.

If $\lambda < q^2$, then (λ, r, u) satisfies the system (1.9)-(1.10) if and only if (λ, w, u) (where $w = r - r_{2c\varepsilon}$) satisfies the system (3.8)-(3.9) which is equivalent to

$$\begin{pmatrix} w \\ u \end{pmatrix} = -\begin{pmatrix} B & 0 \\ 0 & A_\lambda \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} - \begin{pmatrix} H_1(w,u) \\ H_2(\lambda,w,u) \end{pmatrix}. \tag{3.10}$$

We have already shown in Introduction that we cannot expect to have properness for problem (1.9)-(1.10). The counterexample that we have seen is essentially due to the invariance by translations of the system and to the fact that we have localized solutions. Of course that passing from (1.9)-(1.10) to (3.10) should not prevent the same problems to appear. To overcome this difficulty, we shall work on some weighted Sobolev space. As a "weight", we take a function $W : \mathbf{R} \longrightarrow \mathbf{R}$ which satisfies the following properties :

(W1) W is continuous and even, i.e. $W(x) = W(-x)$;

(W2) $W \geq 1$ and $\lim\limits_{x\to\infty} W(x) = \infty$;

(W3) There exists $C_W > 0$ such that $W(a+b) \leq C_W(W(a) + W(b))$.

It follows easily from (W1) and (W3) that there exist $K, s > 0$ such that $W(x) \leq K|x|^s$ for $|x| \geq 1$. Indeed, from (W3) we infer that $\forall a \in \mathbf{R}$, $W(2^n a) \leq (2C_W)^n W(a)$. If $x \in [2^{n-1}, 2^n]$ and $M = \max\limits_{x\in[0,1]} W(x)$, then

$$W(x) \leq (2C_W)^n W(\frac{x}{2^n}) \leq 2C_W M (2C_W)^{n-1} = 2C_W M 2^{(n-1)(1+\log_2 C_W)} \leq 2C_W M x^{1+\log_2 C_W}.$$

In particular, we get

(W4) $\forall a > 0, \quad e^{-a|\cdot|}W(\cdot) \in L^1 \cap L^\infty(\mathbf{R})$.

For a function W satisfying (W1)-(W3) we consider the spaces

$$\mathbf{L}_W = \{\varphi \in \mathbf{L} \mid W\varphi \in \mathbf{L}\},$$

$$\mathbf{H}_W = \{\varphi \in \mathbf{H} \mid W\varphi, W\varphi', W\varphi'' \in \mathbf{L}\},$$

endowed with the norms $||\varphi||_{\mathbf{L}_W} = ||W\varphi||_{L^2}$, respectively $||\varphi||^2_{\mathbf{H}_W} = ||W\varphi||^2_{L^2} + ||W\varphi'||^2_{L^2} + ||W\varphi''||^2_{L^2}$. Equiped with these norms, $\mathbf{L}_W$ and $\mathbf{H}_W$ are Hilbert spaces. It is clear that $||\varphi||_{L^2} \le ||\varphi||_{\mathbf{L}_W}$, $||\varphi||_{H^2} \le ||\varphi||_{\mathbf{H}_W}$ and $\mathbf{L}_W$ (respectively $\mathbf{H}_W$) is a dense subspace of $\mathbf{L}$ (respectively of $\mathbf{H}$).

Lemma 3.3 *The embedding $\mathbf{H}_W \subset C^1_b(\mathbf{R})$ is compact.*

Proof. It is clear that the embeddings $\mathbf{H}_W \subset H^2(\mathbf{R}) \subset C^1_b(\mathbf{R})$ are continuous. To prove compactness, consider an arbitrary sequence $u_n \rightharpoonup 0$ in $\mathbf{H}_W$ and let us show that $u_n \longrightarrow 0$ in $C^1_b(\mathbf{R})$. Fix $\epsilon > 0$. Let $K = \sup_n ||u_n||_{\mathbf{H}_W}$. There exists $M > 0$ such that $W(x) \ge \frac{K}{\epsilon}$ if $|x| \ge M$. It follows that $||u_n||_{H^2((-\infty,M)\cup(M,\infty))} \le \epsilon$. By the Sobolev embedding theorem, we have $||u_n||_{L^\infty((-\infty,M]\cup[M,\infty))} + ||u_n'||_{L^\infty((-\infty,M]\cup[M,\infty))} \le C_S\epsilon$. On the other hand $u_{n|[-M,M]} \rightharpoonup 0$ in $H^2(-M,M)$. Since the embedding $H^2(-M,M) \subset C^1([-M,M])$ is compact, it follows that $u_n \longrightarrow 0$ in $C^1([-M,M])$, so $||u_n||_{L^\infty([-M,M])} + ||u_n'||_{L^\infty([-M,M])} \le \epsilon$ if n is sufficiently big. Thus $||u_n||_{L^\infty(\mathbf{R})} + ||u_n'||_{L^\infty(\mathbf{R})} \le (C_S+1)\epsilon$ for n sufficiently big. As ϵ was arbitrary, we infer that $u_n \longrightarrow 0$ in $C^1_b(\mathbf{R})$ and the lemma is proved. □

Lemma 3.4 *Let W satisfy (W1)-(W3). For any $a > 0$, the operator $-\frac{d^2}{dx^2} + a : \mathbf{H}_W \longrightarrow \mathbf{L}_W$ is bounded and invertible. Moreover, the norm of $(-\frac{d^2}{dx^2} + a)^{-1}$ is uniformly bounded in $\mathcal{L}(\mathbf{L}_W, \mathbf{H}_W)$ when a remains in a compact subinterval of $(0,\infty)$.*

Proof. It is clear that

$$||(-\frac{d^2}{dx^2} + a)v||_{\mathbf{L}_W} = ||-v'' + av||_{\mathbf{L}_W} \le C||v||_{\mathbf{H}_W},$$

so the operator is bounded. Since $-\frac{d^2}{dx^2} + a : \mathbf{H} \longrightarrow \mathbf{L}$ is bounded and invertible, it is clear that the restriction of $-\frac{d^2}{dx^2} + a$ to $\mathbf{H}_W$ is one to one and for any $f \in \mathbf{L}_W \subset \mathbf{L}$ there exists an unique $v \in \mathbf{H}$ such that $(-\frac{d^2}{dx^2} + a)v = f$. It remains only to prove that $v \in \mathbf{H}_W$ and $||v||_{\mathbf{H}_W} \le ||f||_{\mathbf{L}_W}$. Using the Fourier transform we get $(\xi^2 + a)\widehat{v}(\xi) = \widehat{f}(\xi)$ or equivalently $\widehat{v}(\xi) = \frac{1}{\xi^2+a}\widehat{f}(\xi)$. Since $\mathcal{F}(e^{-\sqrt{a}|\cdot|})(\xi) = \frac{2\sqrt{a}}{\xi^2+a}$, we infer that

$$v = \frac{1}{2\sqrt{a}}(e^{-\sqrt{a}|\cdot|}) * f. \tag{3.11}$$

From (3.11) we get

$$\begin{aligned}
&|v(x)W(x)| = \tfrac{1}{2\sqrt{a}}W(x)\Big|\int_{\mathbf{R}} e^{-\sqrt{a}|x-y|}f(y)dy\Big| \\
&\le \tfrac{C_W}{2\sqrt{a}}\int_{\mathbf{R}} W(x-y)e^{-\sqrt{a}|x-y|}|f(y)| + e^{-\sqrt{a}|x-y|}W(y)|f(y)|dy \\
&\le C_1(a)[((We^{-\sqrt{a}|\cdot|}) * |f|)(x) + (e^{-\sqrt{a}|\cdot|}) * (|f|W)(x)],
\end{aligned}$$

that is $|vW| \le C_1(a)[(We^{-\sqrt{a}|\cdot|}) * |f| + e^{-\sqrt{a}|\cdot|} * (|f|W)]$. But

$$||(We^{-\sqrt{a}|\cdot|}) * |f|\ ||_{L^2} \le ||We^{-\sqrt{a}|\cdot|}||_{L^1}||f||_{L^2} \le ||We^{-\sqrt{a}|\cdot|}||_{L^1}||f||_{\mathbf{L}_W}$$

and

$$||e^{-\sqrt{a}|\cdot|} * (|f|W)||_{L^2} \le ||e^{-\sqrt{a}|\cdot|}||_{L^1}||Wf||_{L^2}$$

so we obtain from (3.11) that

$$||v||_{\mathbf{L}_W} \le C_2(a)||f||_{\mathbf{L}_W}, \tag{3.12}$$

where $C_2(a)$ remains bounded if $a \in [d, e]$, $0 < d < e < \infty$.

In the same way, we have $\widehat{v'}(\xi) = i\xi\widehat{v}(\xi) = \frac{i\xi}{\xi^2+a}\widehat{f}(\xi)$, hence $v'(x) = -\frac{1}{2}\zeta_a * f(x)$, where $\zeta_a(x) = \mathrm{sgn}(x)e^{-\sqrt{a}|x|}$. Repeating the above argument we find

$$||v'W||_{L^2} \leq C_3(a)||f||_{\mathbf{L}_W}, \tag{3.13}$$

where $C_3(a)$ remains bounded if a is in a compact interval of $(0, \infty)$.

Finally, using the equation satisfied by v we get $v'' = -f + av$, hence

$$||v''W||_{L^2} \leq ||f||_{\mathbf{L}_W} + a||v||_{\mathbf{L}_W} \leq (1 + aC_2(a))||f||_{\mathbf{L}_W}. \tag{3.14}$$

Lemma 3.4 follows from (3.12), (3.13) and (3.14). □

Note that the operator $-\frac{d^2}{dx^2} + a : \mathbf{H}_W \longrightarrow \mathbf{L}_W$ is not invertible if the weight W increases too fast at infinity. Indeed, if $f \in C_0^\infty(\mathbf{R})$ and $f \geq 0$, it is easily seen (e.g., from (3.11)) that the solution v of $-v'' + av = f$ behaves like $e^{-\sqrt{a}|\cdot|}$ at $\pm\infty$. If we take $W(x) = e^{b|x|}$ and $a < b^2$, then v does not belong to $\mathbf{H}_W$, so $-\frac{d^2}{dx^2} + a : \mathbf{H}_W \longrightarrow \mathbf{L}_W$ is not surjective.

The next lemma shows that we do not loose solutions if we work in $\mathbf{H}_W$ instead of $\mathbf{H}$.

Lemma 3.5 *Let (λ, r, u) be a solution of (1.9)-(1.10) with $r \in \mathbf{H}$, $u \in \mathbf{H}$ and $\lambda < q^2$. Then r and u belong to $\mathbf{H}_W$.*

Proof. We have already seen in Proposition 3.1 that $-1 + \sqrt{2}c\varepsilon < r \leq 0$. Applying Proposition 2.1 iv) (see also Corollary 2.2, iii)) for $V(x) = q^2(r^2(x) + 2r(x))$, we infer that for any $\epsilon > 0$, u, u' and u'' decay at $\pm\infty$ faster than $e^{-\sqrt{q^2-\lambda-\epsilon}|x|}$, hence $u \in \mathbf{H}_W$.

Since $g'_{2c\varepsilon}(0) > 0$ and $r(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$, there exists $M > 0$ such that $r(x)g_{2c\varepsilon}(r(x)) \geq \frac{1}{2}g'_{2c\varepsilon}(0)r^2(x)$ if $|x| > M$.

Consider a symmetric function $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi \equiv 1$ on $[-1, 1]$, χ is non-increasing on $[0, \infty)$ and $supp(\chi) \subset [-2, 2]$. We multiply (1.9) by $xr(x)\chi(\frac{x}{n})$ and integrate on $[0, \infty)$. Integrating by parts, we get :

$$\begin{aligned} &\int_0^\infty |r'|^2(x)x\chi(\tfrac{x}{n})dx - \tfrac{1}{2}r^2(0) - \tfrac{1}{2}\int_0^\infty r^2(x)\Big(\tfrac{2}{n}\chi'(\tfrac{x}{n}) + \tfrac{x}{n^2}\chi''(\tfrac{x}{n})\Big)dx \\ &+\int_0^M g_{2c\varepsilon}(r(x))r(x)x\chi(\tfrac{x}{n})dx + \int_M^\infty g_{2c\varepsilon}(r(x))r(x)x\chi(\tfrac{x}{n})dx \\ &+\int_0^\infty (1 + r(x))u^2(x)r(x)x\chi(\tfrac{x}{n})dx = 0. \end{aligned} \tag{3.15}$$

By the Monotone Convergence Theorem, the first integral in (3.15) tends to $\int_0^\infty |r'(x)|^2 x dx$ as $n \longrightarrow \infty$, while the fourth integral tends to $\int_M^\infty g_{2c\varepsilon}(r(x))r(x)xdx$. The other three integrals converge as $n \longrightarrow \infty$ by Lebesgue's theorem on dominated convergence. Letting $n \longrightarrow \infty$ in (3.15) we obtain :

$$\begin{aligned} &\int_0^\infty |r'|^2(x)xdx - \tfrac{1}{2}r^2(0) + \int_0^M g_{2c\varepsilon}(r(x))r(x)xdx \\ &+\int_M^\infty g_{2c\varepsilon}(r(x))r(x)xdx + \int_0^\infty r(x)(1 + r(x))xu^2(x)dx = 0. \end{aligned} \tag{3.16}$$

Since the second and the last integral in (3.16) are finite (because u decays exponentially at $\pm\infty$), we infer that $\int_0^\infty |r'|^2(x)xdx < \infty$ and $\int_M^\infty g_{2c\varepsilon}(r(x))r(x)xdx < \infty$. Consequently, $|x|^{\frac{1}{2}}r'(x)$ and $|x|^{\frac{1}{2}}r(x)$ belong to $L^2(\mathbf{R})$.

We have $g_{2c\varepsilon}(s) = g'_{2c\varepsilon}(0)s + h(s)s^2$, where h is continuous on $(-1,\infty)$, hence $h(r(x))$ is bounded. Equation (1.9) can be written as

$$-r'' + g'_{2c\varepsilon}(0)r = -(1+r)u^2 - h(r)r^2, \tag{3.17}$$

which gives, as in the proof of Lemma 3.4,

$$r = -\frac{1}{2\sqrt{g'_{2c\varepsilon}(0)}} e^{-\sqrt{g'_{2c\varepsilon}(0)}|\cdot|} * ((1+r)u^2 + h(r)r^2). \tag{3.18}$$

Suppose that $|x|^\alpha r(x) \in L^2(\mathbf{R})$ for some $\alpha > 0$. Since $|x|^\beta u(x) \in L^p(\mathbf{R})$ for any $\beta > 0$ and $1 \le p \le \infty$, we have :

$$\begin{aligned} |x|^{2\alpha}|r(x)| \ \le\ & C[(|\cdot|^{2\alpha} e^{-\sqrt{g'_{2c\varepsilon}(0)}|\cdot|}) * ((1+r)u^2 + h(r)r^2)(x) \\ & + e^{-\sqrt{g'_{2c\varepsilon}(0)}|\cdot|} * ((1+r)u^2|\cdot|^{2\alpha} + h(r)(|\cdot|^\alpha r)^2)](x) \end{aligned} \tag{3.19}$$

and we infer that $|\cdot|^{2\alpha} r \in L^p(\mathbf{R})$ for $1 \le p \le \infty$.

We have already proved that $|x|^{\frac{1}{2}} r(x) \in L^2(\mathbf{R})$, so it follows easily by induction that $|x|^\sigma r(x) \in L^p(\mathbf{R})$ for any $\sigma > 0$ and $1 \le p \le \infty$. Since $W(x) \le K|x|^s$ for some $K, s > 0$, we infer that $(1+r)u^2 + h(r)r^2 \in \mathbf{L}_W$. Now it follows form (3.17) and Lemma 3.4 that $r \in \mathbf{H}_W$ and Lemma 3.5 is proved. □

Now we turn our attention to the operators A, B, H_1 and H_2 appearing in (3.10).

Lemma 3.6 *We have :*

i) For any $\lambda \in (-\infty, q^2)$, $A_\lambda : \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is linear, compact and the mapping $(\lambda, u) \longmapsto A_\lambda(u)$ is continuous from $(-\infty, q^2) \times \mathbf{H}_W$ to $\mathbf{H}_W$ and compact on closed bounded subsets of $[d,e] \times \mathbf{H}_W$ for $-\infty < d < e < q^2$.

ii) The linear operator $B : \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is compact.

iii) $H_1 : ((V - r_{2c\varepsilon}) \cap \mathbf{H}_W) \times \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is continuous, compact on closed bounded subsets ω_1 of $((V - r_{2c\varepsilon}) \cap \mathbf{H}_W) \times \mathbf{H}_W$ such that $dist(\omega_1, (\mathbf{H}_W \setminus (V - r_{2c\varepsilon})) \times \mathbf{H}_W) > 0$ and

$$||H_1(w,u)||_{\mathbf{H}_W} \le C_{\omega_1}(||w||^2_{\mathbf{H}_W} + ||u||^2_{\mathbf{H}_W}). \tag{3.20}$$

iv) $H_2 : (-\infty, q^2) \times \mathbf{H}_W \times \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is continuous, compact on closed bounded subsets of $[d,e] \times \mathbf{H}_W \times \mathbf{H}_W$ for $-\infty < d < e < q^2$ and

$$||H_2(\lambda, w, u)||_{\mathbf{H}_W} \le C_{d,e}(||w||^2_{\mathbf{H}_W} + ||w||^4_{\mathbf{H}_W} + ||u||^2_{\mathbf{H}_W}) \qquad \textit{for any } \lambda \in [d,e]. \tag{3.21}$$

Proof. It is easy to see that $u_n \rightharpoonup u_*$ in $\mathbf{H}_W$ and $v_n \rightharpoonup v_*$ in $\mathbf{H}_W$ imply that $u_n v_n \longrightarrow u_* v_*$ in $\mathbf{L}_W$. Indeed, (u_n) and (v_n) are bounded in $\mathbf{H}_W$ and by Lemma 3.3 we have

$$||u_n v_n - u_* v_*||_{\mathbf{L}_W} \le ||v_n - v_*||_{L^\infty} ||u_n||_{\mathbf{L}_W} + ||u_n - u_*||_{L^\infty} ||v_*||_{\mathbf{L}_W} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.22}$$

i) It is now clear that $u \longmapsto (r^2_{2c\varepsilon} + 2r^2_{2c\varepsilon})u$ is a linear compact mapping from $\mathbf{H}_W$ to $\mathbf{L}_W$ and we get i) by using Lemma 3.4 and the resolvent formula

$$\Big(-\frac{d^2}{dx^2} + q^2 - \lambda_1\Big)^{-1} - \Big(-\frac{d^2}{dx^2} + q^2 - \lambda_2\Big)^{-1} = (\lambda_1 - \lambda_2)\Big(-\frac{d^2}{dx^2} + q^2 - \lambda_1\Big)^{-1}\Big(-\frac{d^2}{dx^2} + q^2 - \lambda_2\Big)^{-1}.$$

ii) is obvious.

iii) Let ω_1 be as in Lemma 3.6. We claim that there exists $\eta > 0$ such that for any $(w,u) \in \omega_1$ we have $\inf_{x\in\mathbf{R}}(w(x)+r_{2c\varepsilon}(x)) \geq -1+\eta$. We argue by contradiction and suppose that there exists a sequence $(w_n,u_n) \in \omega_1$ such that $a_n := \inf_{x\in\mathbf{R}}(w_n(x)+r_{2c\varepsilon}(x)) = (w_n + r_{2c\varepsilon})(x_n)$ tends to -1. The sequence (w_n) is bounded in $\mathbf{H}_W$, hence we may assume (passing to a subsequence if necessary) that $w_n \rightharpoonup w_*$ in $\mathbf{H}_W$. By Lemma 3.3, $w_n + r_{2c\varepsilon} \longrightarrow w_* + r_{2c\varepsilon}$ in $C_b^1(\mathbf{R})$. Since $w_*(x) + r_{2c\varepsilon}(x) \longrightarrow 0$ as $x \longrightarrow \infty$, the sequence (x_n) is bounded, say, $x_n \in [-M,M]$. Take $\chi \in C_0^\infty(\mathbf{R})$ such that $supp(\chi) \subset [-M-1, M+1]$ and $\chi \equiv 1$ on $[-M,M]$. Then $\inf_{x\in\mathbf{R}}(w_n(x)+r_{2c\varepsilon}(x)-(a_n+1)\chi(x)) = w_n(x_n)+r_{2c\varepsilon}(x_n)-(a_n+1)\chi(x_n) = -1$, so that $w_n + r_{2c\varepsilon} - (a_n+1)\chi \notin V$ and

$$dist(w_n, \mathbf{H}_W \setminus (V - r_{2c\varepsilon})) \leq dist(w_n, w_n - (a_n+1)\chi) = |1+a_n| \, ||\chi||_{\mathbf{H}_W} \longrightarrow 0$$

as $n \longrightarrow \infty$, contradicting the fact that $(w_n,u_n) \in \omega_1$. This proves the claim.

For a given $w \in V - r_{2c\varepsilon}$, we have

$$(g_{2c\varepsilon}(r_{2c\varepsilon}+w) - g_{2c\varepsilon}(r_{2c\varepsilon}) - g'_{2c\varepsilon}(r_{2c\varepsilon})w)(x) = \int_0^1 g'_{2c\varepsilon}(r_{2c\varepsilon}+tw)w(x)dt - g'_{2c\varepsilon}(r_{2c\varepsilon})w(x)$$
$$= w^2(x)\int_0^1\int_0^1 g''_{2c\varepsilon}(r_{2c\varepsilon}+tsw)(x)ds\, t\, dt = w^2(x)h_1(w)(x),$$

where $h_1(w)(x) = \int_0^1\int_0^1 g''_{2c\varepsilon}(r_{2c\varepsilon}+tsw)(x)ds\, t\, dt$.

To prove iii) it suffices to show that for any sequence $(w_n,u_n) \in \omega_1$ such that $w_n \rightharpoonup w_*$ and $u_n \rightharpoonup u_*$ in $\mathbf{H}_W$, we have $H_1(w_n,u_n) \longrightarrow H_1(w_*,u_*)$ in $\mathbf{H}_W$. In view of Lemma 3.4, it suffices to show that

$$h_1(w_n)w_n^2 + (1+r_{2c\varepsilon}+w_n)u_n^2 \longrightarrow h_1(w_*)w_*^2 + (1+r_{2c\varepsilon}+w_*)u_*^2 \qquad \text{in } \mathbf{L}_W. \tag{3.23}$$

The sequence (w_n) being bounded in $\mathbf{H}_W$, there exists $K > 0$ such that $-1+\min(\eta,\sqrt{2}c\varepsilon) \leq r_{2c\varepsilon}(x) + stw_n(x) \leq K$ for any $x \in \mathbf{R}$, $n \in \mathbf{N}$ and $s, t \in [0,1]$. Since $g''_{2c\varepsilon}$ is uniformly continuous on $[-1+\min(\eta,\sqrt{2}c\varepsilon), K]$, it is standard to prove that $h_1(w_n) \longrightarrow h_1(w_*)$ in $L^\infty(\mathbf{R})$ and then (3.23) follows from (3.22). Finally, using Lemma 3.4 we have for any $(w,u) \in \omega_1$

$$||H_1(w,u)||_{\mathbf{H}_W} \leq C||h_1(w)w^2 + (1+r_{2c\varepsilon}+w)u^2||_{\mathbf{L}_W} \leq C_{\omega_1}(||w||^2_{\mathbf{H}_W} + ||u||^2_{\mathbf{H}_W}).$$

iv) From the preceeding arguments it is easy to see that the mapping $(w,u) \longmapsto (w^2 + 2wr_{2c\varepsilon} + 2w)u$ is continuous from $\mathbf{H}_W \times \mathbf{H}_W$ to $\mathbf{L}_W$ and the image of any bounded set in $\mathbf{H}_W \times \mathbf{H}_W$ is precompact in $\mathbf{L}_W$, so iv) follows from Lemma 3.4 and the resolvent formula above. The estimate (3.21) is straightforward. □

Lemma 3.7 *For any $\lambda < q^2$ we have :*

i) $Ker(Id_{\mathbf{H}_W} + A_\lambda) \neq \{0\}$ if and only if λ is an eigenvalue of the operator $A = -\frac{d^2}{dx^2} + q^2(1+r_{2c\varepsilon})^2$. In this case we have $Ker(Id_{\mathbf{H}_W} + A_\lambda)^n = Span\{u_\lambda\}$ for any $n \in \mathbf{N}^*$.

ii) If λ is not an eigenvalue of A, then $ind(Id_{\mathbf{H}_W} + A_\lambda, 0) = (-1)^{n(\lambda)}$ (where $n(\lambda)$ is the number of eigenvalues of A less than λ).

Proof. i) It is easy to see that $u \in \mathbf{L}$ and $u + A_\lambda u = 0$ is equivalent to $u \in \mathbf{H}$ and $Au = \lambda u$. Recall that if $\lambda < q^2$ is an eigenvalue of A in $\mathbf{L}$, then the corresponding eigenvector u_λ is in $\mathbf{H}_W$ by Corollary 2.2 iii). Consequently, we have $Ker(Id_{\mathbf{H}_W} + A_\lambda) = Ker(Id_{\mathbf{L}} + A_\lambda) = Ker(\lambda Id_{\mathbf{H}} - A) = Span\{u_\lambda\}$.

To prove i), it suffices to show that $u_\lambda \notin Im(Id_{\mathbf{L}} + A_\lambda)$. Suppose by contradiction that there exists $v \in \mathbf{L}$ such that $v + A_\lambda v = u_\lambda$. This is equivalent to $v \in \mathbf{H}$ and $Av - \lambda v = -u_\lambda'' + (q^2 - \lambda)u_\lambda$, that is $-u_\lambda'' + (q^2 - \lambda)u_\lambda \in Im(A - \lambda)$. Since $A - \lambda$ is self-adjoint on $\mathbf{L}$, $-u_\lambda'' + (q^2 - \lambda)u_\lambda$ must be orthogonal (in $\mathbf{L}$) to $Ker(A - \lambda) = Span\{u_\lambda\}$, which gives $\int_{\mathbf{R}} |u_\lambda'|^2 dx + (q^2 - \lambda) \int_{\mathbf{R}} |u_\lambda|^2 dx = 0$, a contradiction.

ii) A well-known result of Leray and Schauder asserts that if K is a compact operator on a real Banach space X and 1 is not an eigenvalue of K, then

$$ind(Id - K, 0) = (-1)^\beta,$$

where β is the sum of all the (algebraic) multiplicities of eigenvalues of K greater than 1. (see, e.g., [6], Theorem 4.6 p. 133).

Thus, for a given λ which is not an eigenvalue of A, we are interested by the eigenvalues $\mu > 1$ of $-A_\lambda$. Clearly, $-A_\lambda u = \mu u$ is equivalent to

$$q^2 \Big(-\frac{d^2}{dx^2} + q^2 - \lambda \Big)^{-1} ((r_{2c\varepsilon}^2 + 2r_{2c\varepsilon})u) + \mu u = 0,$$

that is

$$-u'' + q^2(1 + r_{2c\varepsilon})^2 u + q^2\Big(1 - \frac{1}{\mu}\Big)[1 - (1 + r_{2c\varepsilon})^2]u = \lambda u.$$

In other words, $\mu > 1$ is an eigenvalue of $-A_\lambda$ if and only if λ is an eigenvalue of the operator

$$M_\mu = -\frac{d^2}{dx^2} + q^2(1 + r_{2c\varepsilon})^2 + q^2\Big(1 - \frac{1}{\mu}\Big)[1 - (1 + r_{2c\varepsilon})^2] = A + q^2\Big(1 - \frac{1}{\mu}\Big)[1 - (1 + r_{2c\varepsilon})^2].$$

Remark that $M_\mu \geq A$ for any $\mu \geq 1$ and $\sigma_{ess}(M_\mu) = [q^2, \infty)$ by Weyl's theorem. By Proposition 2.1 iv), $\lambda \in (-\infty, q^2)$ is an eigenvalue of M_μ considered as an operator on $\mathbf{L}_W$ if and only if λ is an eigenvalue of M_μ considered as an operator on $\mathbf{L}$. We will work on $\mathbf{L}$ because on this space M_μ is self-adjoint.

Given $\lambda < q^2$ not an eigenvalue of A, we will prove that there are exactly $n(\lambda)$ values $\mu \in (1, \infty)$ such that λ is an eigenvalue of M_μ.

For $\mu \in [1, \infty)$, we define

$$\alpha_n(\mu) = \sup_{\varphi_1, \dots, \varphi_{n-1} \in \mathbf{H}} \ \inf_{\psi \in \{\varphi_1, \dots, \varphi_{n-1}\}^\perp} \frac{\langle M_\mu \psi, \psi \rangle_{\mathbf{L}}}{||\psi||_{\mathbf{L}}^2}. \tag{3.24}$$

By the Min-Max Principle ([13], Theorem XIII.1 p. 76), either $\alpha_n(\mu)$ is the n^{th} eigenvalue of M_μ (counted with multiplicity) or $\alpha_n(\mu) = q^2$. By Proposition 2.1 iii), the eigenvalues of M_μ are simple, thus we have $\alpha_p(\mu) < \alpha_n(\mu)$ if $p < n$ and $\alpha_p(\mu) < q^2$.

It is obvious that the functions $\mu \longmapsto \alpha_n(\mu)$ are increasing on $[1, \infty)$ because $M_{\mu_1} \leq M_{\mu_2}$ if $1 \leq \mu_1 < \mu_2$. In fact, α_n is strictly increasing on $\{\mu \in [1, \infty) \mid \alpha_n(\mu) < q^2\}$. To see this, consider $\mu_1 < \mu_2$ such that $\alpha_n(\mu_2) < q^2$. Then $\alpha_1(\mu_2), \dots, \alpha_n(\mu_2)$ are eigenvalues of M_{μ_2}. Let $u_1, \dots, u_n \in \mathbf{H}$ be corresponding eigenvectors with $||u_i||_{\mathbf{L}} = 1$. Clearly, $u_1, \dots, u_n$ are mutually orthogonal in $\mathbf{L}$ and it is easily seen from the definition of M_μ that $\langle M_{\mu_1} u_i, u_i \rangle_{\mathbf{L}} < \langle M_{\mu_2} u_i, u_i \rangle_{\mathbf{L}} = \alpha_i(\mu_2)$, $i = 1, \dots, n$. Remark that the quantity $N(u) =$

$\Big(\int_{\mathbf{R}}[1-(1+r_{2c\varepsilon})^2]|u|^2dx\Big)^{\frac{1}{2}}$ is a norm on $\mathbf{L}$. Since $Span\{u_1,\dots,u_n\}$ is finite-dimensional, there exists $N_1>0$ such that $N(u)\geq N_1||u||_{\mathbf{L}}$ for any $u\in Span\{u_1,\dots,u_n\}$. Therefore

$$
\begin{aligned}
&\langle M_{\mu_1}\Big(\sum_{i=1}^n a_iu_i\Big),\Big(\sum_{i=1}^n a_iu_i\Big)\rangle_{\mathbf{L}}\\
&=\langle M_{\mu_2}\Big(\sum_{i=1}^n a_iu_i\Big),\Big(\sum_{i=1}^n a_iu_i\Big)\rangle_{\mathbf{L}}-\langle (M_{\mu_2}-M_{\mu_1})\Big(\sum_{i=1}^n a_iu_i\Big),\Big(\sum_{i=1}^n a_iu_i\Big)\rangle_{\mathbf{L}}\\
&=\sum_{i=1}^n\alpha_i(\mu_2)|a_i|^2-q^2(\frac{1}{\mu_1}-\frac{1}{\mu_2})\int_{\mathbf{R}}[1-(1+r_{2c\varepsilon})^2]\Big|\sum_{i=1}^n a_iu_i\Big|^2dx\\
&\leq\alpha_n(\mu_2)||\sum_{i=1}^n a_iu_i||^2_{\mathbf{L}}-q^2(\frac{1}{\mu_1}-\frac{1}{\mu_2})N_1^2||\sum_{i=1}^n a_iu_i||^2_{\mathbf{L}}.
\end{aligned}
\tag{3.25}
$$

Thus for any u in the n-dimensional subspace $Span\{u_1,\dots,u_n\}$ we have

$$\langle M_{\mu_1}u,u\rangle_{\mathbf{L}}\leq\Big(\alpha_n(\mu_2)-q^2\Big(\frac{1}{\mu_1}-\frac{1}{\mu_2}\Big)N_1^2\Big)||u||^2_{\mathbf{L}}.$$

By the Min-Max Principle it follows that $\alpha_n(\mu_1)\leq\alpha_n(\mu_2)-q^2\Big(\frac{1}{\mu_1}-\frac{1}{\mu_2}\Big)N_1^2$.

A standard argument shows that each α_n is continuous. Indeed, suppose by contradiction that $\mu_*\in(1,\infty)$ is a discontinuity point. Then necessarily $l_1:=\sup\limits_{\mu<\mu_*}\alpha_n(\mu)<\inf\limits_{\mu>\mu_*}\alpha_n(\mu):=l_2$. Take $0<\epsilon<\frac{l_2-l_1}{4}$ and $\mu_1<\mu_*$, $\mu_2>\mu_*$ such that $q^2\Big(\frac{1}{\mu_1}-\frac{1}{\mu_2}\Big)<\epsilon$. Since $\alpha_n(\mu_2)>l_2-\epsilon$, there exist $\varphi_1,\dots,\varphi_{n-1}\in\mathbf{H}$ such that $\langle M_{\mu_2}\psi,\psi\rangle_{\mathbf{L}}>l_2-\epsilon$ for any $\psi\in\{\varphi_1,\dots,\varphi_{n-1}\}^{\perp}$ with $||\psi||_{\mathbf{L}}=1$. We have

$$
\begin{aligned}
&\langle M_{\mu_2}\psi,\psi\rangle_{\mathbf{L}}-\langle M_{\mu_1}\psi,\psi\rangle_{\mathbf{L}}\\
&=q^2(\tfrac{1}{\mu_1}-\tfrac{1}{\mu_2})\int_{\mathbf{R}}[1-(1+r_{2c\varepsilon})^2]|\psi|^2dx\leq q^2(\frac{1}{\mu_1}-\frac{1}{\mu_2})||\psi||^2_{\mathbf{L}}<\epsilon,
\end{aligned}
$$

thus $\langle M_{\mu_1}\psi,\psi\rangle_{\mathbf{L}}>l_2-2\epsilon$ for any $\psi\in\{\varphi_1,\dots,\varphi_{n-1}\}^{\perp}$ with $||\psi||_{\mathbf{L}}=1$. Therefore $\alpha_n(\mu_1)>l_2-2\epsilon$, which is a contradiction.

We have also for any $u\in\mathbf{H}$,

$$\langle M_\mu u,u\rangle_{\mathbf{L}}=||u'||^2_{L^2}+q^2||u||^2_{\mathbf{L}}-\frac{q^2}{\mu}\int_{\mathbf{R}}[1-(1+r_{2c\varepsilon})^2]|u|^2dx\geq q^2||u||^2_{\mathbf{L}}-\frac{C}{\mu}||u||^2_{\mathbf{L}},$$

hence $\alpha_1(\mu)\geq q^2-\frac{C}{\mu}\longrightarrow q^2$ as $\mu\longrightarrow\infty$. Consequently, $\alpha_n(\mu)\longrightarrow q^2$ as $\mu\longrightarrow\infty$ for any $n\geq 1$.

Note that $\lambda<q^2$ is an eigenvalue of M_μ if and only if $\lambda=\alpha_n(\mu)$ for some $n\in\mathbf{N}^*$. We know that there are exactly $n(\lambda)$ eigenvalues of A less than λ, say, $\lambda_1<\lambda_2<\ldots<\lambda_{n(\lambda)}<\lambda$. We have $\alpha_i(1)=\lambda_i$ because $M_1=A$, the functions α_i are strictly increasing (until they reach the value q^2, if this happens), continuous and tend to q^2 at infinity. We infer that for each $i\in\{1,\dots,n(\lambda)\}$, there exists exactly one value μ_i such that $\alpha_i(\mu_i)=\lambda$. Moreover, $\mu_1>\mu_2>\ldots>\mu_{n(\lambda)}>1$. For any $n>n(\lambda)$, we have $\alpha_n(1)>\lambda$, hence $\alpha_n(\mu)>\lambda$ for $\mu\in(0,\infty)$ because α_n is increasing.

Thus we have shown that the operator $-A_\lambda$ has exactly $n(\lambda)$ eigenvalues greater than 1, $\mu_1>\mu_2>\ldots>\mu_{n(\lambda)}$. Moreover, $Ker(\mu_i+A_\lambda)=Ker(M_{\mu_i}-\lambda)$. We know by Proposition 2.1 iii) that $Ker(M_{\mu_i}-\lambda)$ is one dimensional. If this kernel is spanned by a function v_i, then $v_i\notin Im(\mu_i+A_\lambda)$. Indeed, $\mu_iu+A_\lambda u=v_i$ would imply $(M_{\mu_i}-\lambda)u=\frac{1}{\mu_i}(-v_i''+(q^2-\lambda)v_i)$.

Since M is self-adjoint, $-v_i'' + (q^2 - \lambda)v_i$ would be orthogonal to $Ker(M_{\mu_i} - \lambda) = Span\{v_i\}$, which gives a contradiction. Consequently, we have $Ker(\mu_i + A_\lambda)^n = Span\{v_i\}$ for any $n \in \mathbf{N}^*$, that is μ_i is a simple eigenvalue of $-A_\lambda$.

As a consequence, we have $ind(Id_{\mathbf{H}_W} + A_\lambda, 0) = (-1)^{n(\lambda)}$ and Lemma 3.7 is proved. □

We are now in position to state the main result of this paper.

Theorem 3.8 *Let $\mathcal{S}$ be the set of nontrivial solutions of the system (1.9)-(1.10) in $\mathbf{R} \times (\mathbf{H} \cap V) \times \mathbf{H}$. For any eigenvalue $\lambda_m < q^2$ of $A = -\frac{d^2}{dx^2} + (1 + r_{2c\varepsilon})^2$, the set $\mathcal{S} \cup \{(\lambda_m, r_{2c\varepsilon}, 0)\}$ contains a maximal closed connected subset $\mathcal{C}_m$ in $(-\infty, q^2) \times \mathbf{H}_W \times \mathbf{H}_W$ such that $\mathcal{C}_m \cap \mathcal{C}_p = \emptyset$ if $m \neq p$ and $\mathcal{C}_m$ satisfies at least one of the two properties :*

i) $\mathcal{C}_m$ is unbounded in $\mathbf{R} \times \mathbf{H}_W \times \mathbf{H}_W$ or

ii) there exists a sequence $(\lambda_n, r_n, u_n) \in \mathcal{C}_m$ such that $\lambda_n \longrightarrow q^2$ as $n \longrightarrow \infty$.

Proof.

We have already seen that $(\lambda, r, u) \in (-\infty, q^2) \times (\mathbf{H} \cap V) \times \mathbf{H}$ is a nontrivial solution of (1.9)-(1.10) if and only if $(\lambda, r - r_{2c\varepsilon}, u)$ belongs to $(-\infty, q^2) \times (\mathbf{H}_W \cap (V - r_{2c\varepsilon})) \times \mathbf{H}_W$ and satisfies the system (3.8)-(3.9) (or, equivalently, (3.10)).

Let $E = \mathbf{H}_W \times \mathbf{H}_W$, $\Omega = (-\infty, q^2) \times (\mathbf{H}_W \cap (V - r_{2c\varepsilon})) \times \mathbf{H}_W$, $L_\lambda = \begin{pmatrix} -B & 0 \\ 0 & -A_\lambda \end{pmatrix}$ and $H(\lambda, w, u) = \begin{pmatrix} -H_1(w, u) \\ -H_2(\lambda, w, u) \end{pmatrix}$. Let $G(\lambda, w, u) = L_\lambda(w, u) + H(\lambda, w, u)$. It is obvious that on Ω, (3.10) is equivalent to the equation $(w, u) = G(\lambda, w, u)$. It follows easily from Lemma 3.6 that L and H satisfy the assumptions a) and b) in Proposition 3.2.

We claim that $Id_{\mathbf{H}_W} + B : \mathbf{H}_W \longrightarrow \mathbf{H}_W$ is invertible. Indeed, $(Id_{\mathbf{H}_W} + B)u = v$ is equivalent to $-u'' + g_{2c\varepsilon}'(r_{2c\varepsilon})u = \left(-\frac{d^2}{dx^2} + g_{2c\varepsilon}'(0)\right) v$. By Lemma 2.4, there exists an unique $u \in \mathbf{H}$ satisfying this equation. We have

$$-u'' + g_{2c\varepsilon}'(0)u = \left(-\frac{d^2}{dx^2} + g_{2c\varepsilon}'(0)\right) v + (g_{2c\varepsilon}'(0) - g_{2c\varepsilon}'(r_{2c\varepsilon}))u \in \mathbf{L}_W$$

(recall that $v \in \mathbf{H}_W$ and $g_{2c\varepsilon}'(0) - g_{2c\varepsilon}'(r_{2c\varepsilon})$ decays exponentially at infinity). Using Lemma 3.4, we infer that $u \in \mathbf{H}_W$.

For $\lambda < q^2$, is is clear that $Id_{\mathbf{H}_W \times \mathbf{H}_W} - L_\lambda$ is not invertible if and only if $Id_{\mathbf{H}_W} + A_\lambda$ is not invertible, i.e. if and only if λ is an eigenvalue of A. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_{N_q} < q^2$ be the eigenvalues of A below q^2. If λ is not an eigenvalue of A, we infer using Lemma 3.7 that $i(\lambda) := ind(Id_{\mathbf{H}_W \times \mathbf{H}_W} - L_\lambda, 0) = ind(Id_{\mathbf{H}_W} + A_\lambda, 0) \cdot ind(Id_{\mathbf{H}_W} + B, 0) = (-1)^{n(\lambda)} ind(Id_{\mathbf{H}_W} + B, 0)$ is constant on each of the intervals $(-\infty, \lambda_1), (\lambda_i, \lambda_{i+1}), (\lambda_{N_q}, q^2)$ and changes sign at each λ_i. Consequently, L_λ also satisfies assumption c) in Proposition 3.2 at any point $(\lambda_i, 0, 0)$. Let $\tilde{\mathcal{S}}_0 = \{(\lambda, w, u) \in \Omega \mid (w, u) \neq (0, 0)$ and (λ, w, u) satisfies (3.10)$\}$ and let $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_0 \setminus \{(\lambda, -r_{2c\varepsilon}, 0) \mid \lambda \in (-\infty, q^2)\}$. Note that the solutions $(\lambda, -r_{2c\varepsilon}, 0)$ of (3.10) correspond to the solutions $(\lambda, 0, 0)$ of (1.9)-(1.10) and $\mathcal{S} \cap ((-\infty, q^2) \times (V \cap \mathbf{H}_W) \times \mathbf{H}_W) = \tilde{\mathcal{S}} + (0, r_{2c\varepsilon}, 0)$. We may apply Proposition 3.2 to infer that for any $1 \leq m \leq N_q$, there exists a maximal closed connected subset $\mathcal{D}_m$ (in Ω) of $\tilde{\mathcal{S}}_0 \cup \{(\lambda_m, 0, 0)\}$ which contains $(\lambda_m, 0, 0)$ and satisfies at least one of the following properties :

1°. $\mathcal{D}_m$ is unbounded.

2°. There exists a sequence $(\lambda_n, w_n, u_n) \in \mathcal{D}_m$ such that $\lambda_n \longrightarrow q^2$ as $n \longrightarrow \infty$.

3°. There exists a sequence $(\lambda_n, w_n, u_n) \in \mathcal{D}_m$ such that $dist(w_n, \partial((V - r_{2c\varepsilon}) \cap \mathbf{H}_W)) \longrightarrow 0$, that is $\inf_{x \in \mathbf{R}} (w_n(x) + r_{2c\varepsilon}(x)) \longrightarrow -1$ as $n \longrightarrow \infty$.

4°. The closure in Ω of $\mathcal{D}_m$ contains a point $(\lambda_i, 0, 0)$ with $i \neq m$.

Let us show first that $\mathcal{D}_m$ cannot meet $\{(\lambda, -r_{2c\varepsilon}, 0) \mid \lambda \in (-\infty, q^2)\}$. A straightforward computation gives $d_{(w,u)}(Id_E - G)(\lambda, -r_{2c\varepsilon}, 0) = Id_E$ for any $\lambda < q^2$. By the Implicit Functions Theorem, there exists a neighbourhood N_λ of $(\lambda, -r_{2c\varepsilon}, 0)$ in $\mathbf{R} \times E$ such that the only solutions of the equation $(w, u) = G(\lambda, w, u)$ in N_λ are $(\mu, -r_{2c\varepsilon}, 0)$. Hence $\cup_\lambda N_\lambda$ is a neighbourhood of $\{(\lambda, -r_{2c\varepsilon}, 0) \mid \lambda < q^2\}$ in Ω which contains no other solutions of (3.10). Consequently, we have $\mathcal{D}_m \subset \tilde{\mathcal{S}}$.

By Proposition 3.1, for any $(\lambda, w, u) \in \tilde{\mathcal{S}}_0$ we have $\inf_{x \in \mathbf{R}}(w(x) + r_{2c\varepsilon}(x)) > -1 + \sqrt{2}c\varepsilon$, hence $\mathcal{D}_m$ cannot satisfy property 3° above.

We will also eliminate the alternative 4°. Observe that if $(\lambda, r, u) \in (-\infty, q^2) \times \mathbf{H} \times \mathbf{H}$ is a nontrivial solution of (1.9)-(1.10), then, in particular, u is an eigenvector of the linear operator $-\frac{d^2}{dx^2} + q^2(1+r)^2$ corresponding to the eigenvalue λ. It is easily checked that this operator is a compact perturbation of $-\frac{d^2}{dx^2} + q^2$, so it has the essential spectrum $[q^2, \infty)$. Since $\lambda < q^2$, the operator $-\frac{d^2}{dx^2} + q^2(1+r)^2$ has only a finite number of eigenvalues less than λ, say, p. We define $z(\lambda, r, u) = p$. By Proposition 2.1 v), we know that u has exactly p zeroes in $(0, \infty)$. We also define $z(\lambda_i, r_{2c\varepsilon}, 0) = i - 1$. We have :

Lemma 3.9 *The function z is continuous on* $(\mathcal{S} \cup \{(\lambda_i, r_{2c\varepsilon}, 0) \mid i = 1, \ldots, N_q\}) \cap ((-\infty, q^2) \times \mathbf{H} \times \mathbf{H})$.

Assume for the moment that Lemma 3.9 holds. Obviously, the function z is also continuous for the $\mathbf{R} \times E$ topology. Since z takes values in $\mathbf{N}$, it must be constant on each connected component of $(\mathcal{S} \cup \{(\lambda_i, r_{2c\varepsilon}, 0) \mid i = 1, \ldots, N_q\}) \cap ((-\infty, q^2) \times \mathbf{H} \times \mathbf{H}) = (\tilde{\mathcal{S}} + (0, r_{2c\varepsilon}, 0)) \cup \{(\lambda_i, r_{2c\varepsilon}, 0) \mid i = 1, \ldots, N_q\}$. In particular, it is constant on $\mathcal{D}_m + (0, r_{2c\varepsilon}, 0)$ and we find $z(\mathcal{D}_m + (0, r_{2c\varepsilon}, 0)) = z(\lambda_m, r_{2c\varepsilon}, 0) = m - 1$. We have also $z(\mathcal{D}_i + (0, r_{2c\varepsilon}, 0)) = i - 1$, hence $\mathcal{D}_m$ and $\mathcal{D}_i$ are disjoint if $i \neq m$ (in fact, we see that the closures of $\mathcal{D}_m$ and $\mathcal{D}_i$ in $(-\infty, q^2) \times \mathbf{H} \times \mathbf{H}$ are disjoint if $i \neq m$). Thus $\mathcal{D}_m$ cannot satisfy the alternative 4° above, hence it necessarily satisfies one of the alternatives 1° or 2°. Let $\mathcal{C}_m = \mathcal{D}_m + (0, r_{2c\varepsilon}, 0)$. It is now clear that $\mathcal{C}_m$ satisfies i) or ii) in Theorem 3.8. $\square$

Proof of Lemma 3.9. Let $(\lambda, r, u), (\nu_n, r_n, u_n) \in (\mathcal{S} \cup \{(\lambda_i, r_{2c\varepsilon}, 0) \mid i = 1, \ldots, N_q\}) \cap ((-\infty, q^2) \times \mathbf{H} \times \mathbf{H})$ be such that $z(\lambda, r, u) = p$ and $(\nu_n, r_n, u_n) \longrightarrow (\lambda, r, u)$ as $n \longrightarrow \infty$. Let $\mu_1 < \mu_2 < \ldots < \mu_{p+1} = \lambda$ be the eigenvalues of the operator $B = -\frac{d^2}{dx^2} + q^2(1+r)^2$ in $\mathbf{L}$ and let $u_1^*, \ldots, u_{p+1}^* = u$ be corresponding eigenvectors. Denote $B_n = -\frac{d^2}{dx^2} + q^2(1+r_n)^2$.

We prove that $z(\nu_n, r_n, u_n) \geq p$ if n is sufficiently big. There is nothing to do if $p = 0$. Suppose that $p \geq 1$. Take $0 < \epsilon < \frac{\mu_{p+1} - \mu_p}{4}$ and let n_0 be suficiently big, so that $||(r_n - r)(2 + r_n + r)||_{L^\infty} < \frac{\epsilon}{q^2}$ and $\lambda - \epsilon < \nu_n < \lambda + \epsilon$ for any $n \geq n_0$. For $n \geq n_0$ and $v \in Span\{u_1^*, \ldots u_p^*\}$ we have

$$\begin{aligned}
&\langle B_n v, v\rangle_{\mathbf{L}} = \langle Bv, v\rangle_{\mathbf{L}} + \langle (B_n - B)v, v\rangle_{\mathbf{L}} \\
&\leq \mu_p ||v||_{\mathbf{L}}^2 + q^2 \int_{\mathbf{R}} (r_n - r)(2 + r_n + r)|v|^2 dx \leq (\mu_p + \epsilon)||v||_{\mathbf{L}}^2 < (\nu_n - \epsilon)||v||_{\mathbf{L}}^2.
\end{aligned}$$

By the Min-Max Principle, B_n has at least p eigenvalues less than or equal to $\nu_n - \epsilon$, so $z(\nu_n, r_n, u_n) \geq p$.

Let $\mu_{p+2} = \sup_{\varphi_1, \ldots, \varphi_{p+1} \in \mathbf{H}} \inf_{\psi \in \{\varphi_1, \ldots, \varphi_{p+1}\}^\perp} \frac{\langle B\psi, \psi\rangle_{\mathbf{L}}}{||\psi||_{\mathbf{L}}^2}$. Since $\lambda = \mu_{p+1} < q^2$ and λ is a simple eigenvalue of B by Proposition 2.1 iii), we know by the Min-Max Principle that either $\mu_{p+2} = q^2$ or μ_{p+2} is an eigenvalue of B and $\mu_{p+2} > \mu_{p+1}$. Let $\epsilon \in (0, \frac{\mu_{p+2} - \mu_{p+1}}{4})$. Take n_0 as above and $\varphi_1, \ldots, \varphi_{p+1} \in \mathbf{H}$ such that $\inf_{\psi \in \{\varphi_1, \ldots, \varphi_{p+1}\}^\perp} \frac{\langle B\psi, \psi\rangle_{\mathbf{L}}}{||\psi||_{\mathbf{L}}^2} \geq \mu_{p+2} - \epsilon$. For any

$\psi \in \{\varphi_1, \ldots, \varphi_{p+1}\}^{\perp}$, $\psi \neq 0$ we have :

$$\langle B_n\psi, \psi\rangle_{\mathbf{L}} = \langle B\psi, \psi\rangle_{\mathbf{L}} + \langle (B_n - B)\psi, \psi\rangle_{\mathbf{L}} \geq (\mu_{p+2} - \epsilon)||\psi||^2_{\mathbf{L}} - \epsilon||\psi||^2_{\mathbf{L}} \geq (\nu_n + \epsilon)||\psi||^2_{\mathbf{L}}.$$

It follows from the Min-Max Principle that for $n \geq n_0$, either B_n has at most $p+1$ eigenvalues, or the $(p+2)^{th}$ eigenvalue is greater than $\nu_n + \epsilon$. Since ν_n is an eigenvalue of B_n, there are at most p eigenvalues of B_n less than ν_n, hence $z(\nu_n, r_n, u_n) \leq p$ for any $n \geq n_0$. This finishes the proof of Lemma 3.9 and that of Theorem 3.8. □

We were not able to eliminate one or another of the alternatives in Theorem 3.8.

Up to now, we have proved the existence of branches of nontrivial *symmetric* solutions (λ, r, u) to the system (1.9)-(1.10). For any such solution, $(\tilde{\psi}, \tilde{\varphi})$ is a travelling wave of (1.1) for $\varepsilon^2(c^2\delta^2 + k^2) = \lambda$ and satisfies the boundary condition (1.2), where $\tilde{\varphi}(x) = \frac{1}{\varepsilon}u(\frac{x}{\varepsilon})e^{ic\delta x}$ and $\tilde{\psi}(x) = (1 + r(\frac{x}{\varepsilon}))e^{i\psi_0(x)}$ (with $\psi_0(x) = c\int_0^x \left[1 - \frac{1}{(1+r(\frac{s}{\varepsilon}))^2}\right]ds = c\varepsilon\int_0^{\frac{x}{\varepsilon}} \frac{2r(\tau)+r^2(\tau)}{(1+r(\tau))^2}d\tau$). Note also that $\tilde{\psi}(-x) = \overline{\tilde{\psi}(x)}, \tilde{\varphi}(-x) = \overline{\tilde{\varphi}(x)}$, $|\tilde{\psi}| > \sqrt{2}c\varepsilon$ by Proposition 2.1 and the phase ψ_0 of $\tilde{\psi}$ remains bounded because r decays at infinity faster than $|x|^{\beta}$ for any $\beta > 0$ (see the end of the proof of Lemma 3.5). Since $2c^2\varepsilon^2q^2 < \lambda \leq q^2$, we have bounds on the single-particle impurity energy : $c^2(2q^2 - \delta^2) < k^2 \leq \frac{q^2}{\varepsilon^2} - c^2\delta^2$.

Remark 3.10 It follows from Corollary 2.2 iv)-v) that there is exactly one branch of travelling-waves bifurcating from the trivial solutions if $q \leq \frac{1}{\sqrt{2\ln 2}}$. The number of these branches is the same as the number of eigenvalues of A, so it tends to infinity as $q \longrightarrow \infty$.

It is natural to ask how the branches $\mathcal{C}_m$ given by Theorem 3.8 behave in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$. The topology of $\mathbf{H}_W$ being stronger than that of $\mathbf{H}$, any of the sets $\mathcal{C}_m$ is also connected in $\mathbf{R}\times\mathbf{H}\times\mathbf{H}$. Roughly speaking, either $\mathcal{C}_m$ approaches $\{q^2\}\times(\mathbf{H}\cap V)\times\mathbf{H}$, or $\mathcal{C}_m$ is unbounded in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ or it remains bounded in $\mathbf{R} \times \mathbf{H} \times \mathbf{H}$ but the norm in $\mathbf{R} \times \mathbf{H}_W \times \mathbf{H}_W$ tends to infinity along $\mathcal{C}_m$, i.e. "there is some mass moving to infinity".

Remark 3.11 The importance of Theorem 2.3 is that it gives a precise description of $\mathcal{C}_m$ in a neighbourhood of $(\lambda_m, r_{2c\varepsilon}, 0)$ in $\mathbf{R}\times\mathbf{H}\times\mathbf{H}$. Let $\mathcal{C}_m^+$ (respectively $\mathcal{C}_m^-$) be the maximal subcontinuum in $\mathbf{R} \times \mathbf{H}_W \times \mathbf{H}_W$ of $\mathcal{C}_m \setminus \{(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_m + u(s))) \mid s \in (-\eta, 0)\}$, (respectively of $\mathcal{C}_m \setminus \{(\lambda(s), r_{2c\varepsilon} + sr(s), s(u_m + u(s))) \mid s \in (0, \eta)\}$), where the curve $s \longmapsto (\lambda(s), r(s), u(s))$ is given by Theorem 2.3. It can be proved by using a variant of a classical result of Rabinowitz (Theorem 1.40 p. 500 in [12]) that each of $\mathcal{C}_m^+$ and $\mathcal{C}_m^-$ satisfies i) or ii) in Theorem 3.8.

Remark 3.12 It is not hard to prove that in dimension $N = 1$, 2 or 3 the Cauchy problem for the system (1.1) is globally well-posed in $(1 + H^1(\mathbf{R}^N)) \times H^1(\mathbf{R}^N)$. However, the dynamics associated to (1.1) and the asymptotic behavior of solutions are not yet understood.

Remark 3.13 The existence of solitary waves for (1.1) in dimension greater than 1 is an open problem. Even the existence of "trivial" solitary waves (i.e., solutions of the form $(\psi(x_1 - ct, x_2, \ldots, x_N), 0)$ is a difficult problem. Note that if $\varphi \equiv 0$, the system (1.1) reduces to the Gross-Pitaevskii equation

$$2i\frac{\partial\psi}{\partial t} = -\Delta\psi + (|\psi|^2 - 1)\psi, \quad |\psi| \longrightarrow 1 \text{ as } |x| \longrightarrow \infty$$

The existence of travelling-waves moving with small speed for this equation was proved, for instance, in [2] (in dimension $N = 2$) and [1], [3] (in dimension $N \geq 3$).

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