Mémoire

déposé en vue de l'obtention de

l'habilitation à diriger des recherches

par

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Aspects quantitatifs de l'inconditionnalité

et soutenu le 9 décembre 2008 devant le jury composé de
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- Vern Paulsen et
- Vladimir Peller.

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Liste des articles et notes

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Résumé

1 Trois cas de figure pour l'inconditionnalité

Toutes nos recherches sont liées à la notion d'inconditionnalité, motivée par la question suivante :

Question 1.1. Lorsqu'un élément x d'un espace normé admet une représentation comme combinaison linéaire $\sum c_q \mathrm{e}_q$ d'éléments e_q, de quelle manière la norme de x dépend-elle du signe des coefficients c_q ?

Les réponses que nous obtiendrons seront en termes du *support* de x, c'est-à-dire de l'ensemble I d'indices q pour lesquels $c_q \neq 0$.

Selon la situation, un changement du signe des coefficients c_q

- fait varier la norme de x de manière bornée et on dira que $(\mathrm{e}_q)_{q\in I}$ est une *suite basique inconditionnelle* ;
- multiplie au plus la norme de x par un facteur D explicite et D sera la *constante d'inconditionnalité* de la suite $(\mathrm{e}_q)_{q\in I}$;
- ne change pas la norme de x et on parlera de suite basique $(\mathrm{e}_q)_{q\in I}$ *1-inconditionnelle* ;

Lorsque nous chercherons à déterminer une constante d'inconditionnalié exacte, nous devrons préciser de quelle manière nous nous permettons de changer le signe des coefficients :

- de manière *réelle* en multipliant certains coefficients par -1, ou
- de manière *complexe* en les faisant tourner d'un angle t_q.

Voici trois cas de figure dans lesquels cette question se pose.

(a) Si x est une fonction sur un groupe abélien compact et les e_q sont les caractères de ce groupe, cette représentation est la série de Fourier de x et un changement du signe des coefficients de Fourier est une *convolution* ou *multiplication de Fourier* unimodulaire.

(b) Si x est un opérateur et les e_q sont les matrices élémentaires, cette représentation est la matrice de x et un changement du signe des coefficients matriciels est une *multiplication de Schur* unimodulaire.

(c) Si x est un élément de l'algèbre d'un groupe discret G et les e_q sont les fonctions indicatrices des éléments de G, alors un changement du signe des coefficients est une *multiplication de Herz-Schur* unimodulaire.

2 Définition des suites basiques inconditionnelles

Voici une définition formelle qui reprend la discussion ci-dessus.

Définition 2.1. Soit X un espace vectoriel quasi-normé muni d'une suite distinguée (e_q) dans X. Soit $(\mathrm{e}_q)_{q\in I}$ une sous-suite. Soit $\mathbb{S} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (vs. $\mathbb{S} = \{-1, 1\}$.)

(a) I est *inconditionnelle* dans X s'il y a une constante D telle que

$$\Big\|\sum_{q\in I} \epsilon_q a_q \mathrm{e}_q\Big\| \leqslant D \Big\|\sum_{q\in I} a_q \mathrm{e}_q\Big\| \tag{1}$$

pour tout choix de signes $\epsilon_q \in \mathbb{S}$ et toute suite de coefficients complexes a_q de support fini. Sa *constante d'inconditionnalité complexe (vs. réelle)* est le minimum de telles constantes D.

(b) I est *1-inconditionnelle* complexe (vs. réelle) dans X si sa constante d'inconditionnalité complexe (vs. réelle) vaut 1. Cela veut dire que l'inégalité (1) devient l'égalité

$$\Big\|\sum_{q\in I} \epsilon_q a_q \mathrm{e}_q\Big\| = \Big\|\sum_{q\in I} a_q \mathrm{e}_q\Big\|.$$

3 Matrices et multiplicateurs de Schur pour les classes de Schatten-von-Neumann

Cette thématique de recherche correspond au cas de figure (b) ci-dessus : (e_q) est la suite des matrices élémentaires.

Notons C l'ensemble des indices colonne et R l'ensemble des indices ligne de matrices, en général deux copies de $\mathbb{N}$, et soit I une partie de $R \times C$. La propriété d'inconditionnalité de I peut aussi se formuler ainsi : une suite I est inconditionnelle de constante D si et seulement si, pour toute matrice φ à coefficients complexes (vs. réels) et pour tout x dont les coefficients de matrice sont nuls hors de I (dont l'espace sera noté X_I) on a

$$\|\varphi * x\| \leqslant D \sup |\varphi_{rc}| \, \|x\|,$$

où $\varphi * x$ est le produit de *Schur* (ou de Hadamard) défini par

$$(\varphi * x)_{rc} = \varphi_{rc} x_{rc}.$$

L'opérateur de multiplication par φ est un *multiplicateur de Schur relatif.* On peut aussi décrire les multiplicateurs de Schur relatifs comme les opérateurs diagonaux sur la suite $(\mathrm{e}_q)_{q\in I}$.

Notre étude se concentrera sur les classes de Schatten-von-Neumann $X = \mathrm{S}^p$ dont la quasi-norme est donnée par $\|x\| = \left(\mathrm{tr}\,(x^*x)^{p/2}\right)^{1/p}$: il s'agit de la contrepartie non commutative des espaces ℓ^p. Lorsque $p \geqslant 1$, l'espace de Banach S^p admet une structure d'espace d'opérateurs canonique qui rend la définition suivante naturelle.

Définition 3.1. I est *complètement inconditionnelle* dans S^p s'il y a une constante D telle que (1) vaut pour tout choix de signes $\epsilon_q \in \mathbb{S}$ et toute suite de coefficients *opérateurs* $a_q \in \mathrm{S}^p$ à support fini. Sa constante d'inconditionnalité *complète* complexe (vs. réelle) est le minimum de telles constantes D.

De la même manière, on parle de la norme *complète* de multiplicateurs de Schur relatifs. On ne sait pas si on définit vraiment ainsi une classe nouvelle ; ce serait répondre à la conjecture de Gilles Pisier qu'il existe des multiplicateurs de Schur bornés sur S^p $(p \neq 1, 2, \infty)$ qui ne sont pas complètement bornés.

Une suite inconditionnelle de matrices élémentaires dans S^∞ est en fait un ensemble V-Sidon, classe que Varopoulos a introduite dans l'étude des algèbres tensorielles $\mathrm{c}_0(C) \hat{\otimes} \mathrm{c}_0(R)$ sur des espaces discrets (voir le théorème B.5.1 page 37 qui rassemble les résultats connus : I doit être réunion finie d'ensembles qui soit contiennent au plus un élément par ligne, soit contiennent au plus un élément par colonne.) Notre étude généralise ainsi les résultats de Varopoulos à toutes les classes de Schatten-von-Neumann.

4 Suites de matrices élémentaires et graphes bipartis

Les suites inconditionnelles de matrices élémentaires forment la contrepartie matricielle des ensembles $\Lambda(p)$ de Walter Rudin étudiés en analyse de Fourier (le cas de figure (a) de la section 1.) Alors que l'étude des ensembles $\Lambda(p)$ voit surgir naturellement leurs propriétés arithmétiques (de théorie additive des nombres,) l'inconditionnalité de I se traduit avantageusement en termes de théorie des graphes.

Nous allons donc considérer I comme un graphe biparti dont les deux classes (« couleurs ») de sommets sont C et R, dont les éléments seront appelés respectivement « sommets colonne » et « sommets ligne. » Ses arêtes (non dirigées) relient seulement des sommets ligne $r \in R$ avec des sommets colonne $c \in C$, et cela exactement lorsque $(r, c) \in I$. La matrice $(\chi_I(r, c))_{(r,c)\in R\times C}$ fonction caractéristique de I est la matrice d'incidence de ce graphe biparti.

Voici deux exemples importants.

Exemple 4.1. Soit s un entier. Considérons l'ensemble

$$I = \{\, (r, c) \in \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/s\mathbb{Z} : r - c \in \{0, 1\} \,\}.$$

Le graphe biparti associé est le cycle (ligne 0, colonne 0, ligne 1, colonne 1, . . . , ligne $s-1$, colonne $s-$

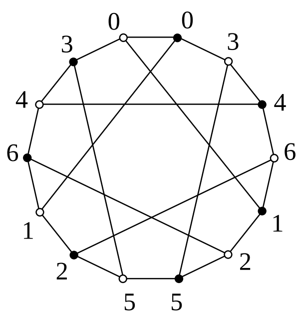
1) de longueur $2s$. La matrice d'incidence de ce graphe est

$$\begin{array}{cc}
 & \begin{matrix} 0 & 1 & \cdots & s-2 & s-1 \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ \vdots \\ s-2 \\ s-1 \end{matrix} & \begin{pmatrix} 1 & 0 & \ddots & 0 & 1 \\ 1 & 1 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \ddots & 1 & 1 \end{pmatrix}
\end{array}.$$

Exemple 4.2. Considérons l'ensemble

$$I = \{\, (r, c) \in \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} : r + c \in \{0, 1, 3\} \,\}.$$

Le graphe biparti associé est le graphe de *Heawood*.



La matrice d'incidence de ce graphe est

$$\begin{array}{cc}
 & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}
\end{array}.$$

5 Matrices lacunaires et inconditionnalité

Dans l'article *Lacunary matrices*, nous montrons que ces sous-suites doivent satisfaire la condition de densité suivante, qui est l'analogue de la condition de maille de Walter Rudin [87, Theorem 3.5] pour les ensembles $\Lambda(p)$.

Théorème 5.1 (page 24)**.** *Si I est inconditionnelle de constante D dans S^p, alors la taille $\#I'$ de tout sous-graphe I' induit par m sommets colonne et n sommets ligne, c'est-à-dire le cardinal de toute partie $I' = I \cap R' \times C'$ avec $\#C' = m$ et $\#R' = n$, satisfait*

$$\begin{aligned} \#I' &\leqslant D^2 \big(m^{1/p} n^{1/2} + m^{1/2} n^{1/p}\big)^2 \\ &\leqslant 4D^2 \min(m, n)^{2/p} \max(m, n). \end{aligned} \tag{2}$$

Les exposants de cette inégalité sont optimaux dans les trois cas suivants :

(*a*) *si m ou n est fixé (trivial ;)*

(*b*) *si $p = 4$ (voir le graphe aléatoire ci-dessous ;)*

(*c*) *si p est un entier pair et $m = n$ (voir [37, Theorem 4.8].)*

Si $m \neq n$, nous construisons des graphes aléatoires qui testent l'inégalité (2) sans en montrer toujours l'optimalité.

Théorème 5.2 (page 26)**.** *Pour tout $\varepsilon > 0$ et tout entier pair $p \geqslant 4$, il existe un graphe I de taille*

$$\#I \sim \begin{cases} \max(m,n)^{1-\varepsilon} \min(m,n)^{1/2} & si\ p = 4 \\ \max(m,n)^{1/2-\varepsilon} \min(m,n)^{1/2+2/p} & si\ p \geqslant 6. \end{cases}$$

et de constante d'inconditionnalité indépendante de m et n lorsque $mn \to \infty$.

Si p est un entier pair, nous donnons aussi une condition suffisante en termes de sentiers sur un graphe biparti : un *sentier* de longueur s dans I est une suite $(v_0, v_1, \ldots, v_s)$ de sommets alternativement dans R et C telle que les arêtes reliant v_0 à v_1, v_1 à v_2, etc. correspondent à des élements deux à deux distincts de I (alors qu'un *chemin* est requis d'avoir même tous ses sommets distincts et que les sommets d'une *promenade* sont admis à se répéter.) Le théorème suivant est aussi l'analogue d'un théorème de Walter Rudin.

Théorème 5.3 (page 22)**.** *Soit p un entier pair. Si le nombre de sentiers dans I de longueur $p/2$ entre deux sommets donnés admet une borne uniforme, alors I est inconditionnelle dans S^p.*

Le calcul suivant montre le lien étroit entre la norme S^p avec $p = 2s$ un entier pair et les promenades fermées de longueur p dans ce graphe.

$$\begin{aligned} \operatorname{tr}\Big|\sum_{q \in I} a_q \mathrm{e}_q\Big|^p &= \operatorname{tr}\Big(\sum_{(r,c),(r',c') \in I} (a_{rc}\mathrm{e}_{rc})^* (a_{r'c'}\mathrm{e}_{r'c'}) \Big)^s \\ &= \operatorname{tr} \sum_{\substack{(r_1,c_1),(r_1',c_1'),\ldots,\\ (r_s,c_s),(r_s',c_s') \in I}} \prod_{i=1}^s (\overline{a_{r_ic_i}} \mathrm{e}_{c_ir_i})(a_{r_i'c_i'} \mathrm{e}_{r_i'c_i'}) \\ &= \sum_{\substack{(r_1,c_1),(r_1,c_2),\ldots,\\ (r_s,c_s),(r_s,c_{s+1}) \in I}} \prod_{i=1}^s \overline{a_{r_ic_i}} a_{r_ic_{i+1}} \quad (\text{où } c_{s+1} = c_1.) \end{aligned}$$

Cette dernière somme est indexée par les promenades fermées $(c_1, r_1, c_2, \ldots, c_s, r_s)$ de longueur p dans le graphe associé à I !

La conjonction des théorèmes 5.1 et 5.3 donne une nouvelle preuve du théorème de Paul Erdős selon lequel un graphe sur v sommets sans circuit de longueur paire p est de taille bornée par $v^{1+2/p}$, à une constante près (un *circuit* est un sentier fermé.) La généralisation de ce théorème des circuits aux *cycles* (chemins fermés) par Bondy et Simonovits [12] échappe à notre méthode. L'existence de graphes qui montreraient l'optimalité de cette estimation est une question ouverte posée par Erdős en 1963.

6 Matrices lacunaires et 1-inconditionnalité

L'article *Cycles and 1-unconditional matrices* aboutit à une caractérisation des suites basiques 1-inconditionnelles dans S^p.

Un des ingrédients est l'étude des multiplicateurs de Schur unimodulaires sur un cycle. Nous obtenons en particulier la proposition suivante.

Proposition 6.1 (page 41)**.** *Si p n'est pas un entier pair, alors ϵ est un multiplicateur de Schur unimodulaire isométrique sur un cycle I pour S^p si et seulement si ϵ peut être interpolée par une matrice de rang 1 : $\epsilon_{rc} = \zeta_c \eta_r$ pour $(r,c) \in I$, où $\zeta \in \mathbb{T}^C$ et $\eta \in \mathbb{T}^R$.*

Esquisse de démonstration. La condition est bien suffisante : on a alors

$$\epsilon * x = \begin{pmatrix} \ddots & & \\ & \eta_r & \\ & & \ddots \end{pmatrix} (x_{rc}) \begin{pmatrix} \ddots & & \\ & \zeta_c & \\ & & \ddots \end{pmatrix}.$$

Étudions la nécessité. On peut supposer que le cycle I soit donné comme dans l'exemple 4.1. Soit $\epsilon \in \mathbb{T}^I$ une matrice de nombres unimodulaires partiellement spécifiée. Il est possible de multiplier les lignes et les colonnes de ϵ par des nombres complexes de module 1 de sorte que ϵ devienne la matrice circulante

$$\tilde{\epsilon} = \begin{array}{c} \\ {\scriptstyle 0} \\ {\scriptstyle 1} \\ \vdots \\ {\scriptstyle s-2} \\ {\scriptstyle s-1} \end{array} \begin{array}{c} \begin{array}{ccccc} {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle \cdots} & {\scriptstyle s-2} & {\scriptstyle s-1} \end{array} \\ \begin{pmatrix} 1 & 0 & \ddots & 0 & \vartheta \\ \vartheta & 1 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \ddots & \vartheta & 1 \end{pmatrix} \end{array} \tag{3}$$

avec ϑ racine $s$ième de $\overline{\epsilon_{00}}\epsilon_{10}\ldots\overline{\epsilon_{s-1,s-1}}\epsilon_{0,s-1}$. Un argument de transfert montre que la norme du multiplicateur de Schur par $\tilde{\epsilon}$ sur I borne le multiplicateur de Fourier relatif $\mu\colon a+bz \mapsto a+\vartheta bz$ dans le groupe G des racines $s$ièmes de l'unité, où on norme $a+bz$ par la norme L^p : $\|a+bz\| = \left(s^{-1}\sum_{z^s=1}|a+bz|^p\right)^{1/p}$: voir la proposition 13.2 (a). Si μ est une isométrie, le théorème d'équimesurabilité de Plotkin-Rudin montre que z et ϑz ont même distribution et donc $\vartheta^s = 1$. $\square$

On peut calculer la norme exacte du multiplicateur de Schur relatif $\tilde{\epsilon}$ sur S^1_I et sur S^∞_I : elle égale la norme de μ sur $\mathrm{L}^1_\Lambda(G)$ et sur $\mathrm{L}^\infty_\Lambda(G)$ avec $\Lambda = \{1, z\}$ et cette norme est

$$\frac{\max_{z^s=-1}|\vartheta+z|}{|1+\mathrm{e}^{\mathrm{i}\pi/s}|}$$

(proposition B.7.1(d) page 41.)

Cette proposition est une des étapes dans la démonstration du théorème suivant.

Théorème 6.2 (page 42)**.** *Soit p un nombre réel strictement positif qui ne soit pas un entier pair. Les propriétés suivantes sont équivalentes.*

- *I est complètement 1-inconditionnelle complexe dans S^p.*
- *I est 1-inconditionnelle réelle dans S^p.*
- *I est une réunion disjointe d'arbres, c'est-à-dire que I ne contient aucun cycle.*
- *Toute suite de signes complexes $\epsilon \in \mathbb{T}^I$ peut être interpolée par une matrice de rang 1.*
- *I est un* ensemble de Varopoulos *de* V-interpolation de constante 1 : *toute suite $\varphi \in \ell^\infty_I$ peut être interpolée par un tenseur $u \in \ell^\infty_C \mathbin{\hat{\otimes}} \ell^\infty_R$ avec $\|u\| = \|\varphi\|$.*
- *I est un ensemble d'interpolation isométrique pour les multiplicateurs de Schur : toute suite $\varphi \in \ell^\infty_I$ est la restriction d'un multiplicateur de Schur sur S^∞ de norme $\|\varphi\|$.*

Dans le cas où p est un entier pair, la combinatoire devient plus compliquée : cela se reflète dans la proposition suivante.

Proposition 6.3 (page 41)**.** *Si p est un entier pair, alors ϵ est un multiplicateur de Schur unimodulaire isométrique sur un cycle I de longueur $2s$ pour S^p si et seulement si $p/2 \in \{1, 2, \ldots, s-1\}$ ou si ϵ peut être interpolée par une matrice de rang 1.*

Cette proposition est une des étapes dans la démonstration de la caractérisation suivante.

Théorème 6.4 (page 44)**.** *Soit p un entier pair. Les propriétés suivantes sont équivalentes.*

- *I est complètement 1-inconditionnelle complexe dans S^p.*
- *I est 1-inconditionnelle réelle dans S^p.*
- *I ne contient aucun cycle de longueur paire inférieure ou égale à p.*

Illustrons ce théorème sur l'exemple 4.2. Le graphe de Heawood ne contient aucun cycle de longueur 4 : donc la norme S^4 de sa matrice d'incidence ne varie pas si on change le signe de ses coefficients.

La propriété de ne pas contenir de cycle de longueur paire donnée a été très étudiée en théorie des graphes. Quelle conséquence a-t-elle pour la taille du graphe ? La section suivante en propose un résumé, à comparer aux résultats du théorème 5.1.

7 Graphes bipartis sans cycle de longueur donnée

Proposition 7.1 (page 50)**.** *Soient $2 \leqslant n \leqslant m$, $I \subseteq R \times C$ avec $\# C = n$ et $\# R = m$, et $e = \# I$.*

(*a*) *Si I est 1-inconditionnelle dans S^4 — I ne contient pas de cycle de longueur 4 — alors*

$$n \geqslant 1 + \Big(\frac{e}{m} - 1\Big) + \Big(\frac{e}{m} - 1\Big)\Big(\frac{e}{n} - 1\Big),$$

c'est-à-dire $e^2 - me - mn(n-1) \leqslant 0$. On a égalité si et seulement si I est le graphe d'incidence d'un système de Steiner $\mathrm{S}(2, e/m; n)$ sur n points et m blocs (voir [9, Def. I.3.1] pour la définition des systèmes de Steiner.)

(*b*) *Si I est 1-inconditionnelle dans S^6 — I ne contient pas de cycle de longueur 4 ni 6 — alors*

$$n \geqslant 1 + \Big(\frac{e}{m} - 1\Big) + \Big(\frac{e}{m} - 1\Big)\Big(\frac{e}{n} - 1\Big) + \Big(\frac{e}{m} - 1\Big)^2\Big(\frac{e}{n} - 1\Big),$$

c'est-à-dire $e^3 - (m+n)e^2 + 2mne - m^2n^2 \leqslant 0$. On a égalité si et seulement si I est le graphe d'incidence du quadrangle *(le cycle de longueur 8) ou d'un quadrangle généralisé avec n points et m lignes (voir [56, Def. 1.3.1] pour la définition des polygones généralisés ; l'exemple 4.2 décrit le plus petit quadrangle généralisé.)*

(*c*) *Si I est 1-inconditionnelle dans S^p avec $p = 2k$ un entier pair — I ne contient pas de cycle de longueur inférieure ou égale à p — alors*

$$n \geqslant \sum_{i=0}^{k} \Big(\frac{e}{m} - 1\Big)^{\lceil \frac{i}{2} \rceil}\Big(\frac{e}{n} - 1\Big)^{\lfloor \frac{i}{2} \rfloor}. \tag{4}$$

*On a égalité si et seulement si I est le graphe d'incidence du $(k+1)$*gone *(le cycle de longueur $2k+2$) ou d'un $(k+1)$gone généralisé avec n points et m lignes.*

Les résultats (*a*) et (*b*) ci-dessus ont été obtenus dans la note *The size of bipartite graphs with girth eight* (voir pages 61 et 62), alors que le cas général résulte de travaux de Noga Alon, Shlomo Hoory et Nathan Linial (voir [44]).

L'inégalité (4) montre que si I est 1-inconditionnelle dans S^{2k}, alors $\# I \leqslant n^{1+1/k} + (s-1)n/s$. Si $k \notin \{2, 3, 5, 7\}$, il n'existe pas de $(k+1)$gones généralisés de taille arbitrairement grande et l'existence de graphes arbitrairement grands sans cycle de longueur $2k$ de taille minorée par $n^{1+1/k}$ à une constante près est une question importante en théorie des graphes extrémaux.

La recherche pratique de graphes extrémaux nous a amenés à écrire un algorithme implémenté en langage `C` qui énumère tous les graphes bipartis d'un nombre de sommets donnés et teste l'existence de cycles. La proposition suivante, démontrée indépendamment par Adolf Mader et Otto Mutzbauer [55], réduit le nombre de matrices d'incidence de graphes bipartis à tester.

Proposition 7.2 (page 67)**.** *Toute matrice à coefficients 0 ou 1 peut être simultanément ordonnée selon les ordres lexicographiques des lignes et des colonnes (c'est-à-dire l'ordre des lignes et des colonnes lues comme des nombres binaires) par une permutation des lignes et des colonnes.*

En effet, une permutation des lignes et des colonnes de la matrice d'incidence d'un graphe biparti consiste juste à réindexer les sommets de ce graphe.

8 Matrices lacunaires et ensembles lacunaires d'un groupe abélien discret

Voici la traduction naturelle entre inconditionnalités de Fourier et matricielle (les cas (*a*) et (*b*) de la section 1.)

Proposition 8.1 (page 49)**.** *Soit $I \subseteq R \times C$. Soit $p \in [1, \infty]$: les propriétés suivantes sont équivalentes.*

- *I est complètement inconditionnelle dans S^p.*
- *La suite de produits de Walsh de longueur deux $\{\epsilon_c \epsilon'_r : (r, c) \in I\}$ est complètement inconditionnelle dans $\mathrm{L}^p(\{-1,1\}^C \times \{-1,1\}^R)$.*

– *La suite de produits de deux fonctions de Steinhaus* $\{z_c z'_r : (r,c) \in I\}$ *est complètement inconditionnelle dans* $\mathrm{L}^p(\mathbb{T}^C \times \mathbb{T}^R)$.

Soit $p \in (0,\infty]$ *: les propriétés suivantes sont équivalentes.*

– I *est 1-inconditionnelle dans* S^p.

– *La suite* $\{\epsilon_c \epsilon'_r : (r,c) \in I\}$ *est 1-inconditionnelle dans* $\mathrm{L}^p(\{-1,1\}^C \times \{-1,1\}^R)$.

– *La suite* $\{z_c z'_r : (r,c) \in I\}$ *est 1-inconditionnelle dans* $\mathrm{L}^p(\mathbb{T}^C \times \mathbb{T}^R)$.

La proposition B.11.1 (page 49) décrit dans quelle mesure cette proposition reste vraie pour d'autres groupes discrets.

9 Sous-espaces S^p_I 1-complémentés

En route pour ces résultats, nous obtenons aussi la caractérisation suivante.

Proposition 9.1 (page 35)**.** *Le sous-espace* S^p_I *de* S^p *formé des opérateurs à support dans* I *est 1-complémenté si et seulement si* I *est la réunion disjointe de graphes bipartis complets* $R_j \times C_j$ *: sa matrice d'incidence est, à une permutation des colonnes et des lignes près, bloc-diagonale :*

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & \cdots \end{array} \\ \begin{pmatrix} (1) & (0) & (0) & \cdots \\ (0) & (1) & (0) & \ddots \\ (0) & (0) & (1) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \end{array}.$$

10 Matrices de rang 1 partiellement spécifiées

Si φ est une matrice de rang 1, $\varphi = x \otimes y$, alors l'opérateur de multiplication de Schur par φ est de norme $\sup|x_r| \sup|y_c|$. Or les exemples de calcul exact de normes de tels opérateurs sont très rares et nous avons voulu savoir comment cette norme change lorsque φ agit sur un sous-espace S^p_I.

Théorème 10.1 (page 86)**.** *Soit* $I \subset R \times C$ *et considérons* $(x_r)_{r\in R}$ *et* $(y_c)_{c\in C}$*. Alors le multiplicateur de Schur relatif* S^p_I *donné par* $(x_r y_c)_{(r,c)\in I}$ *est de norme* $\sup_{(r,c)\in I} |x_r y_c|$.

11 Propriété d'approximation métriquement inconditionnelle dans S^p_I

Même si I n'est pas 1-inconditionnelle dans l'espace S^p, l'espace S^p_I pourrait néanmoins admettre une autre base 1-inconditionnelle. Pour approcher de telles questions, Peter G. Casazza et Nigel J. Kalton ont introduit la propriété (c) ci-dessous.

Définition 11.1. Soit X un espace de Banach séparable et $\mathbb{S} = \mathbb{T}$ (vs. $\mathbb{S} = \{-1,1\}$.)

(a) Une suite (T_k) d'opérateurs sur X est une *suite approximante* si chaque T_k est de rang fini et $T_k x \to x$ pour chaque $x \in X$.

(b) ([72].) Posons $\Delta T_k = T_k - T_{k-1}$. L'espace X a la *propriété d'approximation inconditionnelle* s'il existe une suite approximante (T_k) telle que pour une certaine constante D

$$\left\| \sum_{k=1}^{n} \epsilon_k \Delta T_k \right\| \leqslant D \quad \text{pour tout } n \text{ et tous } \epsilon_k \in \mathbb{S}.$$

La *constante d'inconditionnalité complexe (vs. réelle)* de (T_k) est la plus petite des constantes D.

(c) ([22, § 3], [32, § 8].) L'espace X a la *propriété d'approximation métriquement inconditionnelle complexe (vs. réelle)* si, pour tout $\delta > 0$, X admet une suite approximante de constante d'inconditionnalité complexe (vs. réelle) $1 + \delta$.

Voici notre description des sous-espaces S_I^p métriquement inconditionnels :

Theorem 11.2. *On a deux cas.*

– *Si $p \in [1,\infty] \setminus \{2,4,6,\ldots\}$ et S_I^p a la propriété d'approximation métriquement inconditionnelle réelle, alors la distance d'un sommet colonne à un sommet colonne est* asymptotiquement *infinie dans I : leur distance devient arbitrairement grande en effaçant un nombre fini d'arêtes de I.*

– *Si $p \in \{2,4,6,\ldots\}$, l'espace S_I^p a la propriété d'approximation métriquement inconditionnelle complexe, ou réelle, si et seulement si deux sommets à distance $2j+1 \leqslant p/2$ sont à distance asymptotiquement supérieure ou égale à $p-2j+1$.*

12 Inégalités matricielles

L'article *Matrix inequalities with applications to the theory of iterated kernels* montre l'inégalité matricielle suivante.

Théorème 12.1 (page 52)**.** *Soit A une matrice de taille $n \times m$ à coefficients positifs et notons* somme(A) *la somme de tous ses coefficients. On a*

$$\operatorname{somme}\Bigl(\overbrace{AA^*A\ldots A^{(*)}}^{k\ \text{termes}}\Bigr) \geqslant \frac{\operatorname{somme}(A)^k}{n^{\lfloor\frac{k-1}{2}\rfloor} m^{\lfloor\frac{k}{2}\rfloor}}$$

où $A^{(*)}$ est A^* ou A selon la parité de k.

Cette inégalité est la version discrète d'un théorème sur les itérés d'un noyau (voir la remarque C.1.4 page 53.) Si on applique cette inégalité à la matrice d'incidence d'un graphe biparti I, on obtient une minoration optimale du nombre de promenades de longueur k en termes de la taille de I. Dans le cas $k=3$, une généralisation de cette inégalité (le théorème D.4.4 page 63) donne une minoration optimale du nombre de chemins (voir le corollaire D.4.6 page 64.)

13 Transfert entre multiplicateurs de Schur et de Fourier

Le théorème suivant est bien connu.

Proposition 13.1. *Soit Γ un groupe discret et $R, C \subseteq \Gamma$. À $\Lambda \subset \Gamma$ associons $I = \{\,(r,c) \in R \times C : rc \in \Lambda\,\}$. À $\varrho \in \mathbb{C}^\Lambda$ associons $\varphi \in \mathbb{C}^I$ défini par $\varphi(r,c) = \varrho(rc)$ pour tout $(r,c) \in I$.*

– *Soit $p > 0$. La norme complète du multiplicateur de Schur relatif φ sur S_I^p est bornée par la norme complète du multiplicateur de Fourier relatif ϱ sur $\mathrm{L}_\Lambda^p(\tau)$.*

– *La norme du multiplicateur de Schur relatif φ sur S_I^∞ est bornée par la norme du multiplicateur de Fourier relatif ϱ sur $\mathrm{L}_\Lambda^\infty(\tau)$.*

Une forme de réciproque peut être déduite du théorème limite de Szegő matriciel (voir le théorème F.2.1 page 75.)

Proposition 13.2. *Soit Γ un groupe discret* moyennable *et soit $I \subseteq \Gamma \times \Gamma$ un ensemble* toeplitzien *au sens que $I = \{(r,c) \in \Gamma \times \Gamma : rc^{-1} \in \Lambda\}$ pour une partie Λ de Γ. Soit $\varphi \in \mathbb{C}^I$ une matrice* toeplitzienne *au le sens qu'il existe $\varrho \in \mathbb{C}^\Lambda$ tel que $\varphi(r,c) = \varrho(rc^{-1})$ pour tout $(r,c) \in I$.*

(a) *Soit $p > 0$. La norme du multiplicateur de Fourier relatif ϱ sur $\mathrm{L}_\Lambda^p(\tau)$ est le supremum de la norme du multiplicateur de Schur relatif φ sur des sous-espaces de matrices de Toeplitz tronquées dans S_I^p.*

(b) *De plus, la norme complète du multiplicateur de Fourier relatif ϱ sur $\mathrm{L}_\Lambda^p(\tau)$ et la norme complète du multiplicateur de Schur relatif φ sur S_I^p sont égales.*

(c) *La norme du multiplicateur de Fourier relatif ϱ sur $\mathrm{L}_\Lambda^\infty(\tau)$ et la norme du multiplicateur de Schur relatif φ sur S_I^∞ sont égales.*

14 Ensembles lacunaires somme de deux ensembles infinis

Il est bien connu que les ensembles de Sidon ne peuvent contenir la somme de deux ensembles infinis ; Daniel Li a obtenu la même conclusion pour les ensembles Λ tels que C_Λ admet une suite approximante métriquement inconditionnelle. Nous rassemblons ces deux résultats dans le théorème suivant.

Théorème 14.1. *Soit Γ un groupe abélien discret de caractères sur un groupe abélien compact G. Soit $\Lambda \subset \Gamma$. Si Γ contient la somme $R + C$ de deux ensembles infinis R et C, alors l'espace $\mathrm{C}_\Lambda(G)$ n'admet pas de suite approximante inconditionnelle.*

Esquisse de preuve. On utilise l'hypothèse pour montrer qu'il existe des parties infinies R' et C' sur lesquelles une *somme à blocs sautés* $\sum(T_{k_{j+1}} - T_{k_j})$ agit comme la projection sur la « partie triangulaire supérieure » de $R' + C'$. Or ce multiplicateur de Fourier relatif se transfère en le multiplicateur de Schur qu'est la projection de *Riesz* sur les matrices triangulaires supérieures, qui est notoirement non bornée. □

Nous obtenons ainsi une preuve élémentaire que l'algèbre du disque $\mathrm{C}_{\mathbb{N}}(\mathbb{T})$ n'a pas la propriété d'approximation inconditionnelle, ni l'espace engendré par les fonctions de Walsh de longueur deux (les produits $\{r_i r_j\}$ de deux fonctions de Rademacher) dans $\mathrm{C}(\{-1, 1\}^\infty)$, ni l'espace engendré par les produits $\{s_i s_j\}$ de deux fonctions de Steinhaus dans $\mathrm{C}(\mathbb{T}^\infty)$.

Nous montrons aussi qu'un ensemble « complètement $\Lambda(p)$ » ne peut contenir la somme de deux ensembles infinis (théorème F.4.8 de la page 81.)

La preuve ci-dessus montre aussi que la constante d'inconditionnalité réelle pour les espaces $\mathrm{L}^p_\Lambda(G)$ est minorée par la norme complète de la projection de Riesz sur S^p. Cela nous a motivés pour calculer cette norme et, à défaut, la norme complète de la transformation de Hilbert matricielle.

Théorème 14.2. *La norme complète de la projection de Riesz et de la transformation de Hilbert matricielle sur S^p coïncident avec leur norme.*

- *Si p est un entier pair, la norme de la transformation de Hilbert matricielle est $\cot(\pi/2p)$ (voir page 83).*
- *La norme de la projection de Riesz sur S^4 est $\sqrt{2}$ (voir page 84).*

15 Problèmes extrémaux pour les polynômes trigonométriques

Cette thématique de recherche correspond au cas de figure (a) de la question 1.1, avec $\mathbb{R}/\mathbb{Z}$ comme groupe abélien et le module maximum comme norme.

Soit $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ un ensemble de n entiers. Pour n nombres réels positifs $r_1, r_2, \dots, r_n$ et n nombres réels $t_1, t_2, \dots, t_n$, considérons le polynôme trigonométrique

$$f(x) = r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + \cdots + r_n \mathrm{e}^{\mathrm{i}(t_n+\lambda_n x)}. \tag{5}$$

La question de l'inconditionnalité dans l'espace C des fonctions continues est alors celle de la dépendance du module maximum du polynôme trigonométrique f par rapport aux arguments (phases) $t_1, t_2, \dots, t_n$.

Voici quatre problèmes qui éclairent divers aspects de l'inconditionnalité.

Problème extrémal 15.1 (problème de Mandelʹshtam complexe — voir [26, page 2 et le supplément])**.** Trouver le minimum du module maximum du polynôme trigonometrique f pour des modules de coefficients de Fourier $r_1, r_2, \dots, r_n$ donnés :

$$\min_{t_1,t_2,\dots,t_n} \max_x |r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + \cdots + r_n \mathrm{e}^{\mathrm{i}(t_n+\lambda_n x)}|.$$

Problème extrémal 15.2. Trouver le minimum du module maximum du polynôme trigonometrique f pour un spectre Λ, des arguments $t_1, t_2, \dots, t_n$ et la somme des modules $r_1 + r_2 + \cdots + r_n$ des coefficients de Fourier donnés :

$$\min_{r_1,r_2,\dots,r_n} \max_x \frac{|r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + \cdots + r_n \mathrm{e}^{\mathrm{i}(t_n+\lambda_n x)}|}{r_1 + r_2 + \cdots + r_n}.$$

Problème 15.3. Trouver le maximum de la variation du module maximum du polynôme trigonometrique f pour un spectre Λ et une variation des arguments $\Delta t_1, \Delta t_2, \ldots, \Delta t_n$ donnée. En d'autres termes, trouver la norme du multiplicateur de Fourier relatif unimodulaire par les signes $\mathrm{e}^{\mathrm{i}\Delta t_1}$, $\mathrm{e}^{\mathrm{i}\Delta t_2}$, $\ldots$, $\mathrm{e}^{\mathrm{i}\Delta t_n}$:

$$\min_{\substack{r_1,r_2,\ldots,r_n\\ t_1,t_2,\ldots,t_n}} \frac{\max_x |r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + \cdots + r_n\mathrm{e}^{\mathrm{i}(t_n+\lambda_n x)}|}{\max_x |r_1\mathrm{e}^{\mathrm{i}(t_1+\Delta t_1+\lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(t_2+\Delta t_2+\lambda_2 x)} + \cdots + r_n\mathrm{e}^{\mathrm{i}(t_n+\Delta t_n+\lambda_n x)}|}.$$

Problème extrémal 15.4 (constante de Sidon)**.** Trouver le minimum du module maximum du polynôme trigonometrique f pour un spectre Λ et la somme des modules $r_1 + r_2 + \cdots + r_n$ des coefficients de Fourier donnés :

$$\min_{\substack{r_1,r_2,\ldots,r_n\\ t_1,t_2,\ldots,t_n}} \max_x \frac{|r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + \cdots + r_n\mathrm{e}^{\mathrm{i}(t_n+\lambda_n x)})|}{r_1 + r_2 + \cdots + r_n}.$$

L'inverse de ce minimum est la constante d'inconditionnalité de Λ dans l'espace des fonctions continues : c'est la *constante de Sidon* de Λ.

Littlewood [52] et Salem [90] se sont intéressés à ces problèmes. Ils sont aussi apparus dans la théorie du circuit électrique, comme le raconte N. G. Chebotarëv : « L. I. Mandel′shtam m'a communiqué un problème sur le choix des phases de courants électriques de fréquences différentes de sorte que la capacité du courant résultant de faire sauter les fusibles soit minimal » [24, p. 396]. Ce problème est une de ses motivations pour proposer une formule pour la valeur des dérivées directionnelles d'une fonction maximum en fonction des dérivées des fonctions dont on prend le maximum, qui a été un peu oubliée malgré son caractère naturel.

Formule de N. G. Chebotarëv ([26, Theorem VI.3.2, (3.6)])**.** *Soit I un ouvert de $\mathbb{R}^n$ et soit K un espace compact. Soit $F(t,x)$ une fonction réelle sur $I \times K$ qui soit continue, ainsi que $\frac{\partial F}{\partial t}(t,x)$. Soit*

$$F^*(t) = \max_{x\in K} F(t,x).$$

Alors $F^*(t)$ admet le développement limité suivant en tout $t \in I$:

$$F^*(t+h) = F^*(t) + \max_{F(t,x)=F^*(t)} \Big\langle h, \frac{\partial F}{\partial t}(t,x) \Big\rangle + o(h). \tag{6}$$

Chebotarëv utilise en particulier cette formule pour résoudre le problème d'approximation polynomiale de Chebyshev et le problème du minimum d'une forme quadratique sur les entiers de Korkin et Zolotarev.

16 Problèmes extrémaux pour les trinômes trigonométriques

Ces problèmes sont déjà intéressants dans le cas $n = 3$. Dans l'article *The Sidon constant of sets with three elements*, nous avons résolu les problèmes extrémaux 15.1 et 15.4 pour ce cas. Nous allons supposer, en toute généralité, que Λ est un ensemble de trois entiers $\lambda_1 < \lambda_2 < \lambda_3$ tels que $\lambda_2 - \lambda_1$ et $\lambda_3 - \lambda_2$ sont premiers entre eux.

Il s'avère que les arguments t_1, t_2, t_3 d'un trinôme trigonométrique

$$f(x) = r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)} \tag{7}$$

donnent lieu à un paramètre unique, que nous appellerons l'*argument du trinôme f* : la détermination principale $\tau \in\,]-\pi,\pi]$ de

$$(\lambda_3 - \lambda_2)t_1 + (\lambda_1 - \lambda_3)t_2 + (\lambda_2 - \lambda_1)t_3 \bmod 2\pi\mathbb{Z}. \tag{8}$$

Nous avons ainsi établi que les arguments minimaux du problème extrémal 15.1 correspondent à des multiples de π :

Théorème 16.1 (page 91)**.** *Soit Λ un ensemble de trois entiers. Soient r_1, r_2 et r_3 trois réels strictement positifs. Les arguments t_1, t_2, t_3 résolvent le problème extrémal 15.1 si et seulement si l'argument τ du trinôme égale π. En particulier, t_1, t_2 et t_3 peuvent être choisis parmi 0 et π.*

Ce théorème permet de déterminer les coefficients de Fourier minimaux pour le problème extrémal 15.4 :

Proposition 16.2 (page 92). *Le polynôme suivant résout le problème extrémal 15.4 :*

$$f(x) = \epsilon_1(\lambda_3 - \lambda_2)\mathrm{e}^{\mathrm{i}\lambda_1 x} + \epsilon_2(\lambda_3 - \lambda_1)\mathrm{e}^{\mathrm{i}\lambda_2 x} + \epsilon_3(\lambda_2 - \lambda_1)\mathrm{e}^{\mathrm{i}\lambda_3 x}$$

où $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}$ *sont trois signes réels tels que*

- $\epsilon_1\epsilon_2 = -1$ *si* $\lambda_2 - \lambda_1$ *est pair ;*
- $\epsilon_1\epsilon_3 = -1$ *si* $\lambda_3 - \lambda_1$ *est pair ;*
- $\epsilon_2\epsilon_3 = -1$ *si* $\lambda_3 - \lambda_2$ *est pair.*

La constante de Sidon de Λ *égale donc* $\cos(\pi/2(\lambda_3 - \lambda_1))^{-1}$ *et les constantes d'inconditionnalité complexes et réelles de* Λ *dans l'espace des fonctions continues coïncident donc pour les ensembles* Λ *à trois éléments.*

Nous avions entamé cette direction de recherche pour vérifier que les constantes d'inconditionnalité complexes et réelles d'un ensemble Λ étaient bien différentes ; les ensembles à trois éléments ne fourniront pas de contre-exemple et la question demeure ouverte.

En fait, les problèmes extrémaux 15.2 et 15.4 admettent une solution élémentaire : on ramène le trinôme trigonométrique à la forme « normale »

$$r_1\mathrm{e}^{-\mathrm{i}kx} + r_2\mathrm{e}^{\mathrm{i}\tau/(k+l)} + r_3\mathrm{e}^{\mathrm{i}lx} \tag{9}$$

avec k et l positifs et premiers entre eux et $\tau \in]-\pi, \pi]$. Alors

$$\begin{aligned} \frac{\max_x\left|r_1\mathrm{e}^{-\mathrm{i}kx} + r_2\mathrm{e}^{\mathrm{i}\tau/(k+l)} + r_3\mathrm{e}^{\mathrm{i}lx}\right|}{r_1 + r_2 + r_3} &\geqslant \frac{\left|r_1 + r_2\mathrm{e}^{\mathrm{i}\tau/(k+l)} + r_3\right|}{r_1 + r_2 + r_3} \\ &= \sqrt{1 - \frac{4(r_1 + r_3)r_2}{(r_1 + r_2 + r_3)^2}\sin^2\frac{\tau}{2(k+l)}} \\ &\geqslant \sqrt{1 - \sin^2\frac{\tau}{2(k+l)}} = \cos\frac{\tau}{2(k+l)}, \end{aligned}$$

et si $r_1 : r_2 : r_3 = l : k + l : k$, le trinôme trigonométrique (9) atteint son module maximum en 0 et satisfait $r_1 + r_3 = r_2$.

17 Points extrémaux et points exposés de la boule unité de l'espace C_Λ

Si on cherche à résoudre les problèmes extrémaux 15.1, 15.2 et 15.4 par une application de la formule de Chebotarëv (6), il est utile d'obtenir des informations sur les points x tels que « $F(t, x) = F^*(t)$, » c'est-à-dire les points maximum de $F(t, \cdot) = |f|$. Par exemple, on peut déduire de cette formule qu'il y en a plus d'un car sinon (t, x) serait un point critique de F ; or un petit calcul (lemme G.3.1 page 90) montre que ce n'est pas possible.

Voici un argument d'analyse fonctionnelle qui démontre la même chose. Comme les problèmes ci-dessus sont linéaires, on peut limiter la recherche de polynômes trigonométriques extrémaux aux points exposés de la boule unité K de l'espace C_Λ (rappelons qu'un point P de K est exposé par un hyperplan H si H ne coupe K qu'en P.) Pourquoi un point exposé P de K atteint-il son module maximum en au moins deux points ? parce que la forme linéaire qui définit l'hyperplan H s'étend en une mesure μ qui atteint sa norme sur P et on sait que P doit être de module maximum sur le support de μ ; la mesure μ n'est pas une masse de Dirac puisqu'elle atteint sa norme uniquement en P, de sorte que le support de μ a au moins deux points.

Dans l'article *The maximum modulus of a trigonometric trinomial*, nous obtenons une description très complète des points de module maximum d'un trinôme trigonométrique (voir le théorème H.7.1 page 102) dont voici le point saillant.

Théorème 17.1. *Soit* Λ *un ensemble de trois entiers* $\lambda_1 < \lambda_2 < \lambda_3$ *tels que* $\lambda_2 - \lambda_1$ *et* $\lambda_3 - \lambda_2$ *soient premiers entre eux. Soient* r_1, r_2, r_3 *trois réels strictement positifs. Le trinôme trigonométrique*

$$f(x) = r_1\mathrm{e}^{\mathrm{i}(t_1 + \lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(t_2 + \lambda_2 x)} + r_3\mathrm{e}^{\mathrm{i}(t_3 + \lambda_3 x)}$$

atteint son module maximum en un point unique modulo 2π, de multiplicité 2, sauf si son argument τ égale π : Si f atteint son module maximum en deux points modulo 2π, c'est parce que son graphe admet un axe de symétrie.

Esquisse de démonstration. On ramène le trinôme trigonométrique f à la forme normale (9) avec de plus $\tau \in [0,\pi]$ et on montre alors que f doit atteindre son module maximum sur le petit intervalle $[-\tau/k(k+l), \tau/l(k+l)]$ en trouvant, pour tout y hors de cet intervalle, un point x qui y soit pour lequel $|f(x)| \geqslant |f(y)|$. De plus, on peut rendre cette inégalité stricte sauf si $\tau = \pi$. Il reste alors à étudier les variations de $|f|$ sur $[-\tau/k(k+l), \tau/l(k+l)]$. □

La formule de Chebotarëv donne alors une nouvelle solution pour le problème extrémal 15.1.

Proposition 17.2. *Le module maximum de $r_1\mathrm{e}^{-\mathrm{i}kx} + r_2\mathrm{e}^{\mathrm{i}\tau/(k+l)} + r_3\mathrm{e}^{\mathrm{i}lx}$ est une fonction strictement décroissante de τ sur $[0,\pi]$.*

Démonstration. Restons dans le contexte de l'esquisse de démonstration ci-dessus et soit $\tau \in]0,\pi[$. Soit x^* l'unique point de module maximum pour f : on a vu que $x^* \in [-\tau/k(k+l), \tau/l(k+l)]$. Mais alors

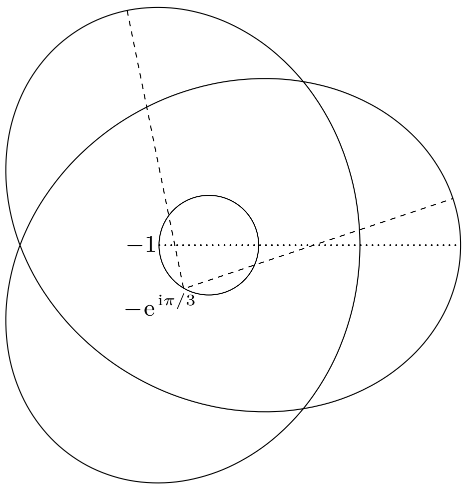
$$|f(x)|^2 = r_1^2 + r_2^2 + r_3^2 + 2r_1r_3\cos\bigl((k+l)x\bigr) + 2r_2\bigl(r_1\cos(\tau/(k+l)+kx) + r_3\cos(\tau/(k+l)-lx)\bigr)$$

et

$$\frac{k+l}{2r_2}\frac{\partial|f|^2}{\partial\tau}(x^*) = -r_1\sin(\tau/(k+l)+kx^*) - r_3\sin(\tau/(k+l)-lx^*) < 0$$

car $\tau/(k+l)+kx^* \in [0,\tau/l]$ et $\tau/(k+l)-lx^* \in [0,\tau/k]$ ne s'annulent pas simultanémanent. □

Illustrons notre propos : le module maximum de $f(x) = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}x}$ est la distance maximum de points de l'hypotrochoïde d'équation $z = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}x}$ à un point donné $-\mathrm{e}^{\mathrm{i}t}$ du plan complexe. Nous avons donc montré que si deux points de H sont simultanément à distance maximum de $-\mathrm{e}^{\mathrm{i}t}$, alors $-\mathrm{e}^{\mathrm{i}t}$ est sur un axe de symétrie de H, c'est-à-dire $t \equiv \pi/3 \bmod 2\pi/3$.



Notre étude aboutit au théorème suivant, dont on peut espérer une généralisation à des ensembles Λ plus grands.

Théorème 17.3 (page 94)**.** *Soit Λ un ensemble à trois éléments. Soit K la boule unité de l'espace C_Λ et soit $P \in K$.*

- *Le point P est un point exposé de K si et seulement si P est un monôme trigonométrique $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\lambda x}$ avec $\alpha \in \mathbb{R}$ et $\lambda \in \Lambda$ ou un trinôme trigonométrique qui atteint son module maximum, 1, en deux points modulo $2\pi/d$. Toute forme linéaire sur C_Λ atteint sa norme en un point exposé de K.*
- *Le point P est un point extrémal de K si et seulement si P est un monôme trigonométrique $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\lambda x}$ avec $\alpha \in \mathbb{R}$ et $\lambda \in \Lambda$ ou un trinôme trigonométrique tel que $1 - |P|^2$ a quatre zéros modulo 2π, comptés avec leur multiplicité.*

18 La variation du module maximum en fonction de l'argument

Nous utilisons aussi la formule de Chebotarëv pour montrer que le module maximum d'un trinôme trigonométrique est une fonction décroissante de la valeur absolue $|\tau|$ de son argument (voir (8)) et pour borner cette décroissance. Nous obtenons les inégalités suivantes.

Théorème 18.1 (pages 95, 95 et 96)**.** *Soit f un trinôme trigonométrique comme en* (7) *et varions ses arguments de $\Delta t_1, \Delta t_2, \Delta t_3$. Notons $\tilde f$ le trinôme qui en résulte, $\tilde\tau$ l'argument du trinôme $\tilde f$, et $\Delta\tau$ la variation de l'argument :*

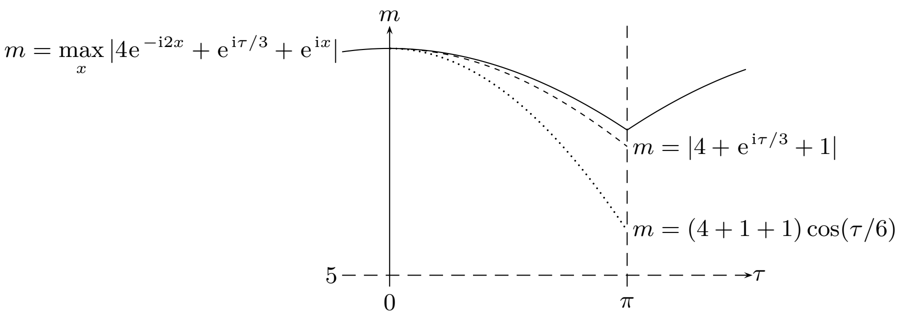
$$\Delta\tau \equiv (\lambda_3-\lambda_2)\Delta t_1 + (\lambda_1-\lambda_3)\Delta t_2 + (\lambda_2-\lambda_1)\Delta t_3 \mod 2\pi\mathbb{Z}.$$

Si $|\tilde\tau| > |\tau|$, alors

$$\begin{aligned}
\max_x|\tilde f(x)| &< \max_x|f(x)| \\
&\leqslant \frac{|r_1 + r_2\mathrm{e}^{\mathrm{i}\tau/|\lambda_3-\lambda_1|} + r_3|}{|r_1 + r_2\mathrm{e}^{\mathrm{i}\tilde\tau|\lambda_3-\lambda_1|} + r_3|}\max_x|\tilde f(x)| && (10)\\
&\leqslant \frac{\cos(\tau/2(\lambda_3-\lambda_1))}{\cos(\tilde\tau/2(\lambda_3-\lambda_1))}\max_x|\tilde f(x)| && (11)\\
&\leqslant \frac{\cos((\pi-|\Delta\tau|)/2(\lambda_3-\lambda_1))}{\cos(\pi/2(\lambda_3-\lambda_1))}\max_x|\tilde f(x)|; && (12)
\end{aligned}$$

l'inégalité (10) *est une égalité si et seulement si $r_1 : r_3 = \lambda_3-\lambda_2 : \lambda_2-\lambda_1$, l'inégalité* (11) *si et seulement si $r_1 : r_2 : r_3 = \lambda_3-\lambda_2 : \lambda_3-\lambda_1 : \lambda_2-\lambda_1$ et l'inégalité* (12) *si et seulement si de plus $\tau = \pi$. La norme du multiplicateur de Fourier relatif unimodulaire par les signes $\mathrm{e}^{\mathrm{i}\Delta t_1}, \mathrm{e}^{\mathrm{i}\Delta t_2}, \dots, \mathrm{e}^{\mathrm{i}\Delta t_n}$ est donc le facteur dans l'inégalité* (12).

En particulier, le module maximum de f comme fonction de τ admet les deux minorants $|r_1 + r_2\mathrm{e}^{\mathrm{i}\tau/|\lambda_3-\lambda_1|} + r_3|$ et $(r_1+r_2+r_3)\cos(\tau/2(\lambda_3-\lambda_1))$ sur l'intervalle $[-\pi,\pi]$. Illustrons-le dans le cas particulier $f(x) = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}\tau/3} + \mathrm{e}^{\mathrm{i}x}$:



19 Problèmes extrémaux pour les quadrinômes trigonométriques

Dans le cas $\Lambda = \{0,1,2,3\}$, le problème extrémal 15.4 est un problème ouvert posé par Harold S. Shapiro en 1951. Par des moyens heuristiques, nous avons conjecturé que les polynômes extrémaux sont de la forme

$$\frac{\mathrm{i}2\sqrt{2}\cos u - 1 - 3\sin u}{15} + \frac{3+\sin u}{10}\mathrm{e}^{\mathrm{i}x} + \frac{3-\sin u}{10}\mathrm{e}^{\mathrm{i}2x} + \frac{\mathrm{i}2\sqrt{2}\cos u - 1 + 3\sin u}{15}\mathrm{e}^{\mathrm{i}3x}$$

où u parcourt $[0,2\pi[$. Ces polynômes sont étudiés dans la note *On the Sidon constant of* $\{0,1,2,3\}$, section I.4. On en déduirait que la constante de Sidon de Λ vaut $5/3$, qui est sa constante d'inconditionnalité réelle (voir la proposition I.5.4 page 115.) Il s'agit de le démontrer et d'étudier plus généralement les polynômes trigonométriques à quatre termes dont le module atteint son maximum en trois points.

Chapter A

Lacunary matrices

with Asma Harcharras and Krzysztof Oleszkiewicz.

We study unconditional subsequences of the canonical basis (e_{rc}) of elementary matrices in the Schatten class S^p. They form the matrix counterpart to Rudin's $\Lambda(p)$ sets of integers in Fourier analysis. In the case of p an even integer, we find a sufficient condition in terms of trails on a bipartite graph. We also establish an optimal density condition and present a random construction of bipartite graphs. As a byproduct, we get a new proof for a theorem of Erdős on circuits in graphs.

1 Introduction

We study the following question on the Schatten class S^p.

(†) How many matrix coefficients of an operator $x \in S^p$ must vanish so that the norm of x has a bounded variation if we change the sign of the remaining nonzero matrix coefficients?

Let C be the set of columns and R be the set of rows for coordinates in the matrix, in general two copies of $\mathbb{N}$. Let $I \subseteq R \times C$ be the set of matrix coordinates of the remaining nonzero matrix coefficients of x. Property (†) means that the subsequence $(\mathrm{e}_{rc})_{(r,c)\in I}$ of the canonical basis of elementary matrices is an unconditional basic sequence in S^p: I forms a $\sigma(p)$ set in the terminology of [37, §4].

It is natural to wonder about the operator valued case, where the matrix coefficients are themselves operators in S^p. As the proof of our main result carries over to that case, we shall state it in the more general terms of complete $\sigma(p)$ sets.

We show that for our purpose, a set of matrix entries $I \subseteq R \times C$ is best understood as a bipartite graph. Its two vertex classes are C and R, whose elements will respectively be termed "column vertices" and "row vertices". Its edges join only row vertices $r \in R$ with column vertices $c \in C$, this occurring exactly if $(r, c) \in I$.

We obtain a generic condition for $\sigma(p)$ sets in the case of even p (Theorem 3.2) that generalises [37, Prop. 6.5]. These sets reveal in fact as a matrix counterpart to Rudin's $\Lambda(p)$ sets and we are able to transfer Rudin's proof of [87, Theorem 4.5(b)] to a non-commutative context: his number $r_s(E, n)$ is replaced by the numbers of Def. 2.4(b) and we count trails between given vertices instead of representations of an integer.

We also establish an upper bound for the intersection of a $\sigma(p)$ set with a finite product set $R' \times C'$ (Theorem 4.2): this is a matrix counterpart to Rudin's [87, Theorem 3.5]. In terms of bipartite graphs, this intersection is the subgraph induced by the vertex subclasses $C' \subseteq C$ and $R' \subseteq R$.

The bound of Theorem 4.2 provides together with Theorem 3.2 a generalisation of a theorem by Erdős [29, p. 33] on graphs without circuits of a given even length. In the last part of this article, we present a random construction of maximal $\sigma(p)$ sets for even integers p.

Terminology. C is the set of columns and R is the set of rows, in general both indexed by $\mathbb{N}$. The set V of all vertices is their disjoint union $R \amalg C$. An edge on V is a pair $\{v, w\} \subseteq V$. A graph on V is given by its set of edges E. A bipartite graph on V with vertex classes C and R has only edges $\{r,c\}$ such that $c \in C$ and $r \in R$ and may therefore be described alternatively by the set $I = \{\, (r,c) \in R \times C : \{r,c\} \in E \,\}$. A trail of length s in a graph is a sequence $(v_0, \dots, v_s)$ of $s+1$ vertices such that $\{v_0, v_1\}, \dots, \{v_{s-1}, v_s\}$ are pairwise distinct edges of the graph. A trail is a path if its vertices are pairwise distinct. A circuit of length p in a graph is a sequence $(v_1, \dots, v_p)$ of p vertices such that $\{v_1, v_2\}, \dots, \{v_{p-1}, v_p\}, \{v_p, v_1\}$ are pairwise distinct edges of the graph. A circuit is a cycle if its vertices are pairwise distinct.

Notation. $\mathbb{T} = \{\, z \in \mathbb{C} : |z| = 1 \,\}$. Let $q = (r,c) \in R \times C$. The transpose of q is $q^* = (c,r)$. The entry (elementary matrix) $\mathrm{e}_q = \mathrm{e}_{rc}$ is the operator on ℓ_2 that maps the cth basis vector on the rth basis vector and all other basis vectors on 0. The matrix coefficient at coordinate q of an operator x on ℓ_2 is $x_q = \operatorname{tr} \mathrm{e}_q^* x$ and its matrix representation is $(x_q)_{q \in R\times C} = \sum_{q \in R\times C} x_q \mathrm{e}_q$. The Schatten class S^p, $1 \leqslant p < \infty$, is the space of those compact operators x on ℓ_2 such that $\|x\|_p^p = \operatorname{tr} |x|^p = \operatorname{tr}(x^*x)^{p/2} < \infty$. For $I \subseteq R \times C$, the entry space S^p_I is the space of those $x \in S^p$ whose matrix representation is supported by I: $x_q = 0$ if $q \notin I$. S^p_I is also the closed subspace of S^p spanned by $(\mathrm{e}_q)_{q\in I}$. The S^p-valued Schatten class $S^p(S^p)$ is the space of those operators x from ℓ_2 to S^p such that $\|x\|_p^p = \operatorname{tr}(\operatorname{tr} |x|^p) < \infty$, where the inner trace is the S^p-valued analogue of the usual trace. The S^p-valued entry space $S^p_I(S^p)$ is the closed subspace spanned by the $x_q \mathrm{e}_q$ with $x_q \in S^p$ and $q \in I$: $x_q = \operatorname{tr} \mathrm{e}_q^* x$ is the operator coefficient of x at matrix coordinate q. Thus, for even integers p and $x = (x_q)_{q\in I} = \sum_{q\in I} x_q \mathrm{e}_q$ with $x_q \in S^p$ and I finite,

$$\|x\|_p^p = \sum_{q_1,\dots q_p \in I} \operatorname{tr} x_{q_1}^* x_{q_2} \dots x_{q_{p-1}}^* x_{q_p} \operatorname{tr} \mathrm{e}_{q_1}^* \mathrm{e}_{q_2} \dots \mathrm{e}_{q_{p-1}}^* e_{q_p}.$$

A Schur multiplier T on S^p_I associated to $(\mu_q)_{q\in I} \in \mathbb{C}^I$ is a bounded operator on S^p_I such that $T\mathrm{e}_q = \mu_q \mathrm{e}_q$ for $q \in I$. T is furthermore completely bounded (c.b. for short) if T is bounded as the operator on $S^p_I(S^p)$ defined by $T(x_q \mathrm{e}_q) = \mu_q x_q \mathrm{e}_q$ for $x_q \in S^p$ and $q \in I$.

We shall stick to this harmonic analysis type notation; let us nevertheless show how these objects are termed with tensor products: $S^p(S^p)$ is also $S^p(\ell_2 \otimes_2 \ell_2)$ endowed with $\|x\|_p^p = \operatorname{tr} \otimes \operatorname{tr} |x|^p$; one should write $x_q \otimes \mathrm{e}_q$ instead of $x_q \mathrm{e}_q$; here $x_q = \mathrm{Id}_{S^p} \otimes \operatorname{tr}((\mathrm{Id}_{\ell_2} \otimes \mathrm{e}_q^*)x)$; T is c.b. if $\mathrm{Id}_{S^p} \otimes T$ is bounded on $S^p(\ell_2 \otimes_2 \ell_2)$.

Acknowledgement. The first-named and last-named authors undertook this research at the Équipe d'Analyse de l'Université Paris 6. It is their pleasure to acknowledge its kind hospitality.

2 Definitions

We use the notion of unconditionality in order to define the matrix analogue of Rudin's "commutative" $\Lambda(p)$ sets.

Definition 2.1. Let X be a Banach space. The sequence $(y_n) \subseteq X$ is an unconditional basic sequence in X if there is a constant D such that

$$\Big\|\sum \vartheta_n c_n y_n\Big\|_X \leqslant D \Big\|\sum c_n y_n\Big\|_X$$

for every real (vs. complex) choice of signs $\vartheta_n \in \{-1, 1\}$ (vs. $\vartheta_n \in \mathbb{T}$) and every finitely supported sequence of scalar coefficients (c_n). The optimal D is the real (vs. complex) unconditionality constant of (y_n) in X.

Real and complex unconditionality are isomorphically equivalent: the complex unconditionality constant is at most $\pi/2$ times the real one. The notions of unconditionality and multipliers are intimately connected: we have

Proposition 2.2. *Let $(y_n) \subseteq X$ be an unconditional basic sequence in X and let Y be the closed subspace of X spanned by (y_n). The real (vs. complex) unconditionality constant of (y_n) in X is exactly the least upper bound for the norms $\|T\|_{\mathscr{L}(Y)}$, where T is the multiplication operator defined by $Ty_n = \mu_n y_n$, and the μ_n range over all real (vs. complex) numbers with $|\mu_n| \leqslant 1$.*

Let us encompass the notions proposed in Question (†).

Definition 2.3. Let $I \subseteq R \times C$ and $p > 2$.

(a) [37, Def. 4.1] I is a $\sigma(p)$ set if $(\mathrm{e}_q)_{q\in I}$ is an unconditional basic sequence in S^p. This amounts to the uniform boundedness of the family of all relative Schur multipliers by signs

$$T_\vartheta\colon S^p_I \to S^p_I,\ x=(x_q)_{q\in I} \mapsto T_\vartheta x = (\vartheta_q x_q)_{q\in I} \text{ with } \vartheta_q \in \{-1,1\}. \tag{A.1}$$

By [37, Lemma 0.5], this means that there is a constant D such that for every finitely supported operator $x=(x_q)_{q\in I}=\sum_{q\in I} x_q \mathrm{e}_q$ with $x_q\in\mathbb{C}$

$$D^{-1}\|x\|_p \leqslant |\!|\!|x|\!|\!|_p \leqslant \|x\|_p, \tag{A.2}$$

where the second inequality is a convexity inequality that is always satisfied (see [95, Theorem 8.9]) and

$$|\!|\!|x|\!|\!|_p^p = \sum_c \Bigl(\sum_r |x_{rc}|^2\Bigr)^{p/2} \vee \sum_r \Bigl(\sum_c |x_{rc}|^2\Bigr)^{p/2}. \tag{A.3}$$

(b) [37, Def. 4.4] I is a complete $\sigma(p)$ set if the family of all relative Schur multipliers by signs (A.1) is uniformly c.b. By [37, Lemma 0.5], I is completely $\sigma(p)$ if and only if there is a constant D such that for every finitely supported operator valued operator $x=(x_q)_{q\in I}=\sum_{q\in I} x_q\mathrm{e}_q$ with $x_q\in S^p$

$$D^{-1}\|x\|_p \leqslant |\!|\!|x|\!|\!|_p \leqslant \|x\|_p, \tag{A.4}$$

where the second inequality is a convexity inequality that is always satisfied and

$$|\!|\!|x|\!|\!|_p^p = \sum_c \Bigl\|\Bigl(\sum_r x_{rc}^* x_{rc}\Bigr)^{1/2}\Bigr\|_p^p \vee \sum_r \Bigl\|\Bigl(\sum_c x_{rc} x_{rc}^*\Bigr)^{1/2}\Bigr\|_p^p.$$

The notion of a complete $\sigma(p)$ set is stronger than that of a $\sigma(p)$ set: Inequality (A.2) amounts to Inequality (A.4) tested on operators of the type $x=\sum_{q\in I} x_q\mathrm{e}_q$ with each x_q acting on the same one-dimensional subspace of ℓ_2. It is an important open problem to decide whether the notions differ. An affirmative answer would solve Pisier's conjecture about completely bounded Schur multipliers [78, p. 113].

Notorious examples of 1-unconditional basic sequences in all Schatten classes S^p are single columns, single rows, single diagonals and single anti-diagonals — and more generally "column sets" (vs. "row sets") I such that for each $(r,c)\in I$, no other element of I is in the column c (vs. row r). These sets are called sections in [99, Def. 4.3]

We shall try to express these notions in terms of trails on bipartite graphs. We proceed as announced in the Introduction: then each example above is a union of disjoint star graphs in which one vertex of one class is connected to some vertices of the other class: trails in a star graph have at most length 2.

Definition 2.4. Let $I\subseteq R\times C$ and $s\geqslant 1$ an integer. We consider I as a bipartite graph: its vertex set is $V=R\amalg C$ and its edge set is $E=\bigl\{\{r,c\}\subseteq V : (r,c)\in I\bigr\}$.

(a) The sets of trails of length s on the graph I from the column (vs. row) vertex v_0 to the vertex v_s are respectively

$$\mathscr{C}^s(I;v_0,v_s) = \{(v_0,\dots,v_s)\in V^{s+1} : v_0\in C \ \&\ \text{all } \{v_i,v_{i+1}\}\in E \text{ are distinct}\},$$

$$\mathscr{R}^s(I;v_0,v_s) = \{(v_0,\dots,v_s)\in V^{s+1} : v_0\in R \ \&\ \text{all } \{v_i,v_{i+1}\}\in E \text{ are distinct}\}.$$

(b) We define the Rudin numbers of trails starting respectively with a column vertex and a row vertex by $c_s(I;v_0,v_s)=\#\mathscr{C}^s(I;v_0,v_s)$ and $r_s(I;v_0,v_s)=\#\mathscr{R}^s(I;v_0,v_s)$.

Remark 2.5. In other words, for an integer $l\geqslant 1$,

$$\begin{aligned} c_{2l-1}(I;v_0,v_{2l-1}) &= \# \begin{bmatrix} (r_1,c_1),(r_1,c_2),(r_2,c_2),(r_2,c_3),\dots,(r_l,c_l) \\ \text{pairwise distinct in } I : c_1=v_0, r_l=v_{2l-1} \end{bmatrix} \\ c_{2l}(I;v_0,v_{2l}) &= \# \begin{bmatrix} (r_1,c_1),(r_1,c_2),\dots,(r_l,c_l),(r_l,c_{l+1}) \\ \text{pairwise distinct in } I : c_1=v_0, c_{l+1}=v_{2l} \end{bmatrix} \end{aligned}$$

and similarly for $r_s(I;v_0,v_s)$. If s is odd, then $c_s(I;v_0,v_s)=r_s(I;v_s,v_0)$ for all $(v_0,v_s)\in C\times R$. But if s is even, one Rudin number may be bounded while the other is infinite: see [37, Rem. 6.4(ii)].

3 $\sigma(p)$ sets as matrix $\Lambda(p)$ sets

We claim the following result.

Theorem 3.1. *Let $I \subseteq R \times C$ and $p = 2s$ be an even integer. If I is a union of sets $I_1, \dots, I_l$ such that one of the Rudin numbers $c_s(I_j; v_0, v_s)$ or $r_s(I_j; v_0, v_s)$ is a bounded function of (v_0, v_s), for each j, then I is a complete $\sigma(p)$ set.*

This follows from Theorem 3.2 below: the union of two complete $\sigma(p)$ sets is a complete $\sigma(p)$ set by [37, Rem. after Def. 4.4]; furthermore the transposed set $I^* = \{\, q^* : q \in I \,\} \subseteq C \times R$ is a complete $\sigma(p)$ set provided I is. Note that the case of $\sigma(\infty)$ sets (see [37, Rem. 4.6(iii)]) provides evidence that Theorem 3.1 might be a characterisation of complete $\sigma(p)$ sets for even p.

Theorem 3.2. *Let $I \subseteq R \times C$ and $p = 2s$ be an even integer. If the Rudin number $c_s(I; v_0, v_s)$ is a bounded function of (v_0, v_s), then I is a complete $\sigma(p)$ set.*

This is proved for $p = 4$ in [37, Prop. 6.5]. We wish to emphasise that the proof below follows the scheme of the proof of [37, Theorem 1.13]. In particular, we make crucial use of Pisier's idea to express repetitions by dependent Rademacher variables ([37, Prop. 1.14]).

Proof. Let $x = \sum_{q \in I} x_q \mathrm{e}_q$ with $x_q \in S^p$. We have the following expression for $\|x\|_p$.

$$\|x\|_p^p = \operatorname{tr} \otimes \operatorname{tr}(x^* x)^s = \|y\|_2^2 \quad \text{with} \quad y = \overbrace{x^* x x^* \cdots x^{(*)}}^{s \text{ terms}},$$

i.e., y is the product of s terms which are alternatively x^* and x, and we set $x^{(*)} = x$ for even s, $x^{(*)} = x^*$ for odd s. Set $C^{(*)} = C$ for even s and $C^{(*)} = R$ for odd s. Let $(v_0, v_s) \in C \times C^{(*)}$ and $y_{v_0 v_s} = \operatorname{tr} \mathrm{e}^*_{v_0 v_s} y$ be the matrix coefficient of y at coordinate (v_0, v_s). Then we obtain by the rule of matrix multiplication

$$y = \sum_{q_1, \dots, q_s \in I} (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s})$$

$$y_{v_0 v_s} = \sum_{(v_1, v_0),\, (v_1, v_2), \dots \in I} x^*_{v_1 v_0} x_{v_1 v_2} x^*_{v_3 v_2} \dots x^{(*)}_{(v_{s-1}, v_s)^{(*)}}. \tag{A.5}$$

Let $\mathscr{E}$ be the set of equivalence relations on $\{1, \dots, s\}$. Then

$$y = \sum_{\sim \in \mathscr{E}} \sum_{i \sim j \Leftrightarrow q_i = q_j} (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s}). \tag{A.6}$$

We shall bound the sum above in two steps.

(a) Let $\sim$ be equality and consider the corresponding term in the sum (A.6). The number of terms in the sum (A.5) such that $\{v_{i-1}, v_i\} \neq \{v_{j-1}, v_j\}$ if $i \neq j$ is $c_s(I; v_0, v_s)$. If c is an upper bound for $c_s(I; v_0, v_s)$, we have by the expression of the Hilbert–Schmidt norm and the Arithmetic-Quadratic Mean Inequality

$$\begin{aligned}
&\Big\| \sum_{\substack{q_1, \dots, q_s \\ \text{pairwise distinct}}} (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s}) \Big\|_2^2 \\
&\quad = \sum_{(v_0, v_s) \in C \times C^{(*)}} \Big\| \sum_{v \in \mathscr{C}^s(I; v_0, v_s)} x^*_{v_1 v_0} x_{v_1 v_2} x^*_{v_3 v_2} \dots x^{(*)}_{(v_{s-1}, v_s)^{(*)}} \Big\|_2^2 \\
&\quad \leqslant c \sum_{(v_0, v_s) \in C \times C^{(*)}} \sum_{v \in \mathscr{C}^s(I; v_0, v_s)} \Big\| x^*_{v_1 v_0} x_{v_1 v_2} x^*_{v_3 v_2} \dots x^{(*)}_{(v_{s-1}, v_s)^{(*)}} \Big\|_2^2 \\
&\quad = c \sum_{\substack{q_1, \dots, q_s \\ \text{pairwise distinct}}} \big\| (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s}) \big\|_2^2 \\
&\quad \leqslant c \sum_{q_1, \dots, q_s} \big\| (x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s}) \big\|_2^2 \\
&\quad = c \Big\| \sum_{q_1, \dots, q_s} |(x^*_{q_1} \mathrm{e}^*_{q_1})(x_{q_2} \mathrm{e}_{q_2}) \dots (x^{(*)}_{q_s} \mathrm{e}^{(*)}_{q_s})|^2 \Big\|_1
\end{aligned}$$

Now this last expression may be bounded accordingly to [37, Cor. 0.9] by

$$c\Big(\Big\|\sum(x_q^*\mathrm{e}_q^*)(x_q\mathrm{e}_q)\Big\|_s \vee \Big\|\sum(x_q\mathrm{e}_q)(x_q^*\mathrm{e}_q^*)\Big\|_s\Big)^s = c\|\!|x|\!\|_p^p : \tag{A.7}$$

see [37, Lemma 0.5] for the last equality.

(*b*) Let $\sim$ be distinct from equality. The corresponding term in the sum (A.6) cannot be bounded directly. Consider instead

$$\Psi(\sim) = \Big\|\sum_{i\sim j\Rightarrow q_i=q_j}(x_{q_1}^*\mathrm{e}_{q_1}^*)(x_{q_2}\mathrm{e}_{q_2})\dots(x_{q_s}^{(*)}\mathrm{e}_{q_s}^{(*)})\Big\|_2 = \Big\|\sum_{i\sim j\Rightarrow q_i=q_j}\prod_{i=1}^s f_i(q_i)\Big\|_2$$

with $f_i(q) = x_q\mathrm{e}_q$ for even i and $f_i(q) = x_q^*\mathrm{e}_q^*$ for odd i. We may now apply Pisier's Lemma [37, Prop. 1.14]: let $0 \leqslant r \leqslant s-2$ be the number of one element equivalence classes modulo $\sim$; then

$$\Psi(\sim) \leqslant \|x\|_p^r(B\|\!|x|\!\|_p)^{s-r}, \tag{A.8}$$

where B is the constant arising in Lust-Piquard's non-commutative Khinchin inequality. In order to finish the proof, one does an induction on the number of atoms of the partition induced by $\sim$, along the lines of step 2 of the proof of [37, Theorem 1.13]. □

The Moebius inversion formula for partitions enabled Pisier [77] to obtain the following explicit bounds in the computation above:

$$\|y\|_2 \leqslant c^{1/2}\|\!|x|\!\|_p^s + \sum_{0\leqslant r\leqslant s-2}\binom{s}{r}(s-r)!\|x\|_p^r\big((3\pi/4)\|\!|x|\!\|_p\big)^{s-r}$$

$$\|x\|_p \leqslant \big((4c)^{1/p} \vee 9\pi p/8\big)\|\!|x|\!\|_p. \tag{A.9}$$

Let us also record the following consequence of his study of p-orthogonal sums. The family $(x_q\mathrm{e}_q)_{q\in I}$ is p-orthogonal in the sense of [77] if and only if the graph associated to I does not contain any circuit of length p, so that we have by [77, Theorem 3.1]:

Theorem 3.3. *Let $p \geqslant 4$ be an even integer. If I does not contain any circuit of length p, then I is a complete $\sigma(p)$ set with constant at most* $3\pi p/2$.

Remark 3.4. Pisier proposed to us the following argument to deduce a weaker version of Theorem 3.2 from [37, Theorem 1.13]. Let $\Gamma = \mathbb{T}^V$ and z_v denote the vth coordinate function on Γ. Associate to I the set $\Lambda = \{\, z_r z_c : (r,c) \in I \,\}$. Let still $p = 2s$ be an even integer. Then I is a complete $\sigma(p)$ set if Λ is a complete $\Lambda(p)$ set as defined in [37, Def. 1.5], which in turn holds if Λ has property $Z(s)$ as given in [37, Def. 1.11]. It turns out that this condition implies the uniform boundedness of

$$c_t(I; v_0, v_t) \vee r_t(I; v_0, v_t) \quad \text{for } t \leqslant s,\ v_0, v_t \in V.$$

For $p \geqslant 8$, this implication is strict: in fact, the countable union of disjoint cycles of length 4 ("quadrilaterals")

$$I = \bigcup\nolimits_{i\geqslant 0}\big\{(2i,2i),(2i,2i+1),(2i+1,2i+1),(2i+1,2i)\big\}$$

satisfies $c_t(I; v_0, v_t) \vee r_t(I; v_0, v_t) \leqslant 2$ whereas Λ does not satisfy $Z(s)$ for any $s \geqslant 4$.

Remark 3.5. Theorem 3.1 is especially useful to construct c.b. Schur multipliers: by [37, Rem. 4.6(*ii*)], if I is a complete $\sigma(p)$ set, there is a constant D (the constant D in (A.4)) such that for every sequence $(\mu_q) \in \mathbb{C}^{R\times C}$ supported by I and every operator $T_\mu : (x_q) \mapsto (\mu_q x_q)$ we have

$$\|T_\mu\|_{\mathscr{L}(S^p(S^p))} \leqslant D \sup_{q\in I}|\mu_q|.$$

4 The intersection of a $\sigma(p)$ set with a finite product set

Let $I \subseteq R\times C$ considered as a bipartite graph as in the Introduction and let $I' \subseteq I$ be the subgraph induced by the vertex set $C' \amalg R'$, with $C' \subseteq C$ a set of m column vertices and $R' \subseteq R$ a set of n row vertices. In other words, $I' = I \cap R' \times C'$. Let $d(v)$ be the degree of the vertex $v \in C' \amalg R'$ in I': in other words,

$$\begin{aligned} \forall c \in C' \quad & d(c) = \#[I' \cap R' \times \{c\}], \\ \forall r \in R' \quad & d(r) = \#[I' \cap \{r\} \times C']. \end{aligned}$$

Let us recall that the dual norm of (A.3) is

$$|||x|||_{p'} = \inf_{\substack{\alpha,\beta \in S^{p'} \\ \alpha+\beta=x}} \Bigl(\sum_c \Bigl(\sum_r |\alpha_{rc}|^2\Bigr)^{p'/2}\Bigr)^{1/p'} + \Bigl(\sum_r \Bigl(\sum_c |\beta_{rc}|^2\Bigr)^{p'/2}\Bigr)^{1/p'},$$

where $p \geqslant 2$ and $1/p + 1/p' = 1$ (see [37, Rem. after Lemma 0.5]).

Lemma 4.1. *Let $1 \leqslant p' \leqslant 2$ and $x = \sum_{q\in I'} x_q$. Then*

$$|||x|||_{p'}^{p'} \geqslant \sum\sum_{(r,c)\in I'} \Bigl(\max\bigl(d(c), d(r)\bigr)^{1/2-1/p'} |x_{rc}|\Bigr)^{p'}.$$

Proof. By the p'-Quadratic Mean Inequality and by Minkowski's Inequality,

$$\begin{aligned} &\Bigl(\sum_{c\in C'}\Bigl(\sum_{(r,c)\in I'} |\alpha_{rc}|^2\Bigr)^{p'/2}\Bigr)^{1/p'} + \Bigl(\sum_{r\in R'}\Bigl(\sum_{(r,c)\in I'} |\beta_{rc}|^2\Bigr)^{p'/2}\Bigr)^{1/p'} \\ &\geqslant \Bigl(\sum_{c\in C'} d(c)^{p'/2-1} \sum_{(r,c)\in I'} |\alpha_{rc}|^{p'}\Bigr)^{1/p'} + \Bigl(\sum_{r\in R'} d(r)^{p'/2-1} \sum_{(r,c)\in I'} |\beta_{rc}|^{p'}\Bigr)^{1/p'} \\ &\geqslant \Bigl(\sum\sum_{(r,c)\in I'} \bigl(d(c)^{1/2-1/p'}|\alpha_{rc}| + d(r)^{1/2-1/p'}|\beta_{rc}|\bigr)^{p'}\Bigr)^{1/p'} \end{aligned}$$

The lemma follows by taking the infimum over all α, β with $\alpha_q + \beta_q = x_q$ for $q \in I'$ as one can suppose that $\alpha_q = \beta_q = 0$ if $q \notin I$; note further that $1/2 - 1/p' \leqslant 0$. $\square$

Theorem 4.2. *If I is a $\sigma(p)$ set with constant D as in* (A.2)*, then the size $\#I'$ of any subgraph I' induced by m column vertices and n row vertices, in other words the cardinal of any subset $I' = I \cap R' \times C'$ with $\#C' = m$ and $\#R' = n$, satisfies*

$$\begin{aligned} \#I' &\leqslant D^2\bigl(m^{1/p}n^{1/2} + m^{1/2}n^{1/p}\bigr)^2 \qquad \text{(A.10)} \\ &\leqslant 4D^2 \min(m,n)^{2/p} \max(m,n). \end{aligned}$$

The exponents in this inequality are optimal even for a complete $\sigma(p)$ set I in the following cases:
(a) if m or n is fixed;
(b) if p is an even integer and $m = n$.

Bound (A.10) holds *a fortiori* if I is a complete $\sigma(p)$ set. Density conditions thus do not so far permit to distinguish $\sigma(p)$ sets and complete $\sigma(p)$ sets. One may conjecture that Inequality (A.10) is also optimal for p not an even integer and $m = n$: this would be a matrix counterpart to Bourgain's theorem [14] on maximal $\Lambda(p)$ sets.

Proof. If (A.2) holds, then $\|x|_{I'}\|_p \leqslant D|||x|||_p$ for all $x \in S^p$ by Remark 3.5 applied to (μ_q) the indicator function of I', and by duality $|||x|_{I'}|||_{p'} \leqslant D\|x\|_{p'}$ for all $x \in S^{p'}$ (compare with [37, Rem. 4.6(iv)]). Let

$$\begin{aligned} y &= \sum\sum_{(r,c)\in I'} d(c)^{1/p'-1/2} \mathrm{e}_{rc}, \\ z &= \sum\sum_{(r,c)\in I'} d(r)^{1/p'-1/2} \mathrm{e}_{rc}, \end{aligned}$$

Then the n rows of y are all equal, as well as the m columns of z: y and z have rank 1 and a single singular value. By the norm inequality followed by the $(2/p'-1)$-Arithmetic Mean Inequality,

$$\begin{aligned} \|y+z\|_{p'} &\leqslant \|y\|_{p'} + \|z\|_{p'} \\ &= n^{1/2}\Big(\sum_{c\in C'} d(c)^{2/p'-1}\Big)^{1/2} + m^{1/2}\Big(\sum_{r\in R'} d(r)^{2/p'-1}\Big)^{1/2} \\ &\leqslant n^{1/2}m^{1-1/p'}(\#I')^{1/p'-1/2} + m^{1/2}n^{1-1/p'}(\#I')^{1/p'-1/2}. \end{aligned}$$

We used that $\sum_{c\in C'} d(c) = \sum_{r\in R'} d(r) = \#I'$. By Lemma 4.1 applied to $x = y+z$,

$$(\#I')^{1/p'} \leqslant D(n^{1/2}m^{1-1/p'} + m^{1/2}n^{1-1/p'})(\#I')^{1/p'-1/2},$$

and we get therefore the first part of the theorem.

Let us show optimality in the given cases.

(a) Suppose that n is fixed and $C' = C$: $I' = R' \times C$ is a complete $\sigma(p)$ set for any p as a union of n rows and $\#I' = n\cdot m$.

(b) is proved in [37, Theorem 4.8]. □

Remark 4.3. If $n \not\approx m$, the method used in [37, Theorem 4.8] does not provide optimal $\sigma(p)$ sets but the following lower bound. Let $p = 2s$ with $s \geqslant 2$ an integer. Consider a prime q and let $k = s^{s-1}q^s$. By [87, 4.7] and [37, Theorem 2.5], there is a subset $F \subseteq \{0,\dots,k-1\}$ with q elements whose complete $\Lambda(2s)$ constant is independent of q. Let $m \geqslant k$ and $0 \leqslant n \leqslant m$ and consider the Hankel set

$$I = \big\{\,(r,c) \in \{0,\dots,n-1\}\times\{0,\dots,m-1\} : r+c \in F+m-k\,\big\}.$$

Then the complete $\sigma(p)$ constant of I is independent of q by [37, Prop. 4.7] and

$$\#I \geqslant \begin{cases} nq & \text{if } n \leqslant m-k+1 \\ (m-k+1)q & \text{if } n \geqslant m-k+1. \end{cases}$$

If we choose $m = (s+1)k-1$, this yields

$$\#I \geqslant \frac{s^{1/s}}{(s+1)^{1+1/s}} \min(n,m)\max(m,n)^{1/s}.$$

Random construction 6.1 provides bigger sets than this deterministic construction; however, it also does not provide sets that would show the optimality of Inequality (A.10) unless $s = 2$.

5 Circuits in graphs

Non-commutative methods yield a new proof to a theorem of Erdős [29, p. 33]. Note that its generalisation by Bondy and Simonovits [12] is stronger than Theorem 5.1 below as it deals with cycles instead of circuits. By Theorem 3.3 and (A.10)

Theorem 5.1. *Let $p \geqslant 4$ be an even integer. If G is a nonempty graph on v vertices with e edges without circuit of length p, then*

$$e \leqslant 18\pi^2p^2\, v^{1+2/p}.$$

If G is furthermore a bipartite graph whose two vertex classes have respectively m and n elements, then

$$e \leqslant 9\pi^2p^2 \min(m,n)^{2/p}\max(m,n). \tag{A.11}$$

Proof. For the first assertion, recall that a graph G with e edges contains a bipartite subgraph with more than $e/2$ edges (see [11, p. xvii]). □

Remark 5.2. Łuczak showed to us that (A.11) cannot be optimal if m and n are of very different order of magnitude. In particular, let p be a multiple of 4. Let e' be the maximal number of edges of a graph on n vertices without circuit of length $p/2$. If $m > pe'$, he shows that (A.11) may be replaced by $e < 3m$.

We also get the following result, which enables us to conjecture a generalisation of the theorems of Erdős and Bondy and Simonovits.

Theorem 5.3. *Let G be a nonempty graph on v vertices with e edges. Let $s \geqslant 2$ be an integer.*

(i) If

$$e > 8D^2\, v^{1+1/s} \quad \text{with } D > 9\pi s/4,$$

then one may choose two vertices v_0 and v_s such that G contains more than $D^{2s}/4$ pairwise distinct trails from v_0 to v_s, each of length s and with pairwise distinct edges.

(ii) One may draw the same conclusion if G is a bipartite graph whose two vertex classes have respectively m and n elements and

$$e > 4D^2 \min(m,n)^{1/s} \max(m,n) \quad \text{with } D > 9\pi s/4.$$

Proof. (i) According to [11, p. xvii], the graph G contains a bipartite subgraph with more than $e/2$ edges, so that we may apply (ii).

(ii) Combining inequalities (A.9) and (A.10), if $D > 9\pi s/4$, then there are vertices v_0 and v_s such that the number c of pairwise distinct trails from v_0 to v_s, each of length s and with pairwise distinct edges, satisfies $(4c)^{1/2s} > D$. □

Two paths with equal endvertices are called independent if they have only their endvertices in common.

Question 5.4. Let G be a graph on v vertices with e edges. Let $s, l \geqslant 2$ be integers. Is it so that there is a constant D such that if $e > Dv^{1+1/s}$, then G contains l pairwise independent paths of length s with equal endvertices?

Remark 5.5. Note that by Theorem 4.2, the exponent $1+1/s$ is optimal in Theorem 5.3(i), whereas optimality of the exponent $1+2/p$ in Theorem 5.1 is an important open question in Graph Theory (see [49]).

One may also formulate Theorem 5.3(ii) in the following way.

Theorem 5.6. *If a bipartite graph $G_2(n,m)$ with n and m vertices in its two classes avoids any union of c pairwise distinct trails along s pairwise distinct edges between two given vertices as a subgraph, where the class of the first vertex is fixed, then the size e of the graph satisfies*

$$e \leqslant 4 \max((4c)^{1/2s}, 9\pi s/4)) \min(m,n)^{1/s} \max(m,n).$$

6 A random construction of graphs

Let us precise our construction of a random graph.

Random construction 6.1. *Let C, R be two sets such that $\#C = m$ and $\#R = n$. Let $0 \leqslant \alpha \leqslant 1$. A random bipartite graph on $V = C \amalg R$ is defined by selecting independently each edge in $E = \big\{ \{r,c\} \subseteq V : (r,c) \in R \times C \big\}$ with the same probability α. The resulting random edge set is denoted by $E' \subseteq E$ and $I' \subseteq R \times C$ denotes the associated random subset.*

Our aim is to construct large sets while keeping down the Rudin number c_s.

Theorem 6.2. *For each $\varepsilon > 0$ and for each integer $s \geqslant 2$, there is an α such that Random construction 6.1 yields subsets $I' \subseteq R \times C$ with size*

$$\#I' \sim \min(m,n)^{1/2+1/s} \max(m,n)^{1/2-\varepsilon}$$

and with $\sigma(2s)$ constant independent of m and n for $mn \to \infty$.

Proof. Let us suppose without loss of generality that $m \geqslant n$. We want to estimate the Rudin number of trails in I'. Set $C^{(*)} = C$ for even s, $C^{(*)} = R$ for odd s and let $(v_0, v_s) \in C \times C^{(*)}$. Let $l \geqslant 1$ be a fixed integer. Then

$$\begin{aligned}
\mathbb{P}[c_s(I'; v_0, v_s) \geqslant l] &= \mathbb{P}[\exists\ l \text{ distinct trails } (v_0^j, \dots, v_s^j) \text{ in } \mathscr{C}^s(I'; v_0, v_s)] \\
&= \mathbb{P}\big[\, E' \supseteq \big\{\{v_{i-1}^j, v_i^j\}\big\}_{i,j} : \{(v_0^j, \dots, v_s^j)\}_{j=1}^l \subseteq \mathscr{C}^s(R \times C; v_0, v_s) \,\big] \\
&\leqslant \sum_{k=\lceil l^{1/s} \rceil}^{ls} \#A_k \cdot \alpha^k,
\end{aligned}$$

where A_k is the following set of l-element subsets of trails in $\mathscr{C}^s(R \times C; v_0, v_s)$ built with k pairwise distinct edges

$$A_k = \Big\{ \{(v_0^j, \ldots, v_s^j)\}_{j=1}^l \subseteq \mathscr{C}^s(R \times C; v_0, v_s) : \#\big\{\{v_{i-1}^j, v_i^j\}\big\}_{i,j} = k \Big\};$$

the lower limit of summation is $\lceil l^{1/s} \rceil$ because one can build at most k^s pairwise distinct trails of length s with k pairwise distinct edges.

In order to estimate $\#A_k$, we now have to bound the number of pairwise distinct vertices and the number of pairwise distinct column vertices in each set of l trails $\{(v_0^j, \ldots, v_s^j)\}_{j=1}^l \in A_k$. We claim that

$$\#\{ v_i^j : 1 \leqslant i \leqslant s-1, 1 \leqslant j \leqslant l \} \quad \leqslant \quad k(s-1)/s, \tag{A.12}$$

$$\#\{ v_{2i}^j : 1 \leqslant i \leqslant \lceil s/2 \rceil - 1, 1 \leqslant j \leqslant l \} \quad \leqslant \quad k/2. \tag{A.13}$$

The second estimate is trivial, because each column vertex v_{2i}^j accounts for two distinct edges $\{v_{2i-1}^j, v_{2i}^j\}$ and $\{v_{2i}^j, v_{2i+1}^j\}$. For the first estimate, note that each maximal sequence of h consecutive pairwise distinct vertices $(v_{a+1}^j, \ldots, v_{a+h}^j)$ accounts for $h+1$ pairwise distinct edges

$$\{v_a^j, v_{a+1}^j\}, \{v_{a+1}^j, v_{a+2}^j\}, \ldots, \{v_{a+h}^j, v_{a+h+1}^j\};$$

as $h \leqslant s-1$, $h+1 \geqslant hs/(s-1)$. By (A.12) and (A.13),

$$\#A_k \leqslant m^{k/2} n^{k/2-k/s} (k-k/s)^{ls-l} \leqslant (ls)^{ls} m^{k/2} n^{k/2-k/s} :$$

each element of A_k is obtained by a choice of at most $k - k/s$ vertices, of which at most $k/2$ are column vertices, and the choice of an arrangement with repetitions of $ls - l$ out of at most $k - k/s$ vertices.

Put $\alpha = m^{-1/2} n^{-1/2+1/s} (\#C \cdot \#C^{(*)})^{-\varepsilon}$. Then

$$\begin{aligned} \mathbb{P}[\sup_{(v_0,v_s)} c_s(I'; v_0, v_s) \geqslant l] \quad &\leqslant \quad \#C \cdot \#C^{(*)} \cdot (ls)^{ls} \sum_{k=\lceil l^{1/s} \rceil}^{ls} (\#C \cdot \#C^{(*)})^{-k\varepsilon} \\ &\leqslant \quad (ls)^{ls} \frac{(\#C \cdot \#C^{(*)})^{1-\lceil l^{1/s} \rceil \varepsilon}}{1 - (\#C \cdot \#C^{(*)})^{-\varepsilon}}. \end{aligned}$$

Choose l such that $\lceil l^{1/s} \rceil \varepsilon > 1$. Then this probability is little for mn large. On the other hand, $\#I'$ is of order $mn\alpha$ with probability close to 1. $\square$

Remark 6.3. This construction yields much better results for $s = 2$. Keeping the notation of the proof above and $m \geqslant n$, we get $k = 2l$, $A_k = \binom{n}{l}$ and

$$\mathbb{P}[\sup_{(v_0,v_2) \in C \times C} c_2(I'; v_0, v_2) \geqslant l] \leqslant m^2 \binom{n}{l} \alpha^{2l}.$$

Let $l \geqslant 2$ and $\alpha = m^{-1/l} n^{-1/2}$. This yields sets $I' \subseteq R \times C$ with size

$$\#I' \sim n^{1/2} m^{1-1/l}$$

and with $\sigma(4)$ constant independent of m and n. This case has been extensively studied in Graph theory as the "Zarankiewicz problem:" if $c_2(I'; v_0, v_2) \leqslant l$ for all $v_0, v_2 \in C$, then the graph I' does not contain a complete bipartite subgraph on any two column vertices v_0, v_2 and $l+1$ row vertices. Reiman (see [11, Theorem VI.2.6]) showed that then

$$\#I' \leqslant \big(lnm(m-1) + n^2/4\big)^{1/2} + n/2 \sim l^{1/2} n^{1/2} m.$$

With use of finite projective geometries, he also showed that this bound is optimal for

$$n = l \frac{q^{r+1}-1}{q^2-1} \frac{q^r - 1}{q-1} \quad , \quad m = \frac{q^{r+1}-1}{q-1}$$

with q a prime power and $r \geqslant 2$ an integer, and thus with $m \leqslant n$: there seems to be no constructive example of extremal graphs with $c_2(I'; v_0, v_2) \leqslant l$ and $m > n$ besides the trivial case of complete bipartite graphs with $m > n = l - 1$.

Remark 6.4. In the case $s = 3$, our result cannot be improved just by refining the estimation of $\#A_k$. If we consider first l distinct paths that have their second vertex in common and then l independent paths, we get

$$\#A_{2l+1} \geqslant \binom{m}{l} n \quad , \quad \#A_{3l} \geqslant \binom{m}{l}\binom{n}{l}.$$

Therefore any choice of α as a monomial $m^{-t}n^{-u}$ in the proof above must satisfy $t \geqslant (l+1)/(2l+1)$, $t + u \geqslant (2l+2)/(3l)$ and this yields sets with

$$\#I' \preccurlyeq m^{1/2-1/2(4l+2)} n^{5/6-(7l+6)/(12l^2+6l)}.$$

Chapter B

Cycles and 1-unconditional matrices

1 Introduction

The starting point for this investigation has been the following isometric question on the Schatten-von-Neumann class S^p.

Question 1.1. Which matrix coefficients of an operator $x \in \mathrm{S}^p$ must vanish so that the norm of x does not depend on the argument, or on the sign, of the remaining nonzero matrix coefficients?

Let C be the set of columns and R be the set of rows for coordinates in the matrix. Let $I \subseteq R \times C$ be the set of matrix coordinates of the nonzero matrix coefficients of x (the *pattern*.) Question 1.1 describes the notion of a complex, or real, 1-unconditional basic sequence $(\mathrm{e}_{rc})_{(r,c)\in I}$ of elementary matrices in S^p (see Definition 4.1.)

By a convexity argument, Question 1.1 is equivalent to the following question on Schur multiplication.

Question 1.2. Which matrix coefficients of an operator $x \in \mathrm{S}^p$ must vanish so that for all matrices φ of complex, or real, numbers

$$\|\varphi * x\| \leqslant \sup |\varphi_{rc}| \; \|x\|,$$

where $\varphi * x$ is the Schur (or Hadamard or entrywise) product defined by

$$(\varphi * x)_{rc} = \varphi_{rc} x_{rc}?$$

In the case $p = \infty$, Grothendieck's inequality yields an estimation for the norm of Schur multiplication by φ in terms of the projective tensor product $\ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$: this norm is equivalent to the supremum of the norm of those elements of $\ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$ whose coefficient matrices are finite submatrices of φ. In the framework of tensor algebras over discrete spaces, Question 1.2 turns out to describe as well the isometric counterpart to Varopoulos' V-Sidon sets as well as to his sets of V-interpolation. The following isometric question has however a different answer.

Question 1.3. Which coefficients of a tensor $u \in \ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$ must vanish so that the norm of u is the maximal modulus of its coefficients?

In our answer to Question 1.2, S^p and Schur multiplication are treated as a noncommutative analogue to L^p and convolution. The main step is a careful study of the Schatten-von-Neumann norm $\|x\| = \left(\mathrm{tr}(x^*x)^{p/2}\right)^{1/p}$ for p an even integer. The rule of matrix multiplication provides an expression for this norm as a series in the matrix coefficients of x and their complex conjugate, indexed by the $p$uples $(v_1, v_2, \ldots, v_p)$ satisfying $(v_{2i-1}, v_{2i}), (v_{2i+1}, v_{2i}) \in I$, where $v_{p+1} = v_1$: see the computation in Eq. (B.11). These are best understood as closed walks of length p on the bipartite graph G canonically associated to I: its vertex classes are C and R and its edges are given by the couples in I. A structure theorem for closed walks and a detailed study of the particular case in which G is a cycle yield the two following theorems that answer Questions 1.1 and 1.2.

Theorem 1.4. *Let $p \in (0, \infty] \setminus \{2, 4, 6, \ldots\}$. If the sequence of elementary matrices $(\mathrm{e}_{rc})_{(r,c)\in I}$ is a real 1-unconditional basic sequence in S^p, then the graph G associated to I contains no cycle. In this* *case, I is even a set of* V*-interpolation with constant 1: every sequence $\varphi \in \ell^\infty_I$ may be interpolated by a tensor $u \in \ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$ such that $\|u\| = \|\varphi\|$.*

Theorem 1.5. *Let $p \in \{2, 4, 6, \ldots\}$. The sequence $(\mathrm{e}_{rc})_{(r,c)\in I}$ is a complex, or real, 1-unconditional basic sequence in S^p if and only if G contains no cycle of length $4, 6, \ldots, p$.*

These theorems hold also for the complete counterparts to 1-unconditional basic sequences in the sense of Def. 4.1(c).

In particular, if we denote by U_p the property that $(\mathrm{e}_{rc})_{(r,c)\in I}$ is a 1-unconditional basic sequence in S^p, then we obtain the following hierarchy:

$$\mathrm{U}_p \text{ for a } p \in (0, \infty] \setminus \{2, 4, 6, \ldots\} \;\Rightarrow\; \cdots \;\Rightarrow\; \mathrm{U}_{2n+2} \;\Rightarrow\; \mathrm{U}_{2n} \;\Rightarrow\; \cdots \;\Rightarrow\; \mathrm{U}_2\ .$$

If C and R are finite, extremal graphs without cycles of given lengths remain an ongoing area of research in graph theory. Finite geometries seem to provide all known examples of such graphs when C and R become large. Proposition 11.6 and Remark 11.7 gather up known facts on this issue.

One may also avoid the terminology of graph theory and give an answer in terms of polygons drawn in a matrix by joining matrix coordinates with sides that follow alternately the row (horizontal) and the column (vertical) direction of the matrix:

- Suppose that p is not an even integer. If a pattern I contains the vertices of such a polygon, then there is an operator $x \in \mathrm{S}^p$ whose matrix coefficients vanish outside I and whose norm depends on the sign of its matrix coefficients. This condition is also necessary.
- If matrix coordinates of nonzero matrix coefficients of x are the vertices of such a polygon with n sides, then the norm of x in S^p depends on the argument of its matrix coefficients for every even integer $p \geqslant n$; if the matrix coefficients of x are real, then the norm of x even depends on the sign of its matrix coefficients. These conditions are also necessary.

An elementary example is given by the set

$$I = \big\{(r, c) \in \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} : r + c \in \{0, 1, 3\}\big\}. \tag{B.1}$$

The associated bipartite graph is known as the Heawood graph (Fig. B.1:) it is the incidence graph of the Fano plane (the finite projective plane $\mathrm{PG}(2, 2)$,) which is the smallest generalised triangle, and corresponds to the Steiner system $\mathrm{S}(2, 3; 7)$. It contains no cycle of length 4, but every pair of vertices is contained in a cycle of length 6.

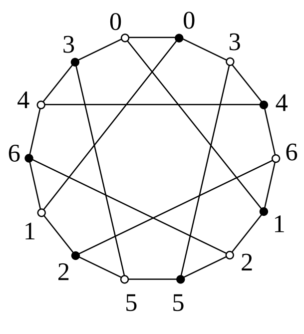


Figure B.1: The Heawood graph

Thus the p-trace norm of every matrix with pattern

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
* & * & 0 & * & 0 & 0 & 0 \\
* & 0 & * & 0 & 0 & 0 & * \\
0 & * & 0 & 0 & 0 & * & * \\
* & 0 & 0 & 0 & * & * & 0 \\
0 & 0 & 0 & * & * & 0 & * \\
0 & 0 & * & * & 0 & * & 0 \\
0 & * & * & 0 & * & 0 & 0
\end{pmatrix} \end{array}$$

does not depend on the sign of its coefficients if and only if $p \in \{2,4\}$.

These results give a complete description of the situation in which $(\mathrm{e}_{rc})_{(r,c)\in I}$ is a 1-unconditional basis of the space S^p_I it spans in S^p. If this is not the case, S^p_I might still admit some other 1-unconditional basis. This leads to the following more general question.

Question 1.6. For which sets I does S^p_I admit some kind of almost 1-unconditional finite dimensional expansion of the identity?

The metric unconditional approximation property (*muap*) provides a formal definition for the object of Question 1.6: see Def. 10.1. We obtain the following results.

Theorem 1.7. *Let $p \in [1,\infty] \setminus \{2,4,6,\ldots\}$. If S^p_I has real (muap), then the distance of any two vertices that are not in the same vertex class is asymptotically infinite in G: their distance becomes arbitrarily large by deleting a finite number of edges from G.*

Theorem 1.8. *Let $p \in \{2,4,6,\ldots\}$. The space S^p_I has complex, or real, (muap) if and only if any two vertices at distance $2j+1 \leqslant p/2$ are asymptotically at distance at least $p-2j+1$.*

We now turn to a detailed description of this article. In Section 2, we provide tools for the computation of Schur multiplier norms. Section 3 characterises idempotent Schur multipliers and 0, 1-tensors in $\ell^\infty_C \overset{\wedge}{\otimes} \ell^\infty_R$ of norm 1. In Section 4, we define the complex and real unconditional constants of basic sequences of elementary matrices and show that they are not equal in general. Section 5 looks back on Varopoulos' results about tensor algebras over discrete spaces. Section 6 puts the connection between p-trace norm and closed walks of length p in the concrete form of closed walk relations. In Section 7, we compute the norm of relative Schur multipliers by signs in the case that G is a cycle, and estimate the corresponding unconditional constants. Section 8 is dedicated to a proof of Th. 1.4 and an answer to Question 1.3. Section 9 establishes Th. 1.5. In Section 10, we study the metric unconditional approximation property for spaces S^p_I. The final section provides four kinds of examples: sets obtained by a transfer of n-independent subsets of a discrete abelian group, Hankel sets, Steiner systems and Tits' generalised polygons.

Terminology. C is the set of *columns* and R is the set of *rows*, both finite or countable and if necessary indexed by natural numbers. V, the set of *vertices*, is their disjoint union $C \amalg R$: if there is a risk of confusion, an element $n \in V$ that is a column (vs. a row) will be referred to as "col n" (vs. "row n".) An *edge* on V is a pair $\{v,w\} \subseteq V$. A *graph* on V is given by a set of edges E. A *bipartite* graph on V with vertex classes C and R has only edges $\{r,c\}$ such that $c \in C$ and $r \in R$ and may therefore be given alternatively by the set of couples $I = \big\{(r,c) \in R \times C : \{r,c\} \in E\big\}$: this will be our custom throughout the article. A bipartite graph on V is *complete* if its set of couples I is the whole of $R \times C$. Two graphs are *disjoint* if so are the sets of vertices of their edges. I is a *column section* if $(r,c),(r',c) \in I \Rightarrow r = r'$, and a *row section* if $(r,c),(r,c') \in I \Rightarrow c = c'$.

A *walk* of length $s \geqslant 0$ in a graph is a sequence $(v_0,\ldots,v_s)$ of $s+1$ vertices such that $\{v_0,v_1\}$, $\ldots$, $\{v_{s-1},v_s\}$ are edges of the graph. A walk is a *path* if its vertices are pairwise distinct. The *distance* of two vertices in a graph is the minimal length of a path in the graph that joins the two vertices; it is infinite if no such path exists. A *closed* walk of length $p \geqslant 0$ in a graph is a sequence $(v_1,\ldots,v_p)$ of p vertices such that $\{v_1,v_2\},\ldots,\{v_{p-1},v_p\},\{v_p,v_1\}$ are edges of the graph. Note that p is necessarily even if the graph is bipartite. A closed walk is a *cycle* if its vertices are pairwise distinct. We take the convention that if a closed walk in a bipartite graph on $V = C \amalg R$ is nonempty, then its first vertex is a column vertex: $v_1 \in C$. We shall identify a path and a cycle with its set of edges $\{r,c\}$ or the corresponding set of couples (r,c).

A bipartite graph on V is a *tree* if there is exactly one path between any two of its vertices. In this case, its vertices may be indexed by finite words over its set of vertices in the following way. Choose any row vertex r as *root* and index it by $\emptyset$. If v is a vertex and $(r,c,\ldots,v)$ is the unique path from r to v, let the word $c^\frown \cdots {}^\frown v$ index v. Let W be the set of all words thus formed. Then

- $\emptyset \in W$ and every beginning of a word in W is also in W: if $w \in W \setminus \{\emptyset\}$, then w is the concatenation $w'^\frown v$ of a word $w' \in W$ with a letter v;
- words of even length index row vertices;
- words of odd length index column vertices;
- a pair of vertices is an edge exactly if their indices have the form w and $w^\frown v$, where w is a word and v is a letter.

A *forest* is a union of pairwise disjoint trees; equivalently, it is a cycle free graph.

Notation. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

The unit ball of a Banach space X is denoted by B_X.

Given an index set I and $q \in I$, e_q is the sequence defined on I as the indicator function $\chi_{\{q\}}$ of the singleton $\{q\}$.

Let $I = R \times C$ and $q = (r, c)$. Then $\mathrm{e}_q = \mathrm{e}_{rc}$ is the *elementary matrix* identified with the operator from ℓ^2_C to ℓ^2_R that maps e_c on e_r and all other basis vectors on 0. The *matrix coefficient* at coordinate q of an operator x from ℓ^2_C to ℓ^2_R is $x_q = \operatorname{tr} \mathrm{e}^*_q x$ and its *matrix representation* is $(x_q)_{q\in R\times C} = \sum_{q\in R\times C} x_q \mathrm{e}_q$. The *support* of x is $\{q \in R \times C : x_q \neq 0\}$.

The *Schatten-von-Neumann class* S^p, $0 < p < \infty$, is the space of those compact operators x from ℓ^2_C to ℓ^2_R such that $\|x\|^p_p = \operatorname{tr} |x|^p = \operatorname{tr}(x^*x)^{p/2} < \infty$. S^∞ is the space of compact operators with the operator norm. S^p is a quasi-normed space, and a Banach space if $p \geqslant 1$. Let $(R_n \times C_n)_{n\geqslant 0}$ be a sequence of finite sets that tends to $R\times C$. Then the sequence of operators $P_n \colon x \mapsto \sum_{q\in R_n\times C_n} x_q \mathrm{e}_q$ tends pointwise to the identity on S^p if $p \geqslant 1$.

For $I \subseteq R \times C$, the *entry space* S^p_I is the subspace of those $x \in \mathrm{S}^p$ whose support is a subset of I. S^p_I is also the closed subspace of S^p spanned by $(\mathrm{e}_q)_{q\in I}$.

The S^p-valued Schatten-von-Neumann class $\mathrm{S}^p(\mathrm{S}^p)$ is the space of those compact operators x from ℓ^2_C to $\ell^2_R(\mathrm{S}^p)$ such that $\|x\|^p_p = \operatorname{tr}(\operatorname{tr} |x|^p) < \infty$, where the inner trace is the S^p-valued analogue of the usual trace: such operators have an S^p-valued matrix representation and their support is defined as in the scalar case. An element $x \in \mathrm{S}^p(\mathrm{S}^p)$ can also be considered as a compact operator from $\ell^2_C(\ell_2) = \ell_2 \otimes_2 \ell^2_C$ to $\ell^2_R(\ell_2) = \ell_2 \otimes_2 \ell^2_R$ such that $\|x\|^p_p = \operatorname{tr} \otimes \operatorname{tr} |x|^p < \infty$; the matrix coefficient of x at q is then $x_q = (\mathrm{Id}_{\mathrm{S}^p} \otimes \operatorname{tr})\big((\mathrm{Id}_{\ell_2} \otimes \mathrm{e}^*_q)x\big)$ and its matrix representation is $\sum_{q\in R\times C} x_q \otimes \mathrm{e}_q$. The entry space $\mathrm{S}^p_I(\mathrm{S}^p)$ is defined in the same way as S^p_I.

A *relative Schur multiplier* on S^p_I is a sequence $\varphi = (\varphi_q)_{q\in I} \in \mathbb{C}^I$ such that the associated *Schur multiplication operator* M_φ defined by $\mathrm{e}_q \mapsto \varphi_q \mathrm{e}_q$ for $q \in I$ is bounded on S^p_I. The Schur multiplier φ is furthermore *completely bounded* (c.b. for short) on S^p_I if $\mathrm{Id}_{\mathrm{S}^p} \otimes \mathrm{M}_\varphi$, the operator defined by $x_q\mathrm{e}_q \mapsto \varphi_q x_q \mathrm{e}_q$ for $x_q \in \mathrm{S}^p$ and $q \in I$, is bounded on $\mathrm{S}^p_I(\mathrm{S}^p)$ (see [76, Lemma 1.7].) The norm of φ is the norm of M_φ and its *complete norm* is the norm of $\mathrm{Id}_{\mathrm{S}^p} \otimes \mathrm{M}_\varphi$. This norm is the supremum of the norm of its restrictions to finite rectangle sets $R' \times C'$. Note that φ is a Schur multiplier on S^∞ if and only if, for every bounded operator $x \colon \ell^2_C \to \ell^2_R$, $(\varphi_q x_q)$ is the matrix representation of a bounded operator; also φ is automatically c.b. on S^∞ [78, Th. 5.1]. We used [76, 78] as a reference.

Let G be a compact abelian group endowed with its normalised Haar measure. Let $\Gamma = \hat{G}$ be the dual group of characters on G. The *spectrum* of an integrable function f on G is $\{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}$. Let $\Lambda \subseteq \Gamma$. If X is a space of integrable functions on G, then X_Λ is the *translation invariant* subspace of those $f \in X$ whose spectrum is a subset of Λ.

Let X be the space of continuous functions $\mathrm{C}(G)$ or the Lebesgue space $\mathrm{L}^p(G)$ with $0 < p < \infty$. Then X_Λ is also the closed subspace of X spanned by Λ. A *relative Fourier multiplier* on X_Λ is a sequence $\mu = (\mu_\gamma)_{\gamma\in\Lambda} \in \mathbb{C}^\Lambda$ such that the associated convolution operator C_μ defined by $\gamma \mapsto \mu_\gamma \gamma$ for $\gamma \in \Lambda$ is bounded on X_Λ. The Fourier multiplier μ is furthermore c.b. if $\mathrm{Id}_{\mathrm{S}^p} \otimes \mathrm{C}_\mu$, the operator defined by $a_\gamma\gamma \mapsto \mu_\gamma a_\gamma \gamma$ for $a_\gamma \in \mathrm{S}^p$ and $\gamma \in \Lambda$, is bounded on the S^p-valued space $X_\Lambda(\mathrm{S}^p)$ (where $p = \infty$ if $X = \mathrm{C}(G)$.) The norm of μ is the norm of C_μ and its complete norm is the norm of $\mathrm{Id}_{\mathrm{S}^p} \otimes \mathrm{C}_\mu$. Note that μ is a Fourier multiplier on $\mathrm{C}_\Lambda(G)$ if and only if, for every $f \in \mathrm{L}^\infty_\Lambda(G)$, $\sum \mu_\gamma \hat{f}(\gamma)\gamma$ is the Fourier series of an element of $\mathrm{L}^\infty_\Lambda(G)$: μ is a *relative Fourier multiplier on* $\mathrm{L}^\infty(G)$; also μ is automatically c.b. on $\mathrm{C}_\Lambda(G)$ [78, Cor. 3.18].

Let X, Y be Banach spaces and $u \in X \otimes Y$. Its *projective* tensor norm is

$$\|u\|_{X \overset{\wedge}{\otimes} Y} = \inf\Big\{\sum_{j=1}^n \|x_j\|\,\|y_j\| : u = \sum_{j=1}^n x_j \otimes y_j\Big\}$$

and $X \overset{\wedge}{\otimes} Y$ is the completion of $X \otimes Y$ with respect to this norm. Note that $\ell^n_\infty \overset{\wedge}{\otimes} \ell^m_\infty \subset \mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0$ because ℓ^n_∞ and ℓ^m_∞ are 1-complemented in c_0, and that $\mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0 \subset \ell_\infty \overset{\wedge}{\otimes} \ell_\infty$ because ℓ_∞ is the bidual of c_0.

Let $\sum x_j \otimes y_j$ be any representation of the tensor u. If $\xi \otimes \eta \in X^* \otimes Y^*$, we define $\langle \xi \otimes \eta, u\rangle = \sum\langle \xi, x_j\rangle\langle \eta, y_j\rangle$. The *injective* tensor norm of u is

$$\|u\|_{X \overset{\vee}{\otimes} Y} = \sup_{(\xi,\eta)\in B_{X^*}\times B_{Y^*}} |\langle \xi \otimes \eta, u\rangle|$$

and $X \overset{\vee}{\otimes} Y$ is the completion of $X \otimes Y$ with respect to this norm.

If X and Y are both finite dimensional, then

$$\big(X \overset{\vee}{\otimes} Y\big)^* = X^* \overset{\wedge}{\otimes} Y^* \quad \text{and} \quad \big(X \overset{\wedge}{\otimes} Y\big)^* = X^* \overset{\vee}{\otimes} Y^*.$$

Further $\big(\mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0\big)^* = \ell_1 \overset{\vee}{\otimes} \ell_1$: in fact, $\big(\mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0\big)^*$ may be identified with the space of bounded operators from c_0 to ℓ_1 and $\ell_1 \overset{\vee}{\otimes} \ell_1$ may be identified with the closure of finite rank operators in that space, and they are the same because every bounded operator from c_0 to ℓ_1 is compact and ℓ_1 has the approximation property.

If X is a sequence space on C and Y is a sequence space on R, then the *coefficient* of the tensor u at (r, c) is $\langle \mathrm{e}_c \otimes \mathrm{e}_r, u\rangle$. Its *support* is the set of coordinates (r, c) of its nonvanishing coefficients. One may use [89] as a reference.

2 Relative Schur multipliers

The following proposition is a straightforward consequence of [71].

Proposition 2.1. *Let $I \subseteq R \times C$ and φ be a Schur multiplier on S^∞_I with norm D. Then φ is also a c.b. Schur multiplier on S^p_I for every $p \in (0, \infty]$, with complete norm bounded by D.*

Proof. We may assume that $D = 1$. Let $R' \times C'$ be any finite subset of $R \times C$. By [71, Th. 3.2], there exist vectors w_c and v_r of norm at most 1 in a Hilbert space H such that $\varphi_{rc} = \langle w_c, v_r\rangle$ for every $(r, c) \in I \cap R' \times C'$. If we define $W\colon \ell^2_{C'} \to \ell^2_{C'}(H)$ and $V\colon \ell^2_{R'} \to \ell^2_{R'}(H)$ by $W\zeta = (\zeta_c w_c)_{c\in C'}$ and $V\eta = (\eta_r v_r)_{r\in R'}$, then V and W have norm at most 1, and the proposition follows from the factorisation

$$\mathrm{M}_\varphi x = V^*(x \otimes \mathrm{Id}_H)W$$

for every x with support in $I \cap R' \times C'$. □

Remark 2.2. Éric Ricard showed us an elementary proof that a Schur multiplier on S^∞_I is automatically c.b., included here by his kind permission. A Schur multiplier φ is bounded on S^∞_I by a constant D if and only if

$$\forall\, \xi \in B_{\mathrm{S}^\infty_I}\ \forall\, \eta \in B_{\ell^2_R}\ \forall\, \zeta \in B_{\ell^2_C} \quad \Big|\sum_{(r,c)\in I} \eta_r \varphi_{rc} \xi_{rc} \zeta_c\Big| \leqslant D. \tag{B.2}$$

It is furthermore completely bounded on S^∞_I by D if

$$\forall\, x \in B_{\mathrm{S}^\infty_I(\mathrm{S}^\infty)}\ \forall\, y \in B_{\ell^2_R(\ell_2)}\ \forall\, z \in B_{\ell^2_C(\ell_2)} \quad \Big|\sum_{(r,c)\in I} \varphi_{rc} \langle y_r, x_{rc} z_c\rangle\Big| \leqslant D. \tag{B.3}$$

Suppose that x, y, z are as quantified in Ineq. (B.3). Let

$$\xi_{rc} = \langle y_r/\|y_r\|, x_{rc} z_c/\|z_c\|\rangle,\ \eta_r = \|y_r\|_{\ell_2} \text{ and } \zeta_c = \|z_c\|_{\ell_2}.$$

Then $\|\eta\|_{\ell^2_R}, \|\zeta\|_{\ell^2_C} \leqslant 1$ and

$$\begin{aligned}\|\xi\| = \sup\Big\{\Big|\sum_{(r,c)\in I} \langle \alpha_r y_r/\|y_r\|_{\ell_2}, x_{rc} \beta_c z_c/\|z_c\|_{\ell_2}\rangle\Big| : \alpha \in B_{\ell^2_R},\ \beta \in B_{\ell^2_C}\Big\}\\ \leqslant \|x\| \sup_{\alpha\in B_{\ell^2_R}} \big\|\big(\alpha_r y_r/\|y_r\|_{\ell_2}\big)\big\|_{\ell^2_R(\ell_2)} \sup_{\beta\in B_{\ell^2_C}} \big\|\big(\beta_c z_c/\|z_c\|_{\ell_2}\big)\big\|_{\ell^2_C(\ell_2)} \leqslant 1,\end{aligned}$$

so that Ineq. (B.2) implies Ineq. (B.3).

The fact that the canonical basis of an ℓ^2 space is 1-unconditional yields that Schatten-von-Neumann norms are *matrix unconditional* in the terminology of [91]:

$$\forall\, \zeta \in \mathbb{T}^C\ \forall\, \eta \in \mathbb{T}^R \quad \Big\|\sum_{(r,c)\in R\times C} \zeta_c \eta_r a_{rc} \mathrm{e}_{rc}\Big\|_p = \Big\|\sum_{(r,c)\in R\times C} a_{rc} \mathrm{e}_{rc}\Big\|_p \tag{B.4}$$

for every finitely supported sequence of complex or S^p-valued coefficients a_{rc}. Let $\zeta \otimes \eta$ denote the *elementary* Schur multiplier $(\zeta_c \eta_r)_{(r,c)\in R\times C}$. Equation (B.4) shows that if $\zeta \in \mathbb{T}^C$ and $\eta \in \mathbb{T}^R$, then

$\mathrm{M}_{\zeta\otimes\eta}$ is an isometry on every S^p. This yields that if $\zeta \in \ell^\infty_C$, $\eta \in \ell^\infty_R$, then the complete norm of $\mathrm{M}_{\zeta\otimes\eta}$ is $\|\zeta\|_{\ell^\infty_C}\|\eta\|_{\ell^\infty_R}$ on every S^p.

Relative Schur multipliers also have a central place among operators on S^p_I because they appear as the range of a contractive projection defined by the following averaging scheme.

Definition 2.3. Let $T\colon \mathrm{S}^p_J \to \mathrm{S}^p_I$ be an operator. Let $R' \times C'$ be a finite subset of $R \times C$ and let $P_{R'\times C'}$ be the contractive projection onto $\mathrm{S}^p_{R'\times C'}$ defined by the Schur multiplier $\chi_{C'} \otimes \chi_{R'}$. Then the *average of T with respect to $R' \times C'$* is given by

$$[T]_{R'\times C'}(x) = \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta\, \mathrm{M}_{\zeta^*\otimes\eta^*} P_{R'\times C'} T(\mathrm{M}_{\zeta\otimes\eta} x), \tag{B.5}$$

where $\zeta^* = (\overline{\zeta_c})_{c\in C}$ and $\eta^* = (\overline{\eta_r})_{r\in R}$.

Proposition 2.4. *Let $T\colon \mathrm{S}^p_J \to \mathrm{S}^p_I$ be an operator and $R' \times C'$ a finite subset of $R \times C$. Then $[T]_{R'\times C'}$ is a Schur multiplication operator from S^p_J to $\mathrm{S}^p_{I\cap R'\times C'}$ such that $\|[T]_{R'\times C'}\| \leqslant \|T\|$. In fact, $[T]_{R'\times C'} = M_{\varphi^{R'\times C'}}$ with*

$$\varphi^{R'\times C'}_{rc} = \begin{cases} \operatorname{tr} \mathrm{e}^*_{rc} T(\mathrm{e}_{rc}) & \text{if } (r,c) \in J \cap R' \times C' \\ 0 & \text{if } (r,c) \in J \setminus R' \times C'. \end{cases}$$

If T is a projection onto S^p_I, then $\varphi^{R'\times C'} = \chi_{I\cap R'\times C'}$, so that $[T]_{R'\times C'}$ is a projection onto $\mathrm{S}^p_{I\cap R'\times C'}$. Let $\varphi = \big(\operatorname{tr} \mathrm{e}^*_q T(\mathrm{e}_q)\big)_{q\in J}$. Then $\|\mathrm{M}_\varphi\| \leqslant \|T\|$ and we define the average of T *by* $[T] = \mathrm{M}_\varphi$.

Proof. Formula (B.5) shows that $\|[T]_{R'\times C'}(x)\| \leqslant \|T\|\,\|x\|$. We have

$$\begin{aligned}
[T]_{R'\times C'}(\mathrm{e}_{rc}) &= \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta\, \mathrm{M}_{\zeta^*\otimes\eta^*} P_{R'\times C'} T(\zeta_c\eta_r \mathrm{e}_{rc}) \\
&= \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta\, \zeta_c\eta_r \mathrm{M}_{\zeta^*\otimes\eta^*} \sum_{(r',c')\in R'\times C'} \operatorname{tr}\big(\mathrm{e}^*_{r'c'} T(\mathrm{e}_{rc})\big)\mathrm{e}_{r'c'} \\
&= \sum_{(r',c')\in R'\times C'} \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta\, \zeta_c\eta_r \operatorname{tr}\big(\mathrm{e}^*_{r'c'} T(\mathrm{e}_{rc})\big)\zeta_{c'}^{-1}\eta_{r'}^{-1} \mathrm{e}_{r'c'} \\
&= \varphi^{R'\times C'}_{rc} \mathrm{e}_{rc}.
\end{aligned}$$

As the norm of a Schur multiplier is the supremum of the norm of its restrictions to finite rectangle sets, this shows that φ is a Schur multiplier on S^p_J and $\|\mathrm{M}_\varphi\| \leqslant \|T\|$. If T is a projection onto S^p_I, note that $\operatorname{tr} \mathrm{e}^*_{rc} T(\mathrm{e}_{rc}) = \chi_I(r,c)$. ☐

The following proposition relates Fourier multipliers to Herz-Schur multipliers in the fashion of [78, Th. 6.4] and will be very useful in the exact computation of the norm of certain relative Schur multipliers.

Proposition 2.5. *Let Γ be a countable discrete abelian group and $\Lambda \subseteq \Gamma$. Let R and C be two copies of Γ and consider $I = \{(r,c) \in R \times C : r - c \in \Lambda\}$. Let $\varphi \in \mathbb{C}^I$ such that there is $\mu \in \mathbb{C}^\Lambda$ with $\varphi(r,c) = \mu(r-c)$ for all $(r,c) \in I$. Let $G = \hat{\Gamma}$, so that Γ is the group of characters on the compact group G. Let $p \in (0,\infty]$.*

- (a) *The complete norm of the relative Schur multiplier φ on S^p_I is bounded by the complete norm of the relative Fourier multiplier μ on $\mathrm{L}^p_\Lambda(G)$.*
- (b) *Suppose that Γ is finite. The norm of the relative Fourier multiplier μ on $\mathrm{L}^p_\Lambda(G)$ is bounded by the norm of the relative Schur multiplier φ on S^p_I. The same holds for complete norms.*

Remark 2.6. Part (b) is just an abstract counterpart to [74, Chapter 6, Lemma 3.8], where the case of the finite cyclic group $\Gamma = \mathbb{Z}/n\mathbb{Z}$ is treated.

Proof. (a) is [76, Lemma 8.1.4]: for all $a_q \in \mathrm{S}^p$, of which only a finite number are nonzero, and all $g \in G$, we have by matrix unconditionality (Eq. (B.4))

$$\begin{aligned}
\Big\|\sum_{q\in I} a_q \mathrm{e}_q\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)} &= \Big\|\sum_{(r,c)\in I} r(g)c(g)^{-1} a_{rc}\mathrm{e}_{rc}\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)} \\
&= \Big\|\sum_{\gamma\in\Lambda}\Big(\sum_{\substack{(r,c)\in I\\ r-c=\gamma}} a_{rc}\mathrm{e}_{rc}\Big)\gamma(g)\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda}\Big(\sum_{\substack{(r,c)\in I\\ r-c=\gamma}} a_{rc}\mathrm{e}_{rc}\Big)\gamma\Big\|_{\mathrm{L}^p_\Lambda(G,\mathrm{S}^p(\mathrm{S}^p))}.
\end{aligned} \tag{B.6}$$

This yields an isometric embedding of $\mathrm{S}^p_I(\mathrm{S}^p)$ in $\mathrm{L}^p_\Lambda(G, \mathrm{S}^p_I(\mathrm{S}^p))$. As $\mathrm{S}^p(\mathrm{S}^p)$ may be identified with $\mathrm{S}^p(\ell^2_\Gamma(\ell^2))$,

$$\Big\|\sum_{q\in I} \varphi_q a_q \mathrm{e}_q\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda} \mu_\gamma\Big(\sum_{\substack{(r,c)\in I\\ r-c=\gamma}} a_{rc}\mathrm{e}_{rc}\Big)\gamma\Big\|_{\mathrm{L}^p_\Lambda(G,\mathrm{S}^p(\mathrm{S}^p))}$$

$$\leqslant \|\mathrm{Id}\otimes \mathrm{C}_\mu\|\Big\|\sum_{q\in I} a_q\mathrm{e}_q\Big\|_{\mathrm{S}^p_I(\mathrm{S}^p)}. \quad \text{(B.7)}$$

(b). Let us embed $\mathrm{L}^p_\Lambda(G)$ into S^p_I by $f\mapsto \mathrm{c}_{\hat f}$, where $\mathrm{c}_{\hat f}\colon \ell^2_C\to\ell^2_R$ is the convolution operator defined by

$$\mathrm{c}_{\hat f}\mathrm{e}_c = \hat f * \mathrm{e}_c = \sum_{\gamma\in\Lambda}\hat f(\gamma)\mathrm{e}_\gamma * \mathrm{e}_c = \sum_{r-c\in\Lambda}\hat f(r-c)\mathrm{e}_r :$$

$\mathrm{c}_{\hat f}$ has the matrix representation $\sum_{(r,c)\in I}\hat f(r-c)\mathrm{e}_{rc}$. The characters $g\in G$ form an orthonormal basis for ℓ^2_C such that $\mathrm{c}_{\hat f}g = f(g)g$: therefore

$$\|\mathrm{c}_{\hat f}\|_p = \Big(\sum_{g\in G}|f(g)|^p\Big)^{1/p} = (\# G)^{1/p}\|f\|_{\mathrm{L}^p(G)}.$$

As $\mathrm{M}_\varphi\mathrm{c}_{\hat f} = \mathrm{c}_{\widehat{\mathrm{C}_\mu f}}$, this shows that the norm of μ on $\mathrm{L}^p_\Lambda(G)$ is the norm of φ on the subspace of circulant matrices in S^p_I. The same holds for complete norms. □

3 Idempotent Schur multipliers of norm 1

A Schur multiplier is *idempotent* if it is the indicator function χ_I of some set $I\subseteq R\times C$; if χ_I is a Schur multiplier on S^p, then it is a projection of S^p onto S^p_I. Idempotent Schur multipliers on S^p and tensors in $\ell^\infty_C\overset{\wedge}{\otimes}\ell^\infty_R$ with $0,1$ coefficients of norm 1 may be characterised by the combinatorics of I.

Proposition 3.1. *Let $I\subseteq R\times C$ be nonempty and $0<p\neq 2<\infty$. The following are equivalent.*

(a) *For every finite rectangle set $R'\times C'$ intersecting I*

$$\Big\|\sum_{(r,c)\in I\cap R'\times C'}\mathrm{e}_c\otimes\mathrm{e}_r\Big\|_{\ell^\infty_C\overset{\wedge}{\otimes}\ell^\infty_R} = 1.$$

(b) *S^p_I is completely 1-complemented in S^p.*

(c) *S^p_I is 1-complemented in S^p.*

(d) *I is a union of pairwise disjoint complete bipartite graphs: there are pairwise disjoint sets $R_j\subseteq R$ and pairwise disjoint sets $C_j\subseteq C$ such that $I=\bigcup R_j\times C_j$.*

Property (d) means that the pattern I is, up to a permutation of columns and rows, block-diagonal:

$$\begin{array}{c c} & \begin{array}{cccc} C_1 & C_2 & C_3 & \cdots \end{array} \\ \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ \vdots \end{array} & \begin{pmatrix} * & 0 & 0 & \cdots \\ 0 & * & 0 & \ddots \\ 0 & 0 & * & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \end{array}$$

Proof. (b) ⇒ (c) is trivial.

(a) ⇒ (b). The complete norm of a Schur multiplier φ on S^p is the supremum of the complete norm of its restrictions $\varphi'=(\varphi_q)_{q\in R'\times C'}$ to finite rectangle sets $R'\times C'$. Furthermore, the complete norm of an elementary Schur multiplier $(\eta_c\zeta_r)_{(r,c)\in R\times C}=\eta\otimes\zeta$ on S^p equals $\|\eta\|_{\ell^\infty_C}\|\zeta\|_{\ell^\infty_R}$.

(c) ⇒ (d). If I is not a union of pairwise disjoint complete bipartite graphs, then there are $r_0,r_1\in R$ and $c_0,c_1\in C$ such that

$$I' = I\cap\{r_0,r_1\}\times\{c_0,c_1\} = \{(r_0,c_0),(r_1,c_0),(r_0,c_1)\}.$$

By Proposition 2.4, the average of a contractive projection of S^p onto S^p_I with respect to $\{r_0, r_1\} \times \{c_0, c_1\}$ would be the contractive projection associated to the Schur multiplier $\chi_{I'}$. Let $x(t)$, $t \in \mathbb{R}$, be the operator from ℓ^2_C to ℓ^2_R whose matrix coefficients vanish except for its $\{r_0, r_1\} \times \{c_0, c_1\}$ submatrix, which is $\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & t \end{pmatrix}$. Its eigenvalues are

$$\frac{1 + t + \sqrt{9 - 2t + t^2}}{2} = 2 + \frac{t}{3} + o(t), \quad \frac{1 + t - \sqrt{9 - 2t + t^2}}{2} = -1 + \frac{2t}{3} + o(t),$$

so that

$$\begin{cases} \|x(t)\|_\infty = 2 + t/3 + o(t) \\ \|x(t)\|_p^p = 2^p + 1 + p(2^p - 4)t/6 + o(t) \quad \text{for } 0 < p < \infty \end{cases}$$

and therefore $\|\chi_{I'} * x(t)\|_p = \|x(0)\|_p > \|x(t)\|_p$ for some $t \neq 0$ if $p \neq 2$.

$(d) \Rightarrow (a)$. Suppose (d) and let $R' \times C'$ intersect I. Then there are pairwise disjoint sets R'_j and pairwise disjoint sets C'_j such that $I \cap R' \times C' = R'_1 \times C'_1 \cup \cdots \cup R'_n \times C'_n$ and

$$\sum_{(r,c) \in I \cap R' \times C'} \mathrm{e}_c \otimes \mathrm{e}_r = \sum_{j=1}^n \chi_{C'_j} \otimes \chi_{R'_j} = \operatorname*{average}_{\epsilon_j = \pm 1} \Big(\sum_{j=1}^n \epsilon_j \chi_{C'_j} \Big) \otimes \Big(\sum_{j=1}^n \epsilon_j \chi_{R'_j} \Big)$$

which is an average of elementary tensors of norm 1, so that its projective tensor norm is bounded by 1, and actually is equal to 1. □

Remark 3.2. Note that the proof of Prop. 3.1 shows that the norm of a projection $\mathrm{M}_{\chi_I} : \mathrm{S}^\infty \to \mathrm{S}^\infty_I$ is either 1 or at least $2/\sqrt{3}$, as

$$\left\| \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \right\|_\infty = \sqrt{3}, \qquad \left\| \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} \right\|_\infty = 2.$$

This is a noncommutative analogue to the fact that an idempotent measure on a locally compact abelian group G has either norm 1 or at least $\sqrt{5}/2$ [88, Th. 3.7.2]. The norm of M_{χ_I} actually equals $2/\sqrt{3}$ for $I = \{(0,0), (0,1), (1,0)\}$, as shown in [53, Lemma 3]. In fact, the following decomposition holds:

$$\mathrm{e}_0 \otimes \mathrm{e}_0 + \mathrm{e}_0 \otimes \mathrm{e}_1 + \mathrm{e}_1 \otimes \mathrm{e}_0 = \\ \big((\mathrm{e}^{-\mathrm{i}\pi/12}, \mathrm{e}^{\mathrm{i}\pi/4}) \otimes (\mathrm{e}^{-\mathrm{i}\pi/12}, \mathrm{e}^{\mathrm{i}\pi/4}) + (\mathrm{e}^{\mathrm{i}\pi/12}, \mathrm{e}^{-\mathrm{i}\pi/4}) \otimes (\mathrm{e}^{\mathrm{i}\pi/12}, \mathrm{e}^{-\mathrm{i}\pi/4}) \big) / \sqrt{3}.$$

Remark 3.3. Results related to the equivalence of (c) with (d) have been obtained independently by Banks and Harcharras [4].

4 Unconditional basic sequences in S^p

Definition 4.1. Let $0 < p \leqslant \infty$ and $I \subseteq R \times C$. Let $\mathbb{S} = \mathbb{T}$ (vs. $\mathbb{S} = \{-1, 1\}$.)

(*a*) I is an *unconditional basic sequence* in S^p if there is a constant D such that

$$\Big\| \sum_{q \in I} \epsilon_q a_q \mathrm{e}_q \Big\|_p \leqslant D \Big\| \sum_{q \in I} a_q \mathrm{e}_q \Big\|_p \tag{B.8}$$

for every choice of signs $\epsilon_q \in \mathbb{S}$ and every finitely supported sequence of complex coefficients a_q. Its *complex (vs. real) unconditional constant* is the least such constant D.

(*b*) I is a *completely* unconditional basic sequence in S^p if there is a constant D such that (B.8) holds for every choice of signs $\epsilon_q \in \mathbb{S}$ and every finitely supported sequence of operator coefficients $a_q \in \mathrm{S}^p$. Its complex (vs. real) *complete* unconditional constant is the least such constant D.

(*c*) I is a complex (vs. real, complex completely, real completely) *1-unconditional* basic sequence in S^p if its complex (vs. real, complex complete, real complete) unconditional constant is 1: Inequality (B.8) turns into the equality

$$\Big\| \sum_{q \in I} \epsilon_q a_q \mathrm{e}_q \Big\|_p = \Big\| \sum_{q \in I} a_q \mathrm{e}_q \Big\|_p.$$

If Inequality (B.8) holds for every real choice of signs, then it also holds for every complex choice of signs at the cost of replacing D by $D\pi/2$ (see [92],) so that there is no need to distinguish between complex and real unconditional basic sequences.

The notions defined in (a) and (b) are called $\sigma(p)$ *sets* and *complete* $\sigma(p)$ *sets* in [37, §4] and [38] (see also the survey [79, § 9].) The notions defined in (c) are their isometric counterparts.

By [91, proof of Cor. 4], the real unconditional constant of any basis of S^p_I cannot be lower than a fourth of the real unconditional constant of I in S^p.

Example 4.2. A single column $R \times \{c\}$, a single row $\{r\} \times C$, the diagonal set $\{(\mathrm{row}\, n, \mathrm{col}\, n)\}_{n\in\mathbb{N}}$ if R and C are copies of $\mathbb{N}$, are 1-unconditional basic sequences in all S^p. In fact, every column section and every row section (this is the terminology of [99, Def. 4.3]) is a 1-unconditional basic sequence; note that the length of every path in the corresponding graph is at most 2.

Note that the set I is a (completely) 1-unconditional basic sequence in S^p if and only if the relative Schur multipliers by signs on S^p_I define (complete) isometries. This yields by Prop. 2.1:

Proposition 4.3. *Let $I \subseteq R \times C$ and $0 < p \leqslant \infty$. If I is a real (vs. complex) 1-unconditional basic sequence in S^∞, then I is also a real (vs. complex) completely 1-unconditional basic sequence in S^p.*

Example 4.4. If $R = C = \{0, \dots, n-1\}$, $1 \leqslant p \leqslant \infty$ and $I = R \times C$, then the complex unconditional constant of the basis of elementary matrices in S^p is $n^{|1/2-1/p|}$ and coincides with its complete unconditional constant (see [76, Lemma 8.1.5].) This is also the real unconditional constant if $n = 2^k$ is a power of 2 as the norm of Schur multiplication by the kth tensor power $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes k}$ (the kth Walsh matrix) on S^p is $\left(2^{|1/2-1/p|}\right)^k = n^{|1/2-1/p|}$. Let us now show that if $n = 3$, the real unconditional constant of the basis of elementary matrices in S^∞ is $5/3$ and differs from its complex unconditional constant, $\sqrt{3}$. In fact, because the canonical bases of ℓ^2_C and ℓ^2_R are symmetric, the norm of a Schur multiplier by real signs turns out to equal the norm of one of the following three Schur multipliers:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

The first one has norm 1: it defines the identity. The second one has the same norm as the Schur multiplier $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which is $\sqrt{2}$, because the norm of that multiplier equals the norm of its tensor product by $\mathrm{Id}_{\ell^2_2}$, which is $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$. By Prop. 2.5 for $\Gamma = \mathbb{Z}/3\mathbb{Z}$, the third one has the same norm as the Fourier multiplier $\varphi = (-1, 1, 1)$ on $\mathrm{L}^\infty(G)$, where $G = \{z \in \mathbb{C} : z^3 = 1\}$: as this multiplier acts by convolution with $f = -1 + z + z^2$, its norm is $\|f\|_{\mathrm{L}^1(G)}$, that is

$$\left(|-1+1+1| + |-1+\mathrm{e}^{2\mathrm{i}\pi/3}+\mathrm{e}^{4\mathrm{i}\pi/3}| + |-1+\mathrm{e}^{4\mathrm{i}\pi/3}+\mathrm{e}^{2\mathrm{i}\pi/3}|\right)/3 = 5/3.$$

Complex interpolation yields that the real unconditional constant of the basis of elementary matrices is in fact strictly less than its complex counterpart in all S^p with $p \neq 2$.

5 Varopoulos' characterisation of unconditional matrices in S^∞

Our results may be seen as the isometric counterpart to results by Varopoulos [99] on tensor algebras over discrete spaces and their generalisation to S^p. He characterised unconditional basic sequences of elementary matrices in S^∞ in his study of the projective tensor product $\mathrm{c}_0 \overset{\wedge}{\otimes} \mathrm{c}_0$. We gather up his results in the next theorem, as they are difficult to extract from the literature.

Theorem 5.1. *Let $I \subseteq R \times C$. The following are equivalent.*

(*a*) *I is an unconditional basic sequence in S^∞.*

(*b*) *I is an* interpolation set *for Schur multipliers on S^∞: every bounded sequence on I is the restriction of a Schur multiplier on S^∞.*

(*c*) *I is a* V-Sidon set *as defined in [99, Def. 4.1]: every null sequence on I is the restriction of the sequence of coefficients of a tensor in* $\mathrm{c}_0(C) \overset{\wedge}{\otimes} \mathrm{c}_0(R)$.

(*d*) *The coefficients of every tensor in $\ell^1_C \overset{\vee}{\otimes} \ell^1_R$ with support in I form an absolutely convergent series.*

(*e*) $(z_c z'_r)_{(r,c)\in I}$ *is a* Sidon set *in the dual of $\mathbb{T}^C \times \mathbb{T}^R$, that is, an unconditional basic sequence in* $\mathrm{C}(\mathbb{T}^C \times \mathbb{T}^R)$.

(*f*) *There is a constant λ such that for all $R' \subseteq R$ and $C' \subseteq C$ with n elements $\#[I \cap R' \times C'] \leqslant \lambda n$.*

(*g*) *I is a finite union of forests.*

(*h*) *I is a finite union of row sections and column sections.*

(*i*) *Every bounded sequence supported by I is a Schur multiplier on* S^∞.

Sketch of proof. (*a*) $\Rightarrow$ (*b*). If (*a*) holds, every sequence of signs $\epsilon \in \{-1,1\}^I$ is a Schur multiplier on S^∞_I. By a convexity argument, this implies that every bounded sequence is a Schur multiplier on S^∞_I, which may be extended to a Schur multiplier on S^∞ with the same norm by [71, Cor. 3.3].

(*b*) $\Rightarrow$ (*c*) holds by Grothendieck's inequality (see [78, § 5]) and an approximation argument.

(*d*) is but the formulation dual to (*c*) (see [98, § 6.2].)

(*d*) $\Rightarrow$ (*e*). A computation yields

$$\Big\| \sum_{(r,c)\in I} a_{rc} \mathrm{e}_c \otimes \mathrm{e}_r \Big\|_{\ell^1_C \overset{\vee}{\otimes} \ell^1_R} = \sup_{|z_c|,|z'_r|=1} \Big| \sum_{(r,c)\in I} a_{rc} z_c z'_r \Big|. \tag{B.9}$$

(*e*) $\Rightarrow$ (*f*) is [99, Th. 4.2]. (The proof can be found in [98, § 6.3] and in [97, § 5].)

(*f*) $\Rightarrow$ (*g*), (*f*) $\Rightarrow$ (*h*) can be found in [97, Th. 6.1].

(*g*) $\Rightarrow$ (*h*). In fact, a forest is the union of a column section I' with a row section I'' (a *bisection* in the terminology of [99, Def. 4.3].) It suffices to prove this for a tree. Let its vertices be indexed by words as described in the Terminology. Then the set I' of couples of the form $(w, w^\frown c)$ with w a word and c a letter is a column section; the set I'' of couples of the form $(w^\frown r, w)$ with w a word and r a letter is a row section.

(*h*) $\Rightarrow$ (*i*) is [97, Th. 4.5]. Note that row sections and column sections form 1-unconditional basic sequences in S^∞ and are 1-complemented in S^∞ by Prop. 3.1.

(*i*) $\Rightarrow$ (*a*) follows from the open mapping theorem.

6 Closed walk relations

We now introduce and study the combinatorial objects that we need in order to analyse the expansion of the function defined by

$$\Phi_I(\epsilon, a) = \mathrm{tr}\Big|\sum_{q\in I} \epsilon_q a_q \mathrm{e}_q\Big|^p \tag{B.10}$$

for $I \subseteq R \times C$, a positive even integer $p = 2k$, signs $\epsilon_q \in \mathbb{T}$ and coefficients $a_q \in \mathbb{C}$, of which only a finite number are nonzero. In fact,

$$\begin{aligned} \Phi_I(\epsilon, a) &= \mathrm{tr}\Big(\sum_{(r,c),(r',c')\in I} (\epsilon_{rc} a_{rc} \mathrm{e}_{rc})^* (\epsilon_{r'c'} a_{r'c'} \mathrm{e}_{r'c'})\Big)^k \\ &= \mathrm{tr} \sum_{\substack{(r_1,c_1),(r'_1,c'_1),\ldots,\\ (r_k,c_k),(r'_k,c'_k)\in I}} \prod_{i=1}^k (\epsilon^{-1}_{r_ic_i} \overline{a_{r_ic_i}} \mathrm{e}_{c_ir_i})(\epsilon_{r'_ic'_i} a_{r'_ic'_i} \mathrm{e}_{r'_ic'_i}) \\ &= \sum_{\substack{(r_1,c_1),(r_1,c_2),\ldots,\\ (r_k,c_k),(r_k,c_{k+1})\in I}} \prod_{i=1}^k \epsilon^{-1}_{r_ic_i} \epsilon_{r_ic_{i+1}} \overline{a_{r_ic_i}} a_{r_ic_{i+1}} \quad \text{(where } c_{k+1} = c_1.) \end{aligned} \tag{B.11}$$

The latter sum runs over all closed walks $(c_1, r_1, c_2, \ldots, c_k, r_k)$ of length p in the graph I. With multinomial notation, its terms have the form $\epsilon^{\beta-\alpha}\overline{a}^\alpha a^\beta$ with $|\alpha| = |\beta| = k$. The attempt to describe those couples (α, β) that effectively arise in this expansion yields the following definition.

Definition 6.1. Let $p = 2k \geqslant 0$ be an even integer and $I \subseteq R \times C$.

(*a*) Let $\mathrm{A}_k^I = \{\alpha \in \mathbb{N}^I : \sum_{q \in I} \alpha_q = k\}$ and set

$$\mathrm{B}_k^I = \big\{(\alpha, \beta) \in \mathrm{A}_k^I \times \mathrm{A}_k^I : \forall\, r \ \textstyle\sum_c \alpha_{rc} = \sum_c \beta_{rc} \text{ and } \forall\, c \ \sum_r \alpha_{rc} = \sum_r \beta_{rc}\big\}.$$

(*b*) Two couples $(\alpha^1, \beta^1) \in \mathrm{B}_{k_1}^I$, $(\alpha^2, \beta^2) \in \mathrm{B}_{k_2}^I$ are *disjoint* if $k_1, k_2 \geqslant 1$ and

$$\alpha_{rc}^1 \geqslant 1 \quad \Rightarrow \quad \forall\, (r', c) \in I \quad \alpha_{r'c}^2 = 0 \quad \text{and} \quad \forall\, (r, c') \in I \quad \alpha_{rc'}^2 = 0. \tag{B.12}$$

(*c*) The set $\mathscr{W}_k^I$ of *closed walk relations* of length p in I is the subset of those $(\alpha, \beta) \in \mathrm{B}_k^I$ that cannot be decomposed into the sum of two disjoint couples.

(*d*) Let W_k^I be the set of closed walks of length p in the graph I. To every closed walk $P = (c_1, r_1, c_2, r_2, \ldots, c_k, r_k)$ of length p we associate the couple $(\alpha, \beta) \in \mathrm{A}_k^I \times \mathrm{A}_k^I$ defined by

$$\alpha_q = \#\big[i \in \{1, \ldots, k\} : (r_i, c_i) = q\big]$$
$$\beta_q = \#\big[i \in \{1, \ldots, k\} : (r_i, c_{i+1}) = q\big] \quad \text{(where } c_{k+1} = c_1\text{.)}$$

We shall write $P \sim (\alpha, \beta)$ and call $n_{\alpha\beta}$ the number of elements of W_k^I mapped onto (α, β).

Note that the conditions in Eq. (B.12) is in fact symmetric and that it may be stated with β^1 and β^2 instead of α^1 and α^2.

Example 6.2. Let $R = C = \{0, 1, 2, 3\}$ and $I = R \times C$. The couple $(\mathrm{e}_{00} + \mathrm{e}_{11} + \mathrm{e}_{22} + \mathrm{e}_{33}, \mathrm{e}_{01} + \mathrm{e}_{10} + \mathrm{e}_{23} + \mathrm{e}_{32})$ is an element of $\mathrm{B}_4^I \setminus \mathscr{W}_4^I$: it is the sum of the two disjoint closed walk relations $(\mathrm{e}_{00} + \mathrm{e}_{11}, \mathrm{e}_{01} + \mathrm{e}_{10})$ and $(\mathrm{e}_{22} + \mathrm{e}_{33}, \mathrm{e}_{23} + \mathrm{e}_{32})$.

Example 6.3. Let $I = R \times C = \{0, 1\} \times \{0, 1\}$. Two closed walks are associated with the closed walk relation $(\mathrm{e}_{00} + \mathrm{e}_{11}, \mathrm{e}_{01} + \mathrm{e}_{10}) \in \mathscr{W}_2^I$: the two cycles $(\mathrm{col}\,0, \mathrm{row}\,0, \mathrm{col}\,1, \mathrm{row}\,1)$ and $(\mathrm{col}\,1, \mathrm{row}\,1, \mathrm{col}\,0, \mathrm{row}\,0)$. Six closed walks are mapped onto the closed walk relation $(2\mathrm{e}_{00} + 2\mathrm{e}_{01}, 2\mathrm{e}_{00} + 2\mathrm{e}_{01})$: the $\dfrac{4!}{2!\,2!}$ concatenations of a permutation of $(\mathrm{col}\,1, \mathrm{row}\,0)$, $(\mathrm{col}\,1, \mathrm{row}\,0)$, $(\mathrm{col}\,0, \mathrm{row}\,0)$, $(\mathrm{col}\,0, \mathrm{row}\,0)$.

The next proposition shows that, for our purpose, closed walk relations describe entirely closed walks.

Proposition 6.4. *Let $p = 2k \geqslant 0$ be an even integer and $I \subseteq R \times C$. The image of the mapping in Def. 6.1(d) is $\mathscr{W}_k^I$:*

(*a*) *if $P \in \mathrm{W}_k^I$ and $P \sim (\alpha, \beta)$, then $(\alpha, \beta) \in \mathscr{W}_k^I$;*

(*b*) *if $(\alpha, \beta) \in \mathscr{W}_k^I$, then there is a $P \in \mathrm{W}_k^I$ such that $P \sim (\alpha, \beta)$, so that $n_{\alpha\beta} \geqslant 1$.*

Proof. (*a*). Let $P = (c_1, r_1, c_2, r_2, \ldots, c_k, r_k)$. In fact,

$$\sum\nolimits_c \alpha_{rc} = \#[i \in \{1, \ldots, k\} : r_i = r] = \sum\nolimits_c \beta_{rc}$$
$$\sum\nolimits_r \alpha_{rc} = \#[i \in \{1, \ldots, k\} : c_i = c] = \sum\nolimits_r \beta_{rc}$$

and $(\alpha, \beta) \in \mathrm{B}_k^I$. If $(\alpha, \beta) = (\alpha^1, \beta^1) + (\alpha^2, \beta^2)$ with $(\alpha^i, \beta^i) \in \mathrm{B}_{k_i}^I$ and $k_i \geqslant 1$, there is an i such that $\alpha_{r_i c_i}^1 \geqslant 1$ and $\alpha_{r_{i+1} c_{i+1}}^2 \geqslant 1$ (where $(r_{i+1}, c_{i+1}) = (r_1, c_1)$ if $i = k$.) If $\beta_{r_i c_{i+1}}^1 \geqslant 1$, then $\sum_r \alpha_{r c_{i+1}}^1 \geqslant 1$, so that there is an r such that $\alpha_{r c_{i+1}}^1 \geqslant 1$. Otherwise $\beta_{r_i c_{i+1}}^2 \geqslant 1$, so that $\sum_c \alpha_{r_i c}^2 \geqslant 1$ and there is a c such that $\alpha_{r_i c}^2 \geqslant 1$. Therefore (α^1, β^1) and (α^2, β^2) are not disjoint and $(\alpha, \beta) \in \mathscr{W}_k^I$.

(*b*). We have to find a closed walk of length p that is mapped onto (α, β). If $k = 0$, the empty closed walk suits. Suppose that $k \geqslant 1$; Consider a walk $(c_1, r_1, c_2, r_2, \ldots, c_j, r_j, c_{j+1})$ in I such that $\alpha_q^1 = \#[i : (r_i, c_i) = q] \leqslant \alpha_q$ and $\beta_q^1 = \#[i : (r_i, c_{i+1}) = q] \leqslant \beta_q$ for every $q \in R \times C$, and furthermore j is maximal. We claim (*A*) that $c_{j+1} = c_1$ and (*B*) that $j = k$. Let $(\alpha^2, \beta^2) = (\alpha, \beta) - (\alpha^1, \beta^1)$.

(*A*). If $c_{j+1} \neq c_1$, then

$$\sum\nolimits_r \alpha_{r c_{j+1}}^1 = \#[i \in \{1, \ldots, j\} : c_i = c_{j+1}]$$
$$\sum\nolimits_r \beta_{r c_{j+1}}^1 = \#[i \in \{1, \ldots, j+1\} : c_i = c_{j+1}] = 1 + \sum\nolimits_r \alpha_{r c_{j+1}}^1,$$

so that there must be r_{j+1} with $\alpha_{r_{j+1} c_{j+1}}^2 \geqslant 1$. But then

$$\sum\nolimits_c \beta_{r_{j+1} c}^2 = \sum\nolimits_c \alpha_{r_{j+1} c}^2 \geqslant 1$$

and there must be c_{j+2} such that $\beta^2_{r_{j+1}c_{j+2}} \geqslant 1$: j is not maximal.

(B). Suppose that $j < k$. Then $(\alpha^1, \beta^1) \in \mathrm{B}^I_j$ and $(\alpha^2, \beta^2) \in \mathrm{B}^I_{k-j}$. By hypothesis, they are not disjoint: there are r, c, c' such that $\alpha^1_{rc}\alpha^2_{rc'} \geqslant 1$ or r, r', c such that $\alpha^1_{rc}\alpha^2_{r'c} \geqslant 1$. By interchanging R and C and by relabelling the vertices if necessary, we may suppose without loss of generality that for $r'_1 = r_j$ there is c'_1 such that $\alpha^2_{r'_1c'_1} \geqslant 1$. Then there is c'_2 such that $\beta^2_{r'_1c'_2} \geqslant 1$. By the argument used in Claim (A), there is a closed walk $(c'_1, r'_1, c'_2, \ldots, c'_{j'}, r'_{j'})$ such that $\#[i : (r'_i, c'_i) = q] \leqslant \alpha^2_q$ and $\#[i : (r'_i, c'_{i+1}) = q] \leqslant \beta^2_q$ (where $c'_{j'+1} = c'_1$.) Then the closed walk

$$(c_1, r_1, c_2, r_2, \ldots, c_j, r_j, c'_2, r'_2, \ldots, c'_{j'}, r'_{j'}, c'_1, r'_1)$$

shows that j is not maximal. □

We are now in position to state the following theorem, a matrix counterpart to the computation presented in [63, Prop. 2.5(ii)].

Theorem 6.5. *Let $p = 2k$ be a positive even integer and $I \subseteq R \times C$.*

(a) *The function Φ_I in Eq.* (B.10) *has the expansion*

$$\Phi_I(\epsilon, a) = \sum_{(\alpha,\beta)\in\mathscr{W}^I_k} n_{\alpha\beta}\epsilon^{\beta-\alpha}\overline{a}^\alpha a^\beta, \tag{B.13}$$

where $n_{\alpha\beta} \geqslant 1$ for every $(\alpha, \beta) \in \mathscr{W}^I_k$.

(b) *If $\epsilon \in \mathbb{T}^I$ and $a \in (\mathrm{S}^p)^I$ is finitely supported, then the function*

$$\Psi_I(\epsilon, a) = \mathrm{tr}\Big|\sum_{q\in I} \epsilon_q a_q \mathrm{e}_q\Big|^p \tag{B.14}$$

has the expansion

$$\sum_{(\alpha,\beta)\in\mathscr{W}^I_k} \epsilon^{\beta-\alpha} \sum_{(c_1,r_1,\ldots,c_k,r_k)\sim(\alpha,\beta)} \prod_{i=1}^k a^*_{r_ic_i}a_{r_ic_{i+1}} \ (\textit{with } c_{k+1} = c_1.) \tag{B.15}$$

Proof. This follows from Def. 6.1 and Prop. 6.4. □

Note that the edges of a closed walk $P \sim (\alpha, \beta)$ are precisely those $\{r, c\}$ such that $\alpha_{rc} + \beta_{rc} \geqslant 1$. P is a cycle if and only if P does not have length 0 or 2 and $\sum_r \alpha_{rc} \leqslant 1$ for all c and $\sum_c \alpha_{rc} \leqslant 1$ for all r. We now show how to decompose closed walks into cycles.

Proposition 6.6. *Let $P = (c_1, r_1, c_2, r_2, \ldots, c_k, r_k) \sim (\alpha, \beta)$ be a closed walk.*

(a) *If $r_i = r_j$ (vs. $c_i = c_j$) for some $i \neq j$, then P is the juxtaposition of two nonempty closed walks $P_1 \sim (\alpha^1, \beta^1)$ and $P_2 \sim (\alpha^2, \beta^2)$ such that $(\alpha, \beta) = (\alpha^1, \beta^1) + (\alpha^2, \beta^2)$ and $\sum_c \alpha^1_{r_ic}, \sum_c \alpha^2_{r_ic} \geqslant 1$ (vs. $\sum_r \alpha^1_{rc_i}, \sum_r \alpha^2_{rc_i} \geqslant 1$.)*

(b) *P is the juxtaposition of nonempty closed walks $P_j \sim (\alpha^j, \beta^j)$ such that $\sum_r \alpha^j_{rc} \leqslant 1$ for all c, $\sum_c \alpha^j_{rc} \leqslant 1$ for all r and $(\alpha, \beta) = \sum(\alpha^j, \beta^j)$.*

(c) *There are cycles $P_j \sim (\alpha^j, \beta^j)$ and a γ such that $(\alpha, \beta) = (\gamma, \gamma) + \sum(\alpha^j, \beta^j)$.*

Proof. (a). If $r_i = r_j$ for $i < j$, we may suppose that $j = k$: consider the closed walks $P_1 = (c_1, r_1, \ldots, c_i, r_i)$ and $P_2 = (c_{i+1}, r_{i+1}, \ldots, c_k, r_k)$. If $c_i = c_j$ for $i < j$, we may suppose that $i = 1$: consider then $P_1 = (c_1, r_1, \ldots, c_{j-1}, r_{j-1})$ and $P_2 = (c_j, r_j, \ldots, c_k, r_k)$.

(b). Use (a) in a maximality argument.

(c). Note that the closed walks P_j in (b) are either cycles or have length 2; in the latter case $P_j = q \sim (\mathrm{e}_q, \mathrm{e}_q)$ for some $q \in I$. □

7 Schur multipliers on a cycle

We can realise a cycle of even length $2s$, $s \geqslant 2$, in the following convenient way. Let $\Gamma = \mathbb{Z}/s\mathbb{Z}$. Then the adjacency relation of integers modulo s turns Γ into the cycle $(0, 1, \ldots, s-1)$ of length s. We double this cycle into the bipartite cycle $(\mathrm{col}\,0, \mathrm{row}\,0, \mathrm{col}\,1, \mathrm{row}\,1, \ldots, \mathrm{col}\,s-1, \mathrm{row}\,s-1)$ on $\Gamma \amalg \Gamma$, corresponding to the set of couples $I = \{(i,i), (i,i+1) : i \in \Gamma\} \subseteq \Gamma \times \Gamma$: I is the pattern

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ s{-}2 \\ s{-}1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & s{-}2 & s{-}1 \end{array} \\ \begin{pmatrix} * & * & 0 & \ddots & 0 & 0 \\ 0 & * & * & \ddots & 0 & 0 \\ 0 & 0 & * & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & * & * \\ * & 0 & 0 & \ddots & 0 & * \end{pmatrix} \end{array}.$$

Γ is the group dual to $G = \hat{\Gamma} = \{z \in \mathbb{C} : z^s = 1\}$. We shall consider the space $\mathrm{L}^p_\Lambda(G)$ spanned by $\Lambda = \{1, z\}$ in $\mathrm{L}^p(G)$, where z is the identical function on G: its norm is given by $\|a + bz\|_{\mathrm{L}^p(G)} = \bigl(s^{-1} \sum_{z^s=1} |a + bz|^p\bigr)^{1/p}$.

Proposition 7.1. *Let $0 < p \leqslant \infty$, $s \geqslant 2$ and $I = \{(i,i), (i,i+1) : i \in \mathbb{Z}/s\mathbb{Z}\}$. Let $\epsilon \in \mathbb{T}^I$ be a Schur multiplier by signs on S^p_I.*

(a) *The Schur multiplier ϵ has the same norm as the Schur multiplier $\hat{\epsilon}$ given by $\hat{\epsilon}_q = 1$ for $q \neq (s-1, 0)$ and $\hat{\epsilon}_{s-1,0} = \overline{\epsilon_{00}}\epsilon_{01} \ldots \overline{\epsilon_{s-1,s-1}}\epsilon_{s-1,0}$.*

(b) *The Schur multiplier ϵ has the same norm as $\check{\epsilon}$ given by $\check{\epsilon}_{ii} = 1$ and $\check{\epsilon}_{i,i+1} = \vartheta$ with ϑ any sth root of $\hat{\epsilon}_{s-1,0}$ or its complex conjugate: without loss of generality, $\vartheta = \mathrm{e}^{\mathrm{i}\alpha}$ with $\alpha \in [0, \pi/s]$.*

(c) *The norm of ϵ on S^p_I is bounded below by the norm of the relative Fourier multiplier $\mu : a+bz \mapsto a + \vartheta bz$ on $\mathrm{L}^p_\Lambda(G)$; their complete norms are equal.*

(d) *The norm of ϵ on S^1_I and on S^∞_I is equal to the norm of μ on $\mathrm{L}^1_\Lambda(G)$ and on $\mathrm{L}^\infty_\Lambda(G)$: this norm is*

$$\frac{\cos(\alpha/2 - \pi/2s)}{\cos \pi/2s} = \frac{\max_{z^s=-1} |\vartheta + z|}{|1 + \mathrm{e}^{\mathrm{i}\pi/s}|}.$$

(e) *The Schur multiplication operator M_ϵ is an isometry on S^p_I if and only if $p/2 \in \{1, 2, \ldots, s-1\}$ or $\overline{\epsilon_{00}}\epsilon_{01} \ldots \overline{\epsilon_{s-1,s-1}}\epsilon_{s-1,0} = 1$.*

Proof. (a) and (b) follow from the matrix unconditionality of Schatten-von-Neumann norms (see Eq. (B.4)) and from the fact that the Schur multipliers ϵ and $\bar{\epsilon} = (\overline{\epsilon_q})_{q \in I}$ have the same norm on S^p_I.

(c) follows from Prop. 2.5.

(d). Let us compute $f(\beta) = \|1 + \mathrm{e}^{\mathrm{i}\beta} z\|_{\mathrm{L}^1(G)}$. As $f(\beta) = f(\beta + 2\pi/s) = f(-\beta)$, we may suppose without loss of generality that $\beta \in [0, \pi/s]$. Then $|\beta/2 + k\pi/s| \leqslant \pi/2$ if $-\lfloor s/2 \rfloor \leqslant k \leqslant \lceil s/2 \rceil - 1$, so that

$$\begin{aligned} f(\beta) &= \frac{1}{s} \sum_{k=-\lfloor s/2 \rfloor}^{\lceil s/2 \rceil - 1} \bigl|1 + \mathrm{e}^{\mathrm{i}\beta} \mathrm{e}^{2\mathrm{i}k\pi/s}\bigr| \\ &= \frac{2}{s} \sum_{k=-\lfloor s/2 \rfloor}^{\lceil s/2 \rceil - 1} \cos(\beta/2 + k\pi/s) \\ &= \frac{2}{s} \Re\biggl(\mathrm{e}^{\mathrm{i}\beta/2} \frac{\mathrm{e}^{\mathrm{i}\lceil s/2 \rceil \pi/s} - \mathrm{e}^{-\mathrm{i}\lfloor s/2 \rfloor/s}}{\mathrm{e}^{\mathrm{i}\pi/s} - 1}\biggr) \\ &= \frac{2}{s \sin(\pi/2s)} \cdot \begin{cases} \cos(\beta/2 - \pi/2s) & \text{if } s \text{ is even} \\ \cos(\beta/2) & \text{if } s \text{ is odd.} \end{cases} \end{aligned}$$

This shows in both cases that the norm of μ on $\mathrm{L}^1_\Lambda(G)$ is bounded below by $\cos(\alpha/2 - \pi/2s)/\cos(\pi/2s)$. The complete norm of μ on $\mathrm{L}^\infty_\Lambda(G)$ is equal to its norm and thus to the maximum of

$g(w) = \|w + \vartheta z\|_{\mathrm{L}^\infty(G)} / \|w + z\|_{\mathrm{L}^\infty(G)}$ for $w \in \mathbb{C}$. Let $w = r\mathrm{e}^{\mathrm{i}\beta}$ with $r \geqslant 0$ and $\beta \in \mathbb{R}$. Note that

$$\|w + z\|_{\mathrm{L}^\infty(G)} = \left|r + \mathrm{e}^{\mathrm{i}\mathrm{d}(\beta,(2\pi/s)\mathbb{Z})}\right|$$

is a decreasing function of $\mathrm{d}(\beta, (2\pi/s)\mathbb{Z})$ and that

$$\mathrm{d}(\alpha - \beta, (2\pi/s)\mathbb{Z}) < \mathrm{d}(\beta, (2\pi/s)\mathbb{Z}) \ \Leftrightarrow\ \beta \in \left]\alpha/2, \pi/s + \alpha/2\right[\bmod 2\pi/s.$$

As $g(w) = g(wz)$ if $z^s = 1$, we may suppose without loss of generality that $\beta \in \left]\alpha/2, \pi/s + \alpha/2\right[$. Therefore

$$g(w) = \begin{cases} \left|\dfrac{w + \mathrm{e}^{\mathrm{i}\alpha}}{w + 1}\right| & \text{if } \beta \in \left]\alpha/2, \pi/s\right] \\ \left|\dfrac{w + \mathrm{e}^{\mathrm{i}\alpha}}{w + \mathrm{e}^{2\mathrm{i}\pi/s}}\right| & \text{if } \beta \in \left[\pi/s, \pi/s + \alpha/2\right[. \end{cases}$$

As g tends to 1 at infinity and $g(w) = 1$ if $\beta \in \{\alpha/2, \pi/s + \alpha/2\}$, the maximum principle shows that g attains its maximum with $\beta = \pi/s$. Finally,

$$\begin{aligned} g(r\mathrm{e}^{\mathrm{i}\pi/s})^2 &= \frac{1 + 2r\cos(\pi/s - \alpha) + r^2}{1 + 2r\cos(\pi/s) + r^2} \\ &= 1 + \frac{\cos(\pi/s - \alpha) - \cos\pi/s}{\cos(\pi/s) + (r + 1/r)/2} \leqslant g(\mathrm{e}^{\mathrm{i}\pi/s})^2 = \left(\frac{\cos(\pi/2s - \alpha/2)}{\cos\pi/2s}\right)^2. \end{aligned}$$

(e). If p is not an even integer and $\vartheta^s \neq 1$, then μ is not an isometry on $\mathrm{L}^p_\Lambda(G)$: otherwise the functions z and ϑz would have the same distribution by the Plotkin-Rudin Equimeasurability Theorem (see [48, Th. 2]). If $p \in \{2, 4, \dots, 2s - 2\}$, then I contains no cycle of length $4, 6, \dots, p$, so that by Prop. 6.6(c) every closed walk $P \sim (\alpha, \beta)$ satisfies $\alpha = \beta$. The function $\Phi_I(\epsilon, a)$ in Eq. (B.10) is therefore constant in ϵ by Th. 6.5(a). If $p \in \{2s, 2s + 2, \dots\}$, the closed walk relation

$$(\alpha, \beta) = \Big(\sum_{i\in\Gamma} \mathrm{e}_{ii}, \sum_{i\in\Gamma} \mathrm{e}_{i,i+1}\Big) + (p/2 - s)(\mathrm{e}_{00}, \mathrm{e}_{00})$$

satisfies $n_{\alpha\beta} \geqslant 1$ by Prop. 6.4. Then the coefficient of $\Phi_I(\epsilon, a)$ in $\bar{a}^\alpha a^\beta$ equals

$$n_{\alpha\beta}\overline{\epsilon_{00}}\epsilon_{01} \dots \overline{\epsilon_{s-1,s-1}}\epsilon_{s-1,0}$$

and must equal the same quantity with ϵ replaced by 1 if ϵ defines an isometry on S^p_I. ☐

Remark 7.2. See [50, p. 245] for a similar application of the Plotkin-Rudin Equimeasurability Theorem in (e).

The real unconditional constant of I is therefore the norm of $\check{\epsilon}$ with $\alpha = \pi/s$, and the complex unconditional constant is the maximum of the norm of $\check{\epsilon}$ for $\alpha \in [0, \pi/s]$. This yields

Corollary 7.3. *Let $0 < p \leqslant \infty$ and $s \geqslant 2$. Let I be the cycle of length $2s$.*

(a) *I is a real 1-unconditional basic sequence in S^p if and only if $p \in \{2, 4, \dots, 2s - 2\}$.*

(b) *The real and complex unconditional constants of I in the spaces S^1 and S^∞ equal $\sec\pi/2s$.*

8 1-unconditional matrices in S^p, p not an even integer

We now state the announced isometric counterpart to Varopoulos' characterisation of unconditional matrices in S^∞ (Section 5) and its generalisation to S^p for p not an even integer.

Theorem 8.1. *Let $I \subseteq R \times C$ be nonempty and $p \in (0, \infty] \setminus 2\mathbb{N}$. The following are equivalent.*

(a) *I is a complex completely 1-unconditional basic sequence in S^p.*

(b) *I is a complex 1-unconditional basic sequence in S^p.*

(c) *I is a real 1-unconditional basic sequence in S^p.*

(d) *I is a forest.*

(e) *For each $\epsilon \in \mathbb{T}^I$ there are $\zeta \in \mathbb{T}^C$ and $\eta \in \mathbb{T}^R$ such that $\epsilon_{rc} = \zeta(c)\eta(r)$ for all $(r, c) \in I$.*

(*f*) *For each* $\epsilon \in \{-1,1\}^I$ *there are* $\zeta \in \{-1,1\}^C$ *and* $\eta \in \{-1,1\}^R$ *such that* $\epsilon_{rc} = \zeta(c)\eta(r)$ *for all* $(r,c) \in I$.

(*g*) *I is a* set of V-interpolation of constant 1*: for all* $\varphi \in \ell^\infty_I$

$$\inf\Big\{\Big\|\sum_{(r,c)\in R\times C}\tilde\varphi_{rc}\mathrm{e}_c\otimes\mathrm{e}_r\Big\|_{\ell^\infty_C\overset{\wedge}{\otimes}\ell^\infty_R} : \tilde\varphi|_I=\varphi\Big\} = \sup_{q\in I}|\varphi_q|. \tag{B.16}$$

(*h*) *I is a* V-Sidon set of constant 1*: for all* $\varphi \in \mathrm{c}_0(I)$

$$\inf\Big\{\Big\|\sum_{(r,c)\in R\times C}\tilde\varphi_{rc}\mathrm{e}_c\otimes\mathrm{e}_r\Big\|_{\mathrm{c}_0(C)\overset{\wedge}{\otimes}\mathrm{c}_0(R)} : \tilde\varphi|_I=\varphi\Big\} = \sup_{q\in I}|\varphi_q|. \tag{B.17}$$

(*i*) *For every tensor* $u = \sum_{(r,c)\in I} a_{rc}\mathrm{e}_c\otimes\mathrm{e}_r$ *in* $\ell^1_C\overset{\vee}{\otimes}\ell^1_R$ *with support in I we have* $\|u\|_{\ell^1_C\overset{\vee}{\otimes}\ell^1_R} = \sum_{(r,c)\in I}|a_{rc}|$.

(*j*) $(z_cz'_r)_{(r,c)\in I}$ *is a* Sidon set of constant 1 *in the dual of* $\mathbb{T}^C\times\mathbb{T}^R$*, that is, a 1-unconditional basic sequence in* $\mathrm{C}(\mathbb{T}^C\times\mathbb{T}^R)$*: if* (a_{rc}) *is finitely supported,*

$$\sup_{(z,z')\in\mathbb{T}^C\times\mathbb{T}^R}\Big|\sum_{(r,c)\in I}a_{rc}z_cz'_r\Big| = \sum_{(r,c)\in I}|a_{rc}|\ .$$

(*k*) *For all* $R'\subseteq R$ *and* $C'\subseteq C$ *with* $k\geqslant 1$ *elements* $\#[I\cap R'\times C']\leqslant 2k-1$.

(*l*) *I is an* isometric interpolation set *for Schur multipliers on* S^∞*: every* $\varphi\in\ell^\infty_I$ *is the restriction of a Schur multiplier on* S^∞ *with norm* $\|\mathrm{M}_\varphi\| = \|\varphi\|_{\ell^\infty_I}$.

Proof. (*a*) ⇒ (*b*) ⇒ (*c*) is trivial.

(*c*) ⇒ (*d*). Suppose that I contains a cycle $(c_0, r_0, \ldots, c_{s-1}, r_{s-1})$ with $s\geqslant 2$. Cor. 7.3(*a*) shows that I is not a real 1-unconditional basic sequence in S^p.

(*d*) ⇔ (*k*). A tree on $2k$ vertices has exactly $2k-1$ edges, so that a forest I satisfies (*k*). Conversely, a cycle of length $2s$ is a graph with s row vertices, s column vertices and $2s$ edges.

(*d*) ⇒ (*e*). Suppose first that I is a tree and index the vertices of its edges by words $w\in W$ as described in the Terminology. Let us define η and ζ inductively. If r is the root of the tree, indexed by $\emptyset$, let $\eta(r)=1$. Suppose that η and ζ have been defined for all vertices indexed by words of length at most $2n$. If c is indexed by a word w of length $2n+1$, let r be the vertex indexed by the word of length $2n$ with which w begins and let $\zeta(c)=\epsilon(r,c)/\eta(r)$. If r is indexed by a word w of length $2n+2$, let c be the vertex indexed by the word of length $2n+1$ with which w begins and let $\eta(r)=\epsilon(r,c)/\zeta(c)$. If I is a union of pairwise disjoint trees, we may define η and ζ on each tree separately. We may finally extend η to R and ζ to C in an arbitrary manner.

(*d*) ⇒ (*f*) may be proved as (*d*) ⇒ (*e*).

(*f*) ⇒ (*c*). If (*f*) holds, then every Schur multiplier by signs $\epsilon\in\{-1,1\}^I$ is elementary in the sense that $\epsilon=\zeta\otimes\eta$. The complete norm of M_ϵ on any S^p_I is therefore $\|\zeta\|_{\ell^\infty_C}\|\eta\|_{\ell^\infty_R}=1$.

(*e*) ⇒ (*g*). If (*e*) holds, every $\varphi\in\mathbb{T}^I\subseteq\ell^\infty_I$ may be extended to an elementary tensor $\zeta\otimes\eta$ of norm 1. (*g*) follows because every element of ℓ^∞_I with norm 1 is the half sum of two elements of $\mathbb{T}^I$: note that $\mathrm{e}^{\mathrm{i}t}\cos u = \big(\mathrm{e}^{\mathrm{i}(t+u)}+\mathrm{e}^{\mathrm{i}(t-u)}\big)/2$.

(*g*) ⇒ (*h*). It suffices to check Equality (B.17) for φ with support contained in a finite rectangle set $R'\times C'$. As $\ell^\infty_{C'}\overset{\wedge}{\otimes}\ell^\infty_{R'}$ is a subspace of $\ell^\infty_C\overset{\wedge}{\otimes}\ell^\infty_R$, Eq. (B.16) yields Eq. (B.17).

(*h*) ⇔ (*i*) because they are dual statements.

(*i*) ⇔ (*j*). Use Equality (B.9).

(*h*) ⇒ (*l*) may be deduced by the argument of Prop. 3.1(*a*) ⇒ (*b*).

(*l*) ⇒ (*a*). Taking sign sequences $\varphi\in\mathbb{T}^I$ in (*l*) shows that all relative Schur multipliers by signs on S^∞_I define isometries. Apply Prop. 4.3. □

Remark 8.2. The equivalence of (*e*) with (*j*) may also be shown as a consequence of the characterisation of Sidon sets of constant 1 in [21].

Let us now answer Question 1.3.

Corollary 8.3. *Let* $I\subseteq R\times C$*. The following are equivalent.*

(*a*) *For all* $\varphi\in\mathrm{c}_0(I)$ *one has* $\big\|\sum_{(r,c)\in I}\varphi_{rc}\mathrm{e}_c\otimes\mathrm{e}_r\big\|_{\mathrm{c}_0(C)\overset{\wedge}{\otimes}\mathrm{c}_0(R)} = \sup_{q\in I}|\varphi_q|$.

(*b*) *There are pairwise disjoint sets $R_j \subseteq R$ and pairwise disjoint sets $C_j \subseteq C$ such that R_j or C_j is a singleton for each j and $I = \bigcup R_j \times C_j$: I is the union of the column section $\bigcup_{\# R_j = 1} R_j \times C_j$ with the disjoint row section $\bigcup_{\# R_j > 1} R_j \times C_j$.*

(*c*) *I is a union of pairwise disjoint star graphs: every path in I has length at most 2.*

Proof. $(a) \Rightarrow (b)$ follows from Prop. 3.1$(a) \Rightarrow (d)$ and Th. 8.1$(g) \Rightarrow (d)$.

$(b) \Leftrightarrow (c)$. (b) holds if and only if $(r,c), (r',c), (r,c') \in I \Rightarrow (r = r'$ or $c = c')$ and therefore if and only if (c) holds.

$(b) \Rightarrow (a)$. Suppose (b) and let $\varphi \in \mathrm{c}_0(I)$. Let $\alpha_j = \sup_{(r,c) \in R_j \times C_j} |\varphi_{rc}|^{1/2}$. If $\alpha_j = 0$, let us define $\varrho^j = 0$ and $\gamma^j = 0$. Otherwise, if R_j is a singleton $\{r\}$, let us define $\varrho^j = \alpha_j \mathrm{e}_r$ and γ^j by $\gamma^j_c = \varphi_{rc}/\alpha_j$ if $c \in C_j$ and $\gamma^j_c = 0$ otherwise. Otherwise, C_j is a singleton $\{c\}$ and we define $\gamma^j = \alpha_j \mathrm{e}_c$ and ϱ^j by $\varrho^j_r = \varphi_{rc}/\alpha_j$ if $r \in R_j$ and $\varrho^j_r = 0$ otherwise. Note that the γ^j have pairwise disjoint support and are null sequences, as well as the ϱ^j. Then

$$\sum_{(r,c) \in I} \varphi_{rc} \mathrm{e}_c \otimes \mathrm{e}_r = \sum_j \gamma^j \otimes \varrho^j = \operatorname*{average}_{\epsilon_j = \pm 1} \Big(\sum_j \epsilon_j \gamma^j \Big) \otimes \Big(\sum_j \epsilon_j \varrho^j \Big)$$

is an average of elementary tensors in $\mathrm{c}_0(C) \overset{\wedge}{\otimes} \mathrm{c}_0(R)$ of norm $\sup_{q \in I} |\varphi_q|$, so that this average is also bounded by this norm, which obviously is a lower bound. $\square$

9 1-unconditional matrices in S^p, p an even integer

Let us now prove Theorem 1.5 as a consequence of Theorem 6.5 together with Proposition 6.6(*c*).

Theorem 9.1. *Let $I \subseteq R \times C$ and $p = 2k$ a positive even integer. The following assertions are equivalent.*

(*a*) *I is a complex completely 1-unconditional basic sequence in S^p.*

(*b*) *I is a complex 1-unconditional basic sequence in S^p.*

(*c*) *For every finite subset $F \subseteq I$ there is an operator $x \in \mathrm{S}^p$, whose support S contains F, such that $\left\| \sum \epsilon_q x_q \mathrm{e}_q \right\|_p$ does not depend on the complex choice of signs $\epsilon \in \mathbb{T}^S$.*

(*d*) *I is a real 1-unconditional basic sequence in S^p.*

(*e*) *For every finite subset $F \subseteq I$ there is an operator $x \in \mathrm{S}^p$ with real matrix coefficients, whose support S contains F, such that $\left\| \sum \epsilon_q x_q \mathrm{e}_q \right\|_p$ does not depend on the real choice of signs $\epsilon \in \{-1,1\}^S$.*

(*f*) *Every closed walk $P \sim (\alpha, \beta)$ of length $2s \leqslant 2k$ in I satisfies $\alpha = \beta$.*

(*g*) *I does not contain any cycle of length $2s \leqslant 2k$ as a subgraph.*

(*h*) *For each $v, w \in V$ there is at most one path in I of length $l \leqslant k$ that joins v to w.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$, $(b) \Rightarrow (d) \Rightarrow (e)$ are trivial.

$(c) \Rightarrow (g)$. Suppose that I contains a cycle $P \sim (\gamma, \delta)$ of length $2s \leqslant 2k$: the corresponding set of couples is $F = \{q : \gamma_q + \delta_q = 1\}$. Let x be as in (c) and let $(\alpha, \beta) = (\gamma, \delta) + (k - s)(\mathrm{e}_q, \mathrm{e}_q)$ for some arbitrary $q \in F$. Then $(\alpha, \beta) \in \mathscr{W}^S_k$. Consider $f(\epsilon) = \left\| \sum \epsilon_q x_q \mathrm{e}_q \right\|^p_p$ as a function on the group $\mathbb{T}^S$. Then the Fourier coefficient $\widehat{f}(\epsilon^{\beta - \alpha})$ of f at the Steinhaus character $\epsilon^{\beta - \alpha}$ is, by Th. 6.5(*a*),

$$\begin{aligned} &\sum \big\{ n_{\varepsilon\zeta} \overline{x}^{\varepsilon} x^{\zeta} : (\varepsilon, \zeta) \in \mathscr{W}^S_k \text{ and } \zeta - \varepsilon = \beta - \alpha \big\} \\ &\qquad = \overline{x}^{\gamma} x^{\delta} \sum \big\{ n_{\varepsilon\zeta} \overline{x}^{\varepsilon - \gamma} x^{\zeta - \delta} : (\varepsilon, \zeta) \in \mathscr{W}^S_k \text{ and } \zeta - \delta = \varepsilon - \gamma \big\}. \end{aligned}$$

(Note that $\beta - \alpha = \delta - \gamma$.) As this last sum has only positive terms and contains at least the term corresponding to (α, β), f cannot be constant.

$(e) \Rightarrow (g)$. Let $P \sim (\gamma, \delta)$, $F = \{q : \gamma_q + \delta_q = 1\}$ and (α, β) be as in the proof of the implication $(c) \Rightarrow (h)$. Let x be as in (e). Consider $f(\epsilon) = \left\| \sum \epsilon_q x_q \mathrm{e}_q \right\|^p_p$ as a function on the group $\{-1,1\}^S$. Then the Fourier coefficient $\widehat{f}(\epsilon^{\beta - \alpha})$ of f at the Walsh character $\epsilon^{\beta - \alpha}$ is, by Th. 6.5(*a*),

$$\begin{aligned} &\sum \big\{ n_{\varepsilon\zeta} x^{\varepsilon + \zeta} : (\varepsilon, \zeta) \in \mathscr{W}^S_k \text{ and } \zeta - \varepsilon \equiv \beta - \alpha \pmod 2 \big\} \\ &\qquad = x^{\gamma + \delta} \sum \big\{ n_{\varepsilon\zeta} x^{\varepsilon + \zeta - \gamma - \delta} : (\varepsilon, \zeta) \in \mathscr{W}^S_k \text{ and } \zeta - \varepsilon \equiv \delta - \gamma \pmod 2 \big\}. \end{aligned}$$

As this last sum has only positive terms and contains at least the term corresponding to (α,β), f cannot be constant.

$(f) \Leftrightarrow (g)$. Apply Prop. 6.6(c).

$(g) \Leftrightarrow (h)$. If I contains a cycle $(v_0,\dots,v_{2s-1})$, then I contains two distinct paths $(v_0,\dots,v_s)$, $(v_0,v_{2s-1},\dots,v_s)$ of length s from v_0 to v_s. If I contains two distinct paths $(v_0,\dots,v_l)$, $(v'_0,\dots,v'_{l'})$ with $v_0=v'_0$, $v_l=v'_{l'}$ and $l,l'\leqslant k$, let a be minimal such that $v_a\neq v'_a$, let $b\geqslant a$ be minimal such that $v_b\in\{v'_a,\dots,v'_{l'}\}$ and let $d\geqslant a$ be minimal such that $v'_d=v_b$. Then $(v_{a-1},\dots,v_b,v'_{d-1},\dots,v'_a)$ is a cycle in I of length $2s\leqslant 2k$.

$(f)\Rightarrow(a)$ holds by Theorem 6.5(b): If each $(\alpha,\beta)\in\mathscr{W}^I_k$ satisfies $\alpha=\beta$, then Eq. (B.15) shows that $\Psi_I(\epsilon,z)$ as defined in Eq. (B.14) is constant in ϵ. ☐

Remark 9.2. The equivalence $(b)\Leftrightarrow(g)$ is a noncommutative analogue to [63, Prop. 2.5(ii)].

Remark 9.3. In [64, Th. 2.7], the condition of Th. 9.1(f) is visualised in another way: a closed walk $P=(c_1,r_1,\dots,c_s,r_s)\sim(\alpha,\beta)$ in $\mathbb{N}\times\mathbb{N}$ is considered as the polygonal closed curve γ in $\mathbb{C}$ with sides parallel to the coordinate axes whose successive vertices are $r_1+\mathrm{i}c_1$, $r_1+\mathrm{i}c_2$, $r_2+\mathrm{i}c_2$, $\dots$, $r_{s-1}+\mathrm{i}c_s$, $r_s+\mathrm{i}c_s$, $r_s+\mathrm{i}c_1$ and again $r_1+\mathrm{i}c_1$. Then $\alpha=\beta$ if and only if the index with respect to γ of every point not on γ is zero, if and only if γ can be shrunk to a point inside of the set of its points.

Remark 9.4. One cannot drop the assumption that x has real matrix coefficients in Th. 9.1(e). Consider a 2×2 matrix x. Then $\operatorname{tr}x^*x=\sum|x_q|^2$ and $\det x^*x=|x_{00}x_{11}-x_{01}x_{10}|^2$. This shows that if $\Re(\overline{x_{00}x_{11}}x_{01}x_{10})=0$, e.g. $x=\begin{pmatrix}1&1\\1&\mathrm{i}\end{pmatrix}$, then the singular values of x do not depend on the real sign of the matrix coefficients of x, whereas $(\mathrm{col}\,0,\mathrm{row}\,0,\mathrm{col}\,1,\mathrm{row}\,1)$ is a cycle of length 4.

Remark 9.5. Theorem 9.1$(h)\Rightarrow(a)$ is the isometric counterpart to [38, Th. 3.1], which shows in particular that I is an unconditional basic sequence in S^{2k} if the number of walks in I between two given vertices of length k and with no edge repeated has a uniform bound. The following combinatorial problem arises naturally: if I satisfies this latter condition, is it so that I is the union of a finite number of sets I_j such that there is at most one path of length at most k in I_j between two given vertices? In the simplest case, $k=2$, William Banks, Ilijas Farah, Asma Harcharras and Dominique Lecomte [5] have deduced from [86] that it is not so.

10 Metric unconditional approximation property for S^p_I

Let R,C be two copies of $\mathbb{N}$. It is well known that, apart from S^2, no S^p has an unconditional basis or just a local unconditional structure (see [79, § 4].) S^1 and S^∞ cannot even be embedded in a space with unconditional basis. If $1<p<\infty$, then S^p has the unconditional finite dimensional decomposition

$$\bigoplus_{n\in\mathbb{N}}\mathrm{S}^p_{\{(r,c):r\leqslant n,c=n\}}\oplus\mathrm{S}^p_{\{(r,c):r=n+1,c\leqslant n\}}$$

because the triangular projection associated to the idempotent Schur multiplier $(\chi_{r\leqslant c})$ is bounded on S^p.

Definition 10.1. Let X be a separable Banach space and $\mathbb{S}=\mathbb{T}$ (vs. $\mathbb{S}=\{-1,1\}$.)

- A sequence (T_k) of operators on X is an *approximating sequence* if each T_k has finite rank and $\|T_kx-x\|\to0$ for every $x\in X$. An approximating sequence of commuting projections is a *finite-dimensional decomposition.*
- ([72].) The *difference sequence* (ΔT_k) of (T_k) is given by $\Delta T_1=T_1$ and $\Delta T_k=T_k-T_{k-1}$ for $k\geqslant2$. X has the *unconditional approximation property* (*uap*) if there is an approximating sequence (T_k) such that for some constant D

$$\Big\|\sum_{k=1}^n\epsilon_k\Delta T_k\Big\|\leqslant D\quad\text{for all } n \text{ and } \epsilon_k\in\mathbb{S}.$$

The *complex (vs. real) unconditional constant* of (T_k) is the least such constant D.

– ([22, § 3], [32, § 8].) X has the *complex (vs. real) metric unconditional approximation property* (*muap*) if, for every $\delta > 0$, X has an approximating sequence with complex (vs. real) unconditional constant $1 + \delta$. By [22, Th. 3.8] and [32, Lemma 8.1], this is the case if and only if there is an approximating sequence (T_k) such that

$$\sup_{\epsilon \in \mathbb{S}} \|T_k + \epsilon(\mathrm{Id} - T_k)\| \longrightarrow 1. \tag{B.18}$$

X has (*muap*) if and only if, for every given $\delta > 0$, X is isometric to a 1-complemented subspace of a space with a $(1 + \delta)$-unconditional finite-dimensional decomposition [31, Cor. IV.4]. If X has (*muap*), then, for any given $\delta > 0$, X is isometric to a subspace of a space with a $(1+\delta)$-unconditional basis.

Example 10.2. The simplest example is the subspace in S^p of operators with an upper triangular matrix. In fact, if $I \subseteq R \times C$ is such that all columns $I \cap R \times \{c\}$ (vs. all rows $I \cap \{r\} \times C$) are finite, then S^p_I admits a 1-unconditional finite-dimensional decomposition in the corresponding finitely supported idempotent Schur multipliers $\chi_{I \cap R \times \{c\}}$ (vs. $\chi_{I \cap \{r\} \times C}$.)

Our results on complete 1-unconditional basic sequences yield the following theorem.

Theorem 10.3. *Let $1 \leqslant p \leqslant \infty$. Let $R_r \subseteq R$, $r \in \mathbb{N}$, be pairwise disjoint and finite. Let $C_c \subseteq C$, $c \in \mathbb{N}$, be pairwise disjoint and finite. Let $J \subseteq \mathbb{N} \times \mathbb{N}$ and $I = \bigcup_{(r,c) \in J} R_r \times C_c$. Then the sequence of Schur multipliers $(\chi_{R_r \times C_c})_{(r,c) \in J}$ forms a complex 1-unconditional finite-dimensional decomposition for S^p_I if and only if J is a forest or p is an even integer and J contains no cycle of length $4, 6, \dots, p$.*

We may always suppose that approximating sequences on spaces S^p_I are associated to Schur multipliers. More precisely, we have

Proposition 10.4. *Let $1 \leqslant p \leqslant \infty$ and $I \subseteq R \times C$. Let (T_n) be an approximating sequence on S^p_I. Then there is a sequence of Schur multipliers (φ_n) such that (M_{φ_n}) is an approximating sequence on S^p_I and such that if (T_n) satisfies* (B.18)*, then so does (M_{φ_n}).*

Proof. Let $\delta_n > 0$ be such that $\delta_n \to 0$. As T_n has finite rank, there is a finite $R_n \times C_n \subseteq R \times C$ such that the projection $P_{R_n \times C_n}$ of S^p onto $\mathrm{S}^p_{R_n \times C_n}$ defined by the Schur multiplier $\chi_{C_n} \otimes \chi_{R_n}$ satisfies $\|P_{R_n \times C_n} T_n - T_n\| < \delta_n$. Let φ_n be the Schur multiplier associated to $[T_n]_{R_n \times C_n}$. With the notation of Eq. (B.5),

$$M_{\varphi_n}(x) - x = \int_{\mathbb{T}^R} \mathrm{d}\eta \int_{\mathbb{T}^C} \mathrm{d}\zeta \, \mathrm{M}_{\zeta^* \otimes \eta^*}(P_{R_n \times C_n} T_n - \mathrm{Id})(\mathrm{M}_{\zeta \otimes \eta} x).$$

As $P_{R_n \times C_n} T_n$ tends to the identity uniformly on compact sets, this shows that M_{φ_n} is an approximating sequence. As

$$\mathrm{M}_{\varphi_n} + \epsilon(\mathrm{Id} - \mathrm{M}_{\varphi_n}) = [P_{R_n \times C_n} T_n + \epsilon(\mathrm{Id} - P_{R_n \times C_n} T_n)] \ ,$$

the norm of this operator is at most $\|T_n + \epsilon(\mathrm{Id} - T_n)\| + 2\delta_n$. □

This proposition shows together with Prop. 2.1 the following results.

Corollary 10.5. *Let $1 \leqslant p \leqslant \infty$ and $I \subseteq R \times C$.*

– *If S^p_I has (muap), then some sequence of Schur multipliers realises it.*
– *Let $J \subseteq I$. If S^p_I has (muap), then so does S^p_J.*
– *If S^∞_I has (muap), then so does S^p_I.*

Let us define the following asymptotic properties.

Definition 10.6. Let $1 \leqslant p \leqslant \infty$, $I \subseteq R \times C$ and $\mathbb{S} = \mathbb{T}$ (vs. $\mathbb{S} = \{-1, 1\}$.)

– S^p_I is *asymptotically unconditional* if for every $x \in \mathrm{S}^p_I$ and for every bounded sequence (y_n) in S^p_I such that each matrix coefficient of y_n tends to 0

$$\max_{\epsilon \in \mathbb{S}} \|x + \epsilon y_n\|_p - \min_{\epsilon \in \mathbb{S}} \|x + \epsilon y_n\|_p \longrightarrow 0.$$

– I enjoys the property $(\mathscr{U})$ of *block unconditionality* in S^p if for each $\delta > 0$ and finite $F \subseteq I$, there is a finite $G \subseteq I$ such that

$$\forall\, x \in B_{\mathrm{S}^p_F}\ \forall\, y \in B_{\mathrm{S}^p_{I\setminus G}} \qquad \max_{\epsilon\in\mathbb{S}} \|x+\epsilon y\|_p - \min_{\epsilon\in\mathbb{S}} \|x+\epsilon y\|_p < \delta.$$

The arguments of [63, § 6.2] show mutatis mutandis

Theorem 10.7. *Let $1 \leqslant p \leqslant \infty$, $I \subseteq R \times C$ and $\mathbb{S} = \mathbb{T}$ (vs. $\mathbb{S} = \{-1, 1\}$.) Consider the following properties.*

(*a*) *S^p_I is asymptotically unconditional.*

(*b*) *I enjoys $(\mathscr{U})$ in S^p.*

(*c*) *S^p_I has (muap).*

Then $(c) \Rightarrow (a) \Leftrightarrow (b)$. If $1 < p < \infty$, then $(b) \Leftrightarrow (c)$. If $p = 1$, S^1_I has (muap) if and only if S^1_I has (uap) and I enjoys $(\mathscr{U})$ in S^1.

The case $p = \infty$ is extreme in the sense that the following properties are equivalent for S^∞_I: to be a dual space, to be reflexive, to have a finite cotype, not to contain c_0, because they are equivalent for I not to contain any sequence (r_n, c_n) with (r_n) and (c_n) injective, that is for I to be contained in the union of a finite set of lines and a finite set of columns, so that S^∞_I is isomorphic to ℓ^2_I.

Let us now introduce the asymptotic property on I that reflects the combinatorics imposed by (*muap*).

Definition 10.8. Let $I \subseteq R \times C$ and $k \geqslant 1$.

– I enjoys property $\mathscr{I}_k$ if for every path $P = (c_0, r_0, \ldots, c_j, r_j)$ of odd length $2j + 1 \leqslant k$ in I there is a finite set $R' \times C'$ such that P cannot be completed with edges in $I \setminus R' \times C'$ to a cycle of length $2s \in \{4j + 2, \ldots, 2k\}$.

– The *asymptotic distance* $d_\infty(r, c)$ of $r \in R$ and $c \in C$ in I is the supremum, over all finite rectangle sets $R' \times C'$, of the distance from r to c in $I \setminus R' \times C'$.

The asymptotic distance takes its values in $\{1, 3, 5, \ldots, \infty\}$. Note that $\mathscr{I}_1$ is true and that $\mathscr{I}_k \Rightarrow \mathscr{I}_{k-1}$. This implication is strict: let R, C be two copies of $\mathbb{N}$ and, given $j \geqslant 1$, consider the union I_j of all paths $(\mathrm{col}\,0, \mathrm{row}\,nj + 1, \mathrm{col}\,nj + 1, \ldots, \mathrm{row}\,nj + j, \mathrm{col}\,nj + j, \mathrm{row}\,0)$ of length $2j + 1$. Then I_j contains no cycle of length $2s \in \{4, \ldots, 4j\}$ and therefore enjoys $\mathscr{I}_{2j}$, but fails $\mathscr{I}_{2j+1}$; $I_j \cup \{(\mathrm{row}\,0, \mathrm{col}\,0)\}$ contains no cycle of length $2s \in \{4, \ldots, 2j\}$ and thus enjoys $\mathscr{I}_j$, but fails $\mathscr{I}_{j+1}$. In particular, the properties $\mathscr{I}_k$, $k \geqslant 2$, are not stable under union with a singleton.

Let us now explicit the relationship between $\mathscr{I}_k$ and d_∞.

Proposition 10.9. *Let $I \subseteq R \times C$ and $k \geqslant 1$.*

(*a*) *I enjoys $\mathscr{I}_k$ if and only if any two vertices $r \in R$ and $c \in C$ at distance $2j + 1 \leqslant k$ satisfy $d_\infty(r, c) \geqslant 2k - 2j + 1$.*

(*b*) *If $d_\infty(r, c) \geqslant 2k + 1$ for all $(r, c) \in R \times C$, then I enjoys $\mathscr{I}_k$.*

(*c*) *If $d_\infty(r, c) \leqslant k$ for some $(r, c) \in R \times C$, then I fails $\mathscr{I}_k$.*

(*d*) *I enjoys $\mathscr{I}_k$ for every k if and only if $d_\infty(r, c) = \infty$ for every $(r, c) \in R \times C$.*

Proof. (a) is but a reformulation of the definition of $\mathscr{I}_k$ and implies (b).

(d) is a consequence of (b) and (c).

(c). If $d_\infty(r, c) \leqslant k$, then there is $0 \leqslant j \leqslant (k - 1)/2$ such that there are infinitely many paths of length $2j + 1$ from c to r: there is a path $(c, r_1, c_1, \ldots, r_j, c_j, r)$ that can be completed with edges outside any given finite set to a cycle of length $4j + 2 \leqslant 2k$. □

Theorem 10.10. *Let $I \subseteq R \times C$ and $1 \leqslant p \leqslant \infty$. If p is an even integer, then S^p_I has complex or real (muap) if and only if I enjoys $\mathscr{I}_{p/2}$. If $p = \infty$ or if p is not an even integer, then S^p_I has real (muap) only if I enjoys $\mathscr{I}_k$ for every k.*

Proof. Suppose that I enjoys $(\mathscr{U})$ in S^p and fails $\mathscr{I}_k$. Then, for some $s \leqslant k$, I contains a sequence of cycles $(c_0, r_0, \ldots, c_{j-1}, r_{j-1}, c^n_j, r^n_j, \ldots, c^n_{s-1}, r^n_{s-1})$ with the property that $\|x-y\|_p \leqslant (1+1/n)\|x+y\|_p$ for all x with support in $\{(r_0, c_0), (r_0, c_1), \ldots, (r_{j-2}, c_{j-1}), (r_{j-1}, c_{j-1})\}$ and all y with support in $\{(r_{j-1}, c^n_j), (r^n_j, c^n_j), \ldots, (r^n_{s-1}, c^n_{s-1}), (r^n_{s-1}, c_0)\}$. With the notation of Section 7, this amounts to stating that the multiplier on $I = \{(i, i), (i, i + 1)\} \subseteq \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/s\mathbb{Z}$ given by $\epsilon_{rc} = 1$ if $r, c \in$

$\{0,\ldots,j-1\}$ and $\epsilon_{rc}=-1$ otherwise actually is an isometry on S^p_I. As $\overline{\epsilon_{00}}\epsilon_{01}\ldots\overline{\epsilon_{s-1\,s-1}}\epsilon_{s-1\,0}=(-1)^{2s-2j+1}=-1$, this implies by Prop. 7.1(e) that $p/2\in\{1,2,\ldots,s-1\}$.

Suppose that I enjoys $\mathscr{J}_k$. We claim that for every finite $F\subseteq I$ there is a finite $G\subseteq I$ such that every closed walk $P\sim(\alpha,\beta)$ of length $2k$ in I satisfies $\sum_{q\in I\setminus G}\beta_q-\alpha_q=0$. This signifies that given a closed walk $(v_0,\ldots,v_{2k-1})$ and $0=a_0<b_0<\cdots<a_m<b_m<a_{m+1}=2k$ such that $v_{a_i},\ldots,v_{b_i-1}\in I\setminus G$ and $v_{b_i},\ldots,v_{a_{i+1}-1}\in F$,

$$\bigl\{i\in\{0,\ldots,m\}:a_i,b_i\text{ even}\bigr\}=\bigl\{i\in\{0,\ldots,m\}:a_i,b_i\text{ odd}\bigr\}.$$

Suppose that this is not true: then there is an $s\leqslant k$, there are $0=a_0<b_0<\cdots<a_m<b_m<2s$ and there are cycles $(v^n_{a_0},\ldots,v^n_{b_0-1},v_{b_0},\ldots,v_{a_1-1},\ldots,v^n_{a_m},\ldots,v^n_{b_m-1},v_{b_m},\ldots,v_{2s-1})$ such that the $(v^n_i)_{n\geqslant0}$ are injective sequences of vertices and b_i-a_i is even for at least one index i: let us suppose so for $i=0$. If $b_0-a_0\geqslant s-1$, consider the path $P=(v_{b_0},\ldots,v_{a_0-1},v^0_{a_0},\ldots,v^0_{b_m-1},v_{b_m},\ldots,v_{2s-1})$ of odd length $2s-1-(b_0-a_0)$; if $b_0-a_0\leqslant s-1$, consider the path $P=(v_{2s-1},v^0_{a_0},\ldots,v^0_{b_0-1},v_{b_0})$ of odd length b_0-a_0+1. Then P can be completed with vertices outside any given finite set to a cycle of length at most $2s$ because $(v_{2s-1},v^n_{a_0},\ldots,v^n_{b_0-1},v_{b_0})$ is a path of length b_0-a_0+1 in I for every n. This proves that I fails $\mathscr{J}_s$.

The claim shows that I enjoys $(\mathscr{U})$ in S^p for $p=2k$. In fact, if $\tilde\epsilon\in\mathbb{T}^{F\cup(I\setminus G)}$ is defined by $\tilde\epsilon_q=1$ for $q\in F$ and $\tilde\epsilon_q=\epsilon\in\mathbb{T}$ for $q\in I\setminus G$, then, with the notation of Th. 6.5,

$$\Phi_{F\cup(I\setminus G)}(\tilde\epsilon,a)=\sum_{(\alpha,\beta)\in\mathscr{W}_k^{F\cup(I\setminus G)}}n_{\alpha\beta}\epsilon^{\sum_{q\in I\setminus G}\beta_q-\alpha_q}\overline{a}^{\alpha}a^{\beta}$$

does not depend on ϵ, so that $\|x+\epsilon y\|_{2k}=\|x+y\|_{2k}$ if $x\in\mathrm{S}^{2k}_F$ and $y\in\mathrm{S}^{2k}_{I\setminus G}$, and S^{2k}_I has complex ($muap$) by Th. 10.7(b) $\Rightarrow$ (c). $\square$

Remark 10.11. This theorem is a noncommutative analogue to [63, Th. 7.5].

11 Examples

One of Varopoulos' motivations for the study of the projective tensor product $\ell_\infty\mathbin{\hat\otimes}\ell_\infty$ are lacunary sets in a locally compact abelian group.

Let Γ be a discrete abelian group and $\Lambda\subseteq\Gamma$. Let us say that Λ is *n-independent* if every element of Γ admits at most one representation as the sum of n terms in Λ, up to a permutation. For example, the geometric sequence $\{j^k\}_{k\geqslant0}$ with $j\in\{2,3,\ldots\}$ is n-independent in $\mathbb{Z}$ if and only if $j\geqslant n$ [63, § 3]. If Λ is n-independent for all n, then Λ is *independent*. Let

$$\mathrm{Z}_n=\{\zeta\in\mathbb{Z}^\Lambda:\sum_{\gamma\in\Lambda}\zeta_\gamma=0\text{ and }\sum_{\gamma\in\Lambda}|\zeta_\gamma|\leqslant 2n]$$

and $\mathrm{Z}=\bigcup\mathrm{Z}_n$. Then Λ is n-independent if and only if, for every $\zeta\in\mathrm{Z}_n$,

$$\sum_{\gamma\in\Lambda}\zeta_\gamma\gamma=0\ \Rightarrow\ \zeta=0$$

and Λ is independent if and only if this holds for every $\zeta\in\mathrm{Z}$.

Let us say that Λ is *n-independent modulo 2* if in every representation of an element of Γ as the sum of n terms in Λ, each element of Λ appears the same number of times modulo 2. In other words, for every $\zeta\in\mathrm{Z}_n$,

$$\sum_{\gamma\in\Lambda}\zeta_\gamma\gamma=0\quad\Longrightarrow\quad\forall\,\gamma\in\Lambda\quad\zeta_\gamma=0\pmod 2\ ;$$

Λ is *independent modulo 2* if this holds for every $\zeta\in\mathrm{Z}$. If Γ contains no element of order 2, then one may always suppose that at least one coefficient ζ_γ of a nontrivial relation $\sum\zeta_\gamma\gamma=0$ is odd, so that these two latter notions "modulo 2" coincide with the two former ones.

Let $G=\hat\Gamma$, so that Γ is the group of characters on G. Then the computation presented in [63, Prop. 2.5(ii)] for the case $\Gamma=\mathbb{Z}$ shows that Λ is a complex (vs. real) 1-unconditional basic sequence in $\mathrm{L}^p(G)$ with $p\in2\mathbb{N}^*$ if and only if Λ is $p/2$-independent (vs. modulo 2). Furthermore Λ is a complex (vs. real) 1-unconditional basic sequence in $\mathrm{L}^p(G)$ with $p \in (0,\infty] \setminus 2\mathbb{N}^*$ if and only if Λ is independent (vs. modulo 2). If Γ contains no element of order 2, then a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$ is also complex 1-unconditional. All these results hold also for the complete counterparts to 1-unconditional basic sequences.

Results on lacunary sets in a discrete abelian group transfer to lacunary matrices in the following way, as in [97, Th. 4.2].

Proposition 11.1. *Let Γ be a discrete abelian group and R, C be countable subsets of Γ. To every $\Lambda \subseteq R + C$ associate $I_\Lambda = \{(r,c) \in R \times C : r + c \in \Lambda\}$. Let $G = \hat{\Gamma}$.*

- (a) *If Λ is a complex 1-unconditional basic sequence in $\mathrm{L}^4(G)$, then I_Λ is a 1-unconditional basic sequence in S^4.*
- (b) *Suppose that each element of Γ admits at most one representation as the sum of an element of R with an element of C. Then every $I \subseteq R \times C$ has the form $I = I_\Lambda$ with $\Lambda = \{r + c : (r,c) \in I\}$. If Λ is a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$, then I_Λ is a 1-unconditional basic sequence in S^p.*
- (c) *Let $p = 2k$ be a positive even integer. Suppose that $R \cap C = \emptyset$ and $R \cup C$ is k-independent modulo 2. I_Λ is a 1-unconditional basic sequence in S^p if and only if Λ is a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$.*

Proof. (a). Let $P = (c, r, c', r')$ be a closed walk in I_Λ. Then $r + c$, $r' + c'$, $r + c'$ and $r' + c$ are in Λ while $(r + c) + (r' + c') = (r + c') + (r' + c)$: if Λ is 2-independent, then $r + c \in \{r + c', r' + c\}$, so that $c = c'$ or $r = r'$ and P is not a cycle.

(b). For each $\gamma \in \Lambda$, let $q_\gamma = (r_\gamma, c_\gamma)$ be the unique element of I such that $r_\gamma + c_\gamma = \gamma$. If Λ is a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$, then it is also a complete real 1-unconditional basic sequence in $\mathrm{L}^p(G)$. Let $\varphi \in \{-1,1\}^{I_\Lambda}$, so that $\varphi_{q_\gamma} \in \{-1,1\}$ for all $\gamma \in \Lambda$. Then, as in Eq. (B.6),

$$\Big\|\sum_{q\in I_\Lambda} a_q \mathrm{e}_q\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)} = \Big\|\sum_{(r,c)\in I_\Lambda} r(g)c(g)a_{rc}\mathrm{e}_{rc}\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda} a_{q_\gamma}\mathrm{e}_{q_\gamma}\gamma(g)\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda} a_{q_\gamma}\mathrm{e}_{q_\gamma}\gamma\Big\|_{\mathrm{L}^p_\Lambda(G,\mathrm{S}^p(\mathrm{S}^p))}$$

so that as in Eq. (B.7), by complete real 1-unconditionality of Λ in $\mathrm{L}^p(G)$,

$$\Big\|\sum_{q\in I_\Lambda} \varphi_q a_q \mathrm{e}_q\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)} = \Big\|\sum_{\gamma\in\Lambda} \varphi_{q_\gamma} a_{q_\gamma}\mathrm{e}_{q_\gamma}\gamma\Big\|_{\mathrm{L}^p_\Lambda(G,\mathrm{S}^p(\mathrm{S}^p))} = \Big\|\sum_{q\in I_\Lambda} a_q \mathrm{e}_q\Big\|_{\mathrm{S}^p_{I_\Lambda}(\mathrm{S}^p)}.$$

(c). Each element of Γ admits at most one representation as the sum of an element of R with an element of C, so that (b) yields sufficiency. Suppose that Λ is not a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$ and let $\zeta \in \mathrm{Z}_k$ such that $\sum_{\gamma\in\Lambda} \zeta_\gamma \gamma = 0$ and $J = \{(r,c) \in I_\Lambda : \zeta_{r+c} \neq 0 \pmod 2\}$ is nonempty; J has at most $2k$ elements. Let $P = (v_1, \ldots, v_j)$ be a path in J of maximal length. Then $\zeta_{v_{j-1}+v_j}$ is odd and $\sum\{\zeta_{v_j+v} : v_j + v \in \Lambda\}$ is even because it is the coefficient of v_j in the relation $\sum_{\gamma\in\Lambda} \zeta_\gamma\gamma = 0$ and $R \cup C$ is k-independent modulo 2. There is therefore v_{j+1} distinct from v_{j-1} such that $\zeta_{v_j+v_{j+1}}$ is odd. As j is maximal and $R \cap C = \emptyset$, $v_{j+1} = v_{j+1-2i}$ for some $2 \leqslant i \leqslant k$, so that $(v_{j+1-2i}, \ldots, v_j)$ is a cycle of length $2i$ in J: I_Λ is not a 1-unconditional basic sequence in S^p. □

Let R and C be any countable sets. Consider $G = \{-1,1\}^C \times \{-1,1\}^R$. If we denote by $\big((\epsilon_c)_{c\in C}, (\epsilon'_r)_{r\in R}\big)$ a generic point in G, then the set of Rademacher functions $\{\epsilon_c\}_{c\in C} \cup \{\epsilon'_r\}_{r\in R}$ is a real 1-unconditional basic sequence in $\mathrm{C}(G)$, so that it is independent modulo 2 in $\hat{G}$. Similarly, the set of Steinhaus functions $\{z_c\}_{c\in C} \cup \{z'_r\}_{r\in R}$ is independent in the dual of $\mathbb{T}^C \times \mathbb{T}^R$. This yields:

Corollary 11.2. *Let $I \subseteq R \times C$ and $p \in (0,\infty]$. The following are equivalent:*

- *I is a 1-unconditional basic sequence in S^p.*
- *$\{\epsilon_c\epsilon'_r : (r,c) \in I\}$ is a real 1-unconditional basic sequence in $\mathrm{L}^p(G)$.*
- *$\{z_c z'_r : (r,c) \in I\}$ is a 1-unconditional basic sequence in $\mathrm{L}^p(\mathbb{T}^C \times \mathbb{T}^R)$.*

Remark 11.3. The isomorphic counterpart is also true: I is a completely unconditional basic sequence in S^p (i.e., a complete $\sigma(p)$ set) if and only if $\{\epsilon_c\epsilon'_r : (r,c) \in I\}$ is a completely unconditional basic sequence in $\mathrm{L}^p(G)$ (a $\Lambda(p)_{\mathrm{cb}}$ *set* in $\hat{G}$, see [37] and [76, § 8.1],) if and only if $\{z_c z'_r : (r,c) \in I\}$ is a completely unconditional basic sequence in $\mathrm{L}^p(\mathbb{T}^C \times \mathbb{T}^R)$. This follows e.g. from the proof of Prop. 11.1(b) and the iterated noncommutative Khinchin inequality [76, Eq. (8.4.11)].

Harcharras [37] used Peller's discovery [73] of the link between Fourier and Hankel Schur multipliers to produce unconditional basic sequences in S^p that are unions of antidiagonals in $\mathbb{N} \times \mathbb{N}$. We have in our context the rather disappointing

Proposition 11.4. *Let $\Lambda \subseteq \mathbb{N} \subseteq \mathbb{Z}$ and $I = \{(r,c) \in \mathbb{N} \times \mathbb{N} : r + c \in \Lambda\}$.*

(a) *I is a 1-unconditional basic sequence in S^4 if and only if $\{z^\lambda\}_{\lambda \in \Lambda}$ is a 1-unconditional basic sequence in $\mathrm{L}^4(\mathbb{T})$.*

(b) *If Λ contains three elements $\lambda < \mu < \nu$ such that $\lambda + \mu \geqslant \nu$, then I is not a 1-unconditional basic sequence in S^p if $p \in (0, \infty] \setminus \{2, 4\}$.*

(c) *If $\Lambda = \{\lambda_k\}$ with $\lambda_{k+1} > 2\lambda_k$ for all k, then I is a 1-unconditional basic sequence in S^p for every p.*

Proof. (a). Sufficiency follows from Prop. 11.1(a) with $R = C = \mathbb{N}$. Conversely, if Λ contains a solution to $\lambda + \mu = \lambda' + \mu'$ with $\lambda < \lambda' \leqslant \mu' < \mu$, then I contains the cycle $(\mathrm{col}\, 0, \mathrm{row}\, \lambda, \mathrm{col}\, \lambda' - \lambda, \mathrm{row}\, \mu')$.

(b). Consider the cycle $(\mathrm{col}\, 0, \mathrm{row}\, \lambda, \mathrm{col}\, \nu - \lambda, \mathrm{row}\, \mu - \nu + \lambda, \mathrm{col}\, \nu - \mu, \mathrm{row}\, \mu)$.

(c). In fact, I is a forest. Let $P = (c_1, r_1, \ldots, c_k, r_k)$ be a closed walk in I. We may suppose without loss of generality that $r_1 + c_2$ is a maximal element of $\{r_1 + c_1, r_1 + c_2, \ldots, r_k + c_k, r_k + c_1\}$. Then $r_1 + c_1 \leqslant r_1 + c_2$ and $r_2 + c_2 \leqslant r_1 + c_2$. One of these inequalities must be an equality and P is not a cycle: for otherwise $2(r_1 + c_1) < r_1 + c_2$ and $2(r_2 + c_2) < r_1 + c_2$ because $r_1 + c_1, r_1 + c_2, r_2 + c_2 \in \Lambda$, so that $2(r_1 + c_1 + r_2 + c_2) < 2(r_1 + c_2)$ and $c_1 + r_2 < 0$. ☐

Remark 11.5. Further computations yield the following result. If $\{z^\lambda\}_{\lambda \in \Lambda}$ is a 1-unconditional basic sequence in $\mathrm{L}^6(\mathbb{T})$ and if $\{\lambda < \mu < \nu\} \subseteq \Lambda \Rightarrow \lambda + \mu < \nu$, then I is a 1-unconditional basic sequence in S^6; the converse does not hold.

Let us now give an overview of the known extremal bipartite graphs without cycle of length $4, 6, \ldots, 2k$ and their size. Look up [9, Def. I.3.1] for the definition of a Steiner system and [56, Def. 1.3.1] for the definition of a generalised polygon. An elementary example is given in the introduction with (B.1).

Proposition 11.6. *Let $2 \leqslant n \leqslant m$, $I \subseteq R \times C$ with $\# C = n$ and $\# R = m$, and $e = \# I$.*

(a) *If I is a 1-unconditional basic sequence in S^4, then*

$$n \geqslant 1 + \Big(\frac{e}{m} - 1\Big) + \Big(\frac{e}{m} - 1\Big)\Big(\frac{e}{n} - 1\Big),$$

that is $e^2 - me - mn(n-1) \leqslant 0$. Equality holds if and only if I is the incidence graph of a Steiner system $\mathrm{S}(2, e/m; n)$ on n points and m blocks.

(b) *If I is a 1-unconditional basic sequence in S^6, then*

$$n \geqslant 1 + \Big(\frac{e}{m} - 1\Big) + \Big(\frac{e}{m} - 1\Big)\Big(\frac{e}{n} - 1\Big) + \Big(\frac{e}{m} - 1\Big)^2\Big(\frac{e}{n} - 1\Big),$$

that is $e^3 - (m+n)e^2 + 2mne - m^2n^2 \leqslant 0$. Equality holds if and only if I is the incidence graph of the quadrangle *(the cycle of length 8) or of a generalised quadrangle with n points and m lines.*

(c) *If I is a 1-unconditional basic sequence in S^{2k} with $k \geqslant 1$ an integer, then*

$$n \geqslant \sum_{i=0}^{k} \Big(\frac{e}{m} - 1\Big)^{\lceil \frac{i}{2} \rceil}\Big(\frac{e}{n} - 1\Big)^{\lfloor \frac{i}{2} \rfloor}. \tag{B.19}$$

Equality holds if I is the incidence graph of the $(k+1)$-gon *(the cycle of length $2k+2$) or of a generalised $(k+1)$-gon with n points and m lines.*

Proof. By Theorem 9.1$(b) \Rightarrow (g)$, I is a 1-unconditional basic sequence in S^{2k}, with $k \geqslant 1$ an integer, if and only if I is a graph of girth $2k+2$ in the sense of [44]. Therefore (a) and (b) are shown in [65, Prop. 4, Th. 8, Rem. 10]. Inequality (B.19) is [44, Eq. (1)] and the sufficient condition for equality follows from [56, Lemma 1.5.4]. □

Consult [9, Tables A1.1, A5.1] for examples of Steiner systems and [56, Table 2.1] for examples of generalised polygons. In both cases, the corresponding incidence graph is biregular: every vertex in R has same degree $s+1$ and every vertex in C has same degree $t+1$. Arbitrarily large generalised $(k+1)$-gons exist only if $2k \in \{4, 6, 10, 14\}$ [56, Lemma 1.7.1]; for $2k \in \{6, 10, 14\}$, it follows from [56, Lemma 1.5.4] that

$$n = (s+1)\frac{(st)^{(k+1)/2} - 1}{st - 1}, \ m = (t+1)\frac{(st)^{(k+1)/2} - 1}{st - 1}.$$

Remark 11.7. Let $I \subseteq R \times C$ with $\# C = \# R = n$. Inequality (B.19) shows that if I is a 1-unconditional basic sequence in S^{2k}, then $\# I \leqslant n^{1+1/k} + (s-1)n/s$. If $p \notin \{4, 6, 10\}$, the existence of 1-unconditional basic sequences in S^{2k} such that $\# I \succcurlyeq n^{1+1/k}$ is in fact an important open problem in graph theory: extremal graphs cannot correspond to generalised polygons and necessarily have less structure.

Acknowledgement. I would like to thank Éric Ricard for his comments and suggestions.

Keywords. Schatten-von-Neumann class, Schur product, graph with a given girth, 1-unconditional basic sequence, metric unconditional approximation property, V-Sidon set, lacunary set.

2000 *Mathematics Subject Classification.* 47B10, 46B15, 46B04, 43A46, 05C38, 46B28.

Chapter C

Matrix inequalities with applications to the theory of iterated kernels

with William Banks, Asma Harcharras and Éric Ricard.

For an $m \times n$ matrix A with nonnegative real entries, Atkinson, Moran and Watterson proved the inequality $s(A)^3 \leqslant mns(AA^tA)$, where A^t is the transpose of A, and $s(\cdot)$ is the sum of the entries. We extend this result to finite products of the form $AA^tAA^t \dots A$ or $AA^tAA^t \dots A^t$ and give some applications to the theory of iterated kernels.

1 Introduction

For any matrix A, let $s(A)$ denote the sum of its entries. For any integer $k \geq 1$, we define

$$A^{(2k)} = (AA^t)^k, \qquad A^{(2k+1)} = (AA^t)^k A,$$

where A^t denotes the transpose of A. In Section 2, we prove the following sharp inequalities:

Theorem 1.1. *Let A be an $m \times n$ matrix with nonnegative real entries. Then for every integer $k \geqslant 1$, the following matrix inequalities hold:*

$$s(A)^{2k} \leqslant m^{k-1}n^k s(A^{(2k)}), \qquad s(A)^{2k+1} \leqslant m^k n^k s(A^{(2k+1)}).$$

For the special case of symmetric matrices, this theorem was proved in 1959 by Mulholland and Smith [59], thus settling an earlier conjecture of Mandel and Hughes [57] that had been based on the study of certain genetical models. For arbitrary matrices (with nonnegative entries), Theorem 1.1 also generalizes the matrix inequality

$$s(A)^3 \leqslant mns(AA^tA),$$

which was first proved in 1960 by Atkinson, Moran and Watterson [3] using methods of perturbation theory.

Theorem 1.1 has a graph theoretic interpretation when applied to matrices with entries in $\{0, 1\}$. Let G be a graph with red vertices labeled $1, \dots, m$ and blue vertices labeled $1, \dots, n$ such that every edge connects only vertices of distinct colours: G is a bipartite graph. Its reduced incidence matrix is an $m \times n$ matrix A such that $a_{i,j} = 1$ if red vertex i is adjacent to blue vertex j, and $a_{i,j} = 0$ otherwise. Then $s(A)$ is the size of G, while $s(A^{(\ell)})$ is the number of walks on G of length ℓ starting from a red vertex, i.e., the number of sequences $(v_0, \dots, v_\ell)$ such that v_0 is a red vertex and every pair $\{v_i, v_{i+1}\}$ is an edge in G. Theorem 1.1 then yields the optimal lower bound of the number of walks in terms of the size of G. We do not know of a corresponding lower bound for the number of trails (walks with no edge repeated) or paths (walks with no vertex repeated).

Recall that an $m \times n$ matrix A is said to be *bistochastic* if every row sum of A is equal to $s(A)/m$, and every column sum of A is equal to $s(A)/n$. In Section 3 we prove the following asymptotic form of Theorem 1.1:

Theorem 1.2. *Let A be an $m \times n$ matrix with nonnegative real entries. If A is bistochastic, then for all $k \geqslant 1$,*

$$s(A)^{2k} = m^{k-1} n^k s(A^{(2k)}), \qquad s(A)^{2k+1} = m^k n^k s(A^{(2k+1)}).$$

If A is not bistochastic, then there exist constants $c > 0$ and $\gamma > 1$ (depending only on A) such that for all $\ell \geqslant 1$,

$$s(A)^{\ell} < c\gamma^{-\ell}(mn)^{\ell/2} s(A^{(\ell)}).$$

As we show in Sections 2 and 3, both of the above theorems, though stated for arbitrary rectangular matrices with nonnegative entries, follow from the special case of *square* matrices.

Theorem 1.2 has an immediate application. Atkinson, Moran and Watterson [3] conjectured that for a nonnegative symmetric kernel function $K(x,y)$ that is integrable (in a suitable sense) over the square $0 \leqslant x, y \leqslant a$, the inequality

$$\int_0^a \int_0^a K_\ell(x,y)\,\mathrm{d}x\,\mathrm{d}y \geqslant \frac{1}{a^{\ell-1}} \left(\int_0^a \int_0^a K(x,y)\,\mathrm{d}x\,\mathrm{d}y \right)^{\ell} \tag{C.1}$$

holds for all $\ell \geqslant 1$. Here $K_\ell(x,y)$ denotes the *ℓ-th order iterate of* $K(x,y)$, which is defined recursively by

$$K_1(x,y) = K(x,y), \qquad K_\ell(x,y) = \int_0^a K_{\ell-1}(x,t)K(t,y)\,\mathrm{d}t.$$

Beesack [7] showed that the Atkinson-Moran-Watterson conjecture follows from the matrix identities of Mulholland and Smith described above. Using Beesack's ideas together with Theorem 1.2, we prove in Section 4 the following asymptotic form of the Atkinson-Moran-Watterson inequality (C.1):

Theorem 1.3. *Let $K(x,y)$ be a nonnegative symmetric kernel function that is integrable over the square $0 \leqslant x, y \leqslant a$, and consider the function $f(x) = \int_0^a K(x,y)\,\mathrm{d}y$ defined on the interval $0 \leqslant x \leqslant a$. If $f(x)$ is constant almost everywhere, then for all $\ell \geqslant 1$*

$$\int_0^a \int_0^a K_\ell(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{a^{\ell-1}} \left(\int_0^a \int_0^a K(x,y)\,\mathrm{d}x\,\mathrm{d}y \right)^{\ell}.$$

If not, there exist constants $c > 0$ and $\gamma > 1$ (depending only on K) such that for all $\ell \geqslant 1$

$$\int_0^a \int_0^a K_\ell(x,y)\,\mathrm{d}x\,\mathrm{d}y > \frac{c\gamma^{\ell}}{a^{\ell-1}} \left(\int_0^a \int_0^a K(x,y)\,\mathrm{d}x\,\mathrm{d}y \right)^{\ell}.$$

Remark 1.4. Using an approximation argument as in the proof of Theorem 1.3, Theorem 1.1 can be also applied to establish an analogue to inequalities (C.1) and Theorem 1.3 in the case of nonsymmetric kernel functions. Let $K(x,y)$ be any nonnegative kernel function that is integrable on the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ and let K_ℓ be the ℓ-th order iterate of K defined by $K_1(x,y) = K(x,y)$ and for each integer $k \geq 1$,

$$K_{2k}(x,x') = \int_0^b K_{2k-1}(x,y)K(x',y)\,\mathrm{d}y,$$

$$K_{2k+1}(x,y) = \int_0^a K_{2k}(x,x')K(x',y)\,\mathrm{d}x'.$$

In this case, inequalities (C.1) become

$$\int_0^a \int_0^b K_{2k+1}(x,y)\,\mathrm{d}x\,\mathrm{d}y \geq \frac{1}{a^k b^k} \left(\int_0^a \int_0^b K(x,y)\,\mathrm{d}x\,\mathrm{d}y \right)^{2k+1}$$

$$\int_0^a \int_0^a K_{2k}(x,x')\,\mathrm{d}x\,\mathrm{d}x' \geq \frac{1}{a^{k-1} b^k} \left(\int_0^a \int_0^b K(x,y)\,\mathrm{d}x\,\mathrm{d}y \right)^{2k}.$$

The analogue of Theorem 1.3 is then obvious.

2 Matrix inequality

Given a matrix $A = (a_{i,j})$ and an integer $\ell \geqslant 0$, we denote by $a_{i,j}^{(\ell)}$ the (i,j)-th entry of $A^{(\ell)}$, so that $A^{(\ell)} = (a_{i,j}^{(\ell)})$. This notation will be used often in the sequel.

Lemma 2.1. *Let $B = (b_{i,j})$ be a $d \times d$ matrix with nonnegative real entries. For any two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ of nonnegative real numbers, the following inequality holds:*

$$(I_2') : \qquad \sum_{i,j=1}^{d} \alpha_i \beta_i b_{i,j} \leqslant d^{\frac{1}{2}} \Big(\sum_{i,j=1}^{d} \alpha_i^2 \beta_j^2 b_{i,j}^{(2)} \Big)^{\frac{1}{2}}.$$

Proof. To prove the lemma, we apply the Cauchy-Schwarz inequality twice as follows:

$$\sum_{i,j=1}^{d} \alpha_i \beta_i b_{i,j} = \sum_{i,k=1}^{d} \alpha_i \beta_i b_{i,k} \leqslant d^{\frac{1}{2}} \Big(\sum_{k=1}^{d} \Big(\sum_{i=1}^{d} \alpha_i \beta_i b_{i,k} \Big)^2 \Big)^{\frac{1}{2}}. \tag{C.2}$$

$$\begin{aligned}
\sum_{i,j=1}^{d} \alpha_i \beta_i b_{i,j} &\leq d^{\frac{1}{2}} \Big(\sum_{i,j,k=1}^{d} \alpha_i \alpha_j \beta_i \beta_j b_{i,k} b_{j,k} \Big)^{\frac{1}{2}} \\
&= d^{\frac{1}{2}} \Big(\sum_{i,j=1}^{d} \alpha_i \alpha_j \beta_i \beta_j b_{i,j}^{(2)} \Big)^{\frac{1}{2}} \\
&= d^{\frac{1}{2}} \Big(\sum_{i,j=1}^{d} \alpha_i \beta_j (b_{i,j}^{(2)})^{\frac{1}{2}} \cdot \alpha_j \beta_i (b_{j,i}^{(2)})^{\frac{1}{2}} \Big)^{\frac{1}{2}} \\
&\leqslant d^{\frac{1}{2}} \Big(\sum_{i,j=1}^{d} \alpha_i^2 \beta_j^2 b_{i,j}^{(2)} \Big)^{\frac{1}{2}}.
\end{aligned}$$

Here we have used the fact that $B^{(2)} = BB^t$ is a symmetric matrix. $\square$

Theorem 2.2. *Let $B = (b_{i,j})$ be a square $d \times d$ matrix with nonnegative real entries, and let $\{\alpha_i\}$ be any sequence of nonnegative real numbers. Then for each integer $\ell \geqslant 1$, we have*

$$(I_\ell) : \qquad \sum_{i,j=1}^{d} \alpha_i b_{i,j} \leqslant d^{\frac{\ell-1}{\ell}} \Big(\sum_{i,j=1}^{d} \alpha_i^\ell b_{i,j}^{(\ell)} \Big)^{\frac{1}{\ell}}.$$

Proof of Theorem 2.2. The case $\ell = 1$ is trivial while the case $\ell = 2$ is a consequence of the lemma above. We prove the general case by induction. Suppose that $p \geqslant 2$, and the inequalities $(I_1), (I_2), \ldots, (I_p)$ hold for all square matrices with nonnegative real entries. If $p = 2k-1$ is an odd integer, then the inequality (I_{p+1}) follows immediately from (I_2) and (I_k). Indeed, since $B^{(2k)} = B^{(2)(k)}$, we have

$$\sum_{i,j=1}^{d} \alpha_i b_{i,j} \leqslant d^{\frac{1}{2}} \Big(\sum_{i,j=1}^{d} \alpha_i^2 b_{i,j}^{(2)} \Big)^{\frac{1}{2}} \leqslant d^{\frac{1}{2}} \Big(d^{\frac{k-1}{k}} \Big(\sum_{i,j=1}^{d} \alpha_i^{2k} b_{i,j}^{(2)(k)} \Big)^{\frac{1}{k}} \Big)^{\frac{1}{2}}. \tag{C.3}$$

Thus

$$\sum_{i,j=1}^{d} \alpha_i b_{i,j} \leqslant d^{\frac{2k-1}{2k}} \Big(\sum_{i,j=1}^{d} \alpha_i^{2k} b_{i,j}^{(2k)} \Big)^{\frac{1}{2k}}.$$

If $p = 2k$ is an even integer, then the inequality (I_{p+1}) follows from Hölder's inequality, and the inequalities (I_k) and (I_2'). Indeed, by Hölder's inequality, we have

$$\sum_{i,j=1}^{d} \alpha_i b_{i,j} \leqslant d^{\frac{1}{2k+1}} \Big(\sum_{i=1}^{d} \alpha_i^{\frac{2k+1}{2k}} \Big(\sum_{j=1}^{d} b_{i,j} \Big)^{\frac{2k+1}{2k}} \Big)^{\frac{2k}{2k+1}}. \tag{C.4}$$

Let $\mathcal{I}$ denote the term between parentheses, and set $\beta_i = \sum_{j=1}^{d} b_{i,j}$ for each i. Then

$$\mathcal{I} = \sum_{i=1}^{d} \alpha_i^{\frac{2k+1}{2k}} \Big(\sum_{j=1}^{d} b_{i,j} \Big)^{\frac{2k+1}{2k}} = \sum_{i,j=1}^{d} \alpha_i^{\frac{2k+1}{2k}} \beta_i^{\frac{1}{2k}} b_{i,j}.$$

Applying (I_k), it follows that

$$\mathcal{I} \leqslant d^{\frac{k-1}{k}} \Bigg(\sum_{i,j=1}^{d} \alpha_i^{\frac{2k+1}{2}} \beta_i^{\frac{1}{2}} b_{i,j}^{(k)} \Bigg)^{\frac{1}{k}}.$$

Applying the lemma to the sequences $\{\alpha_i^{\frac{2k+1}{2}}\}$ and $\{\beta_i^{\frac{1}{2}}\}$, and using the fact that $B^{(k)(2)} = B^{(2k)}$, we see that

$$\mathcal{I} \leqslant d^{\frac{k-1}{k}} \Bigg(d^{\frac{1}{2}} \Big(\sum_{i,j=1}^{d} \alpha_i^{2k+1} \beta_j b_{i,j}^{(k)(2)} \Big)^{\frac{1}{2}} \Bigg)^{\frac{1}{k}} = d^{\frac{2k-1}{2k}} \Bigg(\sum_{i,j=1}^{d} \alpha_i^{2k+1} \beta_j b_{i,j}^{(2k)} \Bigg)^{\frac{1}{2k}}.$$

Putting everything together, we have therefore shown that

$$\sum_{i,j=1}^{d} \alpha_i b_{i,j} \leqslant d^{\frac{2k}{2k+1}} \Bigg(\sum_{i,j=1}^{d} \alpha_i^{2k+1} \beta_j b_{i,j}^{(2k)} \Bigg)^{\frac{1}{2k+1}}.$$

Finally, note that

$$\sum_{j=1}^{d} \beta_j b_{i,j}^{(2k)} = \sum_{\ell=1}^{d} b_{i,\ell}^{(2k)} \beta_\ell = \sum_{j,\ell=1}^{d} b_{i,\ell}^{(2k)} b_{\ell,j} = \sum_{j=1}^{d} b_{i,j}^{(2k+1)}$$

since $B^{(2k+1)} = B^{(2k)} B$. Consequently,

$$\sum_{i,j=1}^{d} \alpha_i b_{i,j} \leqslant d^{\frac{2k}{2k+1}} \Bigg(\sum_{i,j=1}^{d} \alpha_i^{2k+1} b_{i,j}^{(2k+1)} \Bigg)^{\frac{1}{2k+1}} \tag{C.5}$$

and (I_{p+1}) holds for the case $p = 2k$. Theorem 2.2 now follows by induction. □

Proof of Theorem 1.1. For the case of square matrices, Theorem 1.1 follows immediately from Theorem 2.2. Indeed, taking $\alpha_i = 1$ for each i, the inequality (I_ℓ) yields the corresponding inequality in Theorem 1.1.

Now, let A be an $m \times n$ matrix with nonnegative real entries, put $d = mn$, and let B be the $d \times d$ matrix with nonnegative real entries defined as the tensor product $B = A \otimes J_{n,m}$, where $J_{n,m}$ is the $n \times m$ matrix with every entry equal to 1. For any integers $\ell, k \geqslant 0$, the relations

$$B^{(\ell)} = A^{(\ell)} \otimes J_{n,m}^{(\ell)}, \quad s\big(B^{(\ell)}\big) = s\big(A^{(\ell)}\big) s\big(J_{n,m}^{(\ell)}\big),$$

$$s\big(J_{n,m}^{(2k)}\big) = m^k n^{k+1}, \quad s\big(J_{n,m}^{(2k+1)}\big) = m^{k+1} n^{k+1}.$$

are easily checked. In particular, $s(B) = mns(A)$. Applying Theorem 1.1 to the matrix B and using these identities, the inequalities of Theorem 1.1 follow for the matrix A. □

3 Asymptotic matrix inequality

As will be shown below, Theorem 1.2 is a consequence of the following more precise theorem for square matrices:

Theorem 3.1. *Let B be a square $d \times d$ matrix with nonnegative real entries and $s(B) \neq 0$. Let λ be the largest eigenvalue of $B^{(2)} = BB^t$, and put $\gamma = \lambda d^2 / s(B)^2$. Then $\gamma \geqslant 1$, and there exists a constant $c > 0$ (depending only on B) such that for all integers $\ell \geqslant 0$,*

$$s(B)^\ell < c\gamma^{-\frac{\ell}{2}} d^{\ell-1} s(B^{(\ell)}). \tag{C.6}$$

Moreover, the following assertions are equivalent:

(*a*) $\gamma = 1$,

(*b*) $s(B)^\ell = d^{\ell-1} s(B^{(\ell)})$ *for every integer* $\ell \geqslant 0$,

(*c*) $s(B)^\ell = d^{\ell-1} s(B^{(\ell)})$ *for some integer* $\ell \geqslant 3$,

(*d*) B *is bistochastic.*

Proof. We express $B^{(2)} = BB^t$ in the form $B^{(2)} = U^t DU$, where $U = (u_{i,j})$ is an orthogonal matrix, and D is a diagonal matrix $\operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ with $\lambda_1 \geqslant \ldots \geqslant \lambda_d \geqslant 0$. Here $\lambda = \lambda_1$. For each $\nu = 1, \ldots, d$, let E_ν be the projection matrix whose (ν, ν)-th entry is 1, and all other entries are equal to 0. Put $A_\nu = U^t E_\nu U$ for each ν. Then for all integers $k \geqslant 0$,

$$B^{(2k)} = \sum_{\nu=1}^{d} \lambda_\nu^k A_\nu, \qquad B^{(2k+1)} = \sum_{\nu=1}^{d} \lambda_\nu^k A_\nu B.$$

By a straightforward calculation, we see that for each ν

$$s(A_\nu) = \Big(\sum_{i=1}^{d} u_{\nu,i} \Big)^2, \qquad s(A_\nu B) = \Big(\sum_{i=1}^{d} u_{\nu,i} \Big) \Big(\sum_{j,k=1}^{d} u_{\nu,k} b_{k,j} \Big). \tag{C.7}$$

In particular, $s(A_\nu) \geqslant 0$. By Theorem 2.2, it follows that

$$\frac{s(B)^2}{d} \leqslant s(B^{(2)}) = \sum_{\nu=1}^{d} \lambda_\nu s(A_\nu) \leqslant \lambda \sum_{\nu=1}^{d} s(A_\nu) = \lambda d. \tag{C.8}$$

Therefore, $\gamma = \dfrac{\lambda d^2}{s(B)^2} \geqslant 1$. Now, from the definition of γ, we have

$$\frac{\gamma^{\frac{\ell}{2}} s(B)^\ell}{d^{\ell-1} s(B^{(\ell)})} = d \frac{\lambda^{\frac{\ell}{2}}}{s(B^{(\ell)})}.$$

Then, in order to show inequality (C.6), we will show that the $\lambda^{\frac{\ell}{2}}/s(B^{(\ell)})$ are bounded above by a constant that is independent of ℓ. Indeed, let $C_\ell = B^{(\ell)}/s(B^{(\ell)})$ for every $\ell \geqslant 0$. Since each C_ℓ has nonnegative real entries, and $s(C_\ell) = 1$, the entries of C_ℓ all lie in the closed interval $[0, 1]$. Thus the entries of the matrices $UC_{2k}U^t$ and $UC_{2k+1}B^tU^t$ are bounded by a constant that depends only on B. Noting that for each nonnegative integer k, we have

$$UC_{2k}U^t = \frac{D^k}{s(B^{(2k)})}, \qquad UC_{2k+1}B^tU^t = \frac{D^{k+1}}{s(B^{(2k+1)})},$$

and on examining the $(1, 1)$th entry for each of these matrices, we see that $\lambda^k/s(B^{(2k)})$ and $\lambda^{k+1}/s(B^{(2k+1)})$ are both bounded above by a constant that is independent of k. Consequently, inequality (C.6) holds.

$(a) \Rightarrow (b)$: If $\gamma = 1$, then $\lambda d = s(B)^2/d$, hence from (C.8) we see that $s(A_\nu) = 0$ whenever $\lambda_\nu \neq \lambda$. By (C.7), we also have that $s(A_\nu B) = 0$ whenever $\lambda_\nu \neq \lambda$. Thus

$$\begin{aligned} s(B^{(2k)}) &= \sum_{\nu=1}^{d} \lambda_\nu^k s(A_\nu) = \lambda^k \sum_{\nu:\lambda_\nu=\lambda} s(A_\nu) \\ &= \lambda^k \sum_{\nu=1}^{d} s(A_\nu) = \lambda^k d = \frac{s(B)^{2k}}{d^{2k-1}}, \\ s(B^{(2k+1)}) &= \sum_{\nu=1}^{d} \lambda_\nu^k s(A_\nu B) = \lambda^k \sum_{\nu:\lambda_\nu=\lambda} s(A_\nu B) \\ &= \lambda^k \sum_{\nu=1}^{d} s(A_\nu B) = \lambda^k s(B) = \frac{s(B)^{2k+1}}{d^{2k}}. \end{aligned}$$

$(b) \Rightarrow (a)$: If (b) holds, then inequality (C.6) implies $1 < c\gamma^{-\frac{\ell}{2}}$ for some $\gamma \geq 1$ and all integers $\ell \geq 0$. This forces $\gamma = 1$.

$(b) \Rightarrow (c)$: Trivial.

$(c) \Rightarrow (d)$: Suppose that $\ell = 2k + 1 \geqslant 3$ is an odd integer such that $s(B)^\ell = d^{\ell-1} s(B^{(\ell)})$. Taking every $\alpha_i = 1$ in the proof of Theorem 2.2, our hypothesis means that equality holds in (C.5), hence (C.4) must also hold with equality:

$$\sum_{i,j=1}^{d} b_{i,j} = d^{\frac{1}{2k+1}} \left(\sum_{i=1}^{d} \Big(\sum_{j=1}^{d} b_{i,j} \Big)^{\frac{2k+1}{2k}} \right)^{\frac{2k}{2k+1}}.$$

By Hölder's inequality, this is only possible if all of the row sums of B are equal. Since ℓ is odd and s is transpose-invariant, we also have

$$s(B^t)^\ell = d^{\ell-1}s\big((B^{(\ell)})^t\big) = d^{\ell-1}s\big((B^t)^{(\ell)}\big).$$

Thus all of the row sums of B^t are equal as well, and B is bistochastic.

Now suppose that $\ell = 2k \geqslant 4$ is an even integer such that $s(B)^\ell = d^{\ell-1}s(B^{(\ell)})$. By taking every $\alpha_i = 1$ in (C.3), we see that $s(B)^2 = ds(B^{(2)})$. Then, taking every $\alpha_i = \beta_i = 1$ in the proof of the lemma, we see that equality holds in (C.2) which is only possible if all of the column sums of B are equal. Therefore $s(BA) = \beta s(A)$ for every $d \times d$ matrix A, where $\beta = s(B)/d$ is the sum of each column of B. In particular,

$$\begin{aligned} s(B)^\ell &= d^{\ell-1}s\big(B^{(\ell)}\big) = d^{\ell-1}\beta s\big((B^t)^{(\ell-1)}\big) \\ &= d^{\ell-1}\beta s\big((B^{(\ell-1)})^t\big) = d^{\ell-1}\beta s(B^{(\ell-1)}), \end{aligned}$$

thus $s(B)^{\ell-1} = d^{\ell-2}s(B^{(\ell-1)})$. Since $\ell - 1$ is odd, we can apply the previous result to conclude that B is bistochastic.

$(d)\Rightarrow(b)$: Suppose B is bistochastic, with every row or column sum equal to $\beta = s(B)/d$. For any $d\times d$ matrix A, one has $s(AB) = \beta s(A)$ and $s(AB^t) = \beta s(A)$. In particular, $s(B^{(2k+1)}) = \beta s(B^{(2k)})$ and $s(B^{(2k+2)}) = \beta s(B^{(2k+1)})$ for all $k \geqslant 0$. Consequently,

$$s(B^{(\ell)}) = \beta^{\ell-1}s(B) = \frac{s(B)^\ell}{d^{\ell-1}}, \qquad \ell \geqslant 0.$$

This completes the proof. □

Corollary 3.2. *Let B be a square $d \times d$ matrix with nonnegative real entries and $s(B) \neq 0$. Let β_j be the j-th column sum of B for each j, and put*

$$\delta = 1 + \frac{1}{2s(B)^2}\sum_{i,j=1}^{d}(\beta_i - \beta_j)^2.$$

Then there exists a constant $c > 0$ (depending only on B) such that for all $\ell \geqslant 0$, we have

$$s(B)^\ell < c\delta^{-\frac{\ell}{2}}d^{\ell-1}s(B^{(\ell)}).$$

Proof. Note first that for *any* $d \times d$ matrix B, if β_j denotes the j-th column sum of B, then it is easily seen that

$$s(B^{(2)}) = \frac{s(B)^2}{d} + \frac{1}{2d}\sum_{i,j=1}^{d}(\beta_i - \beta_j)^2. \tag{C.9}$$

Using the notation of Theorem 3.1 and applying the relations (C.8) and (C.9), we have

$$\gamma = \frac{\lambda d^2}{s(B)^2} \geqslant \frac{ds(B^{(2)})}{s(B)^2} = 1 + \frac{1}{2s(B)^2}\sum_{i,j=1}^{d}(\beta_i - \beta_j)^2 = \delta.$$

The corollary therefore follows from (C.6). □

Proof of Theorem 1.2. Given an $m \times n$ matrix A with nonnegative real entries, we proceed as in the proof of Theorem 1.1: put $d = mn$, and let $B = A \otimes J_{n,m}$. Note that A is bistochastic if and only if B is bistochastic. Applying the corollary above to B, Theorem 1.2 follows immediately for the matrix A. The details are left to the reader. □

4 Asymptotic kernel inequality

Proof of Theorem 1.3. By changing variables if necessary, we can assume that $a = 1$. For simplicity, we will also assume that $K(x, y)$ is continuous. Consider the function $f(x)$ defined by

$$f(x) = \int_0^1 K(x, y)\,\mathrm{d}y, \qquad x \in [0, 1].$$

If $f(x)$ is a constant function, then since $K(x,y)$ is symmetric, the equality

$$\int_0^1\int_0^1 K_\ell(x,y)\,\mathrm{d}x\,\mathrm{d}y = \Bigl(\int_0^1\int_0^1 K(x,y)\,\mathrm{d}x\,\mathrm{d}y\Bigr)^\ell$$

for all $\ell \geqslant 1$ follows from an easy inductive argument.

Now suppose that $f(x)$ is not constant, and let m and M denote respectively the minimum and maximum value of $f(x)$ on $[0,1]$. Choose $\varepsilon > 0$ such that $4\varepsilon < M - m$. For every integer $d \geqslant 1$, let $\mathscr{U}_i^{[d]}$ be the open interval

$$\mathscr{U}_i^{[d]} = \Bigl(\frac{i-1}{d}, \frac{i}{d}\Bigr), \qquad 1 \leqslant i \leqslant d,$$

and let $\mathscr{U}_{i,j}^{[d]}$ be the rectangle $\mathscr{U}_i^{[d]} \times \mathscr{U}_j^{[d]}$ for $1 \leqslant i,j \leqslant d$. Let $K^{[d]}(x,y)$ be defined on $[0,1]\times[0,1]$ as follows:

$$\begin{cases} \min\Bigl\{K(s,t) : (s,t) \in \overline{\mathscr{U}_{i,j}^{[d]}}\Bigr\} & \text{if } (x,y) \in \mathscr{U}_{i,j}^{[d]} \text{ for some } 1 \leqslant i,j \leqslant d \\ K(x,y) & \text{otherwise.} \end{cases}$$

Here $\overline{\mathscr{U}_{i,j}^{[d]}}$ denotes the closure of $\mathscr{U}_{i,j}^{[d]}$. Noting that $K^{[d]}(x,y)$ is constant on each rectangle $\mathscr{U}_{i,j}^{[d]}$, let $B_{[d]}$ be the $d \times d$ matrix whose (i,j)-th entry is equal to $K^{[d]}(\mathscr{U}_{i,j}^{[d]})$. Let $K_\ell^{[d]}(x,y)$ denote the ℓ-th order iterate of $K^{[d]}(x,y)$ for each $\ell \geqslant 1$. Then

$$K_\ell^{[d]}(x,y) = \int_0^1 K_{\ell-1}^{[d]}(x,t)K^{[d]}(t,y)\,\mathrm{d}t = \sum_{k=1}^d \int_{\mathscr{U}_k^{[d]}} K_{\ell-1}^{[d]}(x,t)K^{[d]}(t,y)\,\mathrm{d}t.$$

It follows by induction that $K_\ell^{[d]}(x,y)$ is also constant on each rectangle $\mathscr{U}_{i,j}^{[d]}$, and

$$K_\ell^{[d]}(\mathscr{U}_{i,j}^{[d]}) = \frac{1}{d}\sum_{k=1}^d K_{\ell-1}^{[d]}(\mathscr{U}_{i,k}^{[d]})K^{[d]}(\mathscr{U}_{k,j}^{[d]});$$

by induction, this is the (i,j)-th entry of the matrix $\dfrac{1}{d^{\ell-1}}B_{[d]}^{(\ell)}$. In other words,

$$\Bigl(K_\ell^{[d]}(\mathscr{U}_{i,j}^{[d]})\Bigr) = \frac{1}{d^{\ell-1}}B_{[d]}^{(\ell)}, \qquad \text{for all } \ell, d \geqslant 1. \tag{C.10}$$

Now since $f(x)$ is continuous, we can choose d sufficiently large such that for some integers $1 \leqslant i_m, i_M \leqslant d$, we have

$$f(x) < m + \varepsilon, \qquad \text{for all } x \in \mathscr{U}_{i_m}^{[d]},$$

$$f(x) > M - \varepsilon, \qquad \text{for all } x \in \mathscr{U}_{i_M}^{[d]}.$$

Taking d larger if necessary, we can further assume that

$$0 \leqslant K(x,y) - K^{[d]}(x,y) < \varepsilon$$

for all $0 \leqslant x,y \leqslant 1$. Fixing this value of d, we define

$$\gamma = 1 + \frac{\varepsilon^2}{2d^2\Bigl(\int_0^1\int_0^1 K(x,y)\,\mathrm{d}x\,\mathrm{d}y\Bigr)^2}.$$

Finally, since $\gamma^{-\frac14} < 1$, we can choose e sufficiently large so that $K^{[de]}(x,y) > \gamma^{-\frac14}K(x,y)$ for all $0 \leqslant x,y \leqslant 1$. For this value of e, we therefore have

$$\int_0^1\int_0^1 K^{[de]}(x,y)\,\mathrm{d}x\,\mathrm{d}y > \gamma^{-\frac14}\int_0^1\int_0^1 K(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

By the corollary to Theorem 3.1 applied to the matrix $B_{[de]}$, there exists a constant $c>0$, which is independent of ℓ, such that

$$s\big(B_{[de]}\big)^{\ell} < c\delta^{-\frac{\ell}{2}}(de)^{\ell-1}s\big(B_{[de]}^{(\ell)}\big)$$

for all integers $\ell \geqslant 0$, where

$$\delta = 1 + \frac{1}{2s\big(B_{[de]}\big)^2}\sum_{i,j=1}^{de}\big(\beta_{[de],i}-\beta_{[de],j}\big)^2.$$

Here $\beta_{[de],j}$ denotes the j-th column sum of $B_{[de]}$ for each j. We now claim that $\delta > \gamma$. Granting this fact for the moment, we apply (C.10) to $K^{[de]}(x,y)$ and obtain:

$$\begin{aligned}
\int_0^1\int_0^1 K_\ell(x,y)\,\mathrm{d}x\,\mathrm{d}y &\geqslant \int_0^1\int_0^1 K_\ell^{[de]}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{(de)^2}\sum_{i,j=1}^{de}K_\ell^{[de]}\Big(\mathscr{U}_{i,j}^{[de]}\Big)\\
&= \frac{1}{(de)^{\ell+1}}s\big(B_{[de]}^{(\ell)}\big) > c^{-1}\delta^{\frac{\ell}{2}}(de)^{-2\ell}s\big(B_{[de]}\big)^{\ell}\\
&= c^{-1}\delta^{\frac{\ell}{2}}\Bigg(\frac{1}{(de)^2}\sum_{i,j=1}^{de}K^{[de]}\Big(\mathscr{U}_{i,j}^{[de]}\Big)\Bigg)^{\ell} = c^{-1}\delta^{\frac{\ell}{2}}\Bigg(\int_0^1\int_0^1 K^{[de]}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Bigg)^{\ell}\\
&> c^{-1}\delta^{\frac{\ell}{2}}\gamma^{-\frac{\ell}{4}}\Bigg(\int_0^1\int_0^1 K(x,y)\,\mathrm{d}x\,\mathrm{d}y\Bigg)^{\ell} > c^{-1}\gamma^{\frac{\ell}{4}}\Bigg(\int_0^1\int_0^1 K(x,y)\,\mathrm{d}x\,\mathrm{d}y\Bigg)^{\ell}.
\end{aligned}$$

This completes the proof of the theorem modulo our claim that $\delta > \gamma$. To see this, let $\mathscr{V}$ be any interval of the form $\mathscr{U}_i^{[de]}$ such that $\mathscr{V} \subset \mathscr{U}_{i_m}^{[d]}$. Note that there are e such intervals. Since $B^{[de]}$ is a symmetric matrix, the column sum $\beta_{[de],\mathscr{V}}$ of $B_{[de]}$ corresponding to the interval $\mathscr{V}$ is equal to the "$\mathscr{V}$-th" row sum, which can be bounded as follows:

$$\begin{aligned}
\beta_{[de],\mathscr{V}} = \sum_{j=1}^{de}K^{[de]}\big(\mathscr{V},\mathscr{U}_j^{[de]}\big) &= (de)^2\int_{\mathscr{V}}\int_0^1 K^{[de]}(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\leqslant (de)^2\int_{\mathscr{V}}\int_0^1 K(x,y)\,\mathrm{d}y\,\mathrm{d}x = (de)^2\int_{\mathscr{V}}f(x)\,\mathrm{d}x < de(m+\varepsilon).
\end{aligned}$$

Similarly, let $\mathscr{W}$ be any interval of the form $\mathscr{U}_i^{[de]}$ such that $\mathscr{W} \subset \mathscr{U}_{i_M}^{[d]}$. Again, there are e such intervals, and by a similar calculation, the column sum $\beta_{[de],\mathscr{W}}$ satisfies the bound

$$\beta_{[de],\mathscr{W}} = \sum_{j=1}^{de}K^{[de]}\big(\mathscr{W},\mathscr{U}_j^{[de]}\big) > de(M-2\varepsilon).$$

Thus

$$\sum_{i,j=1}^{de}\big(\beta_{[de],i}-\beta_{[de],j}\big)^2 \geqslant \sum_{\mathscr{V},\mathscr{W}}\big(\beta_{[de],\mathscr{W}}-\beta_{[de],\mathscr{V}}\big)^2 > d^2e^4(M-m-3\varepsilon)^2 > d^2e^4\varepsilon^2.$$

On the other hand, we have

$$s\big(B_{[de]}\big) = (de)^2\int_0^1\int_0^1 K^{[de]}(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant (de)^2\int_0^1\int_0^1 K(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

and the claim follows. □

Keywords. Matrix inequality, iterated kernel.

2000 *Mathematics Subject Classification.* 15A45.

Chapter D

The size of bipartite graphs with girth eight

Reiman's inequality for the size of bipartite graphs of girth six is generalised to girth eight. It is optimal in as far as it admits the algebraic structure of generalised quadrangles as case of equality. This enables us to obtain the optimal estimate $e \sim v^{4/3}$ for balanced bipartite graphs. We also get an optimal estimate for very unbalanced graphs.

1 Introduction

De Caen and Székely recently proposed a new bound for the size of a bipartite graph of girth eight, that is a bipartite graph without cycle of length four and six. We adapt their method to obtain the following cubic inequality.

Theorem 1.1. *Let G be a bipartite graph on $v + w$ vertices.*

(i) *If G contains no cycle of length 4 and 6, then its size e satisfies*

$$e^3 - (v + w)e^2 + 2vwe - v^2w^2 \leqslant 0.$$

(ii) *If $v \geqslant \lfloor w^2/4 \rfloor$, then furthermore $e \leqslant v + \lfloor w^2/4 \rfloor$.*

Part (i) is the right generalisation of Reiman's inequality for bipartite graphs of girth 6 (see Prop. 3.1) to girth 8. It is optimal in the sense that it is an equality for all known extremal graphs constructed via finite fields. Part (ii) describes the case of very unbalanced bipartite graphs and is optimal: there is a graph, constructed by hand, for which it is an equality.

Let us give a brief description of this article. Section 2 describes a way to translate uncoloured graphs into bipartite graphs and its converse. This permits to get two propositions on very unbalanced graphs.

Section 3 summarises facts about bipartite graphs of girth six that should be folklore and well known although I did not see them printed.

Section 4 is the core of the paper. We adapt an inequality of Atkinson et al. to get an optimal lower bound on the number of paths of length 3 in a bipartite graph (Cor. 4.6). This enables us to bypass the final step in the proof of [20, Th. 1] and to get our theorem.

2 Uncoloured graphs and bipartite graphs

2.1 Expanding a graph to a bipartite graph

We propose the following construction of a bipartite graph out of an uncoloured graph. Let G' be an uncoloured graph with set of vertices V. Then the bipartite graph G is defined as follows:

- the first class of vertices of G is V;
- the second class W of vertices of G is the set of edges of G';
- the set of edges of G is $\{\, \{x, y\} : y \text{ is an edge of } G' \text{ with endpoint } x \,\}$.

Thus every vertex of W has degree 2 and the size of G is twice the size of G'.

2.2 Contracting a bipartite graph to an uncoloured graph

Let us describe an inverse construction. Let G be a bipartite graph with colour classes V and W. Let G' be the following graph:

- its set of vertices is V;
- its set of edges is $\{ \{x,z\} \subseteq V : \exists y \ \{x,y\} \text{ and } \{z,y\} \text{ are edges of } G \}$.

The size of G' is at most half the size of G. If G contains no cycle of length 4, then, given $\{x,z\}$, there is at most one y such that $\{x,y\}$ and $\{z,y\}$ are edges of G, so that the size of G' is exactly

$$\sum_{y \in W} \binom{d(y)}{2} \leqslant \binom{\#V}{2}. \tag{D.1}$$

(We recognise here [11, Inequality (2), p. 310] for $s = t = 2$.) Thus each vertex $y \in W$ of degree at least 2 contributes at least 1 to sum (D.1). This yields

Proposition 2.1. *Let G be a bipartite graph on $v + w$ vertices that contains no cycle of length 4.*

(i) If $w > \binom{v}{2}$, then there are at least $w - \binom{v}{2}$ vertices in W of degree 0 or 1.

(ii) If its minimal degree is at least 2, then $w \leqslant \binom{v}{2}$ and $v \leqslant \binom{w}{2}$.

If G contains no cycle of length 4 nor 6, then G' contains no triangle and its size is at most $\lfloor v^2/4 \rfloor$. This argument proves

Proposition 2.2. *Let G be a bipartite graph on $v + w$ vertices that contains no cycle of length 4 or 6.*

(i) If $w > \lfloor v^2/4 \rfloor$, then there are at least $\lceil w - v^2/4 \rceil$ vertices in W with degree 0 or 1.

(ii) If its minimal degree is at least 2, then $w \leqslant \lfloor v^2/4 \rfloor$ and $v \leqslant \lfloor w^2/4 \rfloor$.

3 Bipartite graphs of girth six

The following estimate is well known as Reiman's inequality, but its cases of equality were not written down explicitly. Reading the proof of [11, Th. VI.2.6], one gets with [9, Def. I.3.1]

Proposition 3.1. *Let $v \leqslant w$. A graph of girth at least 6 on $v + w$ vertices with e edges satisfies*

$$O(v,w,e) = e^2 - we - vw(v-1) \leqslant 0$$

$$e \leqslant \sqrt{vw(v-1) + w^2/4} + w/2.$$

We have equality if and only if it is the incidence graph of a Steiner system $S(2,k;v)$ on v points with block degree k given by $wk(k-1) = v(v-1)$.

Note that by symmetry, we also get $O(w,v,e) \leqslant 0$, but this is superfluous by

Lemma 3.2. *Let $v \leqslant w$. Let e be the positive root of $X^2 - vX - vw(w-1)$. Then $O(v,w,e) \geqslant 0$.*

Proof. As $(vw)^2 - vvw - vw(w-1) = vw(vw - v - w + 1) \geqslant 0$, we have $e \leqslant vw$. Therefore

$$\begin{aligned} e^2 - we - vw(v-1) &= e^2 - ve - vw(w-1) + (v-w)e + vw(w-v) \\ &= (vw - e)(w - v) \geqslant 0 \ . \end{aligned}$$

□

Remark 3.3. The case of equality in Prop. 3.1 may be described further as follows. By [9, Cor. I.2.11], every vertex in V has same degree r and every vertex in W has same degree k with

$$k - 1 \mid v - 1 \text{ and } k(k-1) \mid v(v-1), \tag{D.2}$$

so that $v = 1 + r(k-1)$ and $k \mid r(r-1)$. For given k, this set of conditions is in fact sufficient for the existence of an extremal graph for large r: this is Wilson's Theorem [9, Th. XI.3.8]. For example, we have the following complete sets of parameters (v,w,r,k):

$$(1 + r(k-1), r(1 + r(k-1))/k, r, k) \text{ for } 1 \leqslant k \leqslant 5 \text{ and } k \mid r(r-1).$$

The first set of parameters satisfying (D.2) for which an extremal graph does not exist is $(36, 42, 7, 6)$. Consult [9, Table A1.1] for all known block designs with $r \leqslant 17$. [9, Table A5.1] provides the following sets of parameters (v, w, r, k) for block designs: given any prime power q and natural number n, given $t \leqslant s$,

$$\Big(q^n, q^{n-1}\frac{q^n-1}{q-1}, \frac{q^n-1}{q-1}, q\Big)\ ,\ \Big(\frac{q^{n+1}-1}{q-1}, \frac{q^{n+1}-1}{q^2-1}\frac{q^n-1}{q-1}, \frac{q^n-1}{q-1}, q+1\Big)\ ,$$

$$\big(q^3+1, q^2(q^2-q+1), q^2, q+1\big)\ ,\ \big(2^{t+s}-2^s+2^t, (2^s+1)(2^s-2^{s-t}+1), 2^s+1, 2^t\big)\ .$$

The following proposition provides a simpler but coarser bound.

Proposition 3.4. *Let G be a bipartite graph on vertex classes V and W with $\#V = v$ and $\#W = w$ without cycles of length 4. Its size satisfies*

$$e \leqslant \begin{cases} \sqrt{2vw(v-1)} & \text{if } w \leqslant v(v-1)/2 \\ v(v-1)/2 + w & \text{otherwise.} \end{cases}$$

We have optimality in the second alternative for the bipartite expansion of a complete graph on V as described in Section 2.1, on which we add $w - v(v-1)/2$ new edges by connecting any vertex of V to $w - v(v-1)/2$ new vertices in colour class W.

Proof. By Proposition 2.1, if $w > v(v-1)/2$, then $w - v(v-1)/2$ vertices in W have degree 0 or 1. If we remove them, we remove at most $w - v(v-1)/2$ edges and the remaining graph has at most $v(v-1)$ edges because $O(v, v(v-1)/2, v(v-1)) = 0$. The first alternative follows from

$$O(v, w, \sqrt{2vw(v-1)}) = w\sqrt{v(v-1)}(\sqrt{v(v-1)} - \sqrt{2w}).$$ □

4 Bipartite graphs of girth eight

4.1 Statement of the theorem

Consult [56, Def. 1.3,1] for the definition of generalised polygons.

Theorem 4.1. *Let G be a bipartite graph on vertex classes V and W with $\#V = v$ and $\#W = w$. If G contains no cycle of length 4 or 6, then its size e satisfies*

$$P(v, w, e) = e^3 - (v+w)e^2 + 2vwe - v^2w^2 \leqslant 0. \tag{D.3}$$

We have equality exactly in two cases:

(i) if G is the complete bipartite graph and $v = 1$ or $w = 1$;

(ii) if G is the incidence graph of a generalised quadrangle.

Remark 4.2. Let us first note that this polynomial has exactly one positive root in e for positive v, w. It suffices to this purpose to show that its discriminant is negative. This is $-v^2w^2D$ with

$$D = 27p^2 + 4s^3 - 36sp - 4s^2 + 32p,\ \ s = v + w,\ p = vw.$$

Let us study this quantity for $s \geqslant 2$, $p \geqslant s - 1$. We have

$$\frac{dD}{dp} = 54p - 36s + 32 \geqslant 54p - 36(p+1) + 32 = 18p - 4 > 0,$$

so that its minimum satisfies $p = s-1$, which implies $D = (4s-5)(s-1)^2 \geqslant 3$. Therefore Inequality (D.3) is equivalent to an inequality of form $e \leqslant e(v, w)$.

Remark 4.3. The case of equality in Th. 4.1 implies the following: every vertex in V has same degree $s + 1$ and every vertex in W has same degree $t + 1$. By [56, Cor. 1.5.5, Th. 1.7.1], $s + t \mid st(1 + st)$ and

$$v = (t+1)(1+st)\ ,\ w = (s+1)(1+st)\ ,\ e = (s+1)(t+1)(1+st).$$

Let us suppose, by symmetry, that $s \leqslant t$. If $s = 0$, we get case (i). If $s = 1$, we obtain exactly the examples of extremal graphs produced by de Caen and Székely: W consists of $t+1$ horizontal lines and as much vertical lines and V is the set of $(t+1)^2$ intersection points and G is the point-line incidence graph of this grid (this is also the bipartite expansion of a complete bipartite graph on $(t+1)+(t+1)$ vertices.) Otherwise $s, t \geqslant 2$ and G is in fact the incidence graph of a generalised quadrangle, so that by [56, Th. 1.7.2], $t \leqslant s^2$. Let q be a prime power. Then there are generalised quadrangles with set of parameters (s,t) any of (q,q), (q,q^2), (q^2,q^3), $(q-1,q+1)$; all known ones fit in this list. In particular, by [56, Th. 1.7.9], if $t \geqslant s = 2$, then $t = 2$ or $t = 4$ and in each case there is exactly one extremal graph. By [56, Sec. 1.7.11], if $t \geqslant s = 3$, then there is a (unique) extremal graph exactly if $t = 3, 5, 9$. There is a unique extremal graph with $s = t = 4$. It is open whether there exists a generalised quadrangle with $s = 4$ and $t \in \{11, 12\}$.

4.2 A generalisation of an inequality of Atkinson et al.

We first need an optimal lower bound on the number of paths of length 3. Let us prove the following inequality.

Theorem 4.4. *Let $(a_{ij})_{1\leqslant i\leqslant v, 1\leqslant j\leqslant w}$ be a matrix of nonnegative coefficients and $\rho, \gamma \geqslant 0$. Let*

$$a_{i\star} = \sum_{j=1}^{w} a_{ij}, \;\; a_{\star j} = \sum_{i=1}^{v} a_{ij}, \;\; e = \sum_{i=1}^{n}\sum_{j=1}^{v} a_{ij}. \tag{D.4}$$

If $a_{i\star} \geqslant 2\rho$ and $a_{\star j} \geqslant 2\gamma$, then

$$\phi = \sum_{i=1}^{v}\sum_{j=1}^{w} a_{ij}(a_{i\star} - \rho)(a_{\star j} - \gamma) \geqslant e(e/v - \rho)(e/w - \gamma), \tag{D.5}$$

equality holding exactly if $a_{i\star}$ and $a_{\star j}$ are constant.

This refines the inequality in [3], which states

$$\psi = \sum_{i=1}^{v}\sum_{j=1}^{w} a_{ij} a_{i\star} a_{\star j} \geqslant e^3/vw \tag{D.6}$$

as, by the Arithmetic-Quadratic Mean Inequality,

$$\phi - \psi = -\gamma \sum_{i=1}^{v} a_{i\star}^2 - \rho \sum_{j=1}^{w} a_{\star j}^2 + \rho\gamma e \leqslant e(-\gamma e/v - \rho e/w + \rho\gamma) \tag{D.7}$$

Remark 4.5. If $v = w$ and a is diagonal, Inequality (D.6) is the Arithmetic-Cubic Mean Inequality and Inequality (D.5) becomes

$$\frac{1}{v}\sum_{i=1}^{v} a_{ii}(a_{ii} - \rho)(a_{ii} - \gamma) \geqslant \frac{e}{v}\frac{e - v\rho}{v}\frac{e - v\gamma}{v},$$

which is true by Chebyshev's Inequality [39, Th. 43] if $a_{ii} \geqslant \rho$ and $a_{ii} \geqslant \gamma$. For our "non commutative Chebyshev Inequality", the conditions $a_{i\star} \geqslant 2\rho$ and $a_{\star j} \geqslant 2\gamma$ cannot be weakened to $a_{i\star} \geqslant \rho$ and $a_{\star j} \geqslant \gamma$, as we have the following counterexamples:

$$\begin{pmatrix} 2 & 5 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof. If (D.5) is an equality, then so are (D.7) and (D.6) and our case of equality follows from the identical case of equality in [3], whose proof we now imitate. We shall suppose that $a_{i\star} > 2\rho$ or $a_{\star j} > 2\gamma$, so that the whole inequality follows by continuity. Fix e and suppose that under this condition the a_{ij} are chosen so to minimise ϕ. We may suppose that the rows and the columns have been permuted such that the sequences $(a_{i\star})$ and $(a_{\star j})$ are nondecreasing:

$$a_{1\star} \leqslant \cdots \leqslant a_{v\star}, \;\; a_{\star 1} \leqslant \cdots \leqslant a_{\star w}. \tag{D.8}$$

If one of these sequences is constant, the inequality follows by the Arithmetic-Quadratic inequality (and the case of equality is easy). Let us suppose that this is not so.

One can suppose that a_{1w} and a_{v1} are positive. Let us show the argument for a_{1w}. If $a_{1w} = 0$, there are k, l such that $a_{1k}, a_{lw} > 0$. Make a perturbation by adding α to a_{1w} and to a_{lk} and subtracting α to a_{1k} and to a_{lw}. The row and column sums $a_{i\star}$ and $a_{\star j}$ are unaltered and ϕ increases of

$$\begin{aligned}
\Delta\phi &= \alpha\big((a_{1\star}-\rho)(a_{\star w}-\gamma)+(a_{l\star}-\rho)(a_{\star k}-\gamma) \\
&\quad -(a_{1\star}-\rho)(a_{\star k}-\gamma)-(a_{l\star}-\rho)(a_{\star m}-\gamma)\big) \\
&= \alpha(a_{1\star}a_{\star w}+a_{l\star}a_{\star k}-a_{1\star}a_{\star k}-a_{l\star}a_{\star w}) \\
&= \alpha(a_{1\star}-a_{l\star})(a_{\star w}-a_{\star k}),
\end{aligned}$$

so that ϕ does not increase.

Now make the following perturbation: add 2α to a_{11} and subtract α to a_{1w} and to a_{v1}. Let us compute the differential of ϕ: as

$$\frac{d\phi}{da_{rc}} = (a_{r\star}-\rho)(a_{\star c}-\gamma)+\sum_{i=1}^{v} a_{ic}(a_{i\star}-\rho)+\sum_{j=1}^{w} a_{rj}(a_{\star j}-\gamma),$$

$$\begin{aligned}
d\phi &= d\alpha\Big(2\frac{d\phi}{da_{11}}-\frac{d\phi}{da_{1w}}-\frac{d\phi}{da_{v1}}\Big) \\
&= d\alpha\Big((a_{1\star}-\rho)(a_{\star 1}-a_{\star w})+(a_{1\star}-a_{v\star})(a_{\star 1}-\gamma)+\sum_{i=1}^{v} a_{i1}(a_{i\star}-\rho) \\
&\qquad +\sum_{j=1}^{w} a_{1j}(a_{\star j}-\gamma)-\sum_{i=1}^{v} a_{iw}(a_{i\star}-\rho)-\sum_{j=1}^{w} a_{vj}(a_{\star j}-\gamma)\Big)
\end{aligned}$$

For positive $d\alpha$, we have by (D.8)

$$\begin{aligned}
d\phi &\leqslant d\alpha\big((a_{1\star}-\rho)(a_{\star 1}-a_{\star w})+(a_{1\star}-a_{v\star})(a_{\star 1}-\gamma)+a_{\star 1}(a_{v\star}-\rho) \\
&\qquad +a_{1\star}(a_{\star w}-\gamma)-a_{\star w}(a_{1\star}-\rho)-a_{v\star}(a_{\star 1}-\gamma)\big) \\
&= d\alpha\big((a_{1\star}-2\rho)(a_{\star 1}-a_{\star w})+(a_{\star 1}-2\gamma)(a_{1\star}-a_{v\star})\big) \\
&< 0\,,
\end{aligned}$$

which contradicts the minimum hypothesis. □

Corollary 4.6. *Let G be a bipartite graph on $v+w$ vertices and of minimal degree 2. Then the number of paths of length 3 in G is at least $e(e/v-1)(e/w-1)$. This bound is achieved exactly if the graph is regular for each of its two colours.*

Proof. A path of length 3 is a sequence of 4 vertices (x, y, z, t) with no repetition such that

$$\{x, y\}, \{y, z\}, \{z, t\} \in G.$$

Given two adjacent vertices y and z, the number of paths (x, y, z, t) makes $(d(y)-1)(d(z)-1)$, where d denotes the degree of a vertex. Therefore the number of all paths of length 3 is

$$\sum_{\{y,z\}\in G} (d(y)-1)(d(z)-1).$$

Let $(a_{ij})_{1\leqslant i\leqslant v, 1\leqslant j\leqslant w}$ be the reduced incidence matrix of G: $a_{ij} = 1$ if the ith vertex of the first class is adjacent to the jth vertex of the second class; otherwise $a_{ij} = 0$. Then this sum is

$$\sum_{i=1}^{v}\sum_{j=1}^{w} a_{ij}(a_{i\star}-1)(a_{\star j}-1), \tag{D.9}$$

so that it suffices to take $\rho = \gamma = 1$ in Th. 4.4. □

4.3 Proof of Theorem 4.1

The case of equality follows from [56, Lemma 1.4.1] because its axiom (i) is exactly what makes Bound (D.10) an equality.

I now follow the proof of [20, Th. 1]. If G contains no cycle of length 4, there is no path of length 3 between two adjacent vertices; if G contain no cycle of length 6, there is at most one path of length 3 between non-adjacent vertices of different colour. Therefore the sum (D.9) is bounded by

$$vw - e \text{ with } e = \sum_{i=1}^{n}\sum_{j=1}^{v} a_{ij}. \tag{D.10}$$

By Corollary 4.6, if all the vertices of G have degree at least two, one has

$$vw - e \geqslant e^3/vw - (1/v + 1/w)e^2 + e$$

and therefore (D.3). In order to get rid of this degree condition, we have to do an induction on the sum $s = v + w$ of the number of columns and the number of rows of the incidence matrix. If $v = 1$, then $P(v, w, e) = (e - w)(e^2 - e + w)$, so that the inequality states $e \leqslant w$, which is trivial; symmetrically for $w = 1$. Suppose the result is true for all $v \times w$ incidence matrices with $v + w = s$. Consider now a $v \times w$ incidence matrix with $v + w = s + 1$ and $v, w \geqslant 2$. If each vertex has degree at least two, the result is true; otherwise there is a column or a row containing only zeroes or exactly one "1". Apply the induction hypothesis on the matrix without this row or column: we get $P(v - 1, w, e - 1) \leqslant 0$ or $P(v, w - 1, e - 1) \leqslant 0$ and we may apply the following growth lemma to conclude.

Lemma 4.7. *Let $v, w \geqslant 1$. If $P(v, w, e) \leqslant 0$, then $P(v + 1, w, e + 1) \leqslant 0$.*

Proof. In fact, one has

$$P(v + 1, w, e + 1) - P(v, w, e) = 2e^2 + (1 - 2v)e + (w - w^2)(2v + 1) - v,$$

which is negative as long as

$$0 \leqslant e \leqslant e_0 = \left(2v - 1 + \sqrt{(2v + 1)\,(2v + 8w^2 - 8w + 1)}\right)/4 = (2v - 1 + \Delta)/4.$$

Let us use that $P(v, w, e)$ has a unique root in e and compute $P(v, w, e_0)$. This makes

$$(4vw^2 + 2w^2 + 1)\Delta/16 + (-16vw^3 - 8v^2w^2 - 8w^3 + 8vw^2 + 2w^2 - 2v + 4w - 1)/16.$$

Then either the second term in this sum is positive and $P(v, w, e_0)$ is positive, or the conjugate expression of this sum is positive, and the product of the sum with this conjugate expression is

$$(w - 1)^2w^2(8v^3w^2 + 4v^2w^2 - 2vw^2 + 2v^2 - w^2 - 4vw + 2v - 2w)/8,$$

which is positive if $v, w \geqslant 1$. □

4.4 Further remarks

Remark 4.8. Theorem 4.1 does not always give the right order of magnitude for the maximal size of a graph of girth 8: as

$$P(v, w, (vw)^{2/3}) = 2(vw)^{5/3} - (vw)^{4/3}(v + w)) \leqslant 2(vw)^{5/3} - 2(vw)^{4/3+1/2} \leqslant 0,$$

we expect to find maximal graphs of size $(vw)^{2/3}$: De Caen and Székely [20, Th. 4] find a counterexample to this expectation if v "lies in an interval just slightly below" w^2. They conjecture [19] that this is the case as soon as $v \gg w^{5/4}$ and $v \ll w^2$.

In the case of $v = w$, let us give the following approximation for the real root of the polynomial. For

$$e = v^{4/3} + \frac{2}{3}v - \frac{2}{9}v^{2/3} - \frac{20}{81}v^{1/3},$$

$$P(v, v, e) = \frac{40}{243}v^{7/3} + \frac{376}{2187}v^2 - \frac{80}{2187}v^{5/3} - \frac{800}{19683}v^{4/3} - \frac{8000}{531441}v \geqslant \frac{129808}{531441},$$

$$\begin{aligned} P(v,v,e-16/81) &= -\frac{8}{531441}(v^{1/3}-1)\big(39366v^{7/3}+28431v^2+8262v^{5/3} \\ &\quad -8748v^{4/3}-11880v-6560v^{2/3}-2432v^{1/3}-512\big) \\ &\leqslant 0. \end{aligned}$$

In particular,

Corollary 4.9. *Let G be a bipartite graph of size e with v vertices in each vertex class. If the girth of G is at least 8, then*

$$e < v^{4/3} + \frac{2}{3}v - \frac{2}{9}v^{2/3} - \frac{20}{81}v^{1/3}.$$

Let us now show that we generalise the following estimations for the size of bipartite graphs of girth 8 in [20, Th. 1]:

(i) *if the minimal degree of G is at least 2, then $e \leqslant 2^{1/3}(vw)^{2/3}$;*

(ii) *if $v \preccurlyeq w^2$ or $w \preccurlyeq v^2$, then $e \preccurlyeq (vw)^{2/3}$.*

In fact,

$$P(v,w,2^{1/3}(vw)^{2/3}) = (vw)^{4/3}(w^{2/3}-2^{2/3}v^{1/3})(v^{2/3}-2^{2/3}w^{1/3}),$$

which is nonnegative exactly if $v \leqslant w^2/4$ and $w \leqslant v^2/4$ or if (v,w) is among $\big\{(1,1),(1,2),(2,1),(2,2),(3,3)\big\}$, and this is the case by Prop. 2.2 if the minimal degree is at least 2.

Furthermore, by Prop. 2.2, if $w > \lfloor v^2/4 \rfloor$, then $\lceil w - v^2/4 \rceil$ vertices in W have degree 0 or 1. If we remove them, we remove at most $\lceil w - v^2/4 \rceil$ edges and the remaining graph has at most $\lfloor v^2/2 \rfloor$ edges because $P(v, \lfloor v^2/4 \rfloor, \lfloor v^2/2 \rfloor + 1) > 0$. This yields

Proposition 4.10. *Let G be a bipartite graph on vertex classes V and W with $\#V = v$ and $\#W = w$ without cycles of length 4 and 6. Its size satisfies*

$$e \leqslant \begin{cases} 2^{1/3}(vw)^{2/3} & \text{if } \max(v,w) \leqslant \lfloor \min(v,w)^2/4 \rfloor \\ \lfloor \min(v,w)^2/4 \rfloor + \max(v,w) & \text{otherwise.} \end{cases}$$

We have optimality in the second alternative: make a bipartition $V = V_1 \cup V_2$ with $V_1 = \lceil v/2 \rceil$ and $V_2 = \lfloor v/2 \rfloor$, let G' be the complete bipartite graph on the colour classes V_1 and V_2, which has $\lfloor v^2/4 \rfloor$ edges. Now consider the bipartite expansion of G', add $\lceil w - v^2/4 \rceil$ new vertices to colour class W, and connect each of them to some vertex of V.

Note that this estimate yields another proof of [35, Th. 1] by means of [35, Th. 3].

Remark 4.11. Our inequality condenses the following facts about the behaviour of e for fixed w and large v. If $w \leqslant 3$, then extremal graphs of girth 8 do not contain any cycle at all, so that their size is $e = v + w - 1$; if $v \geqslant w = 4$ and if $v = w = 5$, then extremal graphs of girth 8 contain exactly one cycle, so that their size is $e = v + w$; if $v > w = 5$, then extremal graphs of girth 8 contain exactly one "θ-graph", so that their size is $e = v + w + 1$.

Chapter E

Ordering simultaneously the columns and lines of 0,1 matrices

with Nikolai Kosmatov.

Let M be a $0,1$ matrix with m rows and n columns:

$$M = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ with } a_{ij} \in \{0,\, 1\}.$$

There are 2^{m+n} such matrices. We would like to study these matrices up to a permutation of the rows and a permutation of the columns, and therefore propose an order on the set of rows and on the set of columns.

Definition 1. Let $L_i = L_i(M) = (a_{i1}, a_{i2}, \dots, a_{in})$, $1 \leqslant i \leqslant m$, be the lines of M and let $C_j = C_j(M) = (a_{1j}, a_{2j}, \dots, a_{mj})$, $1 \leqslant j \leqslant n$, be its columns. The rows (vs. columns) of M are *ordered* if they form an increasing sequence with respect to the lexicographic order.

Note that if the rows and columns are considered as digit sequences of binary numbers, the lexicographic order is just the natural order on $\mathbb{N}$.

Theorem 2. *There is a permutation of rows and columns that orders them simultaneously.*

Definition 3. The numbers $r\,, s \geqslant 1$, $0 = i_0 < i_1 < i_2 < \dots < i_r = m$, $0 = j_0 < j_1 < j_2 < \dots < j_s = n$ define a *grid* on $m \times n$ that divides M in $r \times s$ block matrices

$$B_{t,u} = B_{t,u}(M) = \begin{pmatrix} a_{i_{t-1}+1, j_{u-1}+1} & \dots & a_{i_{t-1}+1, j_u} \\ \vdots & & \vdots \\ a_{i_t, j_{u-1}+1} & \dots & a_{i_t, j_u} \end{pmatrix},\ 1 \leqslant t \leqslant r,\ 1 \leqslant u \leqslant s,$$

so that

$$M = \begin{pmatrix} B_{1,1} & \dots & B_{1,s} \\ \vdots & & \vdots \\ B_{r,1} & \dots & B_{r,s} \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c|ccc}
a_{i_0+1,j_0+1} & \cdots & a_{i_0+1,j_1} & \cdots & a_{i_0+1,j_{s-1}+1} & \cdots & a_{i_0+1,j_s} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i_1,j_0+1} & \cdots & a_{i_1,j_1} & \cdots & a_{i_1,j_{s-1}+1} & \cdots & a_{i_1,j_s} \\ \hline
a_{i_1+1,j_0+1} & \cdots & a_{i_1+1,j_1} & \cdots & a_{i_1+1,j_{s-1}+1} & \cdots & a_{i_1+1,j_s} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i_2,j_0+1} & \cdots & a_{i_2,j_1} & \cdots & a_{i_2,j_{s-1}+1} & \cdots & a_{i_2,j_s} \\ \hline
\vdots & & \vdots & & \vdots & & \vdots \\ \hline
a_{i_{r-1}+1,j_0+1} & \cdots & a_{i_{r-1}+1,j_1} & \cdots & a_{i_{r-1}+1,j_{s-1}+1} & \cdots & a_{i_{r-1}+1,j_s} \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{i_r,j_0+1} & \cdots & a_{i_r,j_1} & \cdots & a_{i_r,j_{s-1}+1} & \cdots & a_{i_r,j_s}
\end{array}\right).$$

A permutation of lines L_{p_1} and L_{p_2} is *admissible* if $i_{t-1} < p_1 \leqslant i_t$ and $i_{t-1} < p_2 \leqslant i_t$ for some $t \in \{1, 2, \dots, r\}$; otherwise, it is *inadmissible.* A permutation of columns C_{q_1} and C_{q_2} is *admissible* if $j_{u-1} < q_1 \leqslant j_u$ and $j_{u-1} < q_2 \leqslant j_u$ for some $u \in \{1, 2, \dots, s\}$; otherwise, it is *inadmissible.* In other words, admissible permutations of rows and columns only permute matrix coefficients inside the block matrices defined by the grid. Matrix M is *ordered with respect to the grid* if

$$L_{i_0+1} \leqslant L_{i_0+2} \leqslant \cdots \leqslant L_{i_1},\ L_{i_1+1} \leqslant \cdots \leqslant L_{i_2},\ \dots,\ L_{i_{r-1}+1} \leqslant \cdots \leqslant L_{i_r};$$
$$C_{j_0+1} \leqslant C_{j_0+2} \leqslant \cdots \leqslant C_{j_1},\ C_{j_1+1} \leqslant \cdots \leqslant C_{j_2},\ \dots,\ C_{j_{s-1}+1} \leqslant \cdots \leqslant C_{j_s},$$

where $\leqslant$ is the lexicographic order.

Theorem 2 is an immediate consequence of Lemma 4: it suffices to consider the trivial grid given by $r = s = 1$, $0 = i_0 < i_1 = m$, $0 = j_0 < j_1 = n$.

Lemma 4. *The rows and columns of a* $0, 1$ *matrix may be simultaneously ordered with respect to a given grid by admissible permutations of its rows and columns.*

Proof by induction. If $m = 1$ or $n = 1$, this is trivial. Let us suppose $m, n > 1$ and let us construct the first line and then the first column of the reordered matrix. Let

$$A = \begin{pmatrix}
\sum_{q=j_0+1}^{j_1} a_{1q} & \sum_{q=j_1+1}^{j_2} a_{1q} & \cdots & \sum_{q=j_{s-1}+1}^{j_s} a_{1q} \\
\sum_{q=j_0+1}^{j_1} a_{2q} & \sum_{q=j_1+1}^{j_2} a_{2q} & \cdots & \sum_{q=j_{s-1}+1}^{j_s} a_{2q} \\
\vdots & \vdots & & \vdots \\
\sum_{q=j_0+1}^{j_1} a_{i_1q} & \sum_{q=j_1+1}^{j_2} a_{i_1q} & \cdots & \sum_{q=j_{s-1}+1}^{j_s} a_{i_1q}
\end{pmatrix}$$

be the matrix of numbers of ones in the lines of the block matrices $B_{1,u}$, $1 \leqslant u \leqslant s$. Let $L_v(A)$ be the minimal line of A: we permute $L_1(M)$ with $L_v(M)$ and still call M the resulting matrix. Then

$$\sum_{q=j_0+1}^{j_1} a_{1q} \leqslant \sum_{q=j_0+1}^{j_1} a_{iq},$$

for each $i \in \{1,\ 2,\ \dots, i_1\}$, so that either some a_{1j}, $1 \leqslant j \leqslant j_1$, vanishes, either $B_{1,1}$ is a matrix of ones. Let us permute the columns of M in such a way that that the first lines of all the resulting block matrices $B_{1,u}$, $1 \leqslant u \leqslant s$, consist in a block of zeroes followed by a block of ones:

$$B_{1,u} = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 & 1 \\
* & * & \dots & * & * & * & * & \dots & * & * \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
* & * & \dots & * & * & * & * & \dots & * & *
\end{pmatrix}.$$

We still call M the resulting matrix. We now refine the grid on $m \times n$ in such a way that the first lines of the block matrices of M with respect to the new grid either consist only of zeroes or consist only of ones, as follows:

$$\left(\begin{array}{ccc|ccc||c||ccc|ccc} 0 & \dots & 0 & 1 & \dots & 1 & \dots & 0 & \dots & 0 & 1 & \dots & 1 \\ * & \dots & * & * & \dots & * & \dots & * & \dots & * & * & \dots & * \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ * & \dots & * & * & \dots & * & \dots & * & \dots & * & * & \dots & * \\ \hline\hline * & \dots & * & * & \dots & * & \dots & * & \dots & * & * & \dots & * \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots & & \vdots \end{array}\right), \tag{E.1}$$

where the double lines belong to the original grid and the simple ones have been added now. Let this new grid be defined by the numbers $r, s' \geqslant 1$, $0 = i_0 < i_1 < i_2 < \cdots < i_r = m$, $0 = j'_0 < j'_1 < j'_2 < \cdots < j'_{s'} = n$, and let $B'_{t,u}$, $t \in \{1, 2, \dots, r\}$, $u \in \{1, 2, \dots, s'\}$, be the block matrices with respect to it. The block matrix $B'_{1,1}$ has one of the two following forms:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ * & * & \dots & * & * \\ \vdots & \vdots & & \vdots & \vdots \\ * & * & \dots & * & * \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

We now proceed similarly with the first column. Let

$$B = \begin{pmatrix} \sum_{p=i_0+1}^{i_1} a_{p1} & \sum_{p=i_0+1}^{i_1} a_{p2} & \dots & \sum_{p=i_0+1}^{i_1} a_{pj'_1} \\ \sum_{p=i_1+1}^{i_2} a_{p1} & \sum_{p=i_1+1}^{i_2} a_{p2} & \dots & \sum_{p=i_1+1}^{i_2} a_{pj'_1} \\ \vdots & \vdots & & \vdots \\ \sum_{p=i_{r-1}+1}^{i_r} a_{p1} & \sum_{p=i_{r-1}+1}^{i_r} a_{p2} & \dots & \sum_{p=i_{r-1}+1}^{i_r} a_{pj'_1} \end{pmatrix}$$

be the matrix of numbers of ones in the columns of the block matrices $B'_{t,1}$, $1 \leqslant t \leqslant r$. Let $C_v(B)$ be the minimal column of B: we permute $C_1(M)$ with $C_v(M)$ and still call M the resulting matrix. Let us permute the rows of M in such a way that the first columns of all the block matrices $B'_{t,1}$, $1 \leqslant t \leqslant r$, consist in a block of zeroes followed by a block of ones:

$$B'_{t,1}(M) = \begin{pmatrix} 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \\ 1 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 1 & * & \dots & * \end{pmatrix}$$

The form of $B'_{1,1}$ shows that this can be done without permuting the first row, so we do not. We still call the resulting matrix M. We now refine the grid on $m \times n$ in such a way that the first columns of the block matrices with respect to the new grid either consist only of zeroes or consist only of ones, so that also the tranpose of M has a form as in (E.1). Let the new grid be defined by the numbers $r', s' \geqslant 1$, $0 = i'_0 < i'_1 < i'_2 < \cdots < i'_{r'} = m$, $0 = j'_0 < j'_1 < j'_2 < \cdots < j'_{s'} = n$.

Let N be the $m - 1 \times n - 1$ submatrix in the lower right corner of M. By induction hypothesis, N may be ordered by admissible permutations with respect to the grid on $m - 1 \times n - 1$ induced by the grid just defined. These permutations are also admissible with respect to the latter grid and do not permute the first line and the first column of M. Let us again call the resulting matrices M and N.

As the rows and columns of N are ordered, we have for M

$$L_{i_0+2} \leqslant \cdots \leqslant L_{i_1},\ L_{i_1+1} \leqslant \cdots \leqslant L_{i_2},\ \dots,\ L_{i_{r-1}+1} \leqslant \cdots \leqslant L_{i_r};$$
$$C_{j_0+2} \leqslant \cdots \leqslant C_{j_1},\ C_{j_1+1} \leqslant \cdots \leqslant C_{j_2},,\ \dots,\ C_{j_{s-1}+1} \leqslant \cdots \leqslant C_{j_s},$$

and our choice for the first line and the first column of M ensures that

$$L_{i_0+1} \leqslant L_{i_0+2},\ C_{j_0+1} \leqslant C_{j_0+2}. \qquad \square$$

Chapter F

Transfer of Fourier multipliers into Schur multipliers and sumsets in a discrete group

with Éric Ricard.

We inspect the relationship between relative Fourier multipliers on noncommutative Lebesgue-Orlicz spaces of a discrete group Γ and relative Toeplitz-Schur multipliers on Schatten-von-Neumann-Orlicz classes. Four applications are given: lacunary sets; unconditional Schauder bases for the subspace of a Lebesgue space determined by a given spectrum $\Lambda \subseteq \Gamma$; the norm of the Hilbert transform and the Riesz projection on Schatten-von-Neumann classes with exponent a power of 2; the norm of Toeplitz Schur multipliers on Schatten-von-Neumann classes with exponent less than 1.

1 Introduction

Let Λ be a subset of $\mathbb{Z}$ and let x be a bounded measurable function on the circle $\mathbb{T}$ with Fourier spectrum in Λ: we write $x \in \mathrm{L}^\infty_\Lambda$, $x \sim \sum_{k\in\Lambda} x_k z^k$. The matrix of the associated operator $y \mapsto xy$ on L^2 with respect to its trigonometric basis is the Toeplitz matrix

$$
(x_{r-c})_{(r,c)\in\mathbb{Z}\times\mathbb{Z}} = \begin{array}{c c}
 & \begin{array}{ccccc} \cdots & 1 & 0 & -1 & \cdots \end{array} \\
\begin{array}{c} \vdots \\ 1 \\ 0 \\ -1 \\ \vdots \end{array} &
\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & x_0 & x_1 & x_2 & \ddots \\
\ddots & x_{-1} & x_0 & x_1 & \ddots \\
\ddots & x_{-2} & x_{-1} & x_0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\end{array}
$$

with support in $\check{\Lambda} = \{(r,c) : r - c \in \Lambda\}$.

This is an entry point to the interplay between harmonic analysis and operator theory. In the general case of a discrete group Γ, the counterpart to a bounded measurable function is defined as a bounded operator on ℓ^2_Γ whose matrix has the form $(x_{rc^{-1}})_{(r,c)\in\Gamma\times\Gamma}$ for some sequence $(x_\gamma)_{\gamma\in\Gamma}$. This will be the framework of the body of this article, while the introduction sticks to the case $\Gamma = \mathbb{Z}$.

We are concerned with two kinds of multipliers. A sequence $\varphi = (\varphi_k)_{k\in\Lambda}$ defines

This work is partially supported by ANR 06-BLAN-0015.

2000 *Mathematics subject classification:* Primary 47B49; Secondary 43A22, 43A46, 46B28.

Key words and phrases: Fourier multiplier, Toeplitz Schur multiplier, lacunary set, unconditional approximation property, Hilbert transform, Riesz projection.

– the relative Fourier multiplication operator on trigonometric polynomials with spectrum in Λ by

$$\sum_{k\in\Lambda} x_k z^k \mapsto \sum_{k\in\Lambda} \varphi_k x_k z^k; \tag{F.1}$$

– the relative Schur multiplication operator on finite matrices with support in $\ddot{\Lambda}$ by

$$(x_{r,c})_{(r,c)\in\mathbb{Z}\times\mathbb{Z}} \mapsto (\ddot{\varphi}_{r,c} x_{r,c})_{(r,c)\in\mathbb{Z}\times\mathbb{Z}} \tag{F.2}$$

where $\ddot{\varphi}_{r,c} = \varphi_{r-c}$.

Marek Bożejko and Gero Fendler proved that these two multipliers have the same norm. The operator (F.1) is nothing but the restriction of (F.2) to Toeplitz matrices. They noted that it is automatically *completely* bounded: it has the same norm when acting on trigonometric series with operator coefficients x_k, and this permits to remove this restriction. Schur multiplication is also automatically completely bounded.

This observation has been extended by Gilles Pisier to multipliers acting on a translation invariant Lebesgue space L^p_Λ and on the subspace $\mathrm{S}^p_{\ddot{\Lambda}}$ of elements of a Schatten-von-Neumann class supported by $\ddot{\Lambda}$, respectively: it yields that the complete norm of a relative Schur multiplier (F.2) remains bounded by the complete norm of the relative Fourier multiplier (F.1).

But L^p_Λ is not a subspace of $\mathrm{S}^p_{\ddot{\Lambda}}$, so that a relative Fourier multiplier may not be viewed anymore as the restriction of a relative Schur multiplier to Toeplitz matrices. We point out that this difficulty may be overcome by using Szegő's limit theorem: a bounded measurable real function is the weak* limit of the normalised counting measure of eigenvalues of finite truncates of its Toeplitz matrix. Note that other types of approximation are also available, as the completely positive approximation property and Reiter sequences combined with complex interpolation: they are compared in Section 3 in terms of local embeddings of L^p into S^p. They are more canonical than Szegő's limit theorem, but give no access to general Orlicz norms.

Theorem 1.1. *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The norm of the following operators is equal:*

– *the relative Fourier multiplication operator* (F.1) *on the Lebesgue-Orlicz space $\mathrm{L}^\psi_\Lambda(\mathrm{S}^\psi)$ of S^ψ-valued trigonometric series with spectrum in Λ;*

– *the relative Schur multiplication operator* (F.2) *on the Schatten-von-Neumann-Orlicz class $\mathrm{S}^\psi_{\ddot{\Lambda}}(\mathrm{S}^\psi)$ of S^ψ-valued matrices with support in $\ddot{\Lambda}$.*

Look at Theorem 2.5 for the precise statement in the general case of an amenable group Γ, for which a block matrix variant of Szegő's limit theorem in the style of Erik Bédos [6], Theorem 2.1, is available.

An application of this theorem to the class of all unimodular Fourier multipliers yields a transfer of lacunary subsets into lacunary matrix patterns. Call Λ *unconditional in* L^p if $(z^k)_{k\in\Lambda}$ is an unconditional basis of L^p_Λ, and $\ddot{\Lambda}$ *unconditional in* S^p if the sequence $(\mathrm{e}_q)_{q\in\ddot{\Lambda}}$ of elementary matrices is an unconditional basis of $\mathrm{S}^p_{\ddot{\Lambda}}$. These properties are also known as $\Lambda(\max(2,p))$ and $\sigma(p)$, respectively; they have natural "complete" counterparts that are also known as $\Lambda(p)_{\mathrm{cb}}$ ($\mathrm{K}(p)_{\mathrm{cb}}$ if $p \leqslant 2$) and $\sigma(p)_{\mathrm{cb}}$, respectively: see Definitions 4.1 and 4.2.

Corollary 1.2. *If $\ddot{\Lambda}$ is unconditional in S^p, then Λ is unconditional in L^p. $\ddot{\Lambda}$ is completely unconditional in S^p if and only if Λ is completely unconditional in L^p.*

Look at Proposition 4.3 for the precise statement in the general case of a discrete group Γ.

The two most prominent multipliers are the Riesz projection and the Hilbert transform. The first consists in letting φ be the indicator function of nonnegative integers and transfers into the upper triangular truncation of matrices. The second corresponds to the sign function and transfers into the Hilbert matrix transform. We obtain the following partial results.

Theorem 1.3. *The norm of the matrix Riesz projection and of the matrix Hilbert transform on $\mathrm{S}^\psi(\mathrm{S}^\psi)$ coincide with their norm on S^ψ.*

– *If p is a power of 2, the norm of the matrix Hilbert transform on S^p is $\cot(\pi/2p)$.*

– *The norm of the matrix Riesz projection on S^4 is $\sqrt{2}$.*

The transfer technique lends itself naturally to the case where Λ contains a sumset $R + C$: if subsets R' and C' are extracted so that the $r + c$ with $r \in R'$ and $c \in C'$ are pairwise distinct, they may play the rôle of rows and columns. Here are consequences of the conditionality of the sequence of elementary matrices $\mathrm{e}_{r,c}$ in S^p for $p \neq 2$ and of the unboundedness of the Riesz transform on S^1 and S^∞, respectively.

Theorem 1.4. *If $(z^k)_{k\in\Lambda}$ is a completely unconditional basis of L^p_Λ with $p \neq 2$, then Λ does not contain sumsets $R + C$ of arbitrarily large sets.*

- *If L^1_Λ admits some completely unconditional approximating sequence, or*
- *if the space C_Λ of continuous functions with spectrum in Λ admits some unconditional approximating sequence,*

then Λ does not contain the sumset $R + C$ of two infinite sets.

The proof of the second part of this theorem consists in constructing infinite subsets R' and C', and skipped block sums $\sum(T_{k_{j+1}} - T_{k_j})$ of a given approximating sequence that act like the projection on the "upper triangular" part of $R' + C'$. Look at Proposition 4.8 and Theorem 7.4 for the precise statement in the general case of a discrete group Γ.

In the case of quasi-normed Schatten-von-Neumann classes S^p with $p < 1$, the transfer technique yields a new proof for the following result of Alexey Alexandrov and Vladimir Peller.

Theorem 1.5. *Let $0 < p < 1$. The Fourier multiplier φ is contractive on L^p or on $\mathrm{L}^p(\mathrm{S}^p)$ if and only if the Schur multiplier $\check{\varphi}$ is contractive on S^p or on $\mathrm{S}^p(\mathrm{S}^p)$, if and only if the sequence φ is the Fourier transform of an atomic measure of the form $\sum a_g\delta_g$ on $\mathbb{T}$ with $\sum|a_g|^p \leqslant 1$.*

The emphasis put on *relative* Schur multipliers motivates the natural question of how the norm of an elementary Schur multiplier, that is a rank 1 matrix $(\varrho_{r,c}) = (x_r y_c)$, gets affected when the action of ϱ is restricted to matrices with a given support. The surprising answer is the following.

Theorem 1.6. *Let $I \subseteq R \times C$ and consider $(x_r)_{r\in R}$ and $(y_c)_{c\in C}$. The relative Schur multiplier on S^∞_I given by $(x_r y_c)_{(r,c)\in I}$ has norm $\sup_{(r,c)\in I}|x_r y_c|$.*

Let us finally describe the content of this article. Section 2 is devoted to transfer techniques for Fourier and Schur multipliers provided by a block matrix Szegő limit theorem. This theorem provides local embeddings of L^ψ into S^ψ; Section 3 shows how interpolation may be used to define such embeddings for the scale of L^p spaces. Section 4 is devoted to the transfer of lacunary sets into lacunary matrix patterns; the unconditional constant of a set Λ is related to the size of the sumsets it contains. Section 5 deals with Toeplitz Schur multipliers for $p < 1$ and comments on the case $p \geqslant 1$. The Riesz projection and the Hilbert transform are studied in Section 6. In Section 7, the presence of sumsets in a spectrum Λ is shown to be an obstruction for the existence of completely unconditional bases for L^p_Λ. The last section provides a norm-preserving extension for partially specified rank 1 Schur multipliers.

Notation. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Given an index set C and $c \in C$, e_c is the sequence defined on C as the indicator function $\chi_{\{c\}}$ of the singleton $\{c\}$, so that $(\mathrm{e}_c)_{c\in C}$ is the canonical Schauder basis of the Hilbert space of square summable sequences indexed by C, denoted by ℓ^2_C. We will use the notation $\ell^2_n = \ell^2_{\{1,2,\ldots,n\}}$ and $\ell^2 = \ell^2_{\mathbb{N}}$.

Given a product set $I = R \times C$ and $q = (r, c)$, the indicator function $\mathrm{e}_q = \mathrm{e}_{r,c}$ is the *elementary matrix* identified with the linear operator from ℓ^2_C to ℓ^2_R that maps e_c on e_r and all other basis vectors on 0. The *matrix coefficient* at coordinate q of a linear operator x from ℓ^2_C to ℓ^2_R is $x_q = \operatorname{tr} \mathrm{e}^*_q x$ and its *matrix representation* is $(x_q)_{q\in R\times C} = \sum_{q\in R\times C} x_q \mathrm{e}_q$. The *support* or *pattern* of x is $\{q \in R \times C : x_q \neq 0\}$.

The space of all bounded operators from ℓ^2_C to ℓ^2_R is denoted by $\mathbb{B}(\ell^2_C, \ell^2_R)$, and its subspace of compact operators is denoted by S^∞.

Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The *Schatten-von-Neumann-Orlicz class* S^ψ is the space of those compact operators x from ℓ^2_C to ℓ^2_R such that $\operatorname{tr}\psi(|x|/a) < \infty$ for some $a > 0$. If ψ is convex, then S^ψ is a Banach space for the norm given by $\|x\|_{\mathrm{S}^\psi} = \inf\{a > 0 : \operatorname{tr}\psi(|x|/a) \leqslant 1\}$. Otherwise, S^ψ is a Fréchet space for the F-norm given by $\|x\|_{\mathrm{S}^\psi} = \inf\{a > 0 : \operatorname{tr}\psi(|x|/a) \leqslant a\}$ (see [69, Chapter 3].) This space may also be constructed as the noncommutative Lebesgue-Orlicz space $\mathrm{L}^\psi(\mathrm{tr})$ associated to the von Neumann algebra $\mathbb{B}(\ell^2_C, \ell^2_R)$

endowed with the normal faithful semifinite trace tr. If ψ is the power function $t \mapsto t^p$, this space is denoted S^p: if $p \geqslant 1$, then $\|x\|_{\mathrm{S}^p} = (\operatorname{tr}|x|^p)^{1/p}$; if $p < 1$, then $\|x\|_{\mathrm{S}^p} = (\operatorname{tr}|x|^p)^{1/(1+p)}$.

If $\# C = \# R = n$, then $\mathbb{B}(\ell^2_C, \ell^2_R)$ identifies with the space of $n \times n$ matrices denoted S^∞_n, and we write S^ψ_n for S^ψ. Let $(R_n \times C_n)$ be a sequence of finite sets such that each element of $R \times C$ eventually is in $R_n \times C_n$. Then the sequence of operators $P_n \colon x \mapsto \sum_{q \in R_n \times C_n} x_q \mathrm{e}_q$ tends pointwise to the identity on S^ψ.

For $I \subseteq R \times C$, we define the space S^ψ_I as the closed subspace of S^ψ spanned by $(\mathrm{e}_q)_{q \in I}$: this coincides with the subspace of those $x \in \mathrm{S}^\psi$ whose support is a subset of I.

A *relative Schur multiplier* on S^ψ_I is a sequence $\varrho = (\varrho_q)_{q \in I} \in \mathbb{C}^I$ such that the associated *Schur multiplication operator* M_ϱ defined by $\mathrm{e}_q \mapsto \varrho_q \mathrm{e}_q$ for $q \in I$ is bounded on S^ψ_I. The norm $\|\varrho\|_{\mathrm{M}(\mathrm{S}^\psi_I)}$ of ϱ is defined as the norm of M_ϱ. This norm is the supremum of the norm of its restrictions to finite rectangle sets $R' \times C'$. We used [76, 78] as a reference.

Let Γ be a discrete group with identity e. The *reduced* C^**-algebra* of Γ is the closed subspace spanned by the left translations λ_γ (the linear operators defined on ℓ^2_Γ by $\lambda_\gamma \mathrm{e}_\beta = \mathrm{e}_{\gamma\beta}$) in $\mathbb{B}(\ell^2_\Gamma)$; we denote it by C. The *von Neumann algebra* of Γ is its weak* closure, endowed with the normal faithful normalised finite trace τ defined by $\tau(x) = x_{\mathrm{e},\mathrm{e}}$; we denote it by L^∞. Let $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0: we define the noncommutative Lebesgue-Orlicz space L^ψ of Γ as the completion of L^∞ with respect to the norm given by $\|x\|_{\mathrm{L}^\psi} = \inf\{a > 0 : \tau(\psi(|x|/a)) \leqslant 1\}$ if ψ is convex and with respect to the F-norm given by$\|x\|_{\mathrm{L}^\psi} = \inf\{a > 0 : \tau(\psi(|x|/a)) \leqslant a\}$ otherwise. If ψ is the power function $t \mapsto t^p$, this space is denoted L^p: if $p \geqslant 1$, then $\|x\|_{\mathrm{L}^p} = \tau(|x|^p)^{1/p}$; if $p < 1$, then $\|x\|_{\mathrm{L}^p} = \tau(|x|^p)^{1/(1+p)}$. The *Fourier coefficient* of x at γ is $x_\gamma = \tau(\lambda^*_\gamma x) = x_{\gamma,\mathrm{e}}$ and its *Fourier series* is $\sum_{\gamma \in \Gamma} x_\gamma \lambda_\gamma$. The *spectrum* of an element x is $\{\gamma \in \Gamma : x_\gamma \neq 0\}$. Let X be the space C or L^ψ and let $\Lambda \subseteq \Gamma$: then we define X_Λ as the closed subspace of X spanned by the λ_γ with $\gamma \in \Lambda$. We skip the general question for which spaces X this coincides with the subspace of those $x \in X$ whose spectrum is a subset of Λ, but note that this is the case if Γ is an amenable group (or if L^∞ has the QWEP by [45, Theorem 4.4]) and ψ is the power function $t \mapsto t^p$; note also that our definition of X_Λ makes it a subspace of the *heart* of X: if $x \in X_\Lambda$, then $\tau(\psi(|x|/a))$ is finite for all $a > 0$.

A *relative Fourier multiplier* on X_Λ is a sequence $\varphi = (\varphi_\gamma)_{\gamma \in \Lambda} \in \mathbb{C}^\Lambda$ such that the associated Fourier multiplication operator C_φ defined by $\lambda_\gamma \mapsto \varphi_\gamma \lambda_\gamma$ for $\gamma \in \Lambda$ is bounded on X_Λ. The norm $\|\varphi\|_{\mathrm{C}(X_\Lambda)}$ of φ is defined as the norm of C_φ. Fourier multipliers on the whole of C are also called *multipliers of the Fourier algebra* $\mathrm{A}(\Gamma)$ (which may be identified with L^1); they form the set $\mathrm{C}(\mathrm{A}(\Gamma))$.

The space $\mathrm{S}^\psi(\mathrm{S}^\psi)$ is the space of those compact operators x from $\ell^2 \otimes \ell^2_C$ to $\ell^2 \otimes \ell^2_R$ such that $\|x\|_{\mathrm{S}^\psi(\mathrm{S}^\psi)} = \inf\{a : \operatorname{tr} \otimes \operatorname{tr} \psi(|x|/a) \leqslant 1\}$: it is the noncommutative Lebesgue-Orlicz space $\mathrm{L}^\psi(\operatorname{tr} \otimes \operatorname{tr})$ associated to the von Neumann algebra $\mathbb{B}(\ell^2) \otimes \mathbb{B}(\ell^2_C, \ell^2_R)$. One may think of $\mathrm{S}^\psi(\mathrm{S}^\psi)$ as the S^ψ-valued Schatten-von-Neumann class: we define the matrix coefficient of x at q by $x_q = (\mathrm{Id}_{\mathrm{S}^\psi} \otimes \operatorname{tr})\big((\mathrm{Id}_{\ell^2} \otimes \mathrm{e}^*_q) x\big) \in \mathrm{S}^\psi$ and its matrix representation by $\sum_{q \in R \times C} x_q \otimes \mathrm{e}_q$. The support of x and the subspace $\mathrm{S}^\psi_I(\mathrm{S}^\psi)$ are defined in the same way as S^ψ_I.

Similarly, the space $\mathrm{L}^\psi(\operatorname{tr} \otimes \tau)$ is the noncommutative Lebesgue-Orlicz space associated to the von Neumann algebra $\mathbb{B}(\ell^2) \otimes \mathrm{L}^\infty = \mathrm{L}^\infty(\operatorname{tr} \otimes \tau)$. One may think of $\mathrm{L}^\psi(\operatorname{tr} \otimes \tau)$ as the S^ψ-valued noncommutative Lebesgue space: we define the Fourier coefficient of x at γ by $x_\gamma = (\mathrm{Id}_{\mathrm{S}^\psi} \otimes \tau)\big((\mathrm{Id}_{\ell^2} \otimes \lambda^*_\gamma) x\big) \in \mathrm{S}^\psi$ and its Fourier series by $\sum_{\gamma \in \Gamma} x_\gamma \otimes \lambda_\gamma$; the spectrum of x is defined accordingly. The subspace $\mathrm{L}^\psi_\Lambda(\operatorname{tr} \otimes \tau)$ is the closed subspace of $\mathrm{L}^\psi(\operatorname{tr} \otimes \tau)$ spanned by the $x \otimes \lambda_\gamma$ with $x \in \mathrm{S}^\psi$ and $\gamma \in \Lambda$.

An operator T on S^ψ_I is *bounded on* $\mathrm{S}^\psi_I(\mathrm{S}^\psi)$ if the linear operator $\mathrm{Id}_{\mathrm{S}^\psi} \otimes T$ defined by $x \otimes y \mapsto x \otimes T(y)$ for $x \in \mathrm{S}^\psi$ and y in S^ψ_I on finite tensors extends to a bounded operator $\mathrm{Id}_{\mathrm{S}^\psi} \otimes T$ on $\mathrm{S}^\psi_I(\mathrm{S}^\psi)$. The norm of a Schur multiplier ϱ on $\mathrm{S}^\psi_I(\mathrm{S}^\psi)$ is defined as the norm of $\mathrm{Id}_{\mathrm{S}^\psi} \otimes \mathrm{M}_\varrho$. Similar definitions hold for an operator T on L^ψ_Λ; the norm of a Fourier multiplier φ on $\mathrm{L}^\psi_\Lambda(\operatorname{tr} \otimes \tau)$ is the norm of $\mathrm{Id}^\psi_{\mathrm{S}} \otimes \mathrm{C}_\varphi$ on $\mathrm{L}^\psi_\Lambda(\operatorname{tr} \otimes \tau)$.

Let ψ be the power function $t \mapsto t^p$ with $p \geqslant 1$: the norms on $\mathrm{S}^p(\mathrm{S}^p)$ and $\mathrm{L}^p(\operatorname{tr} \otimes \tau)$ describe the canonical operator space structure on S^p and L^p, respectively: see [76, Corollary 1.4]; we should rather use the notation $\mathrm{S}^p[\mathrm{S}^p]$ and $\mathrm{S}^p[\mathrm{L}^p]$. This explains the following terminology. An operator T on S^p_I is *completely bounded* (c.b. for short) if $\mathrm{Id}_{\mathrm{S}^p} \otimes T$ is bounded on $\mathrm{S}^p_I(\mathrm{S}^p)$; the norm of $\mathrm{Id}_{\mathrm{S}^p} \otimes T$ is the *complete norm* of T (compare [76, Lemma 1.7].) The complete norm $\|\varrho\|_{\mathrm{M_{cb}}(\mathrm{S}^p_I)}$ of a Schur multiplier ϱ is defined as the complete norm of M_ϱ. Note that the complete norm of a Schur multiplier ϱ on S^∞_I is equal to its norm [71, Theorem 3.2]: $\|\varrho\|_{\mathrm{M_{cb}}(\mathrm{S}^\infty_I)} = \|\varrho\|_{\mathrm{M}(\mathrm{S}^\infty_I)}$. The complete norm $\|\varphi\|_{\mathrm{C_{cb}(L}^p_\Lambda)}$ of a Fourier multiplier φ is defined as the complete norm of C_φ. The complete norm of an operator T on C_Λ is the norm of $\mathrm{Id}_{\mathrm{S}^\infty} \otimes T$ on the subspace of $\mathrm{S}^\infty \otimes \mathrm{C}$ spanned by the $x \otimes \lambda_\gamma$ with $x \in \mathrm{S}^\infty$ and $\gamma \in \Lambda$; in the case $\Lambda = \Gamma$, φ is also called a *c.b. multiplier of the Fourier algebra* $\mathrm{A}(\Gamma)$ and one writes $\varphi \in \mathrm{C_{cb}}(\mathrm{A}(\Gamma))$; if Γ is amenable, the complete norm of a Fourier multiplier φ on C_Λ is equal to its norm [25, Corollary 1.8]: $\|\varphi\|_{\mathrm{C_{cb}(C}_\Lambda)} = \|\varphi\|_{\mathrm{C(C}_\Lambda)}$.

An element whose norm is at most 1 is *contractive*, and if its complete norm is at most 1, it is *completely* contractive.

If Γ is abelian, let G be its dual group and endow it with its unique normalised Haar measure m: then the Fourier transform identifies C as the space of continuous functions on G, L^∞ as the space of classes of bounded measurable functions on (G, m), L^ψ as the Lebesgue-Orlicz space of classes of ψ-integrable functions on (G, m), $\tau(x)$ as $\int_G x(g)\,\mathrm{d}m(g)$, $\mathrm{L}^\psi(\mathrm{tr} \otimes \tau)$ as the S^ψ-valued Lebesgue-Orlicz space $\mathrm{L}^\psi(\mathrm{S}^\psi)$ and x_γ as $\hat{x}(\gamma)$.

2 Transfer between Fourier and Schur multipliers

Let Λ be a subset of a discrete group Γ and let φ be a relative Fourier multiplier on C_Λ, the closed subspace spanned by $(\lambda_\gamma)_{\gamma \in \Lambda}$ in the reduced C*-algebra of Γ. Let $x \in \mathrm{C}_\Lambda$: the matrix of x is *constant down the diagonals* in the sense that for every $(r, c) \in \Gamma \times \Gamma$, $x_{r,c} = x_{rc^{-1},\mathrm{e}} = x_{rc^{-1}}$; we say that x is a *Toeplitz* operator on ℓ^2_Γ. Furthermore, the matrix of the Fourier product $\mathrm{C}_\varphi x$ of φ with x is given by $(\mathrm{C}_\varphi x)_{r,c} = \varphi_{rc^{-1}} x_{r,c}$. This shows that if we set $\H{\Lambda} = \{(r, c) \in \Gamma \times \Gamma : rc^{-1} \in \Lambda\}$ and $\H{\varphi}_{r,c} = \varphi_{rc^{-1}}$, then $\mathrm{C}_\varphi x$ is the Schur product $\mathrm{M}_{\H{\varphi}} x$ of $\H{\varphi}$ with x. We have transferred the Fourier multiplier φ into the Schur multiplier $\H{\varphi}$: this shows at once that the norm of the Fourier multiplier φ on C_Λ is the norm of the Schur multiplier $\H{\varphi}$ on the subspace of Toeplitz elements of $\mathbb{B}(\ell^2_\Gamma)$ with support in $\H{\Lambda}$, and that the same holds for complete norms.

We shall now give us the means to generalise this identification to the setting of Lebesgue-Orlicz spaces L^ψ: we shall bypass the main obstacle, that L^ψ may not be considered as a subspace of S^ψ, by the Szegő limit theorem as stated by Erik Bédos [6, Theorem 10].

As we want to compute complete norms of multipliers, we shall generalise the Szegő limit theorem to the block matrix case, not considered in [6]. Let us first recall the scalar case. Consider a discrete amenable group Γ: it admits a *Følner averaging net of sets* (Γ_ι), that is,

- each Γ_ι is a finite subset of Γ;
- $\#(\gamma\Gamma_\iota \Delta \Gamma_\iota) = o(\#\,\Gamma_\iota)$ for each $\gamma \in \Gamma$.

Each set Γ_ι corresponds to the orthogonal projection p_ι of ℓ^2_Γ onto its $(\#\,\Gamma_\iota)$-dimensional subspace of sequences supported by Γ_ι. The *truncate* of a selfadjoint operator $y \in \mathbb{B}(\ell^2_\Gamma)$ with respect to Γ_ι is $y_\iota = p_\iota y p_\iota$: it has $\#\,\Gamma_\iota$ eigenvalues α_j, counted with multiplicities, and its *normalised counting measure of eigenvalues* is

$$\mu_\iota = \frac{1}{\#\,\Gamma_\iota} \sum_{j=1}^{\#\,\Gamma_\iota} \delta_{\alpha_j}.$$

If y is a Toeplitz operator, that is, if $y \in \mathrm{L}^\infty$, Erik Bédos [6, Theorem 10] proves that (μ_ι) converges weak* to the *spectral measure* of y with respect to τ, which is the unique Borel probability measure μ on $\mathbb{R}$ such that

$$\tau(\psi(y)) = \int_{\mathbb{R}} \psi(\alpha)\mathrm{d}\mu(\alpha)$$

for every continuous function ψ on $\mathbb{R}$ that tends to zero at infinity. If Γ is abelian, then y may be identified as the class of a real-valued bounded measurable function on the group G dual to Γ and μ is the distribution of y.

The matrix Szegő limit theorem is the analogue of this result for selfadjoint elements $y \in \mathrm{S}^\infty_n \otimes \mathrm{L}^\infty$, whose S^∞_n*-valued spectral measure* μ is defined by

$$\int_{\mathbb{R}} \psi(\alpha)\mathrm{d}\mu(\alpha) = \mathrm{Id}_{\ell^2_n} \otimes \tau(\psi(y)).$$

The orthogonal projection $\tilde{p}_\iota = \mathrm{Id}_{\ell^2_n} \otimes p_\iota$ defines the truncate $y_\iota = \tilde{p}_\iota y \tilde{p}_\iota \in \mathrm{S}^\infty_n \otimes \mathbb{B}(\ell^2_{\Gamma_\iota})$, and the S^∞_n*-valued normalised counting measure of eigenvalues* μ_ι by

$$\int_{\mathbb{R}} \psi(\alpha)\mathrm{d}\mu_\iota(\alpha) = \mathrm{Id}_{\ell^2_n} \otimes \frac{\mathrm{tr}}{\#\,\Gamma_\iota}(\psi(y_\iota)).$$

Theorem 2.1 (Matrix Szegő limit theorem). *Let Γ be a discrete amenable group and let (Γ_ι) be a Følner averaging net for Γ. Let y be a selfadjoint element of $\mathrm{S}^\infty_n \otimes \mathrm{L}^\infty$. The net (μ_ι) of S^∞_n-valued normalised counting measures of eigenvalues of the truncates of y with respect to Γ_ι converges in the weak* topology to the spectral measure of y:*

$$\int_{\mathbb{R}} \psi(\alpha)\mathrm{d}\mu_\iota(\alpha) \to \mathrm{Id}_{\ell^2_n} \otimes \tau(\psi(y))$$

for every continuous function ψ on $\mathbb{R}$ that tends to zero at infinity.

Sketch of proof. Let us first suppose that $y \in \mathrm{S}^\infty_n \otimes \mathrm{C}$: we may suppose that $y = \sum_{\gamma\in\Gamma} y_\gamma \otimes \lambda_\gamma$ with only a finite number of the $y_\gamma \in \mathrm{S}^\infty_n$ nonzero: the S^∞_n-valued matrix of y_ι for the canonical basis of $\ell^2_{\Gamma_\iota}$ is $(y_{rc^{-1}})_{(r,c)\in\Gamma_\iota\times\Gamma_\iota}$. It suffices to prove that

$$\mathrm{Id} \otimes \frac{\mathrm{tr}}{\#\Gamma_\iota}(y_\iota^k) \to \mathrm{Id} \otimes \tau(y^k) \tag{F.3}$$

for every k. This is trivial if $k = 0$. If $k = 1$, then

$$\mathrm{Id} \otimes \frac{\mathrm{tr}}{\#\Gamma_\iota}(y_\iota) = \frac{1}{\#\Gamma_\iota}\sum_{c\in\Gamma_\iota} y_{c,c} = \mathrm{Id} \otimes \tau(y)$$

as $y_{c,c} = y_{cc^{-1}} = y_{\mathrm{e}}$. If $k \geqslant 2$, the same formula holds with y^k instead of y:

$$\mathrm{Id} \otimes \tau(y^k) = \mathrm{Id} \otimes \frac{\mathrm{tr}}{\#\Gamma_\iota}(\tilde{p}_\iota y^k \tilde{p}_\iota),$$

so that we wish to prove

$$\mathrm{Id} \otimes \mathrm{tr}(\tilde{p}_\iota y^k \tilde{p}_\iota - (\tilde{p}_\iota y \tilde{p}_\iota)^k) = o(\#\,\Gamma_\iota).$$

Note that

$$\big\|\mathrm{Id} \otimes \mathrm{tr}\big(\tilde{p}_\iota y^k \tilde{p}_\iota - (\tilde{p}_\iota y \tilde{p}_\iota)^k\big)\big\|_{\mathrm{S}^1_n} \leqslant \|\tilde{p}_\iota y^k \tilde{p}_\iota - (\tilde{p}_\iota y \tilde{p}_\iota)^k\|_{\mathrm{S}^1(\mathrm{S}^1_n)}.$$

Lemma 5 in [6] provides the following estimate: as

$$\tilde{p}_\iota y^k \tilde{p}_\iota - (\tilde{p}_\iota y \tilde{p}_\iota)^k = \tilde{p}_\iota y^{k-1}(y\tilde{p}_\iota - \tilde{p}_\iota y \tilde{p}_\iota) + (\tilde{p}_\iota y^{k-1} \tilde{p}_\iota - (\tilde{p}_\iota y \tilde{p}_\iota)^{k-1}) y \tilde{p}_\iota,$$

an induction yields

$$\|\tilde{p}_\iota y^k \tilde{p}_\iota - (\tilde{p}_\iota y \tilde{p}_\iota)^k\|_{\mathrm{S}^1(\mathrm{S}^1_n)} \leqslant k\|y\|^{k-1}_{\mathrm{S}^\infty_n\otimes\mathrm{L}^\infty}\|y\tilde{p}_\iota - \tilde{p}_\iota y \tilde{p}_\iota\|_{\mathrm{S}^1(\mathrm{S}^1_n)}.$$

It suffices to consider the very last norm for each term $y_\gamma \otimes \lambda_\gamma$ of y: let $h \in \ell^2_n$ and $\beta \in \Gamma$; as

$$\big((y_\gamma \otimes \lambda_\gamma)\tilde{p}_\iota - \tilde{p}_\iota(y_\gamma \otimes \lambda_\gamma)\tilde{p}_\iota\big)(h \otimes \mathrm{e}_\beta) = \begin{cases} y_\gamma(h)\mathrm{e}_{\gamma\beta} & \text{if } \beta \in \Gamma_\iota \text{ and } \gamma\beta \notin \Gamma_\iota \\ 0 & \text{otherwise,} \end{cases}$$

the definition of a Følner averaging net yields

$$\|(y_\gamma \otimes \lambda_\gamma)\tilde{p}_\iota - \tilde{p}_\iota(y_\gamma \otimes \lambda_\gamma)\tilde{p}_\iota\|_{\mathrm{S}^1(\mathrm{S}^1_n)} \leqslant \#(\Gamma_\iota \setminus \gamma^{-1}\Gamma_\iota)\|y_\gamma\|_{\mathrm{S}^1_n} = o(\#\,\Gamma_\iota).$$

An approximation argument as in [6, proof of Proposition 4] permits to conclude for $y \in \mathrm{L}^\infty$. ☐

Let us now describe and prove the L^ψ version of the transfer described at the beginning of this section.

Lemma 2.2. *Let Γ be a discrete amenable group and $p > 0$. Let $\Lambda \subseteq \Gamma$ and $\varphi \in \mathbb{C}^\Lambda$. Consider the associated Toeplitz set $\ddot{\Lambda} = \{(r,c) \in R \times C : rc^{-1} \in \Lambda\}$ and the Toeplitz matrix defined by $\ddot{\varphi}_{r,c} = \varphi_{rc^{-1}}$. Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0.*

(*a*) *The norm of the relative Fourier multiplier φ on L^ψ_Λ is the supremum of the norm of the relative Schur multiplier $\ddot{\varphi}$ on subspaces of truncated Toeplitz matrices in $\mathrm{S}^\psi_{\ddot{\Lambda}}$.*

(*b*) *The norm of the relative Fourier multiplier φ on $\mathrm{L}^\psi_\Lambda(\mathrm{tr}\otimes\tau)$ is the supremum of the norm of the relative Schur multiplier $\ddot{\varphi}$ on subspaces of truncated Toeplitz matrices in $\mathrm{S}^\psi_{\ddot{\Lambda}}(\mathrm{S}^\psi)$.*

While the restriction to truncated Toeplitz matrices may not be removed in case (a) (see Remark 5.2), Theorem 2.5 (a) below will provide the full picture of case (b).

Proof. A Toeplitz matrix has the form $(x_{rc^{-1}})_{(r,c)\in\breve{\Lambda}}$. Our definition of the space $\mathrm{L}^{\psi}_{\Lambda}$ (cf. section on Notation and terminology) ensures that we may suppose that only a finite number of the x_γ are nonzero for the computation of the norm of φ. Then $(x_{rc^{-1}})_{(r,c)\in\breve{\Lambda}}$ is the matrix of the operator $x = \sum_{\gamma\in\Lambda} x_\gamma\lambda_\gamma$ for the canonical basis of ℓ^2_Γ.

Let $y = x^*x$ and let us use the notation of Theorem 2.1. Let $\tilde{\psi}$ be a continuous function with compact support such that $\tilde{\psi}(t) = \psi(t)$ on $[0, \|x\|^2]$.

(a). By Szegő's limit theorem (as given in [6, Theorem 1]),

$$\frac{1}{\#\Gamma_\iota}\operatorname{tr}\psi(p_\iota y p_\iota) = \frac{1}{\#\Gamma_\iota}\operatorname{tr}\tilde{\psi}(p_\iota y p_\iota) \to \tau(\tilde{\psi}(y)) = \tau(\psi(y))$$

Let us describe how $\breve{\varphi}$ acts on xp_ι. Schur multiplication with $\breve{\varphi}$ transforms the matrix of xp_ι, that is the truncated Toeplitz matrix $(x_{rc^{-1}})_{(r,c)\in\breve{\Lambda}\cap\Gamma\times\Gamma_\iota}$, into the matrix $(\breve{\varphi}_{rc^{-1}}x_{rc^{-1}})_{(r,c)\in\breve{\Lambda}\cap\Gamma\times\Gamma_\iota}$, so that it transforms xp_ι into $(\mathrm{C}_\varphi x)p_\iota$.

(b). Combine the argument in (a) with the matrix Szegő limit theorem. ☐

In the case of a finite abelian group, no limit theorem is needed: this case has been considered in [66, Proposition 2.5 (b)].

The following well-known argument has been used (first in [17], see [18, Proposition D.6]) to show that the complete norm of the Fourier multiplier φ on $\mathrm{L}^\infty_\Lambda$ bounds the complete norm of the Schur multiplier $\breve{\varphi}$ on $\mathrm{S}^\infty_{\breve{\Lambda}}$, so that we have in full generality $\|\varphi\|_{\mathrm{C_{cb}}(\mathrm{C}_\Lambda)} = \|\breve{\varphi}\|_{\mathrm{M_{cb}}(\mathrm{S}^\infty_{\breve{\Lambda}})}$. This argument permits to strengthen Lemma 2.2 (b).

Lemma 2.3. *Let Γ be a discrete group and let R and C be subsets of Γ. To $\Lambda\subseteq\Gamma$ associate $\breve{\Lambda} = \{(r,c)\in R\times C : rc^{-1}\in\Lambda\}$; given $\varphi\in\mathbb{C}^\Lambda$ define $\breve{\varphi}\in\mathbb{C}^{\breve{\Lambda}}$ by $\breve{\varphi}_{r,c} = \varphi_{rc^{-1}}$. Let $\psi\colon\mathbb{R}^+\to\mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The norm of the relative Schur multiplier $\breve{\varphi}$ on $\mathrm{S}^\psi_{\breve{\Lambda}}(\mathrm{S}^\psi)$ is bounded by the norm of the relative Fourier multiplier φ on $\mathrm{L}^\psi_\Lambda(\operatorname{tr}\otimes\tau)$.*

Proof. We adapt the argument in [76, Lemma 8.1.4]. Let $x_q\in\mathrm{S}^\psi$, of which only a finite number are nonzero. The space $\mathrm{L}^\psi(\operatorname{tr}\otimes\operatorname{tr}\otimes\tau)$ is a left and right $\mathrm{L}^\infty(\operatorname{tr}\otimes\operatorname{tr}\otimes\tau)$-module and $\sum_{\gamma\in\Gamma}\mathrm{e}_{\gamma\gamma}\otimes\lambda_\gamma$ is a unitary in $\mathrm{L}^\infty(\operatorname{tr}\otimes\tau)$, so that

$$\begin{aligned}\Big\|\sum_{q\in\breve{\Lambda}} x_q\otimes\mathrm{e}_q\Big\|_{\mathrm{S}^\psi_{\breve{\Lambda}}(\mathrm{S}^\psi)} &= \Big\|\Big(\mathrm{Id}\otimes\sum_{r\in R}\mathrm{e}_{r,r}\otimes\lambda_r\Big)\Big(\sum_{q\in\breve{\Lambda}} x_q\otimes\mathrm{e}_q\otimes\lambda_{\mathrm{e}}\Big)\Big(\mathrm{Id}\otimes\sum_{c\in C}\mathrm{e}_{c,c}\otimes\lambda_c^*\Big)\Big\|_{\mathrm{L}^\psi(\operatorname{tr}\otimes\operatorname{tr}\otimes\tau)}\\ &= \Big\|\sum_{(r,c)\in\breve{\Lambda}} x_{r,c}\otimes\mathrm{e}_{r,c}\otimes\lambda_{rc^{-1}}\Big\|_{\mathrm{L}^\psi(\operatorname{tr}\otimes\operatorname{tr}\otimes\tau)}\\ &= \Big\|\sum_{\gamma\in\Lambda}\Big(\sum_{rc^{-1}=\gamma} x_{r,c}\otimes\mathrm{e}_{r,c}\Big)\otimes\lambda_\gamma\Big\|_{\mathrm{L}^\psi_\Lambda(\operatorname{tr}\otimes\operatorname{tr}\otimes\tau)}.\end{aligned}$$

This yields an isometric embedding of $\mathrm{S}^\psi_{\breve{\Lambda}}(\mathrm{S}^\psi)$ in $\mathrm{L}^\psi_\Lambda(\operatorname{tr}\otimes\operatorname{tr}\otimes\tau)$. As $\mathrm{S}^\psi(\mathrm{S}^\psi)$ is the Schatten-von-Neumann-Orlicz class for the Hilbert space $\ell^2\otimes\ell^2_\Gamma$ which may be identified with ℓ^2,

$$\begin{aligned}\Big\|\sum_{q\in\breve{\Lambda}} x_q\otimes\breve{\varphi}_q\mathrm{e}_q\Big\|_{\mathrm{S}^\psi_{\breve{\Lambda}}(\mathrm{S}^\psi)} &= \Big\|\sum_{\gamma\in\Lambda}\Big(\sum_{rc^{-1}=\gamma} x_{r,c}\otimes\mathrm{e}_{r,c}\Big)\otimes\varphi_\gamma\lambda_\gamma\Big\|_{\mathrm{L}^\psi_\Lambda(\operatorname{tr}\otimes\operatorname{tr}\otimes\tau)}\\ &\leqslant \|\mathrm{Id}_{\mathrm{S}^\psi}\otimes\mathrm{C}_\varphi\|\Big\|\sum_{q\in\breve{\Lambda}} x_q\otimes\mathrm{e}_q\Big\|_{\mathrm{S}^\psi_{\breve{\Lambda}}(\mathrm{S}^\psi)}.\end{aligned}$$ ☐

Remark 2.4. The proof of Lemma 2.3 shows also the following transfer. Let (r_i) and (c_j) be sequences in Γ, consider $\check{\Lambda} = \{(i,j)\in\mathbb{N}\times\mathbb{N} : r_ic_j\in\Lambda\}$ and define $\check{\varphi}\in\mathbb{C}^{\check{\Lambda}}$ by $\check{\varphi}(i,j) = \varphi(r_ic_j)$. Then the norm of the relative Schur multiplier $\check{\varphi}$ on $\mathrm{S}^\psi_{\check{\Lambda}}(\mathrm{S}^\psi)$ is bounded by the norm of the relative Fourier multiplier $\mathrm{Id}_{\mathrm{S}^\psi}\otimes\mathrm{C}_\varphi$ on $\mathrm{L}^\psi_\Lambda(\operatorname{tr}\otimes\tau)$ (confer [78, Theorem 6.4].) In particular, if the r_ic_j are pairwise distinct, this permits to transfer every Schur multiplier, not just the Toeplitz ones. See [66, Section 11] for applications of this transfer.

Here is the announced strengthening of Lemma 2.2.

Theorem 2.5. *Let Γ be a discrete amenable group. Let $\Lambda \subseteq \Gamma$ and $\varphi \in \mathbb{C}^\Lambda$. Consider the associated Toeplitz set $\ddot{\Lambda} = \{(r, c) \in R \times C : rc^{-1} \in \Lambda\}$ and the Toeplitz matrix defined by $\ddot{\varphi}_{r,c} = \varphi_{rc^{-1}}$.*

- (a) *Let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nondecreasing function vanishing only at 0. The norm of the relative Fourier multiplier φ on $\mathrm{L}^\psi_{\ddot{\Lambda}}(\mathrm{tr} \otimes \tau)$ and the norm of the relative Schur multiplier $\ddot{\varphi}$ on $\mathrm{S}^\psi_{\ddot{\Lambda}}(\mathrm{S}^\psi)$ are equal.*
- (b) *Let $p \geqslant 1$. The complete norm of the relative Fourier multiplier φ on L^p_Λ and the complete norm of the relative Schur multiplier $\ddot{\varphi}$ on $\mathrm{S}^p_{\ddot{\Lambda}}$ are equal:*

$$\|\varphi\|_{\mathrm{C_{cb}}(\mathrm{L}^p_\Lambda)} = \|\ddot{\varphi}\|_{\mathrm{M_{cb}}(\mathrm{S}^p_{\ddot{\Lambda}})}.$$

- (c) *The norm of the relative Fourier multiplier φ on C_Λ, its complete norm, the norm of the relative Schur multiplier $\ddot{\varphi}$ on $\mathrm{S}^\infty_{\ddot{\Lambda}}$ and its complete norm are equal:*

$$\|\varphi\|_{\mathrm{C}(\mathrm{C}_\Lambda)} = \|\varphi\|_{\mathrm{C_{cb}}(\mathrm{C}_\Lambda)} = \|\ddot{\varphi}\|_{\mathrm{M_{cb}}(\mathrm{S}^\infty_{\ddot{\Lambda}})} = \|\ddot{\varphi}\|_{\mathrm{M}(\mathrm{S}^\infty_{\ddot{\Lambda}})}.$$

- (d) *Suppose that $\Lambda = \Gamma$. The norm of the Fourier algebra multiplier φ, its complete norm, the norm of the Schur multiplier $\ddot{\varphi}$ on S^∞ and its complete norm are equal:*

$$\|\varphi\|_{\mathrm{C}(\mathrm{A}(\Gamma))} = \|\varphi\|_{\mathrm{C_{cb}}(\mathrm{A}(\Gamma))} = \|\ddot{\varphi}\|_{\mathrm{M_{cb}}(\mathrm{S}^\infty)} = \|\ddot{\varphi}\|_{\mathrm{M}(\mathrm{S}^\infty)}.$$

Proof. Combine Proposition 2.2 (b) with Lemma 2.3. Recall that if Γ is amenable, the norm of a Fourier multiplier φ on C_Λ is equal to its complete norm [25, Corollary 1.8] and that the complete norm of a Schur multiplier $\ddot{\varphi}$ on $\mathrm{S}^\infty_{\ddot{\Lambda}}$ is equal to its norm [71, Theorem 3.2]. ☐

3 Local embeddings of L^p into S^p

The proof of Lemma 2.2 can be interpreted as an embedding of L^ψ into an ultraproduct of finite-dimensional spaces S^ψ_n that intertwines Fourier and Toeplitz Schur multipliers. If we restrict ourselves to power functions $\psi\colon t \mapsto t^p$ with $p \geqslant 1$, such embeddings are well known and the proof of Lemma 2.2 does not need the full strength of the Matrix Szegő limit theorem but only the existence of such embeddings. In this section, we explain two ways to obtain them by interpolation.

The first way is to extend the classical result that the reduced C*-algebra C of a discrete group Γ has the completely contractive approximation property if Γ is amenable. We follow the approach of [18, Theorem 2.6.8]. Let Γ be a discrete amenable group and Γ_ι be a Følner averaging net of sets. As above, we denote by p_ι the orthogonal projection from ℓ^2_Γ to $\ell^2_{\Gamma_\iota}$. Define the compression φ_ι and the embedding ψ_ι by

$$\begin{aligned} \varphi_\iota\colon \mathrm{C} &\to \mathbb{B}(\ell^2_{\Gamma_\iota}) &\quad\text{and}\quad \psi_\iota\colon \mathbb{B}(\ell^2_{\Gamma_\iota}) &\to \mathrm{C} \\ x &\mapsto p_\iota x p_\iota & \mathrm{e}_{r,c} &\mapsto (1/\#\,\Gamma_\iota)\lambda_r\lambda_{c^{-1}}. \end{aligned} \tag{F.4}$$

If we endow $\mathbb{B}(\ell^2_{\Gamma_\iota})$ with the normalised trace, these maps are unital completely positive, trace preserving (and normal) and the net $(\psi_\iota\varphi_\iota)$ converges pointwise to the identity of C. One can therefore extend them by interpolation to completely positive contractions on the respective noncommutative Lebesgue spaces: recall that $\mathrm{L}^p(\mathbb{B}(\ell^2_{\Gamma_\iota}), (1/\#\,\Gamma_\iota)\,\mathrm{tr})$ is $(\#\,\Gamma_\iota)^{-1/p}\mathrm{S}^p_{\#\,\Gamma_\iota}$. We get a net of complete contractions

$$\tilde{\varphi}_\iota\colon \mathrm{L}^p \to (\#\,\Gamma_\iota)^{-1/p}\mathrm{S}^p_{\#\,\Gamma_\iota} \quad\text{and}\quad \tilde{\psi}_\iota\colon (\#\,\Gamma_\iota)^{-1/p}\mathrm{S}^p_{\#\,\Gamma_\iota} \to \mathrm{L}^p,$$

such that $(\tilde{\psi}_\iota\tilde{\varphi}_\iota)$ converges pointwise to the identity of L^p. Moreover, the definitions (F.4) show that these maps also intertwine Fourier and Toeplitz Schur multipliers.

Remark 3.1. This approach is more canonical as it allows to extend the transfer to the vector-valued spaces in the sense of [76, Chapter 3]. Recall that for any hyperfinite semifinite von Neumann algebra M and any operator space E, one can define $\mathrm{L}^p(M, E)$: for $p = \infty$, this space is defined as $M \otimes_{\min} E$; for $p = 1$, this space is defined as $M^{\mathrm{op}}_* \hat{\otimes} E$; these spaces form an interpolation scale for the complex method when $1 \leqslant p \leqslant \infty$. For us, M will be $\mathbb{B}(\ell^2)$ or the group von Neumann algebra L^∞. As the maps ψ_ι and φ_ι are unital completely positive and trace preserving and normal, they define simultaneously complete contractions on M and M_*. By interpolation, the maps $\psi_\iota \otimes \mathrm{Id}_E$ and

$\varphi_\iota \otimes \mathrm{Id}_E$ are still complete contractions on the spaces $\mathrm{L}_p(E)$ and $\mathrm{S}^p[E]$. Let $\varphi\colon \mathit{\Gamma} \to \mathbb{C}$: the transfer shows that the norm of $\mathrm{Id}_E \otimes \mathrm{C}_\varphi$ on $\mathrm{L}^p(E)$ is bounded by the norm of $\mathrm{Id}_E \otimes \mathrm{M}_{\check{\varphi}}$ on $\mathrm{S}^p[E]$, and that their complete norms coincide. In formulas,

$$\|\mathrm{Id}_E \otimes \mathrm{C}_\varphi\|_{\mathbb{B}(\mathrm{L}^p(E))} \leqslant \|\mathrm{Id}_E \otimes \mathrm{M}_{\check{\varphi}}\|_{\mathbb{B}(\mathrm{S}^p[E])},$$
$$\|\mathrm{Id}_E \otimes \mathrm{C}_\varphi\|_{\mathrm{cb}(\mathrm{L}^p(E))} = \|\mathrm{Id}_E \otimes \mathrm{M}_{\check{\varphi}}\|_{\mathrm{cb}(\mathrm{S}^p[E])}.$$

This approximation is two-sided whereas the proof of Lemma 2.2 uses only a one-sided approximation. This subtlety makes a difference if one tries to give a direct proof by complex interpolation, as we shall do now.

Proposition 3.2. *Let $\mathit{\Gamma}$ be a discrete amenable group and let (μ_ι) be a* Reiter net of means *for $\mathit{\Gamma}$:*

- *each μ_ι is a positive sequence summing to 1 with finite support $\mathit{\Gamma}_\iota \subseteq \mathit{\Gamma}$ and viewed as a diagonal operator from $\ell^2_{\mathit{\Gamma}_\iota}$ to $\ell^2_{\mathit{\Gamma}}$, so that*

$$\|\mu_\iota\|_{\mathrm{S}^1} = \sum_{\gamma\in\mathit{\Gamma}_\iota} (\mu_\iota)_\gamma = 1;$$

- *the net (μ_ι) satisfies, for each $\gamma \in \mathit{\Gamma}$,* Reiter's Property P_1*:*

$$\sum_{\beta\in\mathit{\Gamma}} \left|(\mu_\iota)_{\gamma^{-1}\beta} - (\mu_\iota)_\beta\right| \to 0. \tag{F.5}$$

Let $f \in \mathrm{S}^\infty_n \otimes \mathrm{L}^\infty = \mathrm{L}^\infty(\mathrm{tr} \otimes \tau)$ and $p \geqslant 1$. Then

$$\limsup \|f\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} = \|f\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}.$$

Proof. Consider $f = \sum_{\gamma\in\mathit{\Gamma}} x_\gamma \otimes \lambda_\gamma$ with only a finite number of the $x_\gamma \in \mathrm{S}^\infty_n$ nonzero. As

$$\sum_{\beta\in\mathit{\Gamma}} \left|(\mu_\iota)^{1/2}_{\gamma^{-1}\beta} - (\mu_\iota)^{1/2}_\beta\right|^2 \leqslant \sum_{\beta\in\mathit{\Gamma}} \left|(\mu_\iota)_{\gamma^{-1}\beta} - (\mu_\iota)_\beta\right|,$$

Property P_1 implies *Property* P_2:

$$\|\lambda_\gamma\mu_\iota^{1/2} - \mu_\iota^{1/2}\lambda_\gamma\|_{\mathrm{S}^2} \to 0,$$

so that

$$\|f\mu_\iota^{1/2} - \mu_\iota^{1/2}f\|_{\mathrm{S}^2(\mathrm{S}^2_n)} \to 0.$$

As the S^∞_n-valued matrix of f for the canonical basis of $\ell^2_{\mathit{\Gamma}}$ is $(x_{rc^{-1}})_{(r,c)\in\mathit{\Gamma}\times\mathit{\Gamma}}$,

$$\begin{aligned}\|f\mu_\iota^{1/2}\|^2_{\mathrm{S}^2(\mathrm{S}^2_n)} &= \sum_{(r,c)\in\mathit{\Gamma}\times\mathit{\Gamma}} \|x_{rc^{-1}}\|^2_{\mathrm{S}^n_2}(\mu_\iota)_c \\ &= \sum_{c\in\mathit{\Gamma}} (\mu_\iota)_c \sum_{r\in\mathit{\Gamma}} \|x_{rc^{-1}}\|^2_{\mathrm{S}^n_2} \\ &= \sum_{c\in\mathit{\Gamma}} (\mu_\iota)_c \|f\|^2_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)} = \|f\|^2_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)}.\end{aligned}$$

By density and continuity, the result extends to all $f \in \mathrm{L}^2(\mathrm{tr}\otimes\tau)$.

Let us prove now that for $f \in \mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$

$$\limsup \|f\mu_\iota\|_{\mathrm{S}^1(\mathrm{S}^1_n)} \leqslant \|f\|_{\mathrm{L}^1(\mathrm{tr}\otimes\tau)}.$$

The polar decomposition $f = u|f|$ yields a factorisation $f = ab$ with $a = u|f|^{1/2}$ and $b = |f|^{1/2}$ in $\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$ such that

$$\|a\|_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)} = \|b\|_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)} = \|f\|^{1/2}_{\mathrm{L}^1(\mathrm{tr}\otimes\tau)}$$
$$\|a\|_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)} = \|f\|^{1/2}_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)}.$$

Then $f\mu_\iota = a(b\mu_\iota^{1/2} - \mu_\iota^{1/2}b)\mu_\iota^{1/2} + a\mu_\iota^{1/2}b\mu_\iota^{1/2}$, so that the Cauchy-Schwarz inequality yields

$$\begin{aligned}\|f\mu_\iota\|_{\mathrm{S}^1(\mathrm{S}^1_n)} &\leqslant \|a\|_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)}\|(b\mu_\iota^{1/2} - \mu_\iota^{1/2}b)\mu_\iota^{1/2}\|_{\mathrm{S}^1(\mathrm{S}^1_n)} + \|a\mu_\iota^{1/2}b\mu_\iota^{1/2}\|_{\mathrm{S}^1(\mathrm{S}^1_n)}\\ &\leqslant \|a\|_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)}\|b\mu_\iota^{1/2} - \mu_\iota^{1/2}b\|_{\mathrm{S}^2(\mathrm{S}^2_n)} + \|a\|_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)}\|b\|_{\mathrm{L}^2(\mathrm{tr}\otimes\tau)}\end{aligned}$$

and therefore our claim. Now complex interpolation yields

$$\limsup \|f\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} \leqslant \|f\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} \tag{F.6}$$

for $f \in \mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$ and $p \in [1,\infty]$. Indeed, let u be the unitary appearing in the polar decomposition of f. Consider the function $F(z) = u|f|^{pz}\mu_\iota^z$ analytic in the strip $0 < \Im z < 1$ and continuous on its closure: then $F(\mathrm{i}t)$ is a product of unitaries for $t \in \mathbb{R}$, so that

$$\|F(\mathrm{i}t)\|_{\mathrm{L}^\infty(\mathrm{tr}\otimes\tau)} = 1.$$

Also

$$\|F(1+\mathrm{i}t)\|_{\mathrm{S}^1(\mathrm{S}^1_n)} = \||f|^p\mu_\iota\|_{\mathrm{S}^1(\mathrm{S}^1_n)}$$

As $\mathrm{S}^p(\mathrm{S}^p_n)$ is the complex interpolation space $(\mathrm{S}^\infty(\mathrm{S}^\infty_n), \mathrm{S}^1(\mathrm{S}^1_n))_{1/p}$,

$$\|f\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} = \|F(1/p)\|_{\mathrm{S}^p(\mathrm{S}^p_n)} \leqslant \||f|^p\mu_\iota\|^{1/p}_{\mathrm{S}^1(\mathrm{S}^1_n)}.$$

Then, taking the upper limit and using the estimate on $\mathrm{S}^1(\mathrm{S}^1_n)$

$$\begin{aligned}\limsup \|f\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} &\leqslant \limsup \||f|^p\mu_\iota\|^{1/p}_{\mathrm{S}^1(\mathrm{S}^1_n)}\\ &\leqslant \||f|^p\|^{1/p}_{\mathrm{L}^1(\mathrm{tr}\otimes\tau)} = \|f\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}.\end{aligned}$$

The reverse inequality is obtained by duality; first note that for $g \in \mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$,

$$\lim \mathrm{tr}\, g\mu_i = \tau(g).$$

With the above notation and the inequality for p',

$$\begin{aligned}\|f\|^p_{\mathrm{L}^p} = \tau(|f|^p) = \lim \mathrm{tr}\,|f|^p\mu_i &= \lim \mathrm{tr}\,\mu_i^{1-1/p}|f|^{1-p}u^*f\mu_i^{1/p}\\ &\leqslant \limsup \|\mu_i^{1-1/p}|f|^{1-p}\|_{\mathrm{S}^{p'}(\mathrm{S}^{p'}_n)}\|f\mu_i^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}\\ &= \limsup \||f|^{1-p}\mu_i^{1-1/p}\|_{\mathrm{S}^{p'}(\mathrm{S}^{p'}_n)}\|f\mu_i^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}\\ &\leqslant \||f|^{1-p}\|_{\mathrm{L}^{p'}} \limsup \|f\mu_i^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)}\end{aligned}$$

so that

$$\limsup \|f\mu_\iota^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} = \|f\|^p_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}.$$

□

Remark 3.3. Let μ be any positive diagonal operator with $\mathrm{tr}\,\mu = 1$ and $p \geqslant 2$: then $\|f\mu^{1/p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} \leqslant \|f\|_{\mathrm{L}^p}$ for all $f \in \mathrm{L}^\infty(\mathrm{tr}\otimes\tau)$. The Reiter condition is only necessary to go below exponent 2.

In the same way, using interpolation, we can come back to approximation on both sides using Reiter means, that is

$$\|\mu_\iota^{1/2p}f\mu_\iota^{1/2p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} \leqslant \|f\|_{\mathrm{L}^p}$$

and we have

$$\limsup \|\mu_\iota^{1/2p}f\mu_\iota^{1/2p}\|_{\mathrm{S}^p(\mathrm{S}^p_n)} = \|f\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}.$$

Note that this formula is in the same spirit as the first approach of this section.

4 Transfer of lacunary sets into lacunary matrix patterns

As a first application of Theorem 2.5, let us mention that it provides a shortcut for some arguments in [37] as it permits to transfer lacunary subsets of discrete group Γ into lacunary matrix patterns in $\Gamma \times \Gamma$. Let us first introduce the following terminology.

Definition 4.1. Let $\varGamma$ be a discrete group and $\varLambda \subseteq \varGamma$. Let X be the reduced C*-algebra C of $\varGamma$ or its noncommutative Lebesgue space L^p for $p \in [1, \infty[$.

(a) The set $\varLambda$ is *unconditional* in X if the Fourier series of every $x \in X_\varLambda$ converges unconditionally: there is a constant D such that

$$\Big\|\sum_{\gamma\in\varLambda'} x_\gamma \varepsilon_\gamma \lambda_\gamma\Big\|_X \leqslant D\|x\|_X$$

finite $\varLambda' \subseteq \varLambda$ and $\varepsilon_\gamma \in \mathbb{T}$. The minimal constant D is the *unconditional constant of* $\varLambda$ *in* X.

(b) If $X = \mathrm{C}$, let $\tilde{X} = \mathrm{S}^\infty \otimes \mathrm{C}$; if $X = \mathrm{L}^p$, let $\tilde{X} = \mathrm{L}^p(\mathrm{tr} \otimes \tau)$. The set $\varLambda$ is *completely unconditional* in X if the Fourier series of every $x \in \tilde{X}_\varLambda$ converges unconditionally: there is a constant D such that

$$\Big\|\sum_{\gamma\in\varLambda'} x_\gamma \otimes \varepsilon_\gamma \lambda_\gamma\Big\|_{\tilde{X}} \leqslant D\|x\|_{\tilde{X}}$$

for finite $\varLambda' \subseteq \varLambda$ and $\varepsilon_\gamma \in \mathbb{T}$. The minimal constant D is the *complete unconditional constant of* $\varLambda$ *in* X.

Unconditional sets in L^p have been introduced as "$\Lambda(p)$ sets" in [37, Definition 1.1] for $p > 2$: if $\varGamma$ is abelian, they are Walter Rudin's $\Lambda(p)$ sets if $p > 2$ and his $\Lambda(2)$ sets if $p < 2$: see [87, 15]. Asma Harcharras [37, Definition 1.5, Comments 1.9] termed completely unconditional sets in L^p "$\Lambda(p)_{\mathrm{cb}}$ sets" if $p \in]2, \infty[$, and "$\mathrm{K}(p)_{\mathrm{cb}}$ sets" if $p \in]1, 2]$; her definitions are equivalent to ours by the noncommutative Khinchin inequality.

Sets that are unconditional in C have been introduced as "unconditional Sidon sets" in [16]. If $\varGamma$ is amenable, Fourier multipliers are automatically c.b. on $\mathrm{C}_\varLambda$, so that such sets are automatically completely unconditional in C, and there are at least three more equivalent definitions for the counterpart of Sidon sets in an abelian group. If $\varGamma$ is nonamenable, these definitions are not all equivalent anymore and our notion of completely unconditional sets in C corresponds to Marek Bożejko's "c.b. Sidon sets."

Definition 4.2. Let $1 \leqslant p \leqslant \infty$ and $I \subseteq R \times C$.

(a) The set I is *unconditional* in S^p if the matrix representation of every $x \in \mathrm{S}^p_I$ converges unconditionally: there is a constant D such that

$$\Big\|\sum_{q\in I'} x_q \varepsilon_q \mathrm{e}_q\Big\|_p \leqslant D\|x\|_p$$

for finite $I' \subseteq I$ and $\varepsilon_q \in \mathbb{T}$. The minimal constant D is the *unconditional constant of* I *in* S^p.

(b) The set I is *completely* unconditional in S^p if the matrix representation of every $x \in \mathrm{S}^p_I(\mathrm{S}^p)$ converges unconditionally: there is a constant D such that

$$\Big\|\sum_{q\in I'} x_q \otimes \varepsilon_q \mathrm{e}_q\Big\|_p \leqslant D\|x\|_p$$

for finite $I' \subseteq I$ and $\varepsilon_q \in \mathbb{T}$. The minimal constant D is the *complete unconditional constant of* I *in* S^p.

Harcharras [37, Definitions 4.1 and 4.4, Remarks 4.6 (iv)] termed unconditional and completely unconditional sets in S^p "$\sigma(p)$ sets" and "$\sigma(p)_{\mathrm{cb}}$ sets," respectively; her definitions are equivalent by the noncommutative Khinchin inequality.

Proposition 4.3. *Let $\varGamma$ be a discrete group. Let $\varLambda \subseteq \varGamma$ and consider the associated Toeplitz set $\H{\varLambda} = \{(r, c) \in R \times C : rc^{-1} \in \varLambda\}$. Let $p \in [1, \infty[$.*

(a) *If $\varGamma$ is amenable, then $\varLambda$ is unconditional in L^p if $\H{\varLambda}$ is unconditional in S^p.*

(b) *If $\varLambda$ is completely unconditional in L^p, then $\H{\varLambda}$ is completely unconditional in S^p. The converse holds if $\varGamma$ is amenable.*

Proof. The first part of (b) follows by the argument of the proof of [37, Proposition 4.7]: let us sketch it. Consider the isometric embedding of the space $\mathrm{S}^p_{\H{\varLambda}}(\mathrm{S}^p)$ in $\mathrm{L}^p_\varLambda(\mathrm{tr} \otimes \mathrm{tr} \otimes \tau)$ that is given in the proof of Lemma 2.3 and apply the equivalent Definition 1.5 in [37] of the complete unconditionality of $\varLambda$: this gives the complete unconditionality of $\H{\varLambda}$ in the equivalent Definition 4.4 in [37].

Unconditionality in L^p expresses the uniform boundedness of relative unimodular Fourier multipliers on L^p_Λ; complete unconditionality expresses their uniform complete boundedness. Unconditionality in S^p expresses the uniform boundedness of relative unimodular Schur multipliers on $\mathrm{S}^p_{\check{\Lambda}}$; complete unconditionality expresses their uniform complete boundedness. The second part of (b) follows therefore from Theorem 2.5 (b) and (a) follows from Lemma 2.2 (a). $\square$

Remark 4.4. This transfer does not pass to the limit $p = \infty$: Nicholas Varopoulos ([99, Theorem 4.2], see [66, § 5] for a reader's guide) proved that unconditional sets in S^∞ are finite unions of patterns whose rows or whose columns contain at most one element, and this excludes sets of the form $\check{\Lambda}$ for any infinite Λ.

Remark 4.5. See [66, Remark 11.3] for an illustration of Proposition 4.3 (b) in a particular context.

Remark 4.6. Let p be an even integer greater or equal to 4. The existence of a $\sigma(p)_{\mathrm{cb}}$ set that is not a $\sigma(q)$ set for any $q > p$ [37, Theorem 4.9] becomes a direct consequence of Walter Rudin's construction [87, Theorem 4.8] of a $\Lambda(p)$ set that is not a $\Lambda(q)$ set for any $q > p$, because this set has property $\mathrm{B}(p/2)$ [37, Definition 2.4] and is therefore $\Lambda(p)_{\mathrm{cb}}$ by [37, Theorem 1.13] (in fact, it is even "1-unconditional" in L^p because $\mathrm{B}(p/2)$ is "$p/2$-independence" [66, § 11].)

Remark 4.7. In the same way, Theorem 5.2 in [37] becomes a mere reformulation of [37, Proposition 3.6] if one remembers that the Toeplitz Schur multipliers are 1-complemented in the Schur multipliers for an amenable discrete group and for all classical norms. Basically results on $\Lambda(p)_{\mathrm{cb}}$ sets produce results on $\sigma(p)_{\mathrm{cb}}$ sets.

Let us now estimate the complete unconditional constant of sumsets. In the case $\Gamma = \mathbb{Z}$, Harcharras [37, Prop. 2.8] proved that a completely unconditional set in L^p cannot contain the sumset of characters $A + A$ for arbitrary large finite sets A: in particular, if $\Lambda \supseteq A + A$ with A infinite, then Λ is not a completely unconditional set. Her proof provided thus examples of $\Lambda(p)$ sets that are not $\Lambda(p)_{\mathrm{cb}}$ sets.

We generalise Harcharras' result in two directions. Compare [54, § 1.4].

Proposition 4.8. *Let Γ be a discrete group and $p \neq 2$. A completely unconditional set in L^p cannot contain the sumset of two arbitrarily large sets. More precisely, let R and C be subsets of Γ with $\# R \geqslant n$ and $\# C \geqslant n^3$. Then, for any $p \geqslant 1$, the complete unconditional constant of the sumset RC in L^p is at least $n^{|1/2-1/p|}$.*

Proof. Let $r_1, \ldots, r_n$ be pairwise distinct elements in R. We shall select inductively elements $c_1, \ldots, c_n$ in C such that the $r_i c_j$ are pairwise distinct. Assume there are $c_1, \ldots, c_{m-1}$ such that the induction hypothesis

$$\forall\, i, k \leqslant n\ \forall\, j, l \leqslant m-1 \quad (i,j) \neq (k,l) \ \Rightarrow\ r_i c_j \neq r_k c_l.$$

holds. We are looking for an element $c_m \in C$ such that

$$\forall\, i, k \leqslant n\ \forall\, l \leqslant m-1 \quad r_i c_m \neq r_k c_l.$$

Such an element exists as long as $m \leqslant n$ because the set $\{r_i^{-1} r_k c_l : i, k \leqslant n,\ l \leqslant m-1\}$ has at most $\big(n(n-1)+1\big)(m-1) < n^3$ elements.

The end of the proof is the same as Harcharras'. The unconditional constant of the canonical basis of elementary matrices in S^p_n is $n^{|1/2-1/p|}$; in particular, there is an unimodular Schur multiplier $\check{\varphi}$ on S^p_n of norm $n^{|1/2-1/p|}$ (which is also its complete norm, by the way): see [76, Lemma 8.1.5]. Let Λ be the sumset $\{r_i c_j : i, j \leqslant n\}$; as the $r_i c_j$ are pairwise distinct, we may define a sequence $\varphi \in \mathbb{C}^\Lambda$ by $\varphi_{r_i c_j} = \check{\varphi}_{i,j}$. By Remark 2.4, the complete norm of the Fourier multiplier φ on L^p_Λ is bounded below by the complete norm of the Schur multiplier $\check{\varphi}$ on S^p_I. $\square$

Example 4.9. $\Lambda = \{2^i - 2^j : i > j\}$ is not a complete $\Lambda(p)$ set for any $p \neq 2$. Indeed, $\{2^i - 2^j\} = \Lambda \cup -\Lambda$ does not and if Λ did, then also $-\Lambda$ and $\Lambda \cup -\Lambda$.

5 Toeplitz Schur multipliers on S^p for $p < 1$

When $0 < p < 1$, a complete characterisation of bounded Schur multiplier of Toeplitz type has been obtained by Alexey Alexandrov and Vladimir Peller in [1, Theorem 5.1]. This result was an easy consequence of their deep results on Hankel Schur multipliers. The transfer approach provides a direct proof.

Corollary 5.1. *Let $0 < p < 1$. Let Γ be a discrete abelian group with dual group G. Let φ be a sequence indexed by Γ and define the associated Toeplitz matrix $\ddot{\varphi} \in \mathbb{C}^{\ddot{A}}$ by $\ddot{\varphi}(r,c) = \varphi(rc^{-1})$ for $(r,c) \in \Gamma \times \Gamma$. Then the following are equivalent:*

(a) *The sequence φ is the Fourier transform of an atomic measure $\mu = \sum a_g \delta_g$ on G with $\sum |a_g|^p \leqslant 1$;*

(b) *The Fourier multiplier φ is contractive on L^p;*

(c) *The Fourier multiplier φ is contractive on $\mathrm{L}^p(\mathrm{S}^p)$;*

(d) *The Schur multiplier $\ddot{\varphi}$ is contractive on S^p;*

(e) *The Schur multiplier $\ddot{\varphi}$ is contractive on $\mathrm{S}^p(\mathrm{S}^p)$.*

Proof. The implication $(d) \Rightarrow (b)$ follows from Lemma 2.2 (a). The equivalence $(c) \Leftrightarrow (e)$ follows from Theorem 2.5 (a). The characterisation $(a) \Leftrightarrow (b)$ is an old result of Daniel Oberlin [67]. It is plain that $(e) \Rightarrow (d)$. At last, $(a) \Rightarrow (c)$ is obvious by the p-triangular inequality. □

Remark 5.2. As a consequence, we get that the norm of a Toeplitz Schur multiplier on $\mathrm{S}^p(\mathrm{S}^p)$ coincides with its norm on S^p when $p < 1$. If $p \in \{1, 2, \infty\}$, this holds for every Schur multiplier. Let $p \in \,]1,2[\, \cup \,]2,\infty[$. Then we still do not know whether Schur multipliers are automatically c.b. on S^p. But from [76, Proposition 8.1.3], we know that (b) and (c) are not equivalent: if Γ is an infinite abelian group, there is a bounded Fourier multiplier on L^p that is not c.b. This example is easy to describe: if an infinite set $A \subseteq \Gamma$ is lacunary enough, the sumset $A+A$ is unconditional in L^p (see [54, Theorem 5.13]); by Proposition 4.8, it cannot be completely unconditional. In particular, this shows that in Lemma 2.2 (a) we cannot remove the restriction to truncated Toeplitz matrices in the computation of the Schur multiplier norm, that is, $(b) \Rightarrow (d)$ does not hold.

Remark 5.3. Our questions may also be addressed in the case of a compact group: a measurable function φ on $\mathbb{T}$ defines

– the Fourier multiplier on measurable functions on $\mathbb{T}$ by $x \mapsto \varphi x$;
– the Schur multiplier on integral operators on $\mathrm{L}^2(\mathbb{T})$ with kernel a measurable function x on $\mathbb{T} \times \mathbb{T}$ by $x \mapsto \ddot{\varphi} x$, where $\ddot{\varphi}(z,w) = \varphi(zw^{-1})$.

Victor Olevskii [68] constructed a continuous function φ that defines a bounded Fourier multiplier on the space of functions with p-summable Fourier series endowed with the norm given by $\|x\| = \left(\sum |\hat{x}(n)|^p\right)^{1/p}$ for every $p \in \,]1,\infty[$, while the corresponding Schur multiplier is not bounded on the Schatten-von-Neumann class S^p of operators on $\mathrm{L}^2(\mathbb{T})$ for any $p \in \,]1,2[\, \cup \,]2,\infty[$.

6 The Riesz projection and the Hilbert transform

In this section, we concentrate on $\Gamma = \mathbb{Z}$, the dual group of $\mathbb{T}$.

Proposition 6.1. *Let ϱ be a linear combination of the identity and the upper triangular projection of $\mathbb{N} \times \mathbb{N}$: there are $z, w \in \mathbb{C}$ so that $\varrho_{i,j} = z$ if $i \leqslant j$ and $\varrho_{i,j} = w$ if $i > j$. Then the norm of the Schur multiplier ϱ on S^ψ coincides with the norm of the Schur multiplier ϱ on $\mathrm{S}^\psi(\mathrm{S}^\psi)$.*

Proof. Let $a \in \mathrm{S}^\psi_m(\mathrm{S}^\psi_n)$: a may be considered as an $m \times m$ matrix (a_{ij}) whose entries a_{ij} are $n \times n$ matrices, and a may be identified with the block matrix

$$\tilde{a} = \begin{pmatrix} 0 & a_{11} & 0 & a_{12} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{21} & 0 & a_{22} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In this identification, $\mathrm{Id}_{\mathrm{S}^\psi_n} \otimes \mathrm{M}_\varrho(a)$ is $\mathrm{M}_\varrho(\tilde{a})$. □

The Hilbert transform $\mathscr{H}$ is the Schur multiplier obtained by choosing $z = -1$ and $w = 1$. The upper triangular operators in S^p can be seen as a noncommutative H^p space, and $\mathscr{H}$ corresponds exactly to the Hilbert transform in this setting (see [82, 58]). Using classical results on H^p spaces, all Hilbert transforms are c.b. for $1 < p < \infty$ (see [100, 82, 58]).

On the torus $\mathbb{T}$, the classical Hilbert transform H corresponds to the Fourier multiplier given by the sign function (with the convention $\mathrm{sgn}(0) = 1$) and its norm on L^p is $\cot(\pi/2\max(p,p')) =$

$\csc(\pi/p) + \cot(\pi/p)$ for $1 < p < \infty$. The story of the computation of this norm starts with a paper by Israel Gohberg and Naum Krupnik [34] for p a power of 2. The remaining cases were handled by Stylianos Pichorides [75] and Brian Cole (see [30]) independently. The most achieved results are those of Brian Hollenbeck, Nigel Kalton and Igor Verbitsky [42], but they rely on complex variable methods that are not available in the operator-valued case. When p is a power of 2 (or its conjugate), a combination of arguments of Gohberg and Krupnik [33] with some of László Zsidó [100] yields the following result.

Theorem 6.2. *Let $p \in \,]1,\infty[$. The norm and the complete norm of the Hilbert transform $\mathscr{H}$ on S^p coincide with the complete norm of the Hilbert transform H on L^p: if $\tilde{\tilde{\mathrm{sgn}}}(i,j) = \mathrm{sgn}(i-j)$ for $i, j \geqslant 1$,*

$$\|\,\tilde{\tilde{\mathrm{sgn}}}\,\|_{\mathrm{M}(\mathrm{S}^p)} = \|\,\tilde{\tilde{\mathrm{sgn}}}\,\|_{\mathrm{M_{cb}}(\mathrm{S}^p)} = \|\,\mathrm{sgn}\,\|_{\mathrm{C_{cb}}(\mathrm{L}^p)}.$$

If p is a power of 2, then these norms coincide with the norm of H on L^p:

$$\|\,\tilde{\tilde{\mathrm{sgn}}}\,\|_{\mathrm{M}(\mathrm{S}^p)} = \|\,\tilde{\tilde{\mathrm{sgn}}}\,\|_{\mathrm{M_{cb}}(\mathrm{S}^p)} = \|\,\mathrm{sgn}\,\|_{\mathrm{C_{cb}}(\mathrm{L}^p)} = \|\,\mathrm{sgn}\,\|_{\mathrm{C}(\mathrm{L}^p)} = \cot(\pi/2p).$$

Proof. Let $p \geqslant 2$. The norm of H on L^p is $\cot(\pi/2p)$ and the three other norms are equal by the transfer theorem 2.5 and the above proposition: we only need to compute the complete norm of H. Let $\tilde{H} = \mathrm{Id}_{\mathrm{S}^p} \otimes H$ be the Hilbert transform on $\mathrm{L}^p(\mathrm{tr} \otimes \tau)$. We shall use Mischa Cotlar's trick to go from L^p to L^{2p}: the equality $\mathrm{sgn}\, i \,\mathrm{sgn}\, j + 1 = \mathrm{sgn}(i+j)(\mathrm{sgn}\, i + \mathrm{sgn}\, j)$ shows that

$$(\tilde{H}f)(\tilde{H}g) + fg = \tilde{H}\big((\tilde{H}f)g + f(\tilde{H}g)\big). \tag{F.7}$$

First step. The function sgn is not odd because of its value in 0: this can be fixed in the following way. Let $\Lambda = 2\mathbb{Z}+1$. The norm of $\tilde{H}$ on $\mathrm{L}^p(\mathrm{tr} \otimes \tau)$ is equal to its norm on $\mathrm{L}^p_\Lambda(\mathrm{tr} \otimes \tau)$. In fact, let D be defined by $Df(z) = zf(z^2)$: D is a complete isometry on L^p with range L^p_Λ that commutes with H.
Second step. Let S be the real subspace of $\mathrm{L}^p_\Lambda(\mathrm{tr} \otimes \tau)$ consisting of functions with values in S^p so that $f(z)$ is selfadjoint for almost all $z \in \mathbb{T}$. Let us apply Vern Paulsen's off-diagonal trick [70, Lemma 8.1] to show that the norm of $\tilde{H}$ on L^p is equal to its norm on S. Let $f \in \mathrm{L}^p_\Lambda(\mathrm{tr} \otimes \tau)$: identifying $\mathrm{S}^p_2(\mathrm{S}^p)$ with S^p,

$$g(z) = \begin{pmatrix} 0 & f(z) \\ f(z)^* & 0 \end{pmatrix}$$

defines an element of S. As adjoining is isometric on S^p,

$$\|g\|_S = 2^{1/p}\|f\|_{\mathrm{L}^p(\mathrm{tr} \otimes \tau)}.$$

Let us now consider

$$\tilde{H}g = \begin{pmatrix} 0 & \tilde{H}f \\ \tilde{H}(f^*) & 0 \end{pmatrix}.$$

As $0 \notin \Lambda$ by step 1, the equality $\mathrm{sgn}(-i) = -\,\mathrm{sgn}\, i$ holds for $i \in \Lambda$: this yields that $\tilde{H}(f^*) = -(\tilde{H}f)^*$. Therefore

$$\|\tilde{H}g\|_S = 2^{1/p}\|\tilde{H}f\|_{\mathrm{L}^p(\mathrm{tr} \otimes \tau)}.$$

Third step. Let u_p be the norm of $\tilde{H}$ on $\mathrm{L}^p(\mathrm{tr} \otimes \tau)$: then $u_{2p} \leqslant u_p + \sqrt{1+u_p}$. It suffices to prove this estimate for $f \in S$, and by approximation we may suppose that f is a finite linear combination of terms $a_i \otimes z^i + a_i^* \otimes z^{-i}$ with a_i finite matrices. Note that $\tilde{H}f = -(\tilde{H}f)^*$. Formula (F.7) with $f = g$ yields, combined with Hölder's inequality,

$$\|(\tilde{H}f)^2\|_{\mathrm{L}^p(\mathrm{tr} \otimes \tau)} \leqslant \|f^2\|_{\mathrm{L}^p(\mathrm{tr} \otimes \tau)} + 2u_p\|f\|_{\mathrm{L}^{2p}(\mathrm{tr} \otimes \tau)}\|\tilde{H}f\|_{\mathrm{L}^{2p}(\mathrm{tr} \otimes \tau)}.$$

Since f and $\tilde{H}f$ take normal values,

$$\|f^2\|_{\mathrm{L}^p(\mathrm{tr} \otimes \tau)} = \|f\|^2_{\mathrm{L}^{2p}(\mathrm{tr} \otimes \tau)}$$
$$\|(\tilde{H}f)^2\|_{\mathrm{L}^p(\mathrm{tr} \otimes \tau)} = \|\tilde{H}f\|^2_{\mathrm{L}^{2p}(\mathrm{tr} \otimes \tau)}.$$

Therefore, if $\|f\|_{\mathrm{L}^{2p}(\mathrm{tr} \otimes \tau)} = 1$, $\|\tilde{H}f\|_{\mathrm{L}^{2p}(\mathrm{tr} \otimes \tau)}$ must be smaller than the bigger root of $t^2 - 2u_pt - 1$, that is

$$\|\tilde{H}f\|^2_{\mathrm{L}^{2p}(\mathrm{tr} \otimes \tau)} \leqslant u_p + \sqrt{u_p^2 + 1}$$

and $u_{2p} \leqslant u_p + \sqrt{u_p^2 + 1}$.
Fourth step. The multiplier H is an isometry on $\mathrm{L}^2(\mathrm{tr} \otimes \tau)$, so that $u_2 = 1 = \cot(\pi/4)$. As $\cot(\vartheta/2) = \cot\vartheta + \sqrt{\cot^2\vartheta + 1}$ for $\vartheta \in \,]0,\pi[$, we conclude by an induction. ◻

Unfortunately, we cannot deal with other values of $p > 2$ by this method.

The Riesz projection $\mathscr{T}$ is the Schur multiplier obtained by choosing $z = 0$ and $w = 1$ in Proposition 6.1: it is the projection on the upper triangular part. On the torus, the classical Riesz projection T, that is the projection onto the analytic part, corresponds to the Fourier multiplier given by the indicator function $\chi_{\mathbb{Z}^+}$ of nonnegative integers; its norm on L^p has been computed by Hollenbeck and Verbitsky [43]: it is $\csc(\pi/p)$. As for the Hilbert transform, we know that the norm and the complete norm of $\mathscr{T}$ on S^p are equal and coincide with the complete norm of T on L^p; but, to the best of our knowledge, there is no simple formula like (F.7) to go from exponent p to $2p$. We only obtained the following computation.

Proposition 6.3. *Let $p \in]1, \infty[$. The norm and the complete norm of the Riesz projection $\mathscr{T}$ on S^p coincide with the complete norm of the Riesz projection T on L^p: if $\check{\chi}_{\mathbb{Z}^+}(i,j) = \chi_{\mathbb{Z}^+}(i-j)$ for $i, j \geqslant 1$,*

$$\|\check{\chi}_{\mathbb{Z}^+}\|_{\mathrm{M}(\mathrm{S}^p)} = \|\check{\chi}_{\mathbb{Z}^+}\|_{\mathrm{M}_{\mathrm{cb}}(\mathrm{S}^p)} = \|\chi_{\mathbb{Z}^+}\|_{\mathrm{C}_{\mathrm{cb}}(\mathrm{L}^p)}.$$

If $p = 4$, then these norms coincide with the norm of T on L^p:

$$\|\check{\chi}_{\mathbb{Z}^+}\|_{\mathrm{M}(\mathrm{S}^4)} = \|\check{\chi}_{\mathbb{Z}^+}\|_{\mathrm{M}_{\mathrm{cb}}(\mathrm{S}^4)} = \|\chi_{\mathbb{Z}^+}\|_{\mathrm{C}_{\mathrm{cb}}(\mathrm{L}^4)} = \|\chi_{\mathbb{Z}^+}\|_{\mathrm{C}(\mathrm{L}^4)} = \sqrt{2}.$$

Proof. We shall compute the norm of $\mathscr{T}$ on S^4. Let x be a finite upper triangular matrix and let y be a finite strictly lower triangular matrix. We have to prove that

$$\sqrt{2}\|x + y\|_{\mathrm{S}^4} \geqslant \|x\|_{\mathrm{S}^4}.$$

Let us make the obvious estimates on S^2 and use the fact that adjoining is isometric:

$$\|\mathscr{T}(xx^*)\|_{\mathrm{S}^2} = \|\mathscr{T}((x+y)x^*)\|_{\mathrm{S}^2} \leqslant \|x+y\|_{\mathrm{S}^4}\|x\|_{\mathrm{S}^4}$$

and similarly,

$$\|(\mathrm{Id} - \mathscr{T})(xx^*)\|_{\mathrm{S}^2} = \|(\mathrm{Id} - \mathscr{T})(x(x+y)^*)\|_{\mathrm{S}^2} \leqslant \|x\|_{\mathrm{S}^4}\|x+y\|_{\mathrm{S}^4}.$$

As $\mathscr{T}$ and $\mathrm{Id} - \mathscr{T}$ have orthogonal ranges,

$$\|x\|_{\mathrm{S}^4}^4 = \|xx^*\|_{\mathrm{S}^2}^2 = \|(\mathrm{Id} - \mathscr{T})(xx^*)\|_{\mathrm{S}^2}^2 + \|\mathscr{T}(xx^*)\|_{\mathrm{S}^2}^2 \leqslant 2\|x\|_{\mathrm{S}^4}^2\|x+y\|_{\mathrm{S}^4}^2. \qquad \square$$

7 Unconditional approximating sequences

The following definition makes sense for general operator spaces, but we chose to state it only in our specific context.

Definition 7.1. Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Let X be the reduced C^*-algebra of Γ or its noncommutative Lebesgue space L^p for $p \in [1, \infty[$.

(*a*) A sequence (T_k) of operators on X_Λ is an *approximating sequence* if each T_k has finite rank and $T_k x \to x$ for every $x \in X_\Lambda$. It is a *complete* approximating sequence if the T_k are uniformly c.b. If X admits a complete approximating sequence, then X_Λ enjoys the *c.b. approximation property.*

(*b*) The *difference sequence* (ΔT_k) of a sequence (T_k) is given by $\Delta T_1 = T_1$ and $\Delta T_k = T_k - T_{k-1}$ for $k \geqslant 2$. An approximating sequence (T_k) is *unconditional* if the operators

$$\sum_{k=1}^{n} \varepsilon_k \Delta T_k \quad \text{with } n \geqslant 1 \text{ and } \varepsilon_k \in \{-1, 1\} \tag{F.8}$$

are uniformly bounded on X_Λ: then X_Λ enjoys the *unconditional approximation property.*

(*c*) An approximating sequence (T_k) is *completely* unconditional if the operators in (F.8) are uniformly c.b. on X_Λ: then X_Λ enjoys the *complete* unconditional approximation property. The minimal uniform bound of these operators is the *complete unconditional constant* of X_Λ.

We may always suppose that a complete approximating sequence on C_Λ is a Fourier multiplier sequence: see [36, Theorem 2.1]. We may also do so on L^p_Λ if L^∞ has the so-called QWEP: see [45, Theorem 4.4]. More precisely, the following proposition holds.

Proposition 7.2. *Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Let X either be its reduced* C^**-algebra or its noncommutative Lebesgue space* L^p*, where $p \in [1,\infty[$ and* L^∞ *has the QWEP. If X_Λ enjoys the completely unconditional approximation property with constant D, then, for every $D' > D$, there is a complete approximating sequence of Fourier multipliers (φ_k) that realises the completely unconditional approximation property with constant D': the Fourier multipliers $\sum_{k=1}^n \varepsilon_k \Delta\varphi_k$ are uniformly completely bounded by D' on X_Λ.*

Let us now describe how to skip blocks in an approximating sequence in order to construct an operator that acts like the Riesz projection on the sumset of two infinite sets. The following trick will be used in the induction below: compare [61, proof of Theorem 4.2]:

$$\left(\begin{array}{cc|c}1 & 1 & 0\\ 0 & 1 & 0\\ \hline 0 & 0 & 0\end{array}\right) - \left(\begin{array}{cc|c}1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 0\end{array}\right) + \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

Lemma 7.3. *Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Suppose that Λ contains the sumset RC of two infinite sets R and C. Let (T_k) be either an approximating sequence on L^p_Λ with $p \in [1,\infty[$, or an approximating sequence of Fourier multipliers on C_Λ. Let $\varepsilon > 0$. There is a sequence (r_i) in R, a sequence (c_i) in C and there are indices $l_1 < k_2 < l_2 < k_3 < \dots$ such that, for every n, the* skipped block sum

$$U_n = T_{l_1} + (T_{l_2} - T_{k_2}) + \dots + (T_{l_n} - T_{k_n}) \tag{F.9}$$

acts, up to ε, as the Riesz projection on the sumset $\{r_i c_j\}_{i,j \leqslant n}$:

$$\begin{cases} \|U_n(\lambda_{r_i c_j}) - \lambda_{r_i c_j}\| < \varepsilon & \text{if } i \leqslant j \leqslant n,\\ \|U_n(\lambda_{r_i c_j})\| < \varepsilon & \text{if } j < i \leqslant n.\end{cases} \tag{F.10}$$

Proof. Let us construct the sequences and indices by induction. If $n = 1$, let r_1 and c_1 be arbitrary; there is l_1 such that $\|T_{l_1}(\lambda_{r_1 c_1}) - \lambda_{r_1 c_1}\| < \varepsilon$. Suppose that $r_1, \dots, r_n$, $c_1, \dots, c_n$, $l_1, \dots, k_n, l_n$ have been constructed. Let $\delta > 0$ to be chosen later.

- The operator U_n defined by Equation (F.9) has finite rank. If it is a Fourier multiplier, one can choose an element $r_{n+1} \in R$ such that $U_n(\lambda_{r_{n+1} c_j}) = 0$ for $j \leqslant n$. If it acts on L^p_Λ with $p \in [1,\infty[$, one can choose an element $r_{n+1} \in R$ such that $\|U_n(\lambda_{r_{n+1} c_j})\| < \delta$ for $j \leqslant n$ because $(\lambda_\gamma)_{\gamma\in\Gamma}$ is weakly null in L^p.
- There is $k_{n+1} > l_n$ such that $\|T_{k_{n+1}}(\lambda_\gamma) - \lambda_\gamma\| < \delta$ for $\gamma \in \{r_i c_j : 1 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n\}$.
- Again, choose $c_{n+1} \in C$ such that $\|(U_n - T_{k_{n+1}})(\lambda_{r_i c_{n+1}})\| < \delta$ for $i \leqslant n+1$.
- Again, choose $l_{n+1} > k_{n+1}$ such that $\|T_{l_{n+1}}(\lambda_\gamma) - \lambda_\gamma\| < \delta$ for $\gamma \in \{r_i c_j : 1 \leqslant i, j \leqslant n+1\}$.

Let $U_{n+1} = U_n + (T_{l_{n+1}} - T_{k_{n+1}})$. If $i \leqslant n+1$ and $j \leqslant n$, then

$$\|\Delta U_{n+1}(\lambda_{r_i c_j})\| \leqslant \|T_{l_{n+1}}(\lambda_{r_i c_j}) - \lambda_{r_i c_j}\| + \|\lambda_{r_i c_j} - T_{k_{n+1}}(\lambda_{r_i c_j})\| < 2\delta,$$

so that

$$\begin{aligned} \|U_{n+1}(\lambda_{r_i c_j}) - \lambda_{r_i c_j}\| &< \varepsilon + 2\delta && \text{if } i \leqslant j \leqslant n\\ \|U_{n+1}(\lambda_{r_i c_j})\| &< \varepsilon + 2\delta && \text{if } j < i \leqslant n\\ \|U_{n+1}(\lambda_{r_{n+1} c_j})\| &< 3\delta && \text{if } j \leqslant n.\end{aligned}$$

If $i \leqslant n+1$, then

$$\|U_{n+1}(\lambda_{r_i c_{n+1}}) - \lambda_{r_i c_{n+1}}\| \leqslant \|(U_n - T_{k_{n+1}})(\lambda_{r_i c_{n+1}})\| + \|T_{l_{n+1}}(\lambda_{r_i c_{n+1}}) - \lambda_{r_i c_{n+1}}\| < 2\delta.$$

This shows that our choice of $r_{n+1}, c_{n+1}, k_{n+1}$ and l_{n+1} is adequate if δ is small enough. □

This construction will provide an obstacle to the unconditionality of sumsets.

Theorem 7.4. *Let Γ be a discrete group and $\Lambda \subseteq \Gamma$. Suppose that Λ contains the sumset RC of two infinite sets R and C.*

(a) Let $1 < p < \infty$. The complete unconditional constant of any approximating sequence for L^p *is bounded below by the norm of the Riesz projection on* S^p*, and thus by $\csc \pi/p$.*

(b) The spaces L^1_Λ and C_Λ do not enjoy the complete unconditional approximation property.

(c) *If Γ is amenable, then the space* C_Λ *does not enjoy the unconditional approximation property.*

Proof. Let (T_k) be an approximating sequence on L^p_Λ. By Lemma 7.3, for every $\varepsilon > 0$ and every n, there are elements $r_1, \ldots, r_n \in R$, $c_1, \ldots, c_n \in C$ such that the Fourier multiplier φ given by the indicator function of $\{r_i c_j\}_{i \leqslant j}$ is near to a skipped block sum U_n of (T_k) in the sense that $\|U_n(\lambda_{r_i c_j}) - \varphi_{r_i c_j}\lambda_{r_i c_j}\| < \varepsilon$. But U_n is the mean of two operators of the form (F.8): its complete norm will provide a lower bound for the complete unconditional constant of X_Λ. Let us repeat the argument of Lemma 2.3 with $x \in \mathrm{S}^p_n$: as

$$\Big\|\sum_{i,j=1}^{n} x_{i,j}\mathrm{e}_{i,j}\Big\|_{\mathrm{S}^p_n} = \Big\|\Big(\sum_{i=1}^{n}\mathrm{e}_{i,i}\otimes\lambda_{r_i}\Big)\Big(\sum_{i,j=1}^{n} x_{i,j}\mathrm{e}_{i,j}\otimes\lambda_{\mathrm{e}}\Big)\Big(\sum_{j=1}^{n}\mathrm{e}_{j,j}\otimes\lambda_{c_j}\Big)\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}$$
$$= \Big\|\sum_{i=1}^{n} x_{i,j}\mathrm{e}_{i,j}\otimes\lambda_{r_i c_j}\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)}$$

and

$$\Big\|\sum_{i=1}^{n} x_{i,j}\mathrm{e}_{i,j}\otimes\big(U_n(\lambda_{r_i c_j}) - \varphi_{r_i c_j}\lambda_{r_i c_j}\big)\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} < n^2\varepsilon\|x\|_{\mathrm{S}^p_n},$$

the complete norm of U_n is nearly bounded below by the norm of the Riesz projection on S^p_n:

$$\Big\|\sum_{i=1}^{n} x_{i,j}\mathrm{e}_{i,j}\otimes U_n(\lambda_{r_i c_j})\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} > \Big\|\sum_{i\leqslant j} x_{i,j}\mathrm{e}_{i,j}\otimes\lambda_{r_i c_j}\Big\|_{\mathrm{L}^p(\mathrm{tr}\otimes\tau)} - n^2\varepsilon\|x\|_{\mathrm{S}^p_n}$$
$$= \|\mathscr{T}(x)\|_{\mathrm{S}^p_n} - n^2\varepsilon\|x\|_{\mathrm{S}^p_n}.$$

This proves as well (a) as the first assertion in (b), because the Riesz projection is unbounded on S^1. Let (T_k) be an approximating sequence on C_Λ: by Lemma 7.2, we may suppose that (T_k) is a sequence of Fourier multipliers. Thus the second assertion in (b) follows from Lemma 7.3 combined with the preceding argument (where S^p_n is replaced by S^∞_n and $\mathrm{L}^p(\mathrm{tr}\otimes\tau)$ by $\mathrm{S}^\infty_n \otimes \mathrm{C}$) and the unboundedness of the Riesz projection on S^∞. For (c), note that the Fourier multipliers T_k are automatically c.b. on C_Λ if Γ is amenable [25, Corollary 1.8]. ☐

Theorem 7.4 (b) has been devised originally to prove that the Hardy space H^1, corresponding to the case $\Lambda = \mathbb{N} \subseteq \mathbb{Z}$ and $p = 1$, admits no completely unconditional basis: see [84, 85]. Theorem 7.4 (c) both generalises the fact that a sumset cannot be a Sidon set (see [54, §§ 1.4, 6.6] for two proofs and historical remarks, or [51, Proposition IV.7]) and Daniel Li's result [50, Corollary 13] that the space C_Λ does not have the "metric" unconditional approximation property if Γ is abelian and Λ contains a sumset. Li [50, Theorem 10] also constructed a set $\Lambda \subseteq \mathbb{Z}$ such that C_Λ has this property while Λ contains the sumset of arbitrarily large sets. This theorem also provides a new proof that the disc algebra has no unconditional basis and answers [64, Question 6.1.6].

Example 7.5. Neither the span of products $\{r_i r_j\}$ of two Rademacher functions in the space of continuous functions on $\{-1,1\}^\infty$ nor the span of products $\{s_i s_j\}$ of two Steinhaus functions in the space of continuous functions on $\mathbb{T}^\infty$ have an unconditional basis.

8 Relative Schur multipliers of rank one

Let ϱ be an *elementary* Schur multiplier on S^∞, that is,

$$\varrho = x \otimes y = (x_r y_c)_{(r,c)\in R\times C} :$$

then its norm is $\sup_{r\in R}|x_r|\sup_{c\in C}|y_c|$. How is this norm affected if ϱ is only partially specified, that is, if the action of ϱ is restricted to matrices with a given support?

Theorem 8.1. *Let $I \subseteq R \times C$ and consider $(x_r)_{r\in R}$ and $(y_c)_{c\in C}$. The relative Schur multiplier on S^∞_I given by $(x_r y_c)_{(r,c)\in I}$ has norm $\sup_{(r,c)\in I}|x_r y_c|$.*

Note that the norm of $(x_r y_c)_{(r,c)\in I}$ is bounded by $\sup_{r\in R}|x_r|\sup_{c\in C}|y_c|$ because the matrix $(x_r y_c)_{(r,c)\in R\times C}$ trivially extends $(x_r y_c)_{(r,c)\in I}$; the proof below provides a constructive nontrivial extension of this Schur multiplier that is a composition of ampliations of the Schur multiplier in the following lemma.

Lemma 8.2. *The Schur multiplier* $\begin{pmatrix} \overline{z} & w \\ \overline{w} & z \end{pmatrix}$ *has norm* $\max(|z|, |w|)$ *on* S_2^∞.

Proof. This follows from the decomposition

$$\begin{pmatrix} \overline{z} & w \\ \overline{w} & z \end{pmatrix} = \frac{|z| + |w|}{2} \begin{pmatrix} \bar{t}u \\ t\bar{u} \end{pmatrix} \otimes \begin{pmatrix} \overline{tu} & tu \end{pmatrix} + \frac{|z| - |w|}{2} \begin{pmatrix} \bar{t}u \\ -t\bar{u} \end{pmatrix} \otimes \begin{pmatrix} \overline{tu} & -tu \end{pmatrix},$$

where $t, u \in \mathbb{T}$ are chosen so that $z = |z|t^2$ and $w = |w|u^2$. □

Proof of Theorem 8.1. We may suppose that C is the finite set $\{1, \ldots, m\}$ and that R is the finite set $\{1, \ldots, n\}$, that each y_c is nonzero and that each row in R contains an element of I. We may also suppose that $(|x_r|)_{r \in R}$ and $(|y_c|)_{c \in C}$ are nonincreasing sequences. For each $r \in R$ let c_r be the least column index of elements of I in or above row r: in other words,

$$c_r = \min_{r' \leqslant r} \min\{c : (r', c) \in I\}.$$

The sequence $(c_r)_{r \in R}$ is nonincreasing. Let us define its inverse $(r_c)_{c \in C}$ in the sense that $r_c \leqslant r \Leftrightarrow c_r \leqslant c$: for each $c \in C$ let $r_c = \min\{r : c_r \leqslant c\}$. Given r, let $r' \leqslant r$ be such that $(r', c_r) \in I$: then $|x_r y_{c_r}| \leqslant |x_{r'} y_{c_r}|$, so that $\sup_{r \in R} |x_r y_{c_r}| \leqslant \sup_{(r,c) \in I} |x_r y_c|$ and the rank 1 Schur multiplier

$$\varrho_0 = (x_r y_{c_r})_{(r,c) \in R \times C}$$

with pairwise equal columns is bounded by $\sup_{(r,c) \in I} |x_r y_c|$ on S_n^∞. We will now "correct" ϱ_0 without increasing its norm so as to make it an extension of $(x_r y_c)_{(r,c) \in I}$. Let $r \in R$ and $c' \geqslant c_r$: then

$$\begin{aligned} x_r y_{c'} = x_r y_{c_r} \frac{y_{c_r+1}}{y_{c_r}} \cdots \frac{y_{c'}}{y_{c'-1}} &= x_r y_{c_r} \prod_{c_r \leqslant c \leqslant c'-1} \frac{y_{c+1}}{y_c} \\ &= x_r y_{c_r} \prod_{\substack{r \geqslant r_c \\ c' \geqslant c+1}} \frac{y_{c+1}}{y_c}. \end{aligned}$$

This shows that it suffices to compose the Schur multiplier ϱ_0 with the $m-1$ rank 2 Schur multipliers with block matrix

$$\varrho_c = \begin{array}{cc} & \begin{array}{cc} 1 \cdots c & c{+}1 \cdots m \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ r_c - 1 \\ r_c \\ \vdots \\ n \end{array} & \left(\begin{array}{c|c} \overline{\left(\dfrac{y_{c+1}}{y_c} \right)} & 1 \\ \hline 1 & \dfrac{y_{c+1}}{y_c} \end{array} \right) \end{array},$$

each of which has norm 1 on S_n^∞ by Lemma 8.2. □

Remark 8.3. As an illustration, let $C = R = \{1, ..., n\}$ and $I = \{(r, c) : r \geqslant c\}$, and let a_i be an increasing sequence of positive numbers. Take $x_r = a_r$ and $y_c = 1/a_c$. Then the relative Schur multiplier $(a_r/a_c)_{r \leqslant c}$ has norm 1. The above proof actually constructs the norm 1 extension $(\min(a_r/a_c, a_c/a_r))_{(r,c)}$. If we put $a_i = \mathrm{e}^{x_i}$, we recover that $(\mathrm{e}^{-|x_r - x_c|})_{(r,c)}$ is positive definite, that is, $|\cdot|$ is a conditionally negative function on $\mathbb{R}$.

Chapter G

The Sidon constant of sets with three elements

We solve an elementary extremal problem on trigonometric polynomials and obtain the exact value of the Sidon constant for sets with three elements $\{n_0, n_1, n_2\}$: it is

$$\sec\left(\pi \gcd(n_1 - n_0, n_2 - n_0)/2\max|n_i - n_j|\right).$$

1 Introduction

Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ be a set of three frequencies and $\varrho_0, \varrho_1, \varrho_2$ three positive intensities. We solve the following extremal problem:

(†) To find $\vartheta_0, \vartheta_1, \vartheta_2$ three phases such that, putting $c_j = \varrho_j \mathrm{e}^{\mathrm{i}\vartheta_j}$, the maximum $\max_t |c_0\mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1\mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2\mathrm{e}^{\mathrm{i}\lambda_2 t}|$ is minimal.

This enables us to generalise a result of D. J. Newman. He solved the following extremal problem for $\Lambda = \{0, 1, 2\}$:

(‡) To find $f(t) = c_0\mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1\mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2\mathrm{e}^{\mathrm{i}\lambda_2 t}$ with $\|f\|_\infty = \max_t |f(t)| \leqslant 1$ such that $\|\widehat{f}\|_1 = |c_0| + |c_1| + |c_2|$ is maximal.

Note that for such an f, $\|\widehat{f}\|_1$ is the Sidon constant of Λ. Newman's argument is the following (see [93, Chapter 3]): by the parallelogram law,

$$\begin{aligned}
\max_t |f(t)|^2 &= \max_t |f(t)|^2 \vee |f(t+\pi)|^2 \\
&\geqslant \max_t \bigl(|f(t)|^2 + |f(t+\pi)|^2\bigr)/2 \\
&= \max_t \bigl(|c_0 + c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{\mathrm{i}2t}|^2 + |c_0 - c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{\mathrm{i}2t}|^2\bigr)/2 \\
&= \max_t |c_0 + c_2\mathrm{e}^{\mathrm{i}2t}|^2 + |c_1|^2 = \bigl(|c_0| + |c_2|\bigr)^2 + |c_1|^2 \\
&\geqslant \bigl(|c_0| + |c_1| + |c_2|\bigr)^2/2
\end{aligned}$$

and equality holds exactly for multiples and translates of $f(t) = 1 + 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}2t}$.

Let us describe this paper briefly. We use a real-variable approach: Problem (†) reduces to studying a function of form

$$\Phi(t, \vartheta) = |1 + r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2 \text{ for } r, s > 0,\ k \neq l \in \mathbb{Z}^*$$

and more precisely $\Phi^*(\vartheta) = \max_t \Phi(t, \vartheta)$. We obtain the variations of Φ^*: the point is that we find "by hand" a local minimum of Φ^* and that any two minima of Φ^* are separated by a maximum of Φ^*, which corresponds to an extremal point of Φ and therefore has a handy description. The solution to Problem (‡) then turns out to derive easily from this.

The initial motivation was twofold. In the first place, we wanted to decide whether sets $\Lambda = \{\lambda_n\}$ such that λ_{n+1}/λ_n is bounded by some q may have a Sidon constant arbitrarily close to 1 and to find evidence among sets with three elements. That there are such sets, arbitrarily large but finite, may in fact be proven by the method of Riesz products in [47, Appendix V, §1.II]. In the second place, we wished to show that the real and complex unconditionality constants are distinct for basic sequences of characters $\mathrm{e}^{\mathrm{i}nt}$; we prove however that they coincide in the space $\mathscr{C}(\mathbb{T})$ for sequences with three terms.

Notation. $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ and $\mathrm{e}_\lambda(z) = z^\lambda$ for $z \in \mathbb{T}$ and $\lambda \in \mathbb{Z}$.

2 Definitions

Definition 2.1. (1) Let $\Lambda \subseteq \mathbb{Z}$. Λ is a Sidon set if there is a constant C such that for all trigonometric polynomials $f(t) = \sum_{\lambda\in\Lambda} c_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$ with spectrum in Λ we have

$$\|\widehat{f}\|_1 = \sum_{\lambda\in\Lambda} |c_\lambda| \leqslant C \max_t |f(t)| = \|f\|_\infty.$$

The optimal C is called the Sidon constant of Λ.

(2) Let X be a Banach space. The sequence $(x_n) \subseteq X$ is a real (vs. complex) unconditional basic sequence in X if there is a constant C such that

$$\Big\|\sum \vartheta_n c_n x_n\Big\|_X \leqslant C\Big\|\sum c_n x_n\Big\|_X$$

for every real (vs. complex) choice of signs $\vartheta_n \in \{-1, 1\}$ (vs. $\vartheta_n \in \mathbb{T}$) and every finitely supported family of coefficients (c_n). The optimal C is the real (vs. complex) unconditionality constant of (x_n) in X.

Let us state the two following well known facts.

Proposition 2.2. (1) *The Sidon constant of Λ is the complex unconditionality constant of the sequence of functions $(\mathrm{e}_\lambda)_{\lambda\in\Lambda}$ in the space $\mathscr{C}(\mathbb{T})$.*

(2) *The complex unconditionality constant is at most $\pi/2$ times the real unconditionality constant.*

Proof. (1) holds because $\|\sum \vartheta_\lambda c_\lambda \mathrm{e}_\lambda\|_\infty = \sum |c_\lambda|$ for $\vartheta_\lambda = \overline{c_\lambda}/|c_\lambda|$.

(2) Because the complex unconditionality constant of the sequence (ϵ_n) of Rademacher functions in $\mathscr{C}(\{-1, 1\}^\infty)$ is $\pi/2$ (see [92]),

$$\begin{aligned}
\sup_{\vartheta_n\in\mathbb{T}}\Big\|\sum \vartheta_n c_n x_n\Big\|_X &= \sup_{x^*\in B_{X^*}} \sup_{\vartheta_n\in\mathbb{T}} \sup_{\epsilon_n=\pm1}\Big|\sum \vartheta_n c_n \langle x^*, x_n\rangle \epsilon_n\Big| \\
&\leqslant \pi/2 \sup_{x^*\in B_{X^*}} \sup_{\epsilon_n=\pm1}\Big|\sum c_n \langle x^*, x_n\rangle \epsilon_n\Big| \\
&= \pi/2 \sup_{\epsilon_n=\pm1}\Big\|\sum \epsilon_n c_n x_n\Big\|_X.
\end{aligned}$$

Furthermore the real unconditionality constant of (ϵ_n) in $\mathscr{C}(\{-1, 1\}^\infty)$ is 1: therefore the factor $\pi/2$ is optimal. ∎

Let us straighten out the expression of the Sidon constant. For

$$f(t) = c_0\mathrm{e}^{\mathrm{i}\lambda_0 t} + c_1\mathrm{e}^{\mathrm{i}\lambda_1 t} + c_2\mathrm{e}^{\mathrm{i}\lambda_2 t},\ c_j = \varrho_j \mathrm{e}^{\mathrm{i}\vartheta_j},$$

the supremum norm $\|f\|_\infty$ of f is equal to

$$\|\varrho_0 + \varrho_1\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}_{\lambda_1-\lambda_0} + \varrho_2\mathrm{e}_{\lambda_2-\lambda_0}\|_\infty,\ \vartheta = \frac{\lambda_1-\lambda_2}{\lambda_2-\lambda_0}\vartheta_0 + \vartheta_1 + \frac{\lambda_0-\lambda_1}{\lambda_2-\lambda_0}\vartheta_2 \tag{G.1}$$

and therefore the Sidon constant C of $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ may be written

$$C = \max_{r,s>0,\vartheta}(1 + r + s)/\|1 + r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}_k + s\mathrm{e}_l\|_\infty \quad \text{with} \begin{cases} k = \lambda_1 - \lambda_0 \\ l = \lambda_2 - \lambda_0. \end{cases} \tag{G.2}$$

By change of variables, we may suppose w.l.o.g. that k and l are coprime.

3 A solution to Extremal problem (†)

Let us first establish

Lemma 3.1. *Let* $\lambda_1, \dots, \lambda_k \in \mathbb{Z}^*$ *and* $\varrho_1, \dots, \varrho_k > 0$. *Let*

$$f(t,\vartheta) = 1 + \varrho_1 \mathrm{e}^{\mathrm{i}(\lambda_1 t + \vartheta_1)} + \dots + \varrho_{k-1}\mathrm{e}^{\mathrm{i}(\lambda_{k-1}t+\vartheta_{k-1})} + \varrho_k \mathrm{e}^{\mathrm{i}\lambda_k t}$$

and $\Phi(t,\vartheta) = |f(t,\vartheta)|^2$. *The critical points* (t,ϑ) *such that* $\nabla\Phi(t,\vartheta) = 0$ *satisfy either* $f(t,\vartheta) = 0$ *or* $\lambda_1 t + \vartheta_1 \equiv \dots \equiv \lambda_{k-1}t + \vartheta_{k-1} \equiv \lambda_k t \equiv 0 \mod \pi$.

Proof. As $\Phi = (\Re f)^2 + (\Im f)^2$, the critical points (t,ϑ) satisfy

$$\begin{cases} \Re\frac{\partial f}{\partial t}(t,\vartheta)\,\Re f(t,\vartheta) + \quad \Im\frac{\partial f}{\partial t}(t,\vartheta)\,\Im f(t,\vartheta) = 0 \\ -\sin(\lambda_i t + \vartheta_i)\,\Re f(t,\vartheta) + \cos(\lambda_i t + \vartheta_i)\,\Im f(t,\vartheta) = 0 \quad (1 \leqslant i \leqslant k-1), \end{cases}$$

which simplifies to

$$-\sin(\lambda_i t + \vartheta_i)\,\Re f(t,\vartheta) + \cos(\lambda_i t + \vartheta_i)\,\Im f(t,\vartheta) = 0 \quad (1 \leqslant i \leqslant k,\ \vartheta_k = 0).$$

Suppose that $f(t,\vartheta) \neq 0$: then the system above implies that

$$-\sin(\lambda_i t + \vartheta_i)\cos(\lambda_j t + \vartheta_j) + \cos(\lambda_i t + \vartheta_i)\sin(\lambda_j t + \vartheta_j) = 0 \ (1 \leqslant i, j \leqslant k,\ \vartheta_k = 0)$$

and it simplifies therefore to

$$\sin(\lambda_i t + \vartheta_i) = 0 \quad (1 \leqslant i \leqslant k,\ \vartheta_k = 0).$$ □

The following result is the core of the paper.

Lemma 3.2. *Let* $r, s > 0$, $k, l \in \mathbb{Z}^*$ *distinct and coprime. Let*

$$\begin{aligned} \Phi(t,\vartheta) &= |1 + r\mathrm{e}^{\mathrm{i}\vartheta}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2 \\ &= 1 + r^2 + s^2 + 2r\cos(kt+\vartheta) + 2s\cos lt + 2rs\cos((l-k)t - \vartheta). \end{aligned}$$

Let $\Phi^*(\vartheta) = \max_t \Phi(t,\vartheta)$. *Then* Φ^* *is an even function with period* $2\pi/|l|$ *that decreases on* $[0, \pi/|l|]$. *Therefore* $\min_\vartheta \Phi^*(\vartheta) = \Phi^*(\pi/l)$.

Proof. Φ^* is continuous (see [80, Chapter 5.4]) and even, as $\Phi(t,-\vartheta) = \Phi(-t,\vartheta)$. Φ^* is $(2\pi/|l|)$-periodical: let $j \in \mathbb{Z}$ be such that $jk \equiv 1$ mod. l. Then

$$\Phi(t + 2j\pi/l, \vartheta) = |1 + r\mathrm{e}^{\mathrm{i}(\vartheta + 2\pi jk/l)}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2 = \Phi(t, \vartheta + 2\pi/l).$$

Thus Φ^* attains its minimum on $[0, \pi/|l|]$. Furthermore, we have

$$\Phi(-t - 2j\pi/l, \pi/l - \vartheta) = \Phi(t + 2j\pi/l, -\pi/l + \vartheta) = \Phi(t, \pi/l + \vartheta),$$

so that Φ^* has an extremum at π/l. Now

$$\Phi^*(\pi/l + \vartheta) = \Phi^*(\pi/l) + |\vartheta| \max_{\Phi(t,\pi/l) = \Phi^*(\pi/l)} \left|\frac{\partial\Phi}{\partial\vartheta}(t,\pi/l)\right| + o(\vartheta).$$

Choose a t such that $\Phi(t,\pi/l) = \Phi^*(\pi/l)$. If $\partial\Phi/\partial\vartheta(t,\pi/l) \neq 0$, then this shows that Φ^* has a local minimum and a cusp at π/l. Let us now suppose that $\partial\Phi/\partial\vartheta(t,\pi/l) = 0$. If Φ^* had a local maximum at π/l, then $(t,\pi/l)$ would be a critical point of Φ, so that by Lemma 3.1 $\cos(kt + \pi/l) = \delta$, $\cos lt = \epsilon$, $\cos((l-k)t - \pi/l) = \delta\epsilon$ for some $\delta, \epsilon \in \{-1, 1\}$. One necessarily would have $(\delta,\epsilon) \neq (1,1)$. Furthermore,

$$\begin{aligned} \frac{\partial^2\Phi}{\partial\vartheta^2}(t,\pi/l) &= -2r\delta(1 + s\epsilon) \leqslant 0 \\ \begin{vmatrix} \partial^2\Phi/\partial t^2 & \partial^2\Phi/\partial t\partial\vartheta \\ \partial^2\Phi/\partial\vartheta\partial t & \partial^2\Phi/\partial\vartheta^2 \end{vmatrix}(t,\pi/l) &= 4rsl^2(\delta\epsilon + r\epsilon + s\delta) \geqslant 0, \end{aligned}$$

which would imply $\epsilon = -1$, $r = 0$, $s = 1$. Therefore Φ^* has a local minimum at π/l. Let us show that then Φ^* must decrease on $[0, \pi/|l|]$. Otherwise there are $0 \leqslant \vartheta_0 < \vartheta_1 \leqslant \pi/|l|$ such that $\Phi^*(\vartheta_1) > \Phi^*(\vartheta_0)$. As $\pi/|l|$ is a local minimum, there is a $\vartheta_0 < \vartheta^* < \pi/|l|$ such that

$$\Phi^*(\vartheta^*) = \max_{\vartheta_0 \leqslant \vartheta \leqslant \pi/|l|} \Phi^*(\vartheta) = \max_{\substack{0 \leqslant t < 2\pi \\ \vartheta_0 \leqslant \vartheta \leqslant \pi/|l|}} \Phi(t,\vartheta),$$

i.e., there further is some t^* such that Φ has a local maximum at (t^*, ϑ^*). But then $kt^* + \vartheta^* \equiv lt^* \equiv 0 \mod \pi$ and $\vartheta^* \equiv 0 \mod \pi/l$ and this is false. □

By Computation (G.1) and Lemma 3.2, we obtain

Theorem 3.3. *Let $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ and $\varrho_0, \varrho_1, \varrho_2 > 0$. The solution to Extremal problem (†) is the following.*

- *If the smallest additive group containing $\lambda_1 - \lambda_0$ and $\lambda_2 - \lambda_0$ is dense in $\mathbb{R}$, then the maximum is independent of the phases $\vartheta_0, \vartheta_1, \vartheta_2$ and makes $\varrho_0 + \varrho_1 + \varrho_2$.*
- *Otherwise let $d = \gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$ be a generator of this group. Then the sought phases $\vartheta_0, \vartheta_1, \vartheta_2$ are given by*

$$\vartheta_0(\lambda_2 - \lambda_1) + \vartheta_1(\lambda_0 - \lambda_2) + \vartheta_2(\lambda_1 - \lambda_0) \equiv d\pi \mod 2d\pi.$$

In particular, these phases may be chosen among 0 and π.

4 A solution to Extremal problem (‡)

There are two cases where one can make explicit computations by Lemma 3.2.

Example 4.1. The real and complex unconditionality constant of $\{0, 1, 2\}$ in $\mathscr{C}(\mathbb{T})$ is $\sqrt{2}$. Indeed, a case study shows that

$$\|1 + \mathrm{i}r\mathrm{e}_1 + s\mathrm{e}_2\|_\infty = \begin{cases} r + |s-1| & \text{if } r|s-1| \geqslant 4s \\ (1+s)(1 + r^2/4s)^{1/2} & \text{if } r|s-1| \leqslant 4s \end{cases}$$

and this permits to compute the maximum (G.2), which is obtained for $r = 2$, $s = 1$. This yields another proof to Newman's result presented in the Introduction.

Example 4.2. The real and complex unconditionality constant of $\{0, 1, 3\}$ in $\mathscr{C}(\mathbb{T})$ is $2/\sqrt{3}$. Indeed, a case study shows that $\|1 + r\mathrm{e}^{\mathrm{i}\pi/3}\mathrm{e}_1 + s\mathrm{e}_3\|_\infty$ makes

$$\begin{cases} 1 + r - s & \text{if } s \leqslant r/(4r+9) \\ \left(\frac{2}{27}s(r^2 + 9 + 3r/s)^{3/2} - \frac{2}{27}r^3 s + \frac{2}{3}r^2 + rs + s^2 + 1\right)^{1/2} & \text{if } s \geqslant r/(4r+9) \end{cases}$$

and this permits to compute the maximum (G.2), which is obtained exactly at $r = 3/2$, $s = 1/2$.

These examples are particular cases of the following theorem.

Theorem 4.3. *Let $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}$ be distinct. Then the Sidon constant of $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ is $\sec(\pi/2n)$, where $n = \max|\lambda_i - \lambda_j|/\gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$.*

Proof. We may suppose $\lambda_0 < \lambda_1 < \lambda_2$. Let $k = (\lambda_1 - \lambda_0)/\gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$ and $l = (\lambda_2 - \lambda_0)/\gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$. By Lemma 3.2, the Arithmetic-Geometric Mean Inequality bounds the Sidon constant C of $\{0, k, l\}$ in the following way:

$$\begin{aligned} C = \max_{r,s>0} \frac{1+r+s}{\|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}_k + s\mathrm{e}_l\|_\infty} &\leqslant \max_{r,s>0} \frac{1+r+s}{|1 + r\mathrm{e}^{\mathrm{i}\pi/l} + s|} \\ &= \max_{r,s>0} \left(1 - \sin^2\frac{\pi}{2l}\frac{4r(1+s)}{(1+r+s)^2}\right)^{-1/2} \\ &\leqslant \left(1 - \sin^2(\pi/2l)\right)^{-1/2} = \sec(\pi/2l). \end{aligned}$$

This inequality is sharp: we have equality for $s = k/(l-k)$ and $r = 1 + s$. In fact the derivative of $|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|^2$ is then

$$\frac{8kl}{k-l}\cos\frac{kt + \pi/l}{2}\sin\frac{lt}{2}\cos\frac{(l-k)t - \pi/l}{2},$$

so that its critical points are

$$\frac{2j+1}{k}\pi - \frac{\pi}{kl},\ \frac{2j}{l}\pi,\ \frac{2j+1}{l-k}\pi + \frac{\pi}{l(l-k)} : j \in \mathbb{Z},$$

where it makes

$$4s^2\sin^2\frac{2j+1+l}{2k}\pi,\ 4r^2\cos^2\frac{2j+1}{2l}\pi,\ 4\cos^2\frac{2j+1+k}{2(l-k)}\pi : j \in \mathbb{Z}.$$

Therefore the maximum of $|1 + r\mathrm{e}^{\mathrm{i}\pi/l}\mathrm{e}^{\mathrm{i}kt} + s\mathrm{e}^{\mathrm{i}lt}|$ is $2r\cos(\pi/2l)$. ◻

This proof and (G.1) yield also the more precise

Proposition 4.4. *Let $\Lambda = \{\lambda_0, \lambda_1, \lambda_2\} \subseteq \mathbb{Z}$. The solution to Extremal problem (‡) is a multiple of*

$$f(t) = \epsilon_0 \left|\lambda_1 - \lambda_2\right| \mathrm{e}^{\mathrm{i}\lambda_0 t} + \epsilon_1 \left|\lambda_0 - \lambda_2\right| \mathrm{e}^{\mathrm{i}\lambda_1 t} + \epsilon_2 \left|\lambda_0 - \lambda_1\right| \mathrm{e}^{\mathrm{i}\lambda_2 t}$$

with $\epsilon_0, \epsilon_1, \epsilon_2 \in \{-1, 1\}$ real signs such that

- $\epsilon_0\epsilon_1 = -1$ *if* $2^j \mid \lambda_1 - \lambda_0$ *and* $2^j \nmid \lambda_2 - \lambda_0$ *for some* j*;*
- $\epsilon_0\epsilon_2 = -1$ *if* $2^j \nmid \lambda_1 - \lambda_0$ *and* $2^j \mid \lambda_2 - \lambda_0$ *for some* j*;*
- $\epsilon_1\epsilon_2 = -1$ *otherwise.*

The Sidon constant of Λ is attained for this f. Therefore the complex and real unconditionality constants of $\{\mathrm{e}_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{C}(\mathbb{T})$ coincide for sets Λ with three elements.

5 Some consequences

Let us underline the following consequences of our computation.

Corollary 5.1. (1) *The Sidon constant of sets with three elements is at most $\sqrt{2}$.*

(2) *The Sidon constant of $\{0, n, 2n\}$ is $\sqrt{2}$, while the Sidon constant of $\{0, n+1, 2n\}$ is at most $\sec(\pi/2n) = 1 + \pi^2/8n^2 + o(n^{-2})$ and thus arbitrarily close to 1.*

(3) *The Sidon constant of $\{\lambda_0 < \lambda_1 < \lambda_2\}$ does not depend on λ_1 but on the g.c.d. of $\lambda_1 - \lambda_0$ and $\lambda_2 - \lambda_0$.*

Theorem 4.3 also shows anew that no set of integers with more than two elements has Sidon constant 1 (see [93, p. 21] or [21]). Recall now that $\Lambda = \{\lambda_n\} \subseteq \mathbb{Z}$ is a Hadamard set if there is a $q > 1$ such that $|\lambda_{n+1}/\lambda_n| \geqslant q$ for all n. By [63, Cor. 9.4], the Sidon constant of Λ is at most $1 + \pi^2/(2q^2 - 2 - \pi^2)$ if $q > \sqrt{\pi^2/2 + 1} \approx 2.44$. On the other hand Theorem 4.3 shows

Corollary 5.2. (1) *If there is an integer $q \geqslant 2$ such that $\Lambda \supseteq \{\lambda, \lambda+\mu, \lambda+q\mu\}$ for some integers λ and μ, then the Sidon constant of Λ is at least*

$$\sec(\pi/2q) > 1 + \pi^2/(8q^2).$$

(2) *In particular, we have the following bounds for the Sidon constant C of the set $\Lambda = \{q^k\}$, $q \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}$:*

$$1 + \frac{\pi^2}{8\max(-q, q+1)^2} < C \leqslant 1 + \frac{\pi^2}{2q^2 - 2 - \pi^2}.$$

6 Three questions

(*a*) Is there a set Λ for which the real and complex unconditionality constants of $\{\mathrm{e}_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{C}(\mathbb{T})$ differ? The same question is open in spaces $L^p(\mathbb{T})$, $1 \leqslant p < \infty$, and even for the case of three element sets if p is not a small even integer, and especially for the set $\{0, 1, 2, 3\}$ in any space but $L^2(\mathbb{T})$.

(*b*) Let $q > 1$. Are there infinite sets $\Lambda = \{\lambda_n\}$ such that $|\lambda_{n+1}/\lambda_n| \leqslant q$ with Sidon constant arbitrarily close to 1? What about the sequence of integer parts of the powers of a transcendental number $\sigma > 1$ (see [63, Cor. 2.10, Prop. 3.2])?

(*c*) The only set with more than three elements with known Sidon constant is $\{0, 1, 2, 3, 4\}$, for which it makes 2 (see [93, Chapter 3]). Can one compute the Sidon constant of sets with four elements? I conjecture that the Sidon constant of $\{0, 1, 2, 3\}$ is $5/3$.

Chapter H

The maximum modulus of a trigonometric trinomial

Let Λ be a set of three integers and let C_Λ be the space of 2π-periodic functions with spectrum in Λ endowed with the maximum modulus norm. We isolate the maximum modulus points x of trigonometric trinomials $T \in \mathrm{C}_\Lambda$ and prove that x is unique unless $|T|$ has an axis of symmetry. This enables us to compute the exposed and the extreme points of the unit ball of C_Λ, to describe how the maximum modulus of T varies with respect to the arguments of its Fourier coefficients and to compute the norm of unimodular relative Fourier multipliers on C_Λ. We obtain in particular the Sidon constant of Λ.

1 Introduction

Let λ_1, λ_2 and λ_3 be three pairwise distinct integers. Let r_1, r_2 and r_3 be three positive real numbers. Given three real numbers t_1, t_2 and t_3, let us consider the *trigonometric trinomial*

$$T(x) = r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)} \tag{H.1}$$

for $x \in \mathbb{R}$. The λ's are the *frequencies* of the trigonometric trinomial T, the r's are the *moduli* or *intensities* and the t's the *arguments* or *phases* of its *Fourier coefficients* $r_1 \mathrm{e}^{\mathrm{i}t_1}$, $r_2 \mathrm{e}^{\mathrm{i}t_2}$ and $r_3 \mathrm{e}^{\mathrm{i}t_3}$.

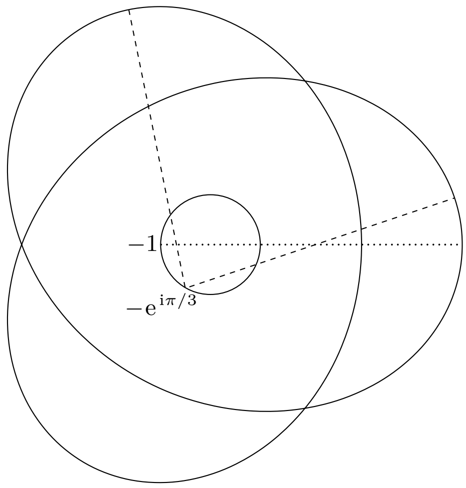


Figure H.1: The unit circle, the hypotrochoid H with equation $z = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}x}$, the segment from -1 to the unique point on H at maximum distance and the segments from $-\mathrm{e}^{\mathrm{i}\pi/3}$ to the two points on H at maximum distance.

The maximum modulus of a trigonometric trinomial has an interpretation in plane geometry. Without loss of generality, we may assume that λ_2 is between λ_1 and λ_3. Let H be the curve with complex equation

$$z = r_1 \mathrm{e}^{\mathrm{i}(t_1-(\lambda_2-\lambda_1)x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+(\lambda_3-\lambda_2)x)} \quad (-\pi < x \leqslant \pi). \tag{H.2}$$

H is a *hypotrochoid*: it is drawn by a point at distance r_3 to the centre of a circle with radius $r_1|\lambda_2-\lambda_1|/|\lambda_3-\lambda_2|$ that rolls inside another circle with radius $r_1|\lambda_3-\lambda_1|/|\lambda_3-\lambda_2|$. The maximum modulus of (H.1) is the maximum distance of points $z \in H$ to a given point $-r_2\mathrm{e}^{\mathrm{i}t_2}$ of the complex plane. If $\lambda_3-\lambda_2 = \lambda_2-\lambda_1$, then H is the ellipse with centre 0, half major axis r_1+r_3 and half minor axis $|r_1-r_3|$. Note that an epitrochoid (Ptolemy's epicycle) amounts also to a hypotrochoid. Figure H.1 illustrates the particular case $T(x) = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}x}$.

We deduce an interval on which T attains its maximum modulus independently of the moduli of its Fourier coefficients (see Theorem 7.1 (a) for a detailed answer.) We prove in particular the following result.

Theorem 1.1. *Let* $d = \gcd(\lambda_2-\lambda_1, \lambda_3-\lambda_2)$ *and let* τ *be the distance of*

$$\frac{\lambda_2-\lambda_3}{d}t_1 + \frac{\lambda_3-\lambda_1}{d}t_2 + \frac{\lambda_1-\lambda_2}{d}t_3 \tag{H.3}$$

to $2\pi\mathbb{Z}$. *The trigonometric trinomial* T *attains its maximum modulus at a unique point modulo* $2\pi/d$*, with multiplicity 2, unless* $\tau = \pi$.

Theorem 1.1 shows that if there are two points of the hypotrochoid H at maximum distance to $-r_2\mathrm{e}^{\mathrm{i}t_2}$, it is so only because $-r_2\mathrm{e}^{\mathrm{i}t_2}$ lies on an axis of symmetry of H.

We obtain a precise description of those trigonometric trinomials that attain their maximum modulus twice modulo $2\pi/d$: see Theorem 7.1 (c). Their rôle becomes clear by the following result in convex geometry: they yield the exposed points of the unit ball of the ambient normed space. Let us first put up the proper functional analytic framework. Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ be the *spectrum* of the trigonometric trinomial T and write $\mathrm{e}_\lambda \colon x \mapsto \mathrm{e}^{\mathrm{i}\lambda x}$. Let C_Λ be the space of functions spanned by $\{\mathrm{e}_\lambda : \lambda \in \Lambda\}$, endowed with the maximum modulus norm. Recall that a point P of a convex set K is *exposed* if there is a hyperplane that meets K only in P; P is *extreme* if it is not the midpoint of any two other points of K.

Theorem 1.2. *Let* K *be the unit ball of the space* C_Λ *and let* $P \in K$.

(a) *The point* P *is an exposed point of* K *if and only if* P *is either a trigonometric monomial* $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_\lambda$ *with* $\alpha \in \mathbb{R}$ *and* $\lambda \in \Lambda$ *or a trigonometric trinomial that attains its maximum modulus, 1, at two points modulo* $2\pi/d$. *Every linear functional on* C_Λ *attains its norm on an exposed point of* K.

(b) *The point* P *is an extreme point of* K *if and only if* P *is either a trigonometric monomial* $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_\lambda$ *with* $\alpha \in \mathbb{R}$ *and* $\lambda \in \Lambda$ *or a trigonometric trinomial such that* $1-|P|^2$ *has four zeroes modulo* $2\pi/d$*, counted with multiplicities.*

We describe the dependence of the maximum modulus of the trigonometric trinomial T on the arguments. The general issue has been studied for a long time; [52, 90] are two early references. In particular, the following problem has been addressed: see [26, page 2 and Supplement].

Extremal problem 1.3 (Complex Mandel′shtam problem)**.** To find the minimum of the maximum modulus of a trigonometric polynomial with given Fourier coefficient moduli.

It appeared originally in electrical circuit theory: "L. I. Mandel′shtam communicated to me a problem on the phase choice of electric currents with different frequencies such that the capacity of the resulting current to blow [the circuit] is minimal", tells N. G. Chebotarëv in [24, p. 396], where he discusses applications of a formula given in Section 9 that we would like to advertise.

Our main theorem solves an elementary case of the complex Mandel′shtam problem.

Theorem 1.4. *The maximum modulus of* T *as defined in* (H.1) *is a strictly decreasing function of* τ *as defined in Theorem 1.1. In particular,*

$$\min_{t_1,t_2,t_3} \max_x \left| r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)} \right| \\ = \max_x \left| \epsilon_1 r_1 \mathrm{e}^{\mathrm{i}\lambda_1 x} + \epsilon_2 r_2 \mathrm{e}^{\mathrm{i}\lambda_2 x} + \epsilon_3 r_3 \mathrm{e}^{\mathrm{i}\lambda_3 x} \right|$$

if ϵ_1, ϵ_2 *and* ϵ_3 *are real signs* $+1$ *or* -1 *such that* $\epsilon_i\epsilon_j = -1$, *where* i, j, k *is a permutation of* $1, 2, 3$ *such that the power of 2 in* $\lambda_i - \lambda_j$ *is greater than the power of 2 in* $\lambda_i - \lambda_k$ *and in* $\lambda_k - \lambda_j$.

This shows that the maximum modulus is minimal when the phases are chosen in opposition, independently of the intensities r_1, r_2 and r_3.

The decrease of the maximum modulus of (H.1) may be bounded as shown in the next result.

Theorem 1.5. *Let d and τ be defined as in Theorem 1.1. Suppose that λ_2 is between λ_1 and λ_3. The quotient of the maximum modulus of T by $\left|r_1 + r_2 \mathrm{e}^{\mathrm{i}\tau d/|\lambda_3-\lambda_1|} + r_3\right|$ is a strictly increasing function of τ unless $r_1 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_2 - \lambda_1|$, in which case it is constantly equal to 1.*

When $r_1 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_2 - \lambda_1|$, the hypotrochoid H with equation (H.2) is a *hypocycloid* with $|\lambda_3 - \lambda_1|/d$ cusps: the rolling point is *on* the rolling circle. Note that an epicycloid amounts also to an hypocycloid. Figure H.2 illustrates the particular case $T(x) = (1/3)\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}t} + (2/3)\mathrm{e}^{\mathrm{i}x}$.

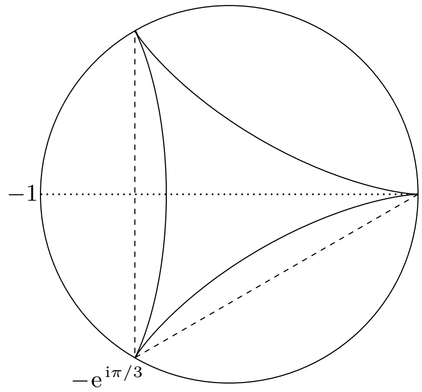


Figure H.2: The unit circle, the deltoid H with equation $z = (1/3)\mathrm{e}^{-\mathrm{i}2x} + (2/3)\mathrm{e}^{\mathrm{i}x}$, the segment from -1 to the unique point on H at maximum distance and the segments from $-\mathrm{e}^{\mathrm{i}\pi/3}$ to the two points on H at maximum distance.

We may deduce from Theorem 1.5 a less precise but handier inequality.

Theorem 1.6. *Let d and τ be defined as in Theorem 1.1. Let*

$$D = \frac{\max(|\lambda_2 - \lambda_1|, |\lambda_3 - \lambda_2|, |\lambda_3 - \lambda_1|)}{\gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_1)} \tag{H.4}$$

be the quotient of the diameter of Λ by d. Let t_1', t_2' and t_3' be another three real numbers and define correspondingly τ'. If $\tau > \tau'$, then

$$\max_x \left|r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)}\right| \geqslant \frac{\cos(\tau/2D)}{\cos(\tau'/2D)} \max_x \left|r_1 \mathrm{e}^{\mathrm{i}(t_1'+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2'+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3'+\lambda_3 x)}\right|$$

with equality if and only if $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$.

Figure H.3 illustrates the inequalities obtained in Theorems 1.5 and 1.6 for the particular case $T(x) = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}x}$, as in Figure H.1.

If we choose $\tau' = 0$ in Theorem 1.6, we get the solution to an elementary case of the following extremal problem.

Extremal problem 1.7. To find the minimum of the maximum modulus of a trigonometric polynomial with given spectrum, Fourier coefficient arguments and moduli sum.

Theorem 1.8. *Let τ be defined as in Theorem 1.1 and D be given by* (H.4)*. Then*

$$\frac{\max_x \left|r_1 \mathrm{e}^{\mathrm{i}(t_1+\lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2+\lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3+\lambda_3 x)}\right|}{r_1 + r_2 + r_3} \geqslant \cos(\tau/2D)$$

with equality if and only if $\tau = 0$ or $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$.

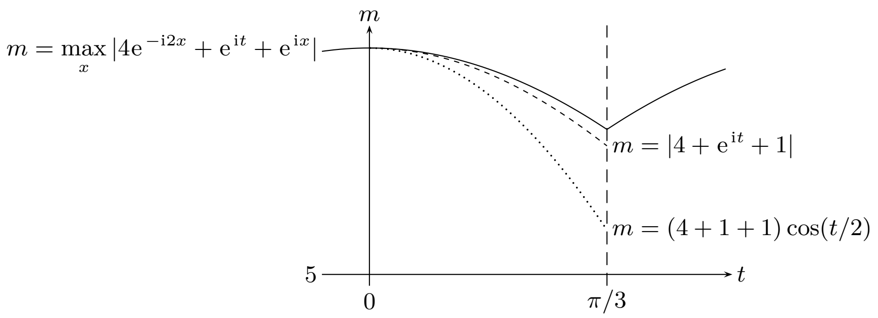


Figure H.3: Let H be the hypotrochoid with equation $z = 4\mathrm{e}^{-\mathrm{i}2x} + \mathrm{e}^{\mathrm{i}x}$. This plot shows the maximum distance m of points $z \in H$ to the point $-\mathrm{e}^{\mathrm{i}t}$ and the two estimates of this maximum distance provided by Theorems 1.5 and 1.6 for $t \in [0, \pi/3]$.

The dependence of the maximum modulus of (H.1) on the arguments may also be expressed as properties of relative multipliers. Given three real numbers t_1, t_2 and t_3, the linear operator on C_Λ defined by $\mathrm{e}_{\lambda_j} \mapsto \mathrm{e}^{\mathrm{i}t_j}\mathrm{e}_{\lambda_j}$ is a *unimodular relative Fourier multiplier*: it multiplies each Fourier coefficient of elements of C_Λ by a fixed unimodular number; let us denote it by (t_1, t_2, t_3). Consult [40] for general background on relative multipliers.

Theorem 1.9. *The unimodular relative Fourier multiplier* (t_1, t_2, t_3) *has norm*

$$\cos\bigl((\pi - \tau)/2D\bigr)\big/\cos(\pi/2D),$$

where τ *is defined as in Theorem 1.1 and* D *is given by* (H.4)*, and attains its norm exactly at functions of the form*

$$r_1\mathrm{e}^{\mathrm{i}(u_1+\lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(u_2+\lambda_2 x)} + r_3\mathrm{e}^{\mathrm{i}(u_3+\lambda_3 x)}$$

with $r_1 : r_2 : r_3 = |\lambda_3 - \lambda_2| : |\lambda_3 - \lambda_1| : |\lambda_2 - \lambda_1|$ *and*

$$\frac{\lambda_2 - \lambda_3}{d}u_1 + \frac{\lambda_3 - \lambda_1}{d}u_2 + \frac{\lambda_1 - \lambda_2}{d}u_3 = \pi \mod 2\pi.$$

The maximum of the norm of unimodular relative Fourier multipliers is the *complex unconditional constant* of the canonical basis $(\mathrm{e}_{\lambda_1}, \mathrm{e}_{\lambda_2}, \mathrm{e}_{\lambda_3})$ of C_Λ. As

$$r_1 + r_2 + r_3 = \max_x\bigl|r_1\mathrm{e}^{\mathrm{i}\lambda_1 x} + r_2\mathrm{e}^{\mathrm{i}\lambda_2 x} + r_3\mathrm{e}^{\mathrm{i}\lambda_3 x}\bigr|,$$

this constant is the minimal constant C such that

$$r_1 + r_2 + r_3 \leqslant C\max_x\bigl|r_1\mathrm{e}^{\mathrm{i}(u_1+\lambda_1 x)} + r_2\mathrm{e}^{\mathrm{i}(u_2+\lambda_2 x)} + r_3\mathrm{e}^{\mathrm{i}(u_3+\lambda_3 x)}\bigr|;$$

it is therefore the *Sidon constant* of Λ. It is also the solution to the following extremal problem.

Extremal problem 1.10 (Sidon constant problem)**.** To find the minimum of the maximum modulus of a trigonometric polynomial with given spectrum and Fourier coefficient moduli sum.

Setting $\tau = \pi$ in Theorem 1.9, we obtain the following result.

Corollary 1.11. *The Sidon constant of* Λ *is* $\sec(\pi/2D)$*, where* D *is given by* (H.4)*. It is attained exactly at functions of the form given in Theorem 1.9.*

Finally, we would like to stress that each of the above results gives rise to open questions if the set Λ is replaced by *any* set of four integers.

Let us now give a brief description of this article. In Sections 2 and 3, we use carefully the invariance of the maximum modulus under rotation, translation and conjugation to reduce the arguments t_1, t_2 and t_3 of the Fourier coefficients of the trigonometric trinomial T to the variable τ. Section 4 shows how to further reduce this study to the trigonometric trinomial

$$r_1\mathrm{e}^{-\mathrm{i}kx} + r_2\mathrm{e}^{\mathrm{i}t} + r_3\mathrm{e}^{\mathrm{i}lx} \tag{H.5}$$

with k and l positive coprime integers and $t \in [0, \pi/(k+l)]$. In Section 5, we prove that (H.5) attains its maximum modulus for $x \in [-t/k, t/l]$. Section 6 studies the variations of the modulus of (H.5) for $x \in [-t/k, t/l]$: it turns out that it attains its absolute maximum only once on that interval. This yields Theorem 1.1. Section 7 restates the results of the two previous sections for a general trigonometric trinomial T. Section 8 is dedicated to the proof of Theorem 1.2. In Section 9, we compute the directional derivative of the maximum modulus of (H.5) with respect to the argument t and prove Theorems 1.4, 1.5, 1.6 and 1.8. In Section 10, we prove Theorem 1.9 and show how to lift unimodular relative Fourier multipliers to operators of convolution with a linear combination of two Dirac measures. Section 11 replaces our computation of the Sidon constant in a general context; it describes the initial motivation for this research.

Part of these results appeared previously, with a different proof, in [64, Chapter II.10] and in [62].

Notation. Throughout this article, λ_1, λ_2 and λ_3 are three pairwise distinct integers, $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ and $d = \gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_2)$. If λ is an integer, e_λ is the function $x \mapsto \mathrm{e}^{\mathrm{i}\lambda x}$ of the real variable x. A *trigonometric polynomial* is a linear combination of functions e_λ; it is a *monomial*, *binomial* or *trinomial* if this linear combination has one, two or three nonzero coefficients, respectively. The normed space C_Λ is the three-dimensional space of complex functions spanned by e_λ with $\lambda \in \Lambda$, endowed with the maximum modulus norm. The Dirac measure δ_x is the linear functional $T \mapsto T(x)$ of evaluation at x on the space of continuous functions. Given three real numbers t_1, t_2 and t_3, the linear operator on C_Λ defined by $\mathrm{e}_{\lambda_j} \mapsto \mathrm{e}^{\mathrm{i}t_j}\mathrm{e}_{\lambda_j}$ is a *unimodular relative Fourier multiplier* denoted by (t_1, t_2, t_3).

2 Isometric relative Fourier multipliers

The rôle of Quantity (H.3) is explained by the following lemma.

Lemma 2.1. *Let t_1, t_2 and t_3 be real numbers. The unimodular relative Fourier multiplier $M = (t_1, t_2, t_3)$ is an isometry on C_Λ if and only if*

$$\frac{\lambda_2 - \lambda_3}{d} t_1 + \frac{\lambda_3 - \lambda_1}{d} t_2 + \frac{\lambda_1 - \lambda_2}{d} t_3 \in 2\pi\mathbb{Z}. \tag{H.6}$$

Then it is a unimodular multiple of a translation: there are real numbers α and v such that $Mf(x) = \mathrm{e}^{\mathrm{i}\alpha} f(x - v)$ for all $f \in \mathrm{C}_\Lambda$ and all $x \in \mathbb{R}$.

Proof. If M is a unimodular multiple of a translation by a real number v, then

$$\left| r_1 \mathrm{e}^{\mathrm{i}(t_1 + \lambda_1 v)} + r_2 \mathrm{e}^{\mathrm{i}(t_2 + \lambda_2 v)} + r_3 \mathrm{e}^{\mathrm{i}(t_3 + \lambda_3 v)} \right| = r_1 + r_2 + r_3,$$

which holds if and only if

$$t_1 + \lambda_1 v = t_2 + \lambda_2 v = t_3 + \lambda_3 v \quad \text{modulo } 2\pi. \tag{H.7}$$

There is a v satisfying (H.7) if and only if Equation (H.6) holds as (H.7) means that there exist integers a_1 and a_3 such that

$$v = \frac{t_2 - t_1 + 2\pi a_1}{\lambda_1 - \lambda_2} = \frac{t_2 - t_3 + 2\pi a_3}{\lambda_3 - \lambda_2}.$$

If t_1, t_2 and t_3 are three real numbers satisfying (H.6), let v be such that (H.7) holds. Then

$$\begin{aligned} &r_1 \mathrm{e}^{\mathrm{i}(t_1 + u_1 + \lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2 + u_2 + \lambda_2 x)} + r_3 \mathrm{e}^{\mathrm{i}(t_3 + u_3 + \lambda_3 x)} \\ &\qquad = \mathrm{e}^{\mathrm{i}(t_2 + \lambda_2 v)} \big(r_1 \mathrm{e}^{\mathrm{i}(u_1 + \lambda_1 (x - v))} + r_2 \mathrm{e}^{\mathrm{i}(u_2 + \lambda_2 (x - v))} + r_3 \mathrm{e}^{\mathrm{i}(u_3 + \lambda_3 (x - v))} \big) \end{aligned}$$

for all real numbers u_1, u_2, u_3 and x. □

3 The arguments of the Fourier coefficients of a trigonometric trinomial

We have used a translation and a rotation to reduce the three arguments of the Fourier coefficients of a trigonometric trinomial to just one variable. Use of the involution $\overline{f(-x)}$ of C_Λ allows us to restrict even further the domain of that variable.

Lemma 3.1. *Let t_1, t_2 and t_3 be real numbers and let $\tilde t_2$ be the representative of*

$$\frac{\lambda_2-\lambda_3}{\lambda_3-\lambda_1}t_1+t_2+\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_1}t_3 \tag{H.8}$$

modulo $2\pi/|\lambda_3-\lambda_1|$ in $\bigl[-\pi d/|\lambda_3-\lambda_1|,\pi d/|\lambda_3-\lambda_1|\bigl[$.

(*a*) *There are real numbers α and v such that*

$$r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1x)}+r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3x)}=\mathrm{e}^{\mathrm{i}\alpha}\bigl(r_1\mathrm{e}^{\mathrm{i}\lambda_1(x-v)}+r_2\mathrm{e}^{\mathrm{i}(\tilde t_2+\lambda_2(x-v))}+r_3\mathrm{e}^{\mathrm{i}\lambda_3(x-v))}\bigr) \tag{H.9}$$

for all x.

(*b*) *Let $t=|\tilde t_2|$ be the distance of* (H.8) *to $(2\pi d/|\lambda_3-\lambda_1|)\mathbb{Z}$. There is a sign $\varepsilon\in\{+1,-1\}$ such that*

$$\bigl|r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1x)}+r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3x)}\bigr|=\bigl|r_1\mathrm{e}^{\mathrm{i}\lambda_1\varepsilon(x-v)}+r_2\mathrm{e}^{\mathrm{i}(t+\lambda_2\varepsilon(x-v))}+r_3\mathrm{e}^{\mathrm{i}\lambda_3\varepsilon(x-v))}\bigr|$$

for all x.

Proof. (*a*). The argument $\tilde t_2$ is chosen so that the relative multiplier $(t_1,t_2-\tilde t_2,t_3)$ is an isometry.
(*b*). If $\tilde t_2$ is negative, take the conjugate under the modulus of the right hand side in (H.9). ☐

Remark 3.2. This proves the following periodicity formula:

$$\bigl|r_1\mathrm{e}^{\mathrm{i}\lambda_1x}+r_2\mathrm{e}^{\mathrm{i}(t+2\pi d/(\lambda_3-\lambda_1)+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}\lambda_3x}\bigr|=\bigl|r_1\mathrm{e}^{\mathrm{i}\lambda_1(x-v)}+r_2\mathrm{e}^{\mathrm{i}(t+\lambda_2(x-v))}+r_3\mathrm{e}^{\mathrm{i}\lambda_3(x-v)}\bigr|$$

for all x and t, where v satisfies $\lambda_1v=2\pi d/(\lambda_3-\lambda_1)+\lambda_2v=\lambda_3v$ modulo 2π, that is

$$v=\frac{2m\pi}{\lambda_3-\lambda_1}\quad\text{with } m \text{ an inverse of } \frac{\lambda_3-\lambda_2}{d} \text{ modulo } \frac{\lambda_3-\lambda_1}{d}.$$

4 The frequencies of a trigonometric trinomial

We may suppose without loss of generality that $\lambda_1<\lambda_2<\lambda_3$. Let $k=(\lambda_2-\lambda_1)/d$ and $l=(\lambda_3-\lambda_2)/d$. Then

$$r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1x)}+r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3x)}=\mathrm{e}^{\mathrm{i}\lambda_2x}\bigl(r_1\mathrm{e}^{\mathrm{i}(t_1-k(dx))}+r_2\mathrm{e}^{\mathrm{i}t_2}+r_3\mathrm{e}^{\mathrm{i}(t_3+l(dx))}\bigr)$$

This defines an isometry between C_Λ and $\mathrm{C}_{\{-k,0,l\}}$ and shows that C_Λ is normed by the maximum modulus norm on $[0,2\pi/d[$. With Lemma 3.1 (*b*), this shows that a homothety by d^{-1} allows us to restrict our study to the function

$$f(t,x)=\bigl|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\bigr|^2$$

for $x\in\mathbb{R}$ with k and l two positive coprime numbers and $t\in\bigl[0,\pi/(k+l)\bigr]$. We have

$$f(-t,x)=f(t,-x) \tag{H.10}$$

and Remark 3.2 shows that

$$f\bigl(t+2\pi/(k+l),x\bigr)=f(t,x-2m\pi/(k+l)) \tag{H.11}$$

for all x and t, where m is the inverse of l modulo $k+l$. In particular, if $t=\pi/(k+l)$, we have the symmetry relation

$$f\bigl(\pi/(k+l),x\bigr)=f\bigl(\pi/(k+l),2m\pi/(k+l)-x\bigr). \tag{H.12}$$

5 Location of the maximum point

The purpose of our first proposition is to deduce the existence of a small interval on which a trigonometric trinomial attains its maximum modulus. Note that a trigonometric binomial attains its maximum modulus at a point that depends only on the phase of its coefficients:

– $\bigl|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}\bigr|$ attains its maximum at $-t/k$ independently of r_1 and r_2,

– $\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_3\mathrm{e}^{\mathrm{i}lx}\right|$ attains its maximum at 0 independently of r_1 and r_3,
– $\left|r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|$ attains its maximum at t/l independently of r_2 and r_3.

The next proposition shows that if the point at which a trigonometric trinomial attains its maximum modulus changes with the modulus of its coefficients, it changes very little; we get bounds for this point that are independent of the intensities.

Proposition 5.1. *Let k and l be positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Let $t\in\bigl[0,\pi/(k+l)\bigr]$. Let*

$$f(x)=\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|^2$$

for $x\in\mathbb{R}$.

(*a*) *The function f attains its absolute maximum in the interval $[-t/k,t/l]$.*

(*b*) *If f attains its absolute maximum at a point y outside of $[-t/k,t/l]$ modulo 2π, then $t=\pi/(k+l)$ and $2m\pi/(k+l)-y$ lies in $[-t/k,t/l]$ modulo 2π, where m is the inverse of l modulo $k+l$.*

Proof. (*a*). We have

$$f(x)=r_1^2+r_2^2+r_3^2+2\cdot\Bigl(r_1r_2\cos(t+kx)+r_1r_3\cos\bigl((k+l)x\bigr)+r_2r_3\cos(t-lx)\Bigr).\tag{H.13}$$

Let us prove that f attains its absolute maximum on $[-t/k,t/l]$. Let y be outside of $[-t/k,t/l]$ modulo 2π. Let I be the set of all $x\in[-t/k,t/l]$ such that

$$\begin{cases}\cos(t+kx)\geqslant\cos(t+ky)\\ \cos\bigl((k+l)x\bigr)\geqslant\cos\bigl((k+l)y\bigr)\\ \cos(t-lx)\geqslant\cos(t-ly).\end{cases}$$

Note that if $x\in[-t/k,t/l]$, then

$$\begin{cases}t+kx\in[0,(k+l)t/l]\\ (k+l)x\in[-(k+l)t/k,(k+l)t/l]\\ t-lx\in[0,(k+l)t/k],\end{cases}$$

and that $(k+l)t/k$, $(k+l)t/l\in[0,\pi]$. Let

– α be the distance of $t/k+y$ to $(2\pi/k)\mathbb{Z}$,
– β be the distance of y to $\bigl(2\pi/(k+l)\bigr)\mathbb{Z}$,
– γ be the distance of $t/l-y$ to $(2\pi/l)\mathbb{Z}$.

Then

$$I=[-t/k,t/l]\cap[-t/k-\alpha,-t/k+\alpha]\cap[-\beta,\beta]\cap[t/l-\gamma,t/l+\gamma].\tag{H.14}$$

Let us check that I is the nonempty interval

$$I=\bigl[\max(-t/k,-\beta,t/l-\gamma),\min(t/l,-t/k+\alpha,\beta)\bigr].\tag{H.15}$$

In fact, we have the following triangular inequalities:

– $-\beta\leqslant-t/k+\alpha$ because t/k is the distance of $(t/k+y)-y$ to $(2\pi/k(k+l))\mathbb{Z}$;
– $t/l-\gamma\leqslant-t/k+\alpha$ because $t/l+t/k$ is the distance of $(t/k+y)+(t/l-y)$ to $(2\pi/kl)\mathbb{Z}$;
– $t/l-\gamma\leqslant\beta$ because t/l is the distance of $(t/l-y)+y$ to $(2\pi/l(k+l))\mathbb{Z}$.

The other six inequalities that are necessary to deduce (H.15) from (H.14) are obvious.

(*b*). We have proved in (*a*) that there is an $x\in[-t/k,t/l]$ such that $\cos(t+kx)\geqslant\cos(t+ky)$, $\cos\bigl((k+l)x\bigr)\geqslant\cos\bigl((k+l)y\bigr)$ and $\cos(t-lx)\geqslant\cos(t-ly)$. In fact, at least one of these inequalities is strict unless there are signs $\delta,\varepsilon,\eta\in\{-1,1\}$ such that $t+kx=\delta(t+ky)$, $t-lx=\varepsilon(t-ly)$ and $(k+l)x=\eta(k+l)y$ modulo 2π. Two out of these three signs are equal and the corresponding two equations imply the third one with the same sign. This system is therefore equivalent to

$$\begin{cases}k(x-y)=0\\ l(x-y)=0\end{cases}\quad\text{or}\quad\begin{cases}k(x+y)=-2t\\ l(x+y)=2t\end{cases}$$

modulo 2π. The first pair of equations yields $x=y$ modulo 2π because k and l are coprime. Let m be an inverse of l modulo $k+l$; then the second pair of equations is equivalent to

$$\begin{cases} 2(k+l)t = 0 \\ \quad x+y = 2mt \end{cases}$$

modulo 2π. Therefore g does not attain its absolute maximum at y unless $t=\pi/(k+l)$ and $2m\pi/(k+l)-y\in[-t/k,t/l]$. ☐

Remark 5.2. This proposition is a complex counterpart to Lemma 2.1. *i*) in [83], where cosine trinomials are investigated.

6 Uniqueness of the maximum point

Note that

$$\begin{aligned} r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx} &= r_3\mathrm{e}^{-\mathrm{i}l(-x)}+r_2\mathrm{e}^{\mathrm{i}t}+r_1\mathrm{e}^{\mathrm{i}k(-x)} \\ &= r_1'\mathrm{e}^{-\mathrm{i}k'x'}+r_2\mathrm{e}^{\mathrm{i}t}+r_3'\mathrm{e}^{\mathrm{i}l'x'} \end{aligned}$$

with $r_1'=r_3$, $r_3'=r_1$, $k'=l$, $l'=k$ and $x'=-x$. We may therefore suppose without loss of generality that $kr_1\leqslant lr_3$.

Our second proposition studies the points at which a trigonometric trinomial attains its maximum modulus. Note that if $k=l=1$, the derivative of $|f|^2$ has at most 4 zeroes, so that the modulus of f has at most two maxima and attains its absolute maximum in at most two points. Proposition 6.1 shows that this is true in general, and that if it may attain its absolute maximum in two points, it is so only because of the symmetry given by (H.12).

Proposition 6.1. *Let k and l be positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers such that $kr_1\leqslant lr_3$. Let $t\in\,]0,\pi/(k+l)]$. Let*

$$f(x)=\left|r_1\mathrm{e}^{-\mathrm{i}kx}+r_2\mathrm{e}^{\mathrm{i}t}+r_3\mathrm{e}^{\mathrm{i}lx}\right|^2$$

for $x\in[-t/k,t/l]$.

(*a*) *There is a point $x^*\in[0,t/l]$ such that $\mathrm{d}f/\mathrm{d}x>0$ on $]-t/k,x^*[$ and $\mathrm{d}f/\mathrm{d}x<0$ on $]x^*,t/l[$.*

(*b*) *There are three cases:*

1. *f attains its absolute maximum at 0 if and only if $kr_1=lr_3$;*
2. *f attains its absolute maximum at t/l if and only if $l=1$, $t=\pi/(k+1)$ and $k^2r_1r_2+(k+1)^2r_1r_3-r_2r_3\leqslant 0$;*
3. *otherwise, f attains its absolute maximum in $]0,t/l[$.*

(*c*) *The function f attains its absolute maximum with multiplicity 2 unless $l=1$, $t=\pi/(k+1)$ and $k^2r_1r_2+(k+1)^2r_1r_3-r_2r_3=0$, in which case it attains its absolute maximum at $\pi/(k+1)$ with multiplicity 4.*

Proof. (*a*). By Proposition 5.1, the derivative of f has a zero in $[-t/k,t/l]$. Let us study the sign of this derivative. Equation (H.13) yields

$$\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}(x)=-kr_1r_2\sin(t+kx)-(k+l)r_1r_3\sin\bigl((k+l)x\bigr)+lr_2r_3\sin(t-lx). \tag{H.16}$$

We wish to compare $\sin(t+kx)$ with $\sin(t-lx)$: note that

$$\sin(t+kx)-\sin(t-lx)=2\sin\bigl((k+l)x/2\bigr)\cos\bigl(t+(k-l)x/2\bigr),$$

and that if $x\in[-t/k,t/l]$, then

$$\begin{gathered} -\pi/2\leqslant-\pi/2k\leqslant-(k+l)t/2k\leqslant(k+l)x/2\leqslant(k+l)t/2l\leqslant\pi/2l\leqslant\pi/2 \\ 0\leqslant t+(k-l)x/2\leqslant\begin{cases} t+(l-k)t/2k=(k+l)t/2k & \text{if } k\leqslant l \\ t+(k-l)t/2l=(k+l)t/2l & \text{if } l\leqslant k \end{cases}\leqslant\pi/2. \end{gathered}$$

Suppose that $x \in [-t/k, 0[$: then it follows that $\sin(t+kx) \leqslant \sin(t-lx)$ and also $\sin\big((k+l)x\big) \leqslant 0$, with equality if and only if $k = 1$ and $-x = t = \pi/(1+l)$. This yields with $kr_1 \leqslant lr_3$ that

$$\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}(x) \geqslant -(k+l)r_1r_3 \sin\big((k+l)x\big) \geqslant 0 \tag{H.17}$$

with equality if and only if $k = 1$ and $-x = t = \pi/(1+l)$.

Suppose that $x \in [0, t/l]$. If $l \geqslant 2$, then

$$\begin{cases} t + kx \in [t, (k+l)t/l] \subset [t, \pi/2] \\ (k+l)x \in [0, (k+l)t/l] \subset [t, \pi/2] \\ t - lx \in [0, t] \subset [0, \pi/3], \end{cases}$$

so that the second derivative of f is strictly negative on $[0, t/l]$: its derivative is strictly decreasing on this interval and (a) is proved. If $l = 1$, let $g(x) = f(t-x)$ for $x \in [0, t]$: we have to prove that there is a point x^* such that $\mathrm{d}g/\mathrm{d}x > 0$ on $]0, x^*[$ and $\mathrm{d}g/\mathrm{d}x < 0$ on $]x^*, t[$. We already know that $(\mathrm{d}g/\mathrm{d}x)(0) \geqslant 0$ and that $\mathrm{d}g/\mathrm{d}x$ has a zero on $[0, t]$. Put $\alpha = (k+1)t$: then

$$\frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}x}(x) = kr_1r_2 \sin(\alpha - kx) + (k+1)r_1r_3 \sin\big(\alpha - (k+1)x\big) - r_2r_3 \sin x$$

and it suffices to prove that

$$\frac{1}{2\sin x}\frac{\mathrm{d}g}{\mathrm{d}x}(x) = kr_1r_2 \frac{\sin(\alpha - kx)}{\sin x} + (k+1)r_1r_3 \frac{\sin\big(\alpha - (k+1)x\big)}{\sin x} - r_2r_3 \tag{H.18}$$

is a strictly decreasing function of x on $]0, \alpha/(k+1)]$. Let us study the sign of

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin(\alpha - kx)}{\sin x} = \frac{-k\cos(\alpha - kx)\sin x - \sin(\alpha - kx)\cos x}{\sin^2 x}$$

for $\alpha \in]0, \pi]$ and $x \in]0, \alpha/k]$. If $k = 1$, then

$$-k\cos(\alpha - kx)\sin x - \sin(\alpha - kx)\cos x = -\sin\alpha \leqslant 0$$

and the inequality is strict unless $\alpha = \pi$. Let us prove by induction on k that

$$k\cos(\alpha - kx)\sin x + \sin(\alpha - kx)\cos x > 0$$

for all $k \geqslant 2$, $\alpha \in]0, \pi]$ and $x \in]0, \alpha/k]$. This will complete the proof of (a). Let $k \geqslant 1$ and $x \in]0, \alpha/(k+1)]$. Then

$$\begin{aligned} &(k+1)\cos\big(\alpha - (k+1)x\big)\sin x + \sin\big(\alpha - (k+1)x\big)\cos x \\ &\qquad = (k+1)\cos(\alpha - kx)\cos x \sin x + (k+1)\sin(\alpha - kx)\sin^2 x \\ &\qquad\qquad + \sin(\alpha - kx)\cos^2 x - \cos(\alpha - kx)\sin x \cos x \\ &\qquad = \big(k\cos(\alpha - kx)\sin x + \sin(\alpha - kx)\cos x\big)\cos x \\ &\qquad\qquad + (k+1)\sin(\alpha - kx)\sin^2 x \\ &\qquad \geqslant (k+1)\sin(\alpha - kx)\sin^2 x > 0 \end{aligned}$$

(b). 1. By Proposition 5.1 and (a), f attains its absolute maximum at 0 if and only if 0 is a critical point for f. We have

$$\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}(0) = (lr_3 - kr_1)r_2 \sin t \geqslant 0$$

and equality holds if and only if $kr_1 = lr_3$.

2. We have

$$\frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x}(t/l) = \big(-kr_1r_2 - (k+l)r_1r_3\big)\sin\big((k+l)t/l\big) \leqslant 0$$

and equality holds if and only if $l = 1$ and $t = \pi/(k+1)$. Let $l = 1$ and $t = \pi/(k+1)$ and let us use the notation introduced in the last part of the proof of (a): we need to characterise the case that g has a maximum at 0. As $\alpha = \pi$ and

$$\frac{1}{2\sin x}\frac{\mathrm{d}g}{\mathrm{d}x}(x) = k^2r_1r_2 + (k+1)^2r_1r_3 - r_2r_3 + o(x) \tag{H.19}$$

is a strictly decreasing function of x on $]0,\pi/(k+1)]$, g has a maximum at 0 if and only if $k^2r_1r_2+(k+1)^2r_1r_3-r_2r_3\leqslant 0$.

(c). If $l\geqslant 2$, then the second derivative of f is strictly negative on $[0,t/l]$. If $l=1$, then the derivative of (H.18) is strictly negative on $]0,\alpha/(k+1)]$: this yields that the second derivative of g can only vanish at 0. By (b) 2., g has a maximum at 0 only if $t=\pi/(k+1)$; then

$$\frac{1}{2}\frac{\mathrm{d}^2g}{\mathrm{d}x^2}(0)=k^2r_1r_2+(k+1)^2r_1r_3-r_2r_3 \tag{H.20}$$

$$\frac{1}{2}\frac{\mathrm{d}^4g}{\mathrm{d}x^4}(0)=-k^4r_1r_2-(k+1)^4r_1r_3+r_2r_3 \tag{H.21}$$

If (H.20) vanishes, then the sum of (H.21) with (H.20) yields

$$\frac{1}{2}\frac{\mathrm{d}^4g}{\mathrm{d}x^4}(0)=-k(k+1)r_1\big((k-1)kr_2+(k+1)(k+2)r_3\big)<0. \qquad \square$$

Remark 6.2. We were able to prove directly that the system

$$\begin{cases} f(x)=f(y)\\ \dfrac{\mathrm{d}f}{\mathrm{d}x}(x)=\dfrac{\mathrm{d}f}{\mathrm{d}x}(y)=0\\ \dfrac{\mathrm{d}^2f}{\mathrm{d}x^2}(x),\ \dfrac{\mathrm{d}^2f}{\mathrm{d}x^2}(y)\leqslant 0\end{cases}$$

implies $x=y$ modulo 2π or $t=\pi/(k+l)$ and $x+y=2m\pi/(k+l)$, but our computations are very involved and opaque.

Remark 6.3. This proposition is a complex counterpart to [83, Lemma 2.1. ii)].

Remark 6.4. Suppose that $l=k=1$. If $t\in]0,\pi/2[$, it is necessary to solve a generally irreducible quartic equation in order to compute the maximum of f. If $t=\pi/2$, it suffices to solve a linear equation and one gets the following expression for $\max_x\left|r_1\mathrm{e}^{-\mathrm{i}x}+\mathrm{i}r_2+r_3\mathrm{e}^{\mathrm{i}x}\right|$:

$$\begin{cases}(r_1+r_3)\sqrt{1+r_2^2/4r_1r_3} & \text{if } \left|r_1^{-1}-r_3^{-1}\right|<4r_2^{-1}\\ r_2+|r_3-r_1| & \text{otherwise.}\end{cases}$$

This formula appears in [2, (3.1)]. In the first case, the maximum is attained at the two points x^* such that $\sin x^*=r_2(r_3-r_1)/4r_1r_3$.

Remark 6.5. Suppose that $l=1$ and $k=2$. If $t\in]0,\pi/3[$, it is necessary to solve a generally irreducible sextic equation in order to compute the maximum of f. If $t=\pi/3$, it suffices to solve a quadratic equation and one gets the following expression for $\max_x\left|r_1\mathrm{e}^{-\mathrm{i}2x}+r_2\mathrm{e}^{\mathrm{i}\pi/3}+r_3\mathrm{e}^{\mathrm{i}x}\right|$: if $r_1^{-1}-4r_3^{-1}<9r_2^{-1}$, then its square makes

$$r_1^2+\frac{2}{3}r_2^2+r_3^2+r_1r_2+2r_1r_3\left[\left(\left(\frac{r_2}{3r_3}\right)^2+\frac{r_2}{3r_1}+1\right)^{3/2}-\left(\frac{r_2}{3r_3}\right)^3\right]$$

and the maximum is attained at the two points x^* such that

$$2\cos(\pi/3-x^*)=\left(\left(\frac{r_2}{3r_3}\right)^2+\frac{r_2}{3r_1}+1\right)^{1/2}-\frac{r_2}{3r_3};$$

otherwise, it makes $-r_1+r_2+r_3$.

7 The maximum modulus points of a trigonometric trinomial

If we undo all the reductions made in Sections 3 and 4 and at the beginning of Section 6, we get the following theorem.

Theorem 7.1. *Let λ_1, λ_2 and λ_3 be three pairwise distinct integers such that λ_2 is between λ_1 and λ_3. Let r_1, r_2 and r_3 be three positive real numbers. Given three real numbers t_1, t_2 and t_3, consider the trigonometric trinomial*

$$T(x)=r_1\mathrm{e}^{\mathrm{i}(t_1+\lambda_1x)}+r_2\mathrm{e}^{\mathrm{i}(t_2+\lambda_2x)}+r_3\mathrm{e}^{\mathrm{i}(t_3+\lambda_3x)}$$

for $x \in \mathbb{R}$. Let $d = \gcd(\lambda_2 - \lambda_1, \lambda_3 - \lambda_2)$ and choose integers a_1 and a_3 such that

$$\tau = \frac{\lambda_2 - \lambda_3}{d}(t_1 - 2\pi a_1) + \frac{\lambda_3 - \lambda_1}{d} t_2 + \frac{\lambda_1 - \lambda_2}{d}(t_3 - 2\pi a_3),$$

satisfies $|\tau| \leqslant \pi$. Let $\tilde{t}_1 = t_1 - 2\pi a_1$ and $\tilde{t}_3 = t_3 - 2\pi a_3$.

(*a*) *The trigonometric trinomial T attains its maximum modulus at a unique point of the interval bounded by $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$ and $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$. More precisely,*
   - *if $r_1|\lambda_2 - \lambda_1| \leqslant r_3|\lambda_3 - \lambda_2|$, then this point is between $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$ and $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$;*
   - *if $r_1|\lambda_2 - \lambda_1| \geqslant r_3|\lambda_3 - \lambda_2|$, then this point is between $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$ and $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$;*
   - *T attains its maximum modulus at $(\tilde{t}_1 - \tilde{t}_3)/(\lambda_3 - \lambda_1)$ if and only if $r_1|\lambda_2 - \lambda_1| = r_3|\lambda_3 - \lambda_2|$ or $\tau = 0$.*

(*b*) *The function T attains its maximum modulus at a unique point modulo $2\pi/d$, and with multiplicity 2, unless $|\tau| = \pi$.*

(*c*) *Suppose that $|\tau| = \pi$, i.e.,*

$$\frac{\lambda_2 - \lambda_3}{d} t_1 + \frac{\lambda_3 - \lambda_1}{d} t_2 + \frac{\lambda_1 - \lambda_2}{d} t_3 = \pi \mod 2\pi. \tag{H.22}$$

Let s be a solution to $2t_1 + \lambda_1 s = 2t_2 + \lambda_2 s = 2t_3 + \lambda_3 s$ modulo 2π: s is unique modulo $2\pi/d$. Then $T(s - x) = \mathrm{e}^{\mathrm{i}(2t_2 + \lambda_2 s)}\overline{T(x)}$ for all x. Suppose that $|\lambda_3 - \lambda_2| \leqslant |\lambda_2 - \lambda_1|$. There are three cases.

1. *If $\lambda_2 - \lambda_1 = k(\lambda_3 - \lambda_2)$ with $k \geqslant 2$ integer and*
$$r_1^{-1} - k^2 r_3^{-1} \geqslant (k+1)^2 r_2^{-1},$$
then T attains its maximum modulus, $-r_1 + r_2 + r_3$, only at $x = (t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$ modulo $2\pi/d$, with multiplicity 2 if the inequality is strict and with multiplicity 4 if there is equality;
2. *if $\lambda_2 - \lambda_1 = \lambda_3 - \lambda_2$ and*
$$\left|r_1^{-1} - r_3^{-1}\right| \geqslant 4 r_2^{-1},$$
then T attains its maximum modulus, $r_2 + |r_3 - r_1|$, at a unique point x modulo $2\pi/d$, with multiplicity 2 if the inequality is strict and with multiplicity 4 if there is equality. This point is $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$ if $r_1 < r_3$, and $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$ if $r_3 < r_1$;
3. *otherwise T attains its maximum modulus at exactly two points x and y modulo $2\pi/d$, with multiplicity 2, where x is strictly between $(\tilde{t}_1 - t_2)/(\lambda_2 - \lambda_1)$ and $(t_2 - \tilde{t}_3)/(\lambda_3 - \lambda_2)$, and $x + y = s$ modulo $2\pi/d$.*

Note that $s - x = x$ modulo $2\pi/d$ in Cases 1 and 2.

8 Exposed and extreme points of the unit ball of C_Λ

The characterisation of the maximum modulus points of a trigonometric trinomial enables us to compute the exposed and the extreme points of the unit ball of C_Λ. We begin with a lemma.

Lemma 8.1. (*a*) *A trigonometric trinomial with a given spectrum that attains its maximum modulus at two given points modulo $2\pi/d$ is determined by its value at these points.*

(*b*) *The trigonometric trinomials with a given spectrum that attain their maximum modulus with multiplicity 4 at a given point and have a given value at this point lie on a parabola.*

Proof. We will use the notation of Theorem 7.1. Without loss of generality, we may suppose that $\lambda_1 = -k$, $\lambda_2 = 0$ and $\lambda_3 = l$ with k and l positive coprime integers. Let x and y be two real numbers that are different modulo $2\pi/d$, let ϑ and ζ be real numbers and let ϱ be a positive real number.

(*a*). Let us prove that at most one trigonometric trinomial T attains its maximum modulus at x and y and satisfies $T(x) = \varrho\mathrm{e}^{\mathrm{i}\vartheta}$ and $T(y) = \varrho\mathrm{e}^{\mathrm{i}\zeta}$. Let us translate T by $(x+y)/2$: we may suppose that $x + y = 0$. Let us divide T by $\mathrm{e}^{\mathrm{i}(\vartheta+\zeta)/2}$: we may suppose that $\vartheta + \zeta = 0$. As T attains its maximum modulus at the two points x and y, we have $s = x + y = 0$ and

$2t_1 - ks = 2t_2 = 2t_3 + ls = \vartheta + \zeta = 0$ modulo 2π. Therefore $t_1 = t_2 = t_3 = 0$ modulo π. Let $p_j = \mathrm{e}^{\mathrm{i}t_j} r_j$: the p_j are nonzero real numbers. We have

$$T(x) = p_1 \mathrm{e}^{-\mathrm{i}kx} + p_2 + p_3 \mathrm{e}^{\mathrm{i}lx} = \varrho \mathrm{e}^{\mathrm{i}\vartheta},$$

so that, multiplying by $\mathrm{e}^{-\mathrm{i}\vartheta}$ and taking real and imaginary parts,

$$p_1 \cos(\vartheta + kx) + p_3 \cos(\vartheta - lx) = \varrho - p_2 \cos\vartheta \tag{H.23}$$

$$p_1 \sin(\vartheta + kx) + p_3 \sin(\vartheta - lx) = -p_2 \sin\vartheta. \tag{H.24}$$

The computation

$$\frac{1}{2}\frac{\mathrm{d}|T|^2}{\mathrm{d}x}(x) = \Re\Bigl(\overline{T(x)}\frac{\mathrm{d}T}{\mathrm{d}x}(x)\Bigr) = \Re\Bigl(\overline{T(x)}\bigl(-\mathrm{i}kp_1\mathrm{e}^{-\mathrm{i}kx} + \mathrm{i}lp_3\mathrm{e}^{\mathrm{i}lx}\bigr)\Bigr),$$

yields

$$kp_1 \sin(\vartheta + kx) - lp_3 \sin(\vartheta - lx) = 0. \tag{H.25}$$

Equations (H.24) and (H.25) yield p_1 and p_3 as linear functions of p_2 because $\sin(\vartheta+kx)\sin(\vartheta-lx) \neq 0$: otherwise both factors would vanish, so that $\vartheta = x = 0$ modulo π and $x = y$ modulo 2π. As $\varrho \neq 0$, Equation (H.23) has at most one solution in p_2.

(*b*). We are necessarily in Case 1 or 2 of Theorem 7.1(*c*), so that we may suppose that $l = 1$. Let us determine all trigonometric trinomials T that attain their maximum modulus at x with multiplicity 4 and satisfy $T(x) = \varrho\mathrm{e}^{\mathrm{i}\vartheta}$. Let us translate T by x: we may suppose that $x = 0$. Let us divide T by $\mathrm{e}^{\mathrm{i}\vartheta}$: we may suppose that $\vartheta = 0$. As T attains its maximum modulus at 0 with multiplicity 4, we have $s - 0 = 0$ and $2t_1 - ks = 2t_2 = 2t_3 + s = 2\vartheta = 0$ modulo 2π. Therefore $t_1 = t_2 = t_3 = 0$ modulo π. Let $p_j = \mathrm{e}^{\mathrm{i}t_j} r_j$: the p_j are nonzero real numbers and satisfy the system

$$\begin{cases} p_1 + p_2 + p_3 = \varrho \\ k^2 p_1 p_2 + (k+1)^2 p_1 p_3 + p_2 p_3 = 0, \end{cases}$$

that is

$$\begin{cases} p_2 = \varrho - p_1 - p_3 \\ (kp_1 - p_3)^2 = \varrho(k^2 p_1 + p_3). \end{cases}$$

This is the equation of a parabola. □

Remark 8.2. The equality

$$\max_x \bigl| r_1 \mathrm{e}^{\mathrm{i}(t_1 + \lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2 + \lambda_2 x)} \bigr| = r_1 + r_2 \tag{H.26}$$

shows that the exposed points of the unit ball of the space $\mathrm{C}_{\{\lambda_1,\lambda_2\}}$ are the trigonometric monomials $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_{\lambda_1}$ and $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_{\lambda_2}$ with $\alpha \in \mathbb{R}$ and that no trigonometric binomial is an extreme point of the unit ball of C_Λ.

Proof of Theorem 1.2 (*a*). *Necessity.* Let $f \in K$. Let us make four remarks:

- if f is an exposed point, then $\|f\| = 1$;
- if l is a nonzero linear functional on C_Λ and z is a complex number such that $l^{-1}(z) \cap K = \{f\}$, then $|l(f)| = |z| = \|l\|$, so that l attains its norm at f;
- f is exposed if and only if $\|f\| = 1$ and there is a nonzero linear functional l on C_Λ that attains its norm only at multiples of f;
- if l is a linear functional on C_Λ that attains its norm at f and μ is the measure on $[0, 2\pi/d[$ identified with the Hahn-Banach extension of l to the space of continuous functions on $[0, 2\pi/d[$, then the support of μ must be a subset of the maximum modulus points of f on $[0, 2\pi/d[$.

Theorem 7.1 (*b*, *c*) tells that a trigonometric trinomial attains its maximum modulus at one or two points modulo $2\pi/d$. We have therefore to show that trigonometric binomials and trigonometric trinomials with only one maximum modulus point modulo $2\pi/d$ are not exposed.

- A linear functional that attains its modulus at a trigonometric binomial attains its norm at a trigonometric monomial because this trigonometric binomial is a convex combination of two trigonometric monomials with same norm by Equation (H.26).

– If f attains its maximum modulus at a unique point $x \in [0, 2\pi/d[$, then l must be a multiple of the Dirac measure δ_x at x, so that l attains also its norm at the monomials in C_Λ.

These arguments show also that every linear functional on C_Λ attains its norm at monomials or at trigonometric trinomials with two maximum modulus points.

Sufficiency. Conversely, the trigonometric monomial $\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}_\lambda$ is exposed to the linear form

$$P \mapsto \frac{1}{2\pi}\int_0^{2\pi} P(x)\mathrm{e}^{-\mathrm{i}(\alpha+\lambda x)}\,\mathrm{d}x.$$

A trigonometric trinomial T that attains its maximum modulus, 1, at two points x_1^* and x_2^* modulo $2\pi/d$ is exposed, by Lemma 8.1 (*a*), to any nontrivial convex combination of the unimodular multiples of Dirac measures $\overline{T(x_1^*)}\delta_{x_1^*}$ and $\overline{T(x_2^*)}\delta_{x_2^*}$. ☐

Remark 8.3. This is a complex counterpart to Lemma 2.3 in [83], dealing with the exposed points of the unit ball of the three-dimensional space spanned by the functions 1, $\cos x$ and $\cos kx$ in the space of continuous functions.

Proof of Theorem 1.2 (*b*). Let K be the unit ball of C_Λ. Straszewicz's Theorem [96] tells that the exposed points of K are dense in the set of its extreme points. Let P be a limit point of exposed points of K. If P is a trigonometric monomial, P is exposed. If P is a trigonometric binomial, P is not an extreme point of K by Remark 8.2. If P is a trigonometric trinomial, it is the limit point of trigonometric trinomials that attain their maximum modulus twice modulo $2\pi/d$, so that either P also attains its maximum modulus twice modulo $2\pi/d$ or, by Rolle's Theorem, P attains its maximum modulus with multiplicity 4. Let us prove that if a trigonometric trinomial T attains its maximum modulus with multiplicity 4 at a point x, then T is an extreme point of K. Suppose that T is the midpoint of two points A and B in K. Then $|A(x)| \leqslant 1$, $|B(x)| \leqslant 1$ and $\big(A(x)+B(x)\big)/2 = T(x)$, so that $A(x) = B(x) = T(x)$. Furthermore

$$\begin{aligned}|T(x+h)| &\leqslant \frac{|A(x+h)|+|B(x+h)|}{2}\\ &= 1+\frac{h^2}{4}\left(\frac{\mathrm{d}^2|A|}{\mathrm{d}x^2}(x)+\frac{\mathrm{d}^2|B|}{\mathrm{d}x^2}(x)\right)+o(h^2)\end{aligned}$$

so that, as $|T(x+h)| = 1 + o(h^3)$,

$$\frac{\mathrm{d}^2|A|}{\mathrm{d}x^2}(x)+\frac{\mathrm{d}^2|B|}{\mathrm{d}x^2}(x) \geqslant 0 \text{ while } \frac{\mathrm{d}^2|A|}{\mathrm{d}x^2}(x), \frac{\mathrm{d}^2|B|}{\mathrm{d}x^2}(x) \leqslant 0$$

and therefore A and B also attain their maximum modulus with multiplicity 4 at x. As this implies that A and B are trigonometric trinomials, Lemma 8.1 (*b*) yields that T, A and B lie on a parabola: this implies $A = B = T$. ☐

Remark 8.4. The set of extreme points of the unit ball of C_Λ is not closed: for example, if λ_2 is between λ_1 and λ_3, every absolutely convex combination of e_{λ_1} and e_{λ_3} is a limit point of exposed points.

Remark 8.5. If λ_2 is between λ_1 and λ_3, and $\lambda_1 - \lambda_2$ is not a multiple of $\lambda_3 - \lambda_2$ nor vice versa, then we obtain that every extreme point of the unit ball of C_Λ is exposed.

Remark 8.6. In particular, compare our description of the extreme points of the unit ball of $\mathrm{C}_{\{0,1,2\}}$ with the characterisation given by K. M. Dyakonov in [27, Theorem 1]. He shows in his Example 1 that it is false that "in order to recognize the extreme points", "one only needs to know 'how often' [the modulus of a trigonometric polynomial] takes the extremal value 1". We show that *with the exception of trigonometric binomials*, the extreme points of the unit ball of $\mathrm{C}_{\{0,1,2\}}$ are characterised by the number of zeroes of $1 - |P|^2$.

Remark 8.7. This is a complex counterpart to [60], dealing with the extreme points of the unit ball of the three-dimensional space spanned by the functions 1, x^n and x^m in the space of real valued continuous functions on $[-1, 1]$.

9 Dependence of the maximum modulus on the arguments

We wish to study how the maximum modulus of a trigonometric trinomial depends on the phase of its coefficients. We shall use the following formula that gives an expression for the directional derivative of a maximum function. It was established in [23]. Elementary properties of maximum functions are addressed in [80, Part Two, Problems 223–226].

N. G. Chebotarëv's formula ([26, Theorem VI.3.2, (3.6)]). *Let $I \subset \mathbb{R}$ be an open interval and let K be a compact space. Let $f(t,x)$ be a function on $I \times K$ that is continuous along with $\dfrac{\partial f}{\partial t}(t,x)$. Let*

$$f^*(t) = \max_{x \in K} f(t,x).$$

Then $f^*(t)$ admits the following expansion at every $t \in I$:

$$f^*(t+h) = f^*(t) + \max_{f(t,x)=f^*(t)} \Big(h \frac{\partial f}{\partial t}(t,x) \Big) + o(h). \tag{H.27}$$

Proposition 9.1. *Let k and l be positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Then*

$$\max_x \big| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \big|$$

is an even $2\pi/(k+l)$-periodic function of t that decreases strictly on $[0, \pi/(k+l)]$: in particular

$$\min_t \max_x \big| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \big| = \max_x \big| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}\pi/(k+l)} + r_3 \mathrm{e}^{\mathrm{i}lx} \big|.$$

Proof. Let

$$f(t,x) = \big| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \big|^2. \tag{H.28}$$

By (H.10) and (H.11), f^* is an even $2\pi/(k+l)$-periodic function.

Let $t \in]0, \pi/(k+l)[$ and choose x^* such that $f(t,x^*) = f^*(t)$: then $x^* \in [-t/k, t/l]$ by Proposition 5.1, so that

$$\frac{1}{2r_2} \frac{\partial f}{\partial t}(t,x^*) = -r_1 \sin(t + kx^*) - r_3 \sin(t - lx^*) < 0$$

because $t + kx^* \in [0, (k+l)t/l]$ and $t - lx^* \in [0, (k+l)t/k]$ do not vanish simultaneously. By Formula (H.27), f^* decreases strictly on $[0, \pi/(k+l)]$. ☐

Proposition 9.2. *Let k and l be positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Then*

$$\frac{\max_x \big| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \big|}{\big| r_1 + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \big|} \tag{H.29}$$

is an increasing function of $t \in [0, \pi/(k+l)]$. If $kr_1 = lr_3$, it is constantly equal to 1; otherwise it is strictly increasing.

Proof. Let $f(t,x)$ be as in (H.28): then the expression (H.29) is $g^*(t)^{1/2}$ with

$$g(t,x) = \frac{f(t,x)}{f(t,0)}.$$

If $kr_1 = lr_3$, then $f(t,0) = f^*(t)$, so that $g^*(t) = 1$. As shown in the beginning of Section 6, we may suppose without loss of generality that $kr_1 < lr_3$. Let $t \in]0, \pi/(k+l)[$ and choose x^* such that $f(t,x^*) = f^*(t)$: then $x^* \in]0, t/l[$ by Propositions 5.1 and 6.1 and

$$\begin{aligned} \frac{f(t,0)^2}{2r_2} \frac{\partial g}{\partial t}(t,x^*) &= \frac{1}{2r_2} \Big(\frac{\partial f}{\partial t}(t,x^*) f(t,0) - f(t,x^*) \frac{\partial f}{\partial t}(t,0) \Big) \\ &= \big(-r_1 \sin(t + kx^*) - r_3 \sin(t - lx^*) \big) f(t,0) + f^*(t)(r_1 + r_3) \sin t \\ &= h(0) f^*(t) - h(x^*) f(t,0) \end{aligned}$$

with

$$h(x) = r_1 \sin(t + kx) + r_3 \sin(t - lx).$$

Let us show that h is strictly decreasing on $[0, t/l]$: in fact, if $x \in]0, t/l[$,

$$\frac{\mathrm{d}h}{\mathrm{d}x}(x) = kr_1 \cos(t + kx) - lr_3 \cos(t - lx) < (kr_1 - lr_3)\cos(t - lx) < 0.$$

As $f^*(t) > f(t, 0)$ and $h(0) > h(x^*)$, $(\partial g/\partial t)(t, x^*) > 0$. By N. G. Chebotarëv's formula, g^* increases strictly on $[0, \pi/(k + l)]$. ☐

It is possible to describe the decrease of the maximum modulus of a trigonometric trinomial independently of the r's as follows.

Proposition 9.3. *Let k and l be two positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Let $0 \leqslant t' < t \leqslant \pi/(k + l)$. Then*

$$\max_x \left| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t'} + r_3 \mathrm{e}^{\mathrm{i}lx} \right| \leqslant \frac{\cos(t'/2)}{\cos(t/2)} \max_x \left| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \right| \tag{H.30}$$

with equality if and only if $r_1 : r_2 : r_3 = l : k + l : k$.

Proof. Let us apply Proposition 9.2. We have

$$\begin{aligned} \frac{\left| r_1 + r_2 \mathrm{e}^{\mathrm{i}t'} + r_3 \right|^2}{\left| r_1 + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \right|^2} &= 1 + \frac{2r_2(r_1 + r_3)(\cos t' - \cos t)}{(r_1 + r_3)^2 + 2r_2(r_1 + r_3)\cos t + r_2^2} \\ &= 1 + \frac{\cos t' - \cos t}{\cos t + \bigl(r_2^2 + (r_1 + r_3)^2\bigr)/2r_2(r_1 + r_3)} \\ &\leqslant 1 + \frac{\cos t' - \cos t}{\cos t + 1} = \frac{\cos t' + 1}{\cos t + 1} \end{aligned}$$

by the arithmetic-geometric inequality, with equality if and only if $r_2 = r_1 + r_3$. Therefore Inequality (H.30) holds, with equality if and only if $kr_1 = lr_3$ and $r_2 = r_1 + r_3$. ☐

We may now find the minimum of the maximum modulus of a trigonometric trinomial with given spectrum, Fourier coefficient arguments and moduli sum. Proposition 9.3 yields with $t' = 0$

Corollary 9.4. *Let k and l be two positive coprime integers. Let r_1, r_2 and r_3 be three positive real numbers. Let $t \in]0, \pi/(k + l)]$. Then*

$$\frac{\max_x \left| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \right|}{r_1 + r_2 + r_3} \geqslant \cos(t/2)$$

with equality if and only if $r_1 : r_2 : r_3 = l : k + l : k$.

Remark 9.5. There is a shortcut proof to Corollary 9.4:

$$\begin{aligned} \frac{\max_x \left| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \right|}{r_1 + r_2 + r_3} &\geqslant \frac{\left| r_1 + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \right|}{r_1 + r_2 + r_3} \\ &= \sqrt{1 - \frac{4(r_1 + r_3)r_2}{(r_1 + r_2 + r_3)^2} \sin^2(t/2)} \\ &\geqslant \sqrt{1 - \sin^2(t/2)} = \cos(t/2) \end{aligned}$$

and equality holds if and only if $\left| r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx} \right|$ is maximal for $x = 0$ and $r_1 + r_3 = r_2$.

10 The norm of unimodular relative Fourier multipliers

We may now compute the norm of unimodular relative Fourier multipliers.

Corollary 10.1. *Let k and l be two positive coprime integers. Let $t \in [0, \pi/(k + l)]$. Let M be the relative Fourier multiplier $(0, t, 0)$ that maps the element*

$$r_1 \mathrm{e}^{\mathrm{i}u_1} \mathrm{e}_{-k} + r_2 \mathrm{e}^{\mathrm{i}u_2} \mathrm{e}_0 + r_3 \mathrm{e}^{\mathrm{i}u_3} \mathrm{e}_l \tag{H.31}$$

of the normed space $\mathrm{C}_{\{-k,0,l\}}$ *on*

$$r_1 \mathrm{e}^{\mathrm{i}u_1} \mathrm{e}_{-k} + r_2 \mathrm{e}^{\mathrm{i}(t+u_2)} \mathrm{e}_0 + r_3 \mathrm{e}^{\mathrm{i}u_3} \mathrm{e}_l.$$

Then M *has norm* $\cos\bigl(\pi/2(k+l) - t/2\bigr)/\cos\bigl(\pi/2(k+l)\bigr)$ *and attains its norm exactly at elements of form* (H.31) *with* $r_1 : r_2 : r_3 = l : k+l : k$ *and*

$$-lu_1 + (k+l)u_2 - ku_3 = \pi \mod 2\pi.$$

Proof. This follows from Proposition 9.3 and the concavity of cos on $[0, \pi/2]$. □

Remark 10.2. This corollary enables us to guess how to lift M to an operator that acts by convolution with a measure μ. Note that μ is a Hahn-Banach extension of the linear form $f \mapsto Mf(0)$. The relative multiplier M is an isometry if and only if $t = 0$ and μ is the Dirac measure in 0. Otherwise, $t \neq 0$; the proof of Theorem 1.2 (a) in Section 8 shows that μ is a linear combination $\alpha\delta_y + \beta\delta_w$ of two Dirac measures such that the norm of M is $|\alpha| + |\beta|$. Let $f(x) = l\mathrm{e}^{-\mathrm{i}kx} + (k+l)\mathrm{e}^{\mathrm{i}\pi/(k+l)} + k\mathrm{e}^{\mathrm{i}lx}$: M attains its norm at f, f attains its maximum modulus at 0 and $2m\pi/(k+l)$, and Mf attains its maximum modulus at $2m\pi/(k+l)$, where m is the inverse of l modulo $k+l$. As

$$\begin{aligned}(|\alpha| + |\beta|) \max_x |f(x)| &= \max_x |Mf(x)| \\ &= \bigl|\mu * f\bigl(2m\pi/(k+l)\bigr)\bigr| \\ &= \bigl|\alpha f\bigl(2m\pi/(k+l) - y\bigr) + \beta f\bigl(2m\pi/(k+l) - w\bigr)\bigr|,\end{aligned}$$

we must choose $\{y, w\} = \{0, 2m\pi/(k+l)\}$. A computation yields then

$$\mu = \mathrm{e}^{\mathrm{i}t/2} \frac{\sin\bigl(\pi/(k+l) - t/2\bigr)}{\sin\bigl(\pi/(k+l)\bigr)} \delta_0 + \mathrm{e}^{\mathrm{i}(t/2 + \pi/(k+l))} \frac{\sin(t/2)}{\sin\bigl(\pi/(k+l)\bigr)} \delta_{2m\pi/(k+l)}.$$

If $k = l = 1$, this is a special case of the formula appearing in [41, proof of Prop. 1]. Consult [94] on this issue.

11 The Sidon constant of integer sets

Let us study the maximum modulus of a trigonometric trinomial with given spectrum and Fourier coefficient moduli sum. We get the following result as an immediate consequence of Corollary 9.4.

Proposition 11.1. *Let* k *and* l *be two positive coprime integers. Let* r_1*,* r_2 *and* r_3 *be three positive real numbers. Let* $t \in [0, \pi/(k+l)]$*. Then*

$$\max_x \bigl|r_1 \mathrm{e}^{-\mathrm{i}kx} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}lx}\bigr| \geqslant \cos\bigl(\pi/2(k+l)\bigr) \cdot (r_1 + r_2 + r_3)$$

with equality if and only if $r_1 : r_2 : r_3 = l : k+l : k$ *and* $t = \pi/(k+l)$*.*

This means that the Sidon constant of $\{-k, 0, l\}$ equals $\sec\bigl(\pi/2(k+l)\bigr)$.

The Sidon constant of integer sets was previously known only in the following three instances:

– The equality

$$\max_x \bigl|r_1 \mathrm{e}^{\mathrm{i}(t_1 + \lambda_1 x)} + r_2 \mathrm{e}^{\mathrm{i}(t_2 + \lambda_2 x)}\bigr| = r_1 + r_2$$

shows that the Sidon constant of sets with one or two elements is 1.

– The Sidon constant of $\{-1, 0, 1\}$ is $\sqrt{2}$ and it is attained for $\mathrm{e}_{-1} + 2\mathrm{i} + \mathrm{e}_1$. Let us give the original argument: if $f(x) = \bigl|r_1 \mathrm{e}^{-\mathrm{i}x} + r_2 \mathrm{e}^{\mathrm{i}t} + r_3 \mathrm{e}^{\mathrm{i}x}\bigr|^2$, the parallelogram identity and the arithmetic-quadratic inequality yield

$$\begin{aligned}\max_x f(x) &\geqslant \max_x \frac{f(x) + f(x+\pi)}{2} \\ &= \max_x \frac{\bigl|r_1 \mathrm{e}^{-\mathrm{i}x} + r_3 \mathrm{e}^{\mathrm{i}x} + r_2 \mathrm{e}^{\mathrm{i}t}\bigr|^2 + \bigl|r_1 \mathrm{e}^{-\mathrm{i}x} + r_3 \mathrm{e}^{\mathrm{i}x} - r_2 \mathrm{e}^{\mathrm{i}t}\bigr|^2}{2} \\ &= \max_x \bigl|r_1 \mathrm{e}^{-\mathrm{i}x} + r_3 \mathrm{e}^{\mathrm{i}x}\bigr|^2 + \bigl|r_2 \mathrm{e}^{\mathrm{i}t}\bigr|^2 \\ &= (r_1 + r_3)^2 + r_2^2 \geqslant \frac{(r_1 + r_2 + r_3)^2}{2}.\end{aligned}$$

– The Sidon constant of $\{0,1,2,3,4\}$ is 2 and it is attained for $1+2\mathrm{e}_1+2\mathrm{e}_2-2\mathrm{e}_3+\mathrm{e}_4$.

These results were obtained by D. J. Newman (see [93].) The fact that the Sidon constant of sets of three integers cannot be 1 had been noted with pairwise different proofs in [93, 21, 50].

Remark 11.2. The real algebraic counterpart is better understood: the maximal absolute value of a real algebraic polynomial of degree at most n with given coefficient absolute value sum is minimal for multiples of the nth Chebyshev polynomial T_n (look up the last paragraph of [28], and [81, Theorem 16.3.3] for a proof). As the sum of the absolute values of T_n's coefficients is the integer t_n nearest to $(1+\sqrt{2})^n/2$, we have for real $a_0, a_1, \ldots, a_n$

$$\max_{x\in[-1,1]}\left|a_0+a_1x+\cdots+a_nx^n\right| \geqslant t_n^{-1}(|a_0|+|a_1|+\cdots+|a_n|).$$

The following estimates for the Sidon constant of large integer sets are known.

– E. Beller and D. J. Newman [8] showed that the Sidon constant of $\{0,1,\ldots,n\}$ is equivalent to $\sqrt{n}$.
– (Hadamard sets.) Let $q>1$ and suppose that the sequence $(\lambda_j)_{j\geqslant 1}$ grows with geometric ratio q: $|\lambda_{j+1}|\geqslant q|\lambda_j|$ for every j. Then the Sidon constant of $\{\lambda_1,\lambda_2,\ldots\}$ is finite; it is at most 4.27 if $q\geqslant 2$ (see [46]), at most 2 if $q\geqslant 3$ (see [54]), and at most $1+\pi^2/\left(2q^2-2-\pi^2\right)$ if $q>\sqrt{1+\pi^2/2}$ (see [63, Corollary 9.4] or the updated [64, Corollary 10.2.1].)

Our computations show that the last estimate of the Sidon constant has the right order in q^{-1} for geometric progressions.

Proposition 11.3. *Let C be the Sidon constant of the geometric progression $\{1,q,q^2,\ldots\}$, where $q\geqslant 3$ is an integer. Then*

$$1+\pi^2/8(q+1)^2 \leqslant \sec\bigl(\pi/2(q+1)\bigr) \leqslant C \leqslant 1+\pi^2/\left(2q^2-2-\pi^2\right).$$

One initial motivation for this work was to decide whether there are sets $\{\lambda_j\}_{j\geqslant 1}$ with $|\lambda_{j+1}|\geqslant q|\lambda_j|$ whose Sidon constant is arbitrarily close to 1 and to find evidence among sets with three elements. That there are such sets, arbitrarily large albeit finite, may in fact be proved by the method of Riesz products in [47, Appendix V, §1.II]; see also [64, Proposition 13.1.3]. The case of infinite sets remains open.

A second motivation was to show that the real and complex unconditional constants of the basis $(\mathrm{e}_{\lambda_1},\mathrm{e}_{\lambda_2},\mathrm{e}_{\lambda_3})$ of C_Λ are different; we prove however that they coincide, and it remains an open question whether they may be different for larger sets. The *real unconditional constant* of $(\mathrm{e}_{\lambda_1},\mathrm{e}_{\lambda_2},\mathrm{e}_{\lambda_3})$ is the maximum of the norm of the eight unimodular relative Fourier multipliers (t_1,t_2,t_3) such that $t_k=0$ modulo π. Let i,j,k be a permutation of $1,2,3$ such that the power of 2 in $\lambda_i-\lambda_k$ and in $\lambda_j-\lambda_k$ are equal. Lemma 2.1 shows that the four relative multipliers satisfying $t_i=t_j$ modulo 2π are isometries and that the norm of any of the four others, satisfying $t_i\neq t_j$ modulo 2π, gives the real unconditional constant. In general, the complex unconditional constant is bounded by $\pi/2$ times the real unconditional constant, as proved in [92]; in our case, they are equal.

Corollary 11.4. *The complex unconditional constant of the basis $(\mathrm{e}_{\lambda_1},\mathrm{e}_{\lambda_2},\mathrm{e}_{\lambda_3})$ of C_Λ is equal to its real unconditional constant.*

Acknowledgement. I would like to thank Henri Comman, Hervé Queffélec and Harold S. Shapiro for their encouragement at crucial moments of this research. The Hungarian-French Scientific and Technological Governmental Cooperation, Project #F-10/04, eased scientific contacts that were helpful to this work.

Keywords. Trigonometric trinomial, maximum modulus, exposed point, extreme point, Mandel′shtam problem, extremal problem, relative Fourier multiplier, Sidon constant, unconditional constant.

2000 *Mathematics Subject Classification.* Primary 42A05; Secondary 26D05, 30C10, 42A45, 42A55, 46B15, 46B20.

Chapter I

On the Sidon constant of $\{0,1,2,3\}$

We study an elementary extremal problem on trigonometric polynomials of degree 3. We discover a distinguished torus of extremal functions.

1 Introduction

Let $\Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_{n-1}\}$ be a set of n frequencies. We study the following extremal problem:

(†) Given n positive intensities $\varrho_0, \varrho_1, \dots, \varrho_{n-1}$, to find n phases $\vartheta_0, \vartheta_1, \dots, \vartheta_{n-1}$ such that the maximum $\max_t |\sum \varrho_j \mathrm{e}^{\mathrm{i}\vartheta_j} \mathrm{e}^{\mathrm{i}\lambda_j t}|$ is minimal.

This should help us to study the following extremal problem:

(‡) To find n complex coefficients $c_0, c_1, \dots, c_{n-1}$ with given moduli sum $|c_0|+|c_1|+\cdots+|c_{n-1}| = 1$ such that the maximum $\max_t |\sum c_j \mathrm{e}^{\mathrm{i}\lambda_j t}|$ is minimal.

Note that this maximum's inverse is the Sidon constant $S(\Lambda)$. D. J. Newman (see [93, Chapter 3]) obtained the following upper bound for $S(\{0,1,\dots,n\})$: by Parseval's theorem on $\ell^2(\{1, \mathrm{e}^{2\mathrm{i}\pi/n}, \dots, \mathrm{e}^{2\mathrm{i}(n-1)\pi/n}\})$, putting $\sum c_j \mathrm{e}^{\mathrm{i}jt} = f(t)$,

$$\begin{aligned}
\max_t |f(t)|^2 &= \max_t |f(t)|^2 \vee |f(t+2\pi/n)|^2 \vee \cdots \vee |f(t+2(n-1)\pi/n)|^2 \\
&\geqslant \max_t \bigl(|f(t)|^2 + |f(t+2\pi/n)|^2 + \cdots + |f(t+2(n-1)\pi/n)|^2\bigr)/n \\
&= \max_t |c_0 + c_n \mathrm{e}^{\mathrm{i}nt}|^2 + |c_1|^2 + \cdots + |c_{n-1}|^2 \\
&= \bigl(|c_0| + |c_n|\bigr)^2 + |c_1|^2 + |c_2|^2 + \cdots + |c_{n-1}|^2 \qquad \text{(I.1)} \\
&\geqslant \bigl(|c_0| + |c_1| + \cdots + |c_n|\bigr)^2/n,
\end{aligned}$$

and H. S. Shapiro showed (ibid.) that equality can hold exactly if $n \in \{1,2,4\}$. If $n=2$, equality holds exactly for multiples and translates of $f(t) = 1 + 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}2t}$. If $n=3$, the functions

$$\frac{\mathrm{i}2\sqrt{2}\cos\tau - 1 - 3\sin\tau}{15} + \frac{3+\sin\tau}{10}\mathrm{e}^{\mathrm{i}t} + \frac{3-\sin\tau}{10}\mathrm{e}^{\mathrm{i}2t} + \frac{\mathrm{i}2\sqrt{2}\cos\tau - 1 + 3\sin\tau}{15}\mathrm{e}^{\mathrm{i}3t}$$

have their modulus bounded by $3/5$ for each τ, so that $5/3 \leqslant S(\{0,1,2,3\}) < \sqrt{3}$.

The motivation is that we wish to know whether the real and complex unconditionality constants are distinct for basic sequences of characters $\mathrm{e}^{\mathrm{i}nt}$.

Notation. $\mathbb{T} = \{\, z \in \mathbb{C} : |z| = 1 \,\}$ and $\mathrm{e}_\lambda(z) = z^\lambda$ for $z \in \mathbb{T}$ and $\lambda \in \mathbb{Z}$.

2 Necessary conditions for solutions to Extremal problem (†)

Let us the state the theorem of Chebotarev (see [26, Th. VI.3.2, (3.6)]) in our context.

Theorem 2.1. *Let $\Phi\colon \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ be a continuous function that is C^1 in the first variable and periodic in the second variable. If $\Phi^*\colon x \mapsto \max_y \Phi(x,y)$ achieves its minimum at x^*, then there are r points $y_1,\ldots,y_r$ such that $1 \leqslant r \leqslant k+1$ and $\Phi(x^*,y_1) = \cdots = \Phi(x^*,y_r) = \Phi^*(x^*)$, and*

$$\exists\,\alpha_1,\ldots,\alpha_r \geqslant 0 \quad \sum \alpha_i = 1 \qquad \sum \alpha_i \frac{\partial \Phi}{\partial x}(x^*,y_i) = 0.$$

Let us apply this theorem to our problem. For n pairwise distinct integers $\lambda_0,\lambda_1,\ldots,\lambda_{n-1}$, n positive real numbers $\varrho_0,\varrho_1,\ldots,\varrho_{n-1}$ and arguments t and $\vartheta = (\vartheta_0,\vartheta_1,\ldots,\vartheta_{n-1})$ let

$$f(t,\varrho,\vartheta) = \frac{\sum \varrho_j \mathrm{e}^{\mathrm{i}(\lambda_j t+\vartheta_j)}}{\sum \varrho_j} = \mathrm{R}(t,\varrho,\vartheta)\mathrm{e}^{\mathrm{i}\Theta(t,\varrho,\vartheta)} \text{ with } \mathrm{R}(t,\varrho,\vartheta) \geqslant 0 \text{ and } \Phi = \mathrm{R}^2.$$

Define

$$\Phi^*(\varrho,\vartheta) = \max_t \Phi(t,\varrho,\vartheta),\ \ \Phi^*_* = \min \Phi^*$$

and, for fixed $\varrho_j > 0$,

$$\Phi_\varrho(t,\vartheta) = \Phi(t,\varrho,\vartheta),\ \ \Theta_\varrho(t,\vartheta) = \Theta(t,\varrho,\vartheta),\ \ \Phi^*_\varrho(\vartheta) = \Phi^*(\varrho,\vartheta),\ \ (\Phi^*_\varrho)_* = \min \Phi^*_\varrho.$$

Note that Φ^* is continuous (see [80, Chapter 5.4]). Then

$$\begin{aligned}
\Phi &= \frac{\sum\sum \rho_j\rho_k \cos((\lambda_k-\lambda_j)t+\vartheta_k-\vartheta_j)}{(\sum \varrho_j)^2}\\
\frac{\partial\Phi}{\partial t} &= \frac{\sum\sum (\lambda_j-\lambda_k)\varrho_j\varrho_k \sin((\lambda_k-\lambda_j)t+\vartheta_k-\vartheta_j)}{(\sum \varrho_j)^2} = 2\Re\Big(\bar f \frac{\partial f}{\partial t}\Big)\\
\frac{\partial\Phi}{\partial\vartheta_k} &= -2\varrho_k \frac{\sum \varrho_j \sin((\lambda_k-\lambda_j)t+\vartheta_k-\vartheta_j)}{(\sum \varrho_j)^2} = \frac{-2\varrho_k}{\sum \varrho_j}\Im(\bar f \mathrm{e}^{\mathrm{i}(\lambda_k t+\vartheta_k)})
\end{aligned} \tag{I.2}$$

Note that our problem depends in fact on $n-2$ phases only; we have

$$\sum \frac{\partial\Phi}{\partial\vartheta_k} = 0 \quad , \quad \sum \lambda_k \frac{\partial\Phi}{\partial\vartheta_k} = \frac{\partial\Phi}{\partial t}. \tag{I.3}$$

We get thus

Lemma 2.2. *Let Φ^*_ϱ achieve its minimum at ϑ and $M = \{\, t : \Phi_\varrho(t,\vartheta) = (\Phi^*_\varrho)_* \,\}$.*

(*a*) *If M is a singleton $\{t\}$, then (t,ϑ) is a critical point of Φ_ϱ. Then all $\lambda_k t+\vartheta_k$ are congruent to $\Theta_\varrho(t,\vartheta)$ modulo π.*

(*b*) *If M is a pair $\{t,u\}$ and $(t,\vartheta),(u,\vartheta)$ are not critical points of Φ_ϱ, then*

$$\exists\,\mu,\nu > 0\ \forall\,k \quad \mu\frac{\partial\Phi}{\partial\vartheta_k}(t,\vartheta) + \nu\frac{\partial\Phi}{\partial\vartheta_k}(u,\vartheta) = 0.$$

Then $\mu\sin(\Theta_\varrho(t,\vartheta)-\lambda_k t-\vartheta_k) + \nu\sin(\Theta_\varrho(u,\vartheta)-\lambda_k u-\vartheta_k) = 0$ for all k.

Furthermore one may compute

$$\frac{\partial\Phi}{\partial\varrho_k} = 2\frac{\sum \varrho_j \cos((\lambda_k-\lambda_j)t+\vartheta_k-\vartheta_j)}{(\sum \varrho_j)^2} - \frac{2\Phi}{\sum \varrho_j} = \frac{2}{\sum \varrho_j}\big(\Re(\bar f \mathrm{e}^{\mathrm{i}(\lambda_k t+\vartheta_k)}) - \Phi\big).$$

Note that our problem depends on $n-1$ intensities only; we have

$$\sum \varrho_k \frac{\partial\Phi}{\partial\varrho_k} = 0.$$

Thus

$$\frac{\sum \varrho_j}{2}\frac{\partial\Phi}{\partial\varrho_k} - \mathrm{i}\frac{\sum \varrho_j}{2\varrho_k}\frac{\partial\Phi}{\partial\vartheta_k} = R(\mathrm{e}^{\mathrm{i}(\lambda_k t+\vartheta_k-\Theta)} - R).$$

Lemma 2.3. *Let Φ^* achieve its minimum at (ϱ,ϑ) and $M = \{\, t : \Phi(t,\varrho,\vartheta) = \Phi^*_* \,\}$.*

(*a*) *If M is a singleton $\{t\}$, then (t,ϱ,ϑ) is a critical point of Φ. Then $\Phi^*_* = 1$.*

(b) *If M is a pair $\{t,u\}$ and $\Phi^*_* \neq 1$, then*

$$\exists\, \mu,\nu > 0\ \forall\, k \quad \mu\Big(\frac{\partial\Phi}{\partial\vartheta_k}, \frac{\partial\Phi}{\partial\varrho_k}\Big)(t,\varrho,\vartheta) + \nu\Big(\frac{\partial\Phi}{\partial\vartheta_k}, \frac{\partial\Phi}{\partial\varrho_k}\Big)(u,\varrho,\vartheta) = 0. \tag{I.4}$$

If in turn $\Phi^*_* \neq 1$ and (I.4) *holds, then there is an α and there are signs $\varepsilon_j \in \{-1,1\}$ not all equal such that, with $d_{ijk} = \gcd(\lambda_k - \lambda_i, \lambda_j - \lambda_i)$ and $\delta_{ijk} \in \{0,1\}$*

$$\begin{cases} \mathrm{R}(t,\varrho,\vartheta) = \mathrm{R}(u,\varrho,\vartheta) = \cos\alpha \\ \sum \varepsilon_i \varrho_i = \sum \varepsilon_i \lambda_i \varrho_i = 0 \\ \forall\, i,j,k \quad \big((\lambda_j - \lambda_k)\varepsilon_i + (\lambda_k - \lambda_i)\varepsilon_j + (\lambda_i - \lambda_j)\varepsilon_k\big)\alpha \equiv \\ \qquad (\lambda_j - \lambda_k)\vartheta_i + (\lambda_k - \lambda_i)\vartheta_j + (\lambda_i - \lambda_j)\vartheta_k \equiv \delta_{ijk} d_{ijk}\pi \mod 2d_{ijk}\pi \\ \forall\, j\ \Theta(t,\varrho,\vartheta) - \lambda_j t - \vartheta_j = \varepsilon_j\alpha = -(\Theta(u,\varrho,\vartheta) - \lambda_j u - \vartheta_j). \end{cases}$$

Proof. Let $\mathrm{R} = \mathrm{R}(t,\varrho,\vartheta) = \mathrm{R}(u,\varrho,\vartheta)$, $\Theta_t = \Theta(t,\varrho,\vartheta)$, $\Theta_u = \Theta(u,\varrho,\vartheta)$. Then

$$\forall\, k \quad \mu(\mathrm{e}^{\mathrm{i}(\lambda_k t + \vartheta_k - \Theta_t)} - \mathrm{R}) + \nu(\mathrm{e}^{\mathrm{i}(\lambda_k u + \vartheta_k - \Theta_u)} - \mathrm{R}) = 0.$$

Let us suppose $\mu + \nu = 1$. By taking moduli, we get then

$$\forall\, k \quad \cos(\lambda_k t + \vartheta_k - \Theta_t) = \frac{\mathrm{R}^2 + \mu^2 - \nu^2}{2\mu\mathrm{R}},\ \cos(\lambda_k u + \vartheta_k - \Theta_u) = \frac{\mathrm{R}^2 + \nu^2 - \mu^2}{2\nu\mathrm{R}};$$

Returning to the definition of f, we have then

$$\forall\, k \quad \cos(\lambda_k t + \vartheta_k - \Theta_t) = \mathrm{R} = \cos(\lambda_k u + \vartheta_k - \Theta_u),$$

so that furthermore $\mu = \nu$. Thus we may choose signs $\varepsilon_k \in \{-1,1\}$ such that

$$\forall\, j,k \quad \varepsilon_j \sin(\lambda_j t + \vartheta_j - \Theta_t) = \varepsilon_k \sin(\lambda_k t + \vartheta_k - \Theta_t) = -\varepsilon_k \sin(\lambda_k u + \vartheta_k - \Theta_u).$$

These sines do not vanish, because otherwise $\mathrm{R} = 1$. The expressions of f and $\partial\Phi/\partial t$ yield therefore

$$\sum \varepsilon_j \varrho_j = 0,\ \sum \varepsilon_j \lambda_j \varrho_j = 0.$$

We can therefore choose α, α' and signs $\varepsilon_j \in \{-1,1\}$ such that

$$\forall\, j \qquad \Theta_t - \lambda_j t - \vartheta_j = \varepsilon_j\alpha \quad , \quad \Theta_u - \lambda_j u - \vartheta_j = \varepsilon_j\alpha'.$$

$$(\lambda_k - \lambda_j)t + \vartheta_k - \vartheta_j = \begin{cases} 0 & \text{if } \varepsilon_j = \varepsilon_k \\ 2\varepsilon_j\alpha & \text{otherwise.} \end{cases}$$

and similarly for u. Finally

$$\frac{\lambda_j - \lambda_k}{d_{ijk}}\vartheta_i + \frac{\lambda_k - \lambda_i}{d_{ijk}}\vartheta_j + \frac{\lambda_i - \lambda_j}{d_{ijk}}\vartheta_k = \pm\Big(\frac{\lambda_j - \lambda_k}{d_{ijk}}\varepsilon_i + \frac{\lambda_k - \lambda_i}{d_{ijk}}\varepsilon_j + \frac{\lambda_i - \lambda_j}{d_{ijk}}\varepsilon_k\Big)\alpha. \qquad \square$$

3 Necessary conditions for solutions to Extremal problem (‡)

Let (ϱ,ϑ) solve the extremal problem. If the point t such that $\Phi^*(\varrho,\vartheta) = \Phi(t,\varrho,\vartheta)$ were unique, then $\Phi^*(\varrho,\vartheta) = 1$. If there are exactly two points t,u such that $\Phi^*(\varrho,\vartheta) = \Phi(t,\varrho,\vartheta) = \Phi(u,\varrho,\vartheta)$, then either $\Phi^*(\varrho,\vartheta) = 1$ or there is an α and there are signs $\varepsilon_0,\varepsilon_1,\varepsilon_2 \in \{-1,1\}$ such that, with $d = \gcd(\lambda_1 - \lambda_0, \lambda_2 - \lambda_0)$,

$$\begin{cases} \mathrm{R}(t,\varrho,\vartheta) = \mathrm{R}(u,\varrho,\vartheta) = \cos\alpha \\ \varepsilon_0\varrho_0 + \varepsilon_1\varrho_1 + \varepsilon_2\varrho_2 = \varepsilon_0\lambda_0\varrho_0 + \varepsilon_1\lambda_1\varrho_1 + \varepsilon_2\lambda_2\varrho_2 = 0 \\ \big((\lambda_1 - \lambda_2)\varepsilon_0 + (\lambda_2 - \lambda_0)\varepsilon_1 + (\lambda_0 - \lambda_1)\varepsilon_2\big)\alpha \equiv \\ \qquad (\lambda_1 - \lambda_2)\vartheta_0 + (\lambda_2 - \lambda_0)\vartheta_1 + (\lambda_0 - \lambda_1)\vartheta_2 \equiv d\pi \mod 2d\pi \end{cases}$$

We need not consider the case $(\lambda_1 - \lambda_2)\vartheta_0 + (\lambda_2 - \lambda_0)\vartheta_1 + (\lambda_0 - \lambda_1)\vartheta_2 \equiv 0 \mod 2d\pi$, for which the system

$$\lambda_0 t + \vartheta_0 = \lambda_1 t + \vartheta_1 = \lambda_2 t + \vartheta_2$$

has a solution, so that $\Phi^*(\varrho,\vartheta)=1$. Then

$$\varepsilon_0=\operatorname{sgn}(\lambda_1-\lambda_2)\varepsilon,\ \varepsilon_1=\operatorname{sgn}(\lambda_2-\lambda_0)\varepsilon,\ \varepsilon_2=\operatorname{sgn}(\lambda_0-\lambda_1)\varepsilon \text{ for some } \varepsilon\in\{-1,1\},$$

so that

$$\varrho_0=|\lambda_1-\lambda_2|\sigma,\ \varrho_1=|\lambda_2-\lambda_0|\sigma,\ \varrho_2=|\lambda_0-\lambda_1|\sigma \quad \text{for some } \sigma>0.$$

Choose $\{i,j,k\}=\{0,1,2\}$ such that $\varepsilon_i\neq\varepsilon_j=\varepsilon_k$. Then $2|\lambda_j-\lambda_k|\alpha\equiv d\pi \mod 2d\pi$, so that $R\leqslant\cos(d\pi/2|\lambda_j-\lambda_k|)$. Further $\varrho_k+\varrho_j=\varrho_i$, and the system

$$\lambda_j t+\vartheta_j=\lambda_k t+\vartheta_k=\lambda_i t+\vartheta_i-d\pi/(\lambda_j-\lambda_k)$$

has a solution t, for which

$$R=\frac{|\varrho_k+\varrho_j+\varrho_i\mathrm{e}^{\mathrm{i}d\pi/(\lambda_j-\lambda_k)}|}{2\varrho_i}=\cos\frac{d\pi}{2|\lambda_j-\lambda_k|}.$$

This solves the problem for $\Lambda=\{0,1,2\}$, as $t\mapsto\Phi(t,\varrho,\vartheta)$ has at most two maxima.

4 The case $\{0,1,2,3\}$: a distinguished family of polynomials

Let $\Lambda=\{0,1,2,3\}$. Let (ϱ,ϑ) solve the extremal problem. If the point t such that $\Phi^*(\varrho,\vartheta)=\Phi(t,\varrho,\vartheta)$ were unique, then $\Phi^*_*=1$. If there were exactly two points t,u such that $\Phi^*(\varrho,\vartheta)=\Phi(t,\varrho,\vartheta)=\Phi(u,\varrho,\vartheta)$, then $\Phi^*_*=\sqrt{2}$. There are therefore exactly three points t,u,t'' such that $\Phi^*(\varrho,\vartheta)=\Phi(t,\varrho,\vartheta)=\Phi(u,\varrho,\vartheta)=\Phi(t'',\varrho,\vartheta)$.

Let $f(t,\tau)$ be given by

$$\frac{\mathrm{i}2\sqrt{2}\cos\tau-1-3\sin\tau}{15}+\frac{3+\sin\tau}{10}\mathrm{e}^{\mathrm{i}t}+\frac{3-\sin\tau}{10}\mathrm{e}^{\mathrm{i}2t}+\frac{\mathrm{i}2\sqrt{2}\cos\tau-1+3\sin\tau}{15}\mathrm{e}^{\mathrm{i}3t}.$$

One computes that the moduli sum of the coefficients is 1, independently of τ. Note that $f(t,-\tau)=\mathrm{e}^{\mathrm{i}3t}f(-t,\tau)$ and $f(t,\tau+\pi)=\mathrm{e}^{\mathrm{i}3t}\overline{f(t,\tau)}$, so that we shall restrict the parameter τ to $[0,\pi/2]$. Let $\Phi(t,\tau)=|f(t,\tau)|^2$. We get

$$\begin{aligned}\Phi(t,\tau)\ &=\ \frac{2\sqrt{2}\sin 2\tau}{75}(\sin t-\sin 2t+2\sin 3t)+\frac{247-13\cos 2\tau}{900}\\ &\quad+(1+\cos 2\tau)\Big(\frac{\cos t}{20}-\frac{\cos 2t}{25}\Big)+\frac{1+17\cos 2\tau}{225}\cos 3t.\end{aligned}$$

Let us put

$$M=\begin{pmatrix}\frac{2\sin 2t}{25}-\frac{\sin t}{20}-\frac{17\sin 3t}{75} & \frac{2\sqrt{2}}{75}(\cos t-2\cos 2t+6\cos 3t)\\ \frac{2\sqrt{2}}{75}(\sin t-\sin 2t+2\sin 3t) & \frac{13}{900}-\frac{\cos t}{20}+\frac{\cos 2t}{25}-\frac{17\cos 3t}{225}\end{pmatrix}$$

The critical points (t,τ) of Φ satisfy

$$M\begin{pmatrix}\cos 2\tau\\ \sin 2\tau\end{pmatrix}=\begin{pmatrix}\frac{\sin t}{20}-\frac{2\sin 2t}{25}+\frac{\sin 3t}{75}\\ 0\end{pmatrix}$$

We have

$$\det M=\frac{1}{6750}\sin t\Big(\cos t-\frac{1}{4}\Big)(4\cos t-11)(16\cos^3 t-72\cos^2 t+33\cos t-41),$$

which vanishes exactly if $\cos t\in\{-1,1/4,1\}$. Otherwise we get

$$\begin{cases}\cos 2\tau &=-\dfrac{272\cos^3 t-72\cos^2 t-159\cos t+23}{16\cos^3 t-72\cos^2 t+33\cos t-41}=C(t)\\ \sin 2\tau &=-\dfrac{24\sqrt{2}\sin t(4\cos t+1)(2\cos t-1)}{16\cos^3 t-72\cos^2 t+33\cos t-41}=S(t)\end{cases}\tag{I.5}$$

Note that this solution is consistent, as $C^2 + S^2 = 1$. For such τ, $\Phi(t,\tau) = 9/25$. Checking the special cases $\cos t \in \{-1, 1/4, 1\}$ yields that all local maxima are given by the above formulas, that Φ attains its global minimum, 0, exactly for $\tau = 0$ and $t = \pi$, and has exactly one other local minimum, of value 49/225, for $\tau = \pi/2$ and $t = 0$. There is exactly one other critical point, of value 5/18, that is a saddle point, given by $\tau = \arccos(17/37)/2$, $t = \arccos 1/4$.

As $C(0) = 1$, $C(\pm\pi/3) = -1$, $C(\pm\arccos(-1/4)) = 1$, $C(\pi) = -1$, the intermediate values theorem shows that for a given τ, there are exactly three solutions t to system (I.5), for which $\Phi(t,\tau)$ achieves then its global maximum, 9/25.

These formulas yield in turn that for a sign $\varepsilon \in \{-1, +1\}$

$$\begin{cases} \cos\tau &= \dfrac{8\varepsilon(2\cos t - 1)\cos t/2}{\sqrt{-16\cos^3 t + 72\cos^2 t - 33\cos t + 41}} = C_\varepsilon(t) \\ \sin\tau &= \dfrac{3\sqrt{2}\varepsilon(4\cos t + 1)\sin t/2}{\sqrt{-16\cos^3 t + 72\cos^2 t - 33\cos t + 41}} = S_\varepsilon(t) \end{cases} \tag{I.6}$$

5 The real unconditional constant of $\{0, 1, 2, 3\}$

If L is a subspace of the space of complex continuous functions on a compact space T with n dimensions, then every functional l on L extends isometrically to a linear combination of at most $2n$ Dirac measures: there are $m \leqslant 2n$ points $t_k \in T$ and coefficients $b_k \in \mathbb{C}$ such that for every $f \in L$ one has $l(f) = \sum b_k f(t_k)$ and $\|l\| = \sum |b_k|$ (see [13, Exercice 6.8].) This implies in particular that there is a function $f \in L$ whose maximum modulus points contain the t_k.

Let us now specialise to the case $L = \mathrm{C}_\Lambda(\mathbb{T})$ with Λ a finite set. Note that a function in L has at most $\max\Lambda - \min\Lambda$ maximum modulus points; a trigonometric trinomial has at most 2 maximum modulus points up to periodicity.

Let us make the ad hoc hypothesis that the t_k are the nth roots of unity, whose set forms the group $\mathbb{U}_n$: this obliges us to restrict our study to those functionals l such that $l(\mathrm{e}_j) = l(\mathrm{e}_{j'})$ if $j \equiv j' \bmod n$. Then the condition $l(f) = \sum b_k f(t_k)$ reads

$$l(\mathrm{e}_j) = \sum_{k=0}^{n-1} b_k \mathrm{e}^{\mathrm{i}2jk\pi/n} \text{ for } j \in \Lambda,$$

which may be interpreted as telling that the $l(\mathrm{e}_j)$ are the Fourier coefficients of the measure μ on $\mathbb{U}_n$ given by

$$\mu = \sum_{k=0}^{n-1} b_k \delta_{\mathrm{e}^{\mathrm{i}2k\pi/n}}$$

(where the Dirac measures act on $\mathbb{U}_n$). A solution to these equations is given by

$$b_k = \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{e}^{-\mathrm{i}2jk\pi/n} \begin{cases} l(\mathrm{e}_{j'}) & \text{if there is } j' \equiv j \bmod n \text{ in } \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

The norm of μ is bounded by

$$\sum_{k=0}^{n-1} |b_k|$$

and is attained at $u \in \mathrm{C}(\mathbb{U}_n)$ if and only if $u(\mathrm{e}^{\mathrm{i}2k\pi/n})b_k = |b_k|$ for every k, up to a nonzero complex number. This yields an upper bound for the norm of l that becomes an equality if there is an $f \in \mathrm{C}(\mathbb{T})$ of norm 1 such that $f(\mathrm{e}^{\mathrm{i}2k\pi/n}) = u(\mathrm{e}^{\mathrm{i}2k\pi/n})$.

Here are two applications.

Proposition 5.1. *Let Λ be a finite subset of $\mathbb{Z}$. The Sidon constant of Λ is at most $(\#\Lambda - 1)^{1/2}$.*

Proof. One may suppose that $\min\Lambda = 0$ and choose $n = \max\Lambda$. Let l be a linear functional l with coefficients $l(\mathrm{e}_j)$ of modulus 1: one may suppose that $l(\mathrm{e}_0) = l(\mathrm{e}_n)$. Then

$$\begin{aligned}\|l\| &\leqslant \frac{1}{n}\sum_{k=0}^{n-1}\Bigl|\sum_{j\in\Lambda\setminus\{n\}} l(\mathrm{e}_j)\mathrm{e}^{-\mathrm{i}2jk\pi/n}\Bigr| \\ &\leqslant \biggl(\frac{1}{n}\sum_{k=0}^{n-1}\Bigl|\sum_{j\in\Lambda\setminus\{n\}} l(\mathrm{e}_j)\mathrm{e}^{-\mathrm{i}2jk\pi/n}\Bigr|^2\biggr)^{1/2} \qquad \text{(I.7)} \\ &= \biggl(\sum_{j\in\Lambda\setminus\{n\}} |l(\mathrm{e}_j)|^2\biggr)^{1/2} = (\#\Lambda - 1)^{1/2}. \qquad \square\end{aligned}$$

Remark 5.2. Hervé Queffélec showed me a more elementary proof of the same fact, that follows the line of D. J. Newman's computation in the introduction: if $f \in \mathrm{C}_\Lambda(\mathbb{T})$, then there are only $\#\Lambda - 1$ squares, and not n, in Inequality (I.1)!

Remark 5.3. If $\Lambda = \{0, 1, \dots, n\}$, then Inequality (I.7) is an equality if and only if $(l(\mathrm{e}_j))_{j=0}^{n-1}$ is a *biunimodular* sequence, that is a unimodular function on $\mathbb{U}_n$ whose Fourier transform is also unimodular. In other words, the matrix $H = (l(\mathrm{e}_{j-k}))_{0\leqslant j,k\leqslant n-1}$ is a *circulant complex Hadamard matrix*, where the indices $j-k$ are computed modulo n: it satisfies $H^*H = n\,\mathrm{Id}$. Such matrices always exist: see [10].

Proposition 5.4. *Let* $\Lambda = \{0, 1, 2, 3\}$. *The real unconditional constant of* $\mathrm{C}_\Lambda(\mathbb{T})$ *is* $5/3$.

Proof. The polynomials in the previous section show that the real unconditional constant of $\mathrm{C}_\Lambda(\mathbb{T})$ is at least $5/3$. This constant is the maximum of the norm of linear functionals l with $l(\mathrm{e}_j) \in \{-1, 1\}$. As l has the same norm as $\tilde{l}\colon f \mapsto l(f(\cdot + \pi))$, for which $\tilde{l}(\mathrm{e}_j) = (-1)^j l(\mathrm{e}_j)$, one may suppose that $l(\mathrm{e}_0) = l(\mathrm{e}_3)$. Let us now try to lift l to a sum of Dirac measures on the third roots of unity. Such a lifting is either the Dirac measure at 0 or

$$\pm(l(\mathrm{e}_j))_{0\leqslant j\leqslant 2} \in \{(-1,1,1), (1,-1,1), (1,1,-1)\}$$

and these six cases yield the same norm

$$\frac{1}{3}(|-1+1+1| + |-1+\mathrm{e}^{\mathrm{i}2\pi/3}+\mathrm{e}^{\mathrm{i}4\pi/3}| + |-1+\mathrm{e}^{\mathrm{i}4\pi/3}+\mathrm{e}^{\mathrm{i}2\pi/3}|) = 5/3. \qquad \square$$

6 Trigonometric polynomials of degree 3 with real coefficients

Suppose c_0, c_1, c_2, c_3 are real and $c_0c_3 \neq 0$. Let $\varepsilon_i = \operatorname{sgn} c_i$ be their sign. Then

$$\|c_0 + c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{2\mathrm{i}t} + c_3\mathrm{e}^{3\mathrm{i}t}\|_\infty = |c_0 + c_2| + |c_1 + c_3|$$

if

$$\left|\begin{array}{ll} & \varepsilon_0 = \varepsilon_2 \text{ and } \varepsilon_1 = \varepsilon_3 \\ \text{or} & \begin{cases} \varepsilon_0\varepsilon_3(c_0c_1 + c_1c_2 + c_2c_3 + 9c_0c_3) + 4(c_0c_2 + c_1c_3) \geqslant 0 \\ 6|c_0c_3| \leqslant |c_0c_2 + c_1c_3| \end{cases} \\ \text{or} & \varepsilon_0\varepsilon_3(c_0c_1 + c_1c_2 + c_2c_3 + 9c_0c_3) \leqslant 4(c_0c_2 + c_1c_3) \\ \text{or} & \varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3 = -1 \text{ and } (|c_0c_2| + |c_1c_3|)^2 \leqslant 4(|c_0| + |c_3|)c_0c_3(\varepsilon_0c_1 + \varepsilon_3c_2). \end{array}\right.$$

Otherwise $\|c_0 + c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{2\mathrm{i}t} + c_3\mathrm{e}^{3\mathrm{i}t}\|_\infty^2$ is equal to

$$2c_0c_3\biggl(\Bigl(\frac{c_1}{3c_0} + \frac{c_2}{3c_3}\Bigr)^3 + \biggl(\Bigl(\frac{c_1}{3c_0} - \frac{c_2}{3c_3}\Bigr)^2 + \Bigl(1 - \frac{c_2}{3c_0}\Bigr)\Bigl(1 - \frac{c_1}{3c_3}\Bigr)\biggr)^{3/2}\biggr) \\ + \Bigl(1 - \frac{c_1c_2}{3c_0c_3}\Bigr)(c_0^2 + c_3^2) - \Bigl(1 + \frac{c_1c_2}{3c_0c_3}\Bigr)(c_0c_2 + c_1c_3) + \frac{2}{3}(c_1^2 + c_2^2).$$

Suppose c_0, c_1, c_2 are real and $c_0c_2 \neq 0$. Let $\varepsilon_i = \operatorname{sgn} c_i$ be their sign. Then

$$\|c_0 + c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{2\mathrm{i}t}\|_\infty = |c_0 + c_2| + |c_1| \quad \text{if } \varepsilon_0 = \varepsilon_2 \text{ or } |c_1|\,|c_0^{-1} + c_2^{-1}| \geqslant 4;$$

otherwise

$$\|c_0 + c_1\mathrm{e}^{\mathrm{i}t} + c_2\mathrm{e}^{2\mathrm{i}t}\|_\infty^2 = (c_0 - c_2)^2(1 - c_1^2/4c_0c_2).$$

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