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Investigating noncommutative structures : quantum groups and dual groups in the context of quantum probability

Defended on the 21st of June 2016.

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I dedicate this work to my fiancée

Ornella Bicchierri

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Introduction

Point de recherche qui ne soit recherche de soi-même et, à quelque degré, introspection[1].

– Alain Besançon, Histoire et expérience du moi, quoted in [18]

0.1 Abstract

The history of Mathematics has been lead in part by the desire for generalization: once an object was given and had been understood, there was the desire to find a more general version of it, to fit it into a broader framework. Noncommutative Mathematics fits into this description, as its interests are objects analoguous to vector spaces, or probability spaces, etc., but without the commonsense interpretation that those latter objects possess. Indeed, a space can be described by its points, but also and equivalently, by the set of functions on this space. This set is actually a commutative algebra, sometimes equipped with some more structure: $*$-algebra, C^*-algebra, von Neumann algebras, Hopf algebras, etc. The idea that lies at the basis of noncommutative Mathematics is to replace such algebras by algebras that are not necessarily commutative any more and to interpret them as "algebras of functions on noncommutative spaces". Of course, these spaces do not exist independently from their defining algebras, but facts show that a lot of the results holding in (classical) probability or (classical) group theory can be extended to their noncommutative counterparts, or find therein powerful analogues.
The extensions of group theory into the realm of noncommutative Mathematics has long been studied and has yielded the various quantum groups. The easiest version of them, the compact quantum groups, consist of C^*-algebras equipped with a $*$-homomorphism Δ with values in the tensor product of the algebra with itself and verifying some coassociativity condition. It is also required that the compact quantum group verifies what is known as quantum cancellation property. It can be shown that (classical) compact groups are indeed a particular case of compact quantum groups. The area of compact quantum groups, and of quantum groups at large, is a fruitful area of research.
Nevertheless, another generalization of group theory could be envisiond, namely by taking a comultiplication Δ taking values not in the tensor product but rather in the free product (in the category of unital $*$-algebras). This leads to the theory of dual groups in the sense of Voiculescu [47], also called H-algebras by [52]. These objects have not been so thoroughly studied as their quantum counterparts. It is true that they are not so flexible and that we therefore do not know many examples of them, see e.g. the remark following Example 2.2.2 and showing that some relations cannot exist in the dual groupcase because they do not pass the coproduct. Nevertheless, I have been interested during a great part of my PhD work by these objects and

[1] There is no research that is not a research of oneself and, to some extent, introspection.

I have made some progress towards their understanding. For instance, Chapter 7 shows in the particular case of the dual groups the existence of traces that are absorbing in the set of traces. Dual groups constitute also objects on which quantum Lévy processes, the noncommutative analogue to the well-known Lévy processes, can be defined. Such processes with values on the dual group are studied in Chapter 6, whereas a particular version of them is already introduced in Chapter 5. More generally, the last part of this work, Part III, is devoted to my results on dual groups.
This work begins naturally with some well-known tools and results that are necessary for the understanding of my research work. Part I is devoted to this, with Chapter 1 devoted specifically to introducing the basic algebraic notions, such as $*$-algebras, C^*-algebras, etc., Chapter 2 to introducing dual groupsand Chapter 3 to introducing the theory of quantum probability.
The remaining Part of the present work, Part II, is devoted to the study of compact quantum groups and specifically of the hypercontractivity properties held by these objects. Though the study of compact quantum groups has taken less time in my research work, it is nonetheless an important part and I believe that it will be worth in the future to investigate more fully the various hypercontractivity properties of such objects in order to further their understanding.
This PhD work having been done in "cotutelle" between the Université de Franche-Comté and the Ernst-Moritz-Arndt Universität Greifswald, the Introduction of this dissertation will also contain an abstract in German ("Zussamenfassung") followed by a detailed presentation of the results in French. After this, the main core of the work follows.

0.2 Zussamenfassung

Die Geschichte der Mathematik ist teilweise vom Streben nach Verallgemeinerung geprägt worden. Sobald ein Objekt gründlich verstanden wurde, gab es immer den Wunsch eine verallgemeinerte Version besagten Objektes zu finden, oder es in einen weiteren Zusammenhang zu stellen. Die nichtkommutative Mathematik entspringt einem solchen Wunsche, denn sie interessiert sich für Räume, die (klassischen) Vektorräumen oder Wahrscheinlichkeitsräumen entsprechen, aber denen es an der "alltäglichen" Interpretation fehlt, die die klassischen Räume haben. Ein Raum kann nämlich immer nicht nur durch seine Punkte beschrieben werden, sondern auch durch die Menge aller Funktionen, die auf besagtem Raum definiert sind. Beide Standpunkte sind äquivalent. Die Menge dieser Funktionen hat die Struktur einer kommutativen Algebra; oft gibt es sogar noch mehr Struktur, wie z.B. $*$-Algebra, C^*-Algebra, von Neumann Algebra, Hopf-algebra usw. Die Idee, die der nichtkommutativen Mathematik zugrunde liegt, ist daher solche Algebren durch Algebren zu ersetzen, die nicht mehr unbedingt kommutativ sind. Diese Algebren werden dann als "Algebren von Funktionen auf nichtkommutativen Räumen" interpretiert. Natürlich haben solche nichtkommutativen Räumen unabhängig von der Algebra, die sie definiert, keinen Bestand. Es stellt sich aber heraus, dass viele Resultate aus der (klassischen) Wahrscheinlichkeits- oder Gruppentheorie eine nichtkommutative Version haben.
Die Verallgemeinerung der Gruppentheorie für eine nichtkommutative Mathematik ist schon lange bekannt und hat die verschiedenen Definitionen von Quantengruppen hervorgebracht. Die einfachste Version davon, die kompakten Quantengruppen, bestehen aus einer C^*-Algebra, die mit einem $*$-Homomorphismus Δ versehen ist, der seine Werte im Tensorprodukt der Algebra mit sich selbst annimmt. Außerdem muss die Algebra auch eine Eigenschaft erfüllen, die als "Quantum Cancellation Property" bezeichnet wird. Es kann bewiesen werden, dass die (klassischen) kompakten Gruppen tatsächlich ein Sonderfall der kompakten Quantengruppen darstellen. Das Gebiet der kompakten Quantengruppen, und der Quantengruppen generell, ist

ein fruchtbarer Bereich der mathematischen Forschung.
Allerdings könnte eine andere Verallgemeinerung der Gruppentheorie in Betracht gezogen werden. Die dualen Gruppen, die von D. Voiculescu erstmals definiert wurden [47], und die H-Algebren von J. J. Zhang genannt wurden [52], entsprechen kompakten Quantengruppen, aber die Komultiplikation Δ nimmt Werte im freien Produkt (in der Kategorie der unitalen $*$-Algebren) anstelle des Tensorprodukts. Diese Objekte wurden nicht so gründlich studiert wie die Quantengruppen. Sie sind zwar nicht so flexibel — siehe z.B. die Anmerkung, die dem Beispiel 2.2.2 folgt und die darauf hinweist, dass einige Relationen im Falle von dualen Gruppen nicht möglich sind, da der Koprodukt nicht respektiert wird — und man kennt daher nur wenige Beispiele davon. Aber es gelang mir trotzdem einige neue Resultate zu finden und somit neue Schritte zum Verstehen dieses Konzeptes zu machen. Zum Beispiel, wird in Kapitel 7 die Existenz von Spuren auf der Unitären Dualen Gruppe bewiesen, die absorbierend in der Menge der Spuren sind. Quanten-Lévy-prozesse können auch in natürlicher Weise auf dualen Gruppen definiert werden. Sie werden im Spezialfall der Unitären Dualen Gruppe im Kapitel 6 studiert. Ein Sonderfall von Lévy-prozessen wird schon im Kapitel 5 eingeführt und studiert, und es wird darauf hingewiesen, dass dieses Prozess ein guter Kandidat wäre, um als Brownsche Bewegung auf der Unitären Dualen Gruppe zu gelten. Der gesamte letzte Teil dieser Arbeit, Teil III ist den dualen Gruppen gewidmet.
Diese Dissertation beginnt natürlicherweise mit der Einführung einiger bekannter Resultate, die für das Verstehen meiner Forschungsarbeit erforderlich sind. Teil I führt diese Konzepte ein, Kapitel 1 ist den algebraischen Grundbegriffen gewidmet, wie $*$-Algebra, C^*-Algebra, usw., Kapitel 2 ist der Einführung dualer Gruppen gewidmet, und Kapitel 3 ist der Einführung in die Quantenwahrscheinlichkeitstheorie gewidmet.
Teil II befasst sich mit den Hyperkontraktivitätseigenschaften kompakter Quantengruppen. Obschon das Studieren von Quantengruppen und ihrer Eigenschaften weniger Zeit während meiner Doktorarbeit in Anspruch genommen hat, glaube ich dennoch, dass es in der Zukunft sinnvoll sein wird, die verschiedenen Hyperkontraktivitätseigenschaften von Quantengruppen zu erforschen, um diese Objekte besser zu verstehen.

0.3 Résumé des résultats

Les Mathématiques non-commutatives sont un domaine de recherche très actif, qui a déjà donné lieu à de nombreux résultats et il est à prévoir qu'elles donneront encore dans le futur de nombreux résultats intéressants, permettant une meilleure compréhension des objets mathématiques, mais certainement aussi une meilleure compréhension du monde de la physique. La géométrie non-commutative, par exemple, semble avoir des liens forts avec la mécanique quantique. On pourra consulter [16] pour plus de détails à ce sujet.
L'idée de base des Mathématiques non-commutatives est de prendre une théorie classique qui s'intéresse à des espaces d'une nature ou une autre, telle que la théorie des probabilités avec ses espaces de probabilités, la théorie des groupes avec ses groupes, la géométrie différentielle avec ses variétés différentielles, etc. Ces espaces peuvent être décrits soit en les voyant comme espaces en tant que tel, comme ensembles de points, soit en considérant l'ensemble des fonctions définies sur ces espaces et à valeurs, par exemple, dans le corps des complexes. Cet ensemble de fonctions possède naturellement une structure d'algèbre, et même d'$*$-algèbre, commutative. Parfois, ils possèdent une structure encore plus riche, telle que C^*-algèbre, algèbre de von Neumann, etc. L'idée est de remplacer l'étude de ces algèbres commutatives par des algèbres ayant des propriétés similaires mais auxquelles on n'impose plus forcément d'être commutatives. On

les interprète alors comme des "algèbres de fonctions sur des espaces non-commutatifs". Bien sûr, ces espaces non-commutatifs n'ont pas d'existence indépendamment de leur algèbre de fonctions. Cependant, les faits démontrent que beaucoup de résultats classiques se généralisent au cas non-commutatif, ou y trouvent des parallèles riches et intéressants.
A titre d'exemple, la théorie classique des Probabilités s'intéresse à des espaces de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. La théorie des Probabilités non-commutatives consiste à remplacer l'algèbre commutative unifère des variables aléatoires complexes définies sur Ω et munie de l'espérance par une algèbre unifère non nécessairement commutative et munie d'une forme linéaire envoyant l'unité de l'algèbre sur 1. Il se trouve que cette théorie permet de généraliser bon nombre de résultats classiques. On peut en particulier montrer qu'il y a cinq notions différentes d'indépendance au lieu de la seule notion d'indépendance utilisée dans le cas classique: l'indépendance tensorielle, qui est juste une transposition pure et simple de la définition classique, la liberté, l'indépendance booléenne, l'indépendance monotone et l'indépendance antimonotone. On peut montrer par exemple un théorème central limite dans le cas des indépendances tensorielle et libre.
La Théorie des groupes se généralise habituellement dans le cas non-commutatif par les différentes notions de groupes quantiques. La notion la plus simple est celle de groupe quantique compact, qui consiste en C^*-algèbres unifères munies d'un $*$-homomorphisme Δ que l'on appelle le coproduit, défini sur l'algèbre et qui prend ses valeurs dans le produit tensoriel de l'algèbre avec elle-même. On demande que ce coproduit vérifie une propriété de coassociativité $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ et que l'algèbre vérifie une propriété de densité appellée "règle de simplification quantique" ou "Quantum Cancellation Property". On peut montrer que les groupes compacts classiques sont en particulier des groupes quantiques compacts.
Le domaine des groupes quantiques a été étudié en profondeur et est encore un sujet de recherche très actif. La partie II de ce travail est consacrée à l'étude des groupes quantiques compacts orthogonal et de permutation, qui généralisent le groupe orthogonal et le groupe symétrique dans le cas classique. Plus précisément, on s'intéresse à des semi-groupes de Markov définis sur ces deux groupes quantiques. Dans le cas du groupe quantique orthogonal, on s'intéresse au semi-groupe dont le générateur est:

$$A(u_{ij}^{(s)}) = \frac{1}{U_s(N)} \left[-bU_s'(N) + \int_{-N}^{N} \frac{U_s(x) - U_s(N)}{N - x} \nu(dx) \right] u_{ij}^{(s)}$$

Et dans le cas du groupe quantique de permutation, le générateur du semi-groupee étudié est:

$$A(u_{ij}^{(s)}) = -\frac{u_{ij}^{(s)} U_{2s}'(\sqrt{N})}{2\sqrt{N} U_{2s}(\sqrt{N})}$$

ces semi-groupes sont particulièrement intéressants car ils vérifient une propriété connue sous le nom d'*ad*-invariance. Il est donc naturel de les considérer dans une première approche de l'étude des semi-groupes sur ces groupes quantiques.
On s'intéresse particulièrement aux propriétés d'hypercontractivité. Il est bien connu qu'un opérateur T est contractant s'il existe $k < 1$ tel que $\|Tx\| \leq k\|x\|$ pour tout x dans le domaine de définition de l'opérateur. L'hypercontractivité quant à elle est un résultat plus fort. Dans les espaces dans lesquels on peut définir des p-normes à l'image des espaces L^p classiques — et l'on donnera un sens précis à cette phrase dans la partie II grâce aux algèbres de von Neumann —, un semi-groupe $(T_t)_t$ est dit hypercontractant si pour $p < q$ il existe τ_{pq} tel que $\|T_t x\|_q \leq \|x\|_p$ pour tout $t \geq \tau_{pq}$. C'est un résultat plus fort que la simple contractivité, car la q-norme est en général beaucoup plus grande que la p-norme. On montre dans le chapitre 4 que les deux semi-groupes précédents sont hypercontractants. Il n'a cependant pas été possible pour le moment

de trouver un temps optimal, c'est-à-dire un τ_{pq} qui soit minimal. La fin du chapitre donne d'autres résultats qui sont liés à l'hypercontractivité, tels que l'existence d'un trou spectral et des inégalités logarithme-Sobolev. Des recherches ultérieures devront certainement tenter de clarifier le temps optimal, ainsi que d'étendre ces résultats à d'autres semi-groupes et à d'autres groupes quantiques compacts.

Si les groupes quantiques sont la généralisation la plus courante de la notion de groupe dans le domaine des Mathématiques non-commutatives, une autre généralisation est possible, celle de groupes duaux, introduits par Voiculescu [47], également appelés H-algèbres par Zhang [52]. L'idée est similaire à celle de groupes quantiques, cependant le coproduit Δ prend à présent ses valeurs dans le produit de l'algèbre avec elle-même et non plus dans le produit tensoriel, le produit libre étant entendu comme étant défini dans la catégorie des $*$-algèbres unifères. Par ailleurs, on n'impose plus la "règle de simplification quantique", mais on requiert l'existence d'une coünité, à savoir d'un $*$-homomorphisme $\epsilon : A \to \mathbb{C}$ qui joue le rôle de l'élément neutre des groupes classiques, et d'un coïnverse, ou antipode, à savoir d'un $*$-homomorphisme $\Sigma : A \to A$ qui joue le rôle de l'opération "inverse" dans les groupes classiques. Il convient de noter que contrairement à l'antipode S des algèbres de Hopf qui est un antihomomorphisme (c'est-à-dire que $S(ab) = S(b)S(a)$), le coïnverse d'un groupe dual est un homomorphisme. Les groupes duaux, quoiqu'introduits déjà dans les années 80, n'ont pas fait l'objet de recherches aussi intensives que leurs analogues quantiques. Un de mes objectifs primaire durant ma thèse a été de mieux comprendre ces objets.

Les processus de Lévy (classiques) sont une classe de processus stochastiques très importants en Probabilités. Ils se définissent de manière la plus générale possible sur des semi-groupes en tant que processus càdlàg (continus à droite, avec limites à gauche) à accroissements stationnaires et indépendants. Le mouvement Brownien dans $\mathbb{R}^d$ par exemple est un processus de Lévy. On peut en définir des analogues non-commutatifs, appelés processus de Lévy quantiques, sur des groupes duaux. En toute généralité, on pourrait les définir sur des semi-groupes duaux, sans beaucoup plus de difficultés, mais l'intérêt premier du travail de thèse s'étant focalisé sur les groupes duaux, c'est dans ce cadre là qu'ils seront définis dans la section 3.4. Ils sont définis comme des processus stochastiques quantiques faiblement continus à accroissement stationnaire et indépendant, mais il faut prendre garde ici que l'indépendance peut être prise dans un des cinq sens mentionnés ci-dessus. Le but est d'étudier les processus de Lévy sur des groupes duaux afin de mieux comprendre ces derniers.

La partie III s'intéresse aux groupes duaux. En particulier, le chapitre 5 s'intéresse à un processus de Lévy particulier sur le groupe dual unitaire, celui obtenu à la limite lorsqu'on regarde le mouvement Brownien à valeurs dans le groupe unitaire (classique) $U(nd)$ bloc $n \times n$ par bloc $n \times n$ et que l'on fait tendre n vers l'infini. On le décrit en particulier par une équation différentielle stochastique quantique et l'on montre qu'il vérifie une propriété de gaussianité telle que définie par Schürmann [39]. On prétend par conséquent qu'il serait un bon candidat au titre de mouvement Brownien sur le groupe dual unitaire. Puis, le chapitre 6 s'intéresse de manière générale à tous les processus de Lévy sur le groupe dual unitaire qui peuvent s'obtenir comme limites de processus de Lévy sur le groupe unitaire lorsqu'on les regarde par blocs et que l'on fait tendre la dimension vers l'infini. On décrit en particulier tous ces processus de Lévy.

Enfin, le chapitre 7 s'intéresse à la question de savoir si l'on peut définir un état de Haar sur les groupes duaux. Les mesures de Haar sont très importantes dans le cas classique et sont à la base de l'Analyse harmonique. Elles se définissent sur des groupes topologiques comme des mesures invariantes par translation à gauche. La mesure de Lebesgue en est l'exemple typique sur $\mathbb{R}^d$. Les groupes quantiques compacts possèdent toujours un état de Haar. Il s'agit d'une forme linéaire $h : A \to \mathbb{C}$ qui vérifie:

$$h(1) = 1, \qquad h(x^*x) \geq 0 \text{ pour tout } x \in A \tag{1}$$

$$(h \otimes id) \circ \Delta(x) = h(x)1_A = (id \otimes h) \circ \Delta \text{ pour tout } x \in A \tag{2}$$

Les états de Haar permettent de construire une théorie de l'Analyse Harmonique sur les groupes quantiques duaux. Il est donc naturel et potentiellement intéressant de se demander si l'on peut définir des états de Haar sur des groupes duaux. On pourrait imaginer de transposer simplement la définition, en remplaçant le produit tensoriel de l'équation 2 par un des produits universels provenant des cinq notions d'indépendance possibles. Le chapitre 7 montre que cela ne convient pas car on peut montrer que, sauf pour $n = 1$, il n'existe pas d'état de Haar sur le groupe dual unitaire $U\langle n\rangle$. Cependant, dans le cas de $U\langle n\rangle$, on peut montrer que l'on peut définir ce que l'on convient d'appeler des traces de Haar, à savoir des états traciaux qui sont absorbant dans l'ensemble des autres états traciaux. La recherche dans ce domaine n'en est certainement qu'à son début, et la suite sera certainement de s'intéresser à d'autres groupes duaux possibles.
Bien sûr, comme dans tout travail de thèse, il est important d'introduire les outils de base et les résultats classiques nécessaires à la compréhension du travail de recherche lui-même. C'est l'objet de la partie I. Plus précisément, le chapitre 1 introduit les notions algébriques fondamentales, telles que celles de $*$-algèbre, C^*-algèbres, produit libre d'algèbres etc., puis le chapitre 2 présente les groupes quantiques compacts et les groupes duaux, et enfin le chapitre 3 présente les bases de la théorie des probabilités quantiques et en particulier les processus de Lévy. A chaque fois, les notions de base nécessaires à la compréhension des résultats obtenus durant la thèse sont présentées ; cependant, il n'est bien sûr pas possible de réintroduire toutes les notions mathématiques. Il convient donc de souligner que l'on supposera que le lecteur est familier des notions les plus fondamentales, en particulier de la théorie des Probabilités et de l'Algèbre.

0.4 Convention, notations and sources

Before entering into the subject itself, we shall give a few remarks about the conventions and notations used and the sources of this paper. As this Doctoral thesis is written in English, the rules of style of the English language are of course used. We shall nevertheless stick to the "French" convention regarding the notation for the set of integer numbers, ie $\mathbb{N}$ represents the set of all integer numbers, including zero. If we want to exclude zero, we denote it by $\mathbb{N}^*$. We make this deliberate choice as we believe that it yields notations that are easier. All the rest is written according to English conventions.
We also want to stress, as was mentionned in the French summary in Section 0.3 that, though we introduce in Part I the basic tools that will be needed in the sequel of the work, we assume that the reader is familiar with the most basic notions and concepts of Algebra and (classical) Probability Theory, as it would not be possible to reintroduce every mathematical concept.
Finally, as every work of research, mathematical research depends heavily on collaboration. Thus, many of the results presented here were obtained in collaboration between me and other researchers. In particular, chapters 6 and 7 were obtained by Cébron and me in [11], and chapter 4 was obtained by Franz, Hong, Lemeux and me in [19]. Chapter 5 is a result that I obtained at the beginning of my PhD studies and that was published in [44]. These chapters are mainly copies of these articles, without the introductions – because this thesis contains a separate part devoted to introducing all the necessary concepts – and with some corrections to make them even clearer.
For the introduction of the basic notions in Part I, I relied on standard works that already

did a very good job. In chapter 1, we have sometimes — but not always — followed the PhD thesis of Voss [49] who presented the matters with great clarity. In chapter 2, we have written a presentation of compact quantum groups inspired from [14] and from the presentation of compact quantum groups that was already in [19], whereas the presentation of dual groups is the one contained in my article with Cebron [11]. In Chapter 3, we relied heavily on [35].

Part I

Entering the noncommutative world

Curiouser and curiouser.

Lewis Carroll, *Alice in Wonderland*

1

Noncommutative Mathematics

1.1 Tools that are necessary

In the course of this work, we will enter, not Alice's Wonderland, but the world of noncommutative Mathematics. To do this, this section shall introduce a certain number of concepts that will be necessary in order to talk about the objects we will be interested in. Among those notions are $*$-algebras, C^*-algebras, von Neumann algebras and free products in these categories. We assume, though, that the basic ideas of linear algebra and of Hilbert space theory are known to the reader. We follow sometimes closely [49], to whom we are thankful for his clear presentation.

1.1.1 Algebras

The basic objects that we will consider are algebras. These can show up in various forms.

Definition 1.1.1. *An algebra $\mathcal{A}$ on the field K is a vector space on K endowed with a multiplication*

$$\begin{aligned} . \quad : \quad & \mathcal{A} \times \mathcal{A} \to \mathcal{A} \\ & (x, y) \mapsto xy \end{aligned}$$

such that for any $x, y, z \in \mathcal{A}$ and any $\lambda, \mu \in K$, we have:

$$\begin{aligned} (x+y)z &= xz + yz \\ x(y+z) &= xy + xz \\ (\lambda x)(\mu y) &= (\lambda\mu)xy \\ (xy)z &= x(yz) \end{aligned}$$

If there exists a unit for the multiplication, the algebra is said to be unital and the unit is denoted $1_{\mathcal{A}}$, or simply 1 when there is no confusion possible. If the multiplication is commutative, the algebra is said to be commutative.

We define a subalgebra of $\mathcal{A}$ as being a subspace that is stable for the multiplication. A unital subalgebra of the unital algebra $\mathcal{A}$ is defined likewise as being a subalgebra which contains the unit of $\mathcal{A}$.
The algebras can have some richer structures.

Definition 1.1.2.
- *An antilinear map f on a vector space V is a map $f : V \to V$ such that for any $x, y \in V$ and any $\lambda \in K$ the relation $f(\lambda x + y) = \overline{\lambda} f(x) + f(y)$ stands, where $\bar{}$ stands for complex conjugation.*
- *An involution $*$ on a vector space V is a antilinear map $* : V \to V$ such that $* \circ * = \mathrm{id}$. We sometimes call x^* the adjoint of x.*
- *A $*$-vector space V is a vector space endowed with an involution*
- *A $*$-vector space homomorphism $f : V \to V$ is a linear map such that for any $v \in V$ we have $f(v^*) = f(v)^*$.*
- *A $*$-algebra $\mathcal{A}$ is an algebra over a $*$-vector space such that the multiplication is a $*$-vector space homomorphism.*
- *A $*$-algebra homomorphism f is an algebra homomorphism that is also a $*$-vector space homomorphism. In the sequel, whenever we speak of a $*$-homomorphism, we will refer to a $*$-algebra homomorphism.*
- *A sub-$*$-algebra of the $*$-algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ that is stable for the involution.*

We can also put some topology on an algebra. We give these definitions for completeness' sake and also because we will need them when we will study hypercontractivity in chapter 4 but this is the only place of the present work where these definitions will be needed. Indeed, most of our work rests on algebraic and combinatoric arguments, without use of analytical arguments.

Definition 1.1.3. *A C^*-algebra $\mathcal{A}$ is a $*$-algebra equipped with a Banach norm (ie a complete norm) $\|.\|$ such that for any $x \in \mathcal{A}$:*

$$\|xx^*\| = \|x\|^2$$

The morphisms in this category are the $*$-homomorphisms.

Let us now introduce von Neumann algebras.

Definition 1.1.4. *Let $\mathcal{A} \subset B(H)$ a $*$-algebra of bounded operators on some Hilbert space H. The algebra $\mathcal{A}$ is called a von Neumann algebra if it contains the identity operator and is equipped with the coarsest topology such that the maps*

$$\begin{aligned} \phi_x \quad : \quad & \mathcal{A} \to \mathbb{C} \\ & M \mapsto M(x) \end{aligned}$$

are continuous for any $x \in H$.

We have some important properties of von Neumann algebras:

Proposition 1.1.5. *A von Neumann algebra A is a C^*-algebra*

Let X be a subset of the algebra $B(H)$ of bounded operators on H. The commutant of X is denoted X' and is the set containing all y such that $xy = yx$ for any $x \in X$.

Theorem 1.1.6 (Von Neumann's bicommutant Theorem)**.** *Let $\mathcal{A}$ be a $*$-algebra of bounded linear operators on some Hilbert space H. let us assume that $\mathcal{A}$ containes the identity operator. Then the smallest von Neumann algebra containing $\mathcal{A}$, called the von Neumann algebra generated by $\mathcal{A}$, is equal to $(A')' = A''$ (bicommutant of $\mathcal{A}$).*

1.1.2 Operations between algebras

We will need for our study the concepts of tensor products and free products.

Definition 1.1.7. *Let V and W be two vector spaces. The tensor product $V \otimes W$ of V and W is a vector space associated to a map $\Phi : V \times W \to V \otimes W$ satisfying an universal property, namely: for any vector space X and any bilinear map $B : V \times W \to Z$, there exists a unique linear map $\tilde{g} : V \otimes W \to Z$ such that $g = \tilde{g} \circ \Phi$.*

$$\begin{tikzcd} V \times W \arrow[d, "g"] \arrow[dr, "\Phi"] & \\ Z & V \otimes W \arrow[l, "\tilde{g}"] \end{tikzcd}$$

The tensor product of V and W can be seen as being the vector space $V \times W$ quotiented by the vector space spanned by the following relations, for any $x, y \in V$, $a, b \in W$ and $\lambda \in K$:

$$\begin{aligned} &(x + y, a) - (x, a) - (y, a) \\ &(x, a + b) - (x, a) - (x, b) \\ &(\lambda x, a) - \lambda(x, a) \\ &(x, \lambda a) - \lambda(x, a) \end{aligned}$$

In this quotient vector space we denote by $x \otimes a$ the image of (x, a) by the quotient map. The set of all $x \otimes a$ generates $V \otimes W$, and, more precisely, if $(x_i)_i$ is a basis of V and $(y_j)_j$ is a basis of W, then $(x_i \otimes y_j)_{i,j}$ is a basis of $V \otimes W$.

Proposition 1.1.8. *Let $f : V \to X$ and $g : W \to Y$ be two linear maps. Then, one can define a linear map*

$$\begin{aligned} f \otimes g \quad : \quad & V \otimes W \to X \otimes Y \\ & v \otimes w \mapsto f(v) \otimes g(w) \end{aligned}$$

If A_1 and A_2 are two algebras, we can equipp $A_1 \otimes A_2$ with an algebra structure.

Proposition 1.1.9. *The vector space $A_1 \otimes A_2$ is endowed with an algebra structure when one takes the following multiplication:*

$$(x \otimes y, a \otimes b) \mapsto (xa) \otimes (yb)$$

Let us remark that in the tensor product $A \otimes B$, we often talk of A as the left leg of the tensor product, whereas B is the right leg.

Definition 1.1.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital $*$-algebras. The* free product *of $\mathcal{A}$ and $\mathcal{B}$ is the unique unital $*$-algebra $\mathcal{A} \sqcup \mathcal{B}$ with $*$-homomorphisms $i_1 : \mathcal{A} \to \mathcal{A} \sqcup \mathcal{B}$ and $i_2 : \mathcal{B} \to \mathcal{A} \sqcup \mathcal{B}$ such that, for all $*$-homomorphisms $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$, there exists a unique $*$-homomorphism $f \sqcup g : \mathcal{A} \sqcup \mathcal{B} \to \mathcal{C}$ such that $f = (f \sqcup g) \circ i_1$ and $g = (f \sqcup g) \circ i_2$.*

$$\begin{tikzcd} & A \sqcup B \arrow[dl, "i_1"'] \arrow[d, "f \sqcup g"] \arrow[dr, "i_2"] & \\ A \arrow[r, "f"'] & C & B \arrow[l, "g"'] \end{tikzcd}$$

Informally, $\mathcal{A} \sqcup \mathcal{B}$ corresponds to the "smallest" $*$-algebra containing $\mathcal{A}$ and $\mathcal{B}$ and such that there is no relation between $\mathcal{A}$ and $\mathcal{B}$ except the fact that the unit elements are identified. We usually say that $\mathcal{A}$ is the left leg of $\mathcal{A} \sqcup \mathcal{B}$, whereas $\mathcal{B}$ is its right leg, and, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we denote $i_1(A)$ by $A^{(1)}$ and $i_2(B)$ by $B^{(2)}$. This terminology is particularly useful when we consider the free product $\mathcal{A} \sqcup \mathcal{A}$ of $\mathcal{A}$ with itself, because, in this case, there exist two different ways of thinking about $\mathcal{A}$ as a subset of $\mathcal{A} \sqcup \mathcal{A}$. Of course, if $\mathcal{A}$ and $\mathcal{B}$ are clearly distinguished, we can avoid this subscript and identify $\mathcal{A}$ with $i_1(\mathcal{A})$ and $\mathcal{B}$ with $i_2(\mathcal{B})$.For $*$-homomorphisms $f : \mathcal{A} \to \mathcal{C}$, $g : \mathcal{B} \to D$, we denote by $f \coprod g$ the $*$-homomorphism $(i_{\mathcal{C}} \circ f) \sqcup (i_D \circ g) : \mathcal{A} \sqcup \mathcal{B} \to \mathcal{C} \sqcup \mathcal{D}$. Let A_1 and A_2 be two unital $*$-algebras. For $n \in \mathbb{N}$, we set

$$\mathbb{A}_n = \{(\epsilon_1, \dots, \epsilon_n) | \epsilon_i \in \{1, 2\}, \epsilon_{i+1} \neq \epsilon_i \forall_{1 \leq i \leq n}\}$$

the set of alternating multi-index in 1 and 2. Then, the free product of A_1 and A_2 can be build as:

$$A_1 \sqcup A_2 = \mathbb{C} \oplus \bigoplus_{n \in \mathbb{N}^*} \bigoplus_{\epsilon \in \mathbb{A}_n} A_{\epsilon_1} \otimes \ldots \otimes A_{\epsilon_n}$$

Remark 1.1.11. We may also consider sometimes nonunital $*$-algebras. In this case we can consider the nonunital free product of $\mathcal{A}$ and $\mathcal{B}$ (the free product in the category of nonunital $*$-algebras), denoted $\mathcal{A} \tilde{\sqcup} \mathcal{B}$, and constructed as being:

$$\mathcal{A} \tilde{\sqcup} \mathcal{B} = \bigoplus_{n \in \mathbb{N}^*} \bigoplus_{\epsilon \in \mathbb{A}_n} A_{\epsilon_1} \otimes \ldots \otimes A_{\epsilon_n}$$

It verifies the same universal property, with the difference that $\mathcal{C}$ is now a nonunital $*$-algebra and the homomorphisms do not preserve the unit (which, anyways, does not exist...)
This notion will be useful only in Section 3.2 for defining boolean, monotone and antimonotone independence in Quantum Probability.

Let us remark that in the free product $A \sqcup B$, similar to what happened in the tensor product case, we often talk of A as the left leg of the free product, whereas B is the right leg.

1.1.3 Bialgebras and Hopf algebras

To finish this section, let us introduce two objects that will be useful at some point later in this work, bialgebras and Hopf algebras.

Definition 1.1.12. *A bialgebra B is an unital algebra equipped with two unital homomorphisms $\Delta : B \to B \otimes B$ and $\epsilon : B \to \mathbb{C}$ such that:*

$$\begin{aligned} (id \otimes \Delta) \circ \Delta &= (\Delta \otimes id) \circ \Delta \\ (id \otimes \epsilon) \circ \Delta &= id = (\epsilon \otimes id) \circ \Delta \end{aligned}$$

The map Δ is called coproduct, whereas ϵ is called counit.
If B has an involution $*$ and the maps Δ and ϵ are $*$-homomorphisms, we say that B is an involutive bialgebra.

Definition 1.1.13. *A bialgebra B is called a Hopf algebra is there exists a linear map $S : B \to B$, called the antipode, and such that:*

$$\mu \circ (S \otimes id) \circ \Delta = 1 \circ \epsilon = \mu \circ (id \otimes S) \circ \Delta$$

where $\mu : B \otimes B \to B$ is the multiplication map and $1 : \mathbb{C} \to B$ the linear map sending the unit of $\mathbb{C}$ to the unit of B.

We define in the same way a Hopf-$*$-algebra when the bialgebra has an involution $*$ and all the maps behave well with regard to this involution.

2

Quantum groups and Dual groups

2.1 From Classical Groups to quantum groups

2.1.1 Compact quantum group: definition

Compact quantum groups are a generalization of compact groups in the context of non-commutative mathematics. For this presentation, we rely on [14]. They are defined in the following way:

Definition 2.1.1. *A compact quantum group is a pair $\mathbb{G} = (A, \Delta)$ such that A is a unital C^*-algebra and $\Delta : A \to A \otimes A$ is a comultiplication, ie it is a unital $*$-algebra homomorphism and it verifies:*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$$

and, moreover, the quantum cancellation properties are verified, ie:

$$\overline{Lin}[(1 \otimes A)\Delta(A)] = \overline{Lin}[(A \otimes 1)\Delta(A)] = A \otimes A$$

where $\overline{Lin}$ is the norm-closure of the linear span.
The C^*-algebra A is also noted $C(\mathbb{G})$.

It is indeed a generalization, because for any compact group G, $A = C(G)$ is a C^*-algebra. We then equipp it with the comultiplication arising from the group multiplication:

$$\Delta_G : \quad \begin{array}{c} C(G) \to C(G \times G) \simeq C(G) \otimes C(G) \\ f \mapsto ((x, y) \mapsto f(x.y)) \end{array}$$

Then (A, Δ_G) is a compact quantum group. The relevant examples for this article were defined by Wang, see [17, 50, 51]:

Example 2.1.2 (Free Orthogonal Quantum Group, see [50])**.** *Let $N \geq 2$ and $C_u(O_N^+)$ be the universal unital C^*-algebra generated by the N^2 self-adjoint elements $u_{ij}, 1 \leq i, j \leq N$ verifying the relations:*

$$\sum_k u_{ki}u_{kj} \quad = \quad \delta_{ij} = \sum_k u_{ik}u_{jk}$$

We define a comultiplication Δ by setting $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$. Then $(C(O_N^+), \Delta)$ is a compact quantum group called the Free Orthogonal Quantum Group. If we impose in addition commutativity, we recover the classical orthogonal group.

Example 2.1.3 (Free Permutation Quantum Group, see [51]). *Let $N \geq 2$ and $C(S_N^+)$ be the universal unital C^*-algebra generated by N^2 elements $u_{ij}, 1 \leq i, j \leq N$ such that for all $1 \leq i, j \leq N$:*

$$u_{ij}^2 = u_{ij} = u_{ij}^*$$
$$\sum_k u_{ik} = 1 = \sum_k u_{kj}$$

We define a comultiplication Δ by setting $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. Then, $(C(S_N^+), \Delta)$ is a compact quantum group called the Free Permutation Quantum Group. If we impose in addition commutativity, we find the classical permutation group.

For $\mathbb{G} = O_N^+, S_N^+$, we denote by $\mathrm{Pol}(\mathbb{G})$ the $*$-algebra generated by the generators $u_{ij}, 1 \leq i, j \leq N$ and contained in $C(\mathbb{G})$. It has a bialgebra structure by setting:

$$\epsilon(u_{ij}) = \delta_{ij}$$

It is called the algebra of polynomials of $\mathbb{G}$.
Moreover, every compact quantum group is endowed with a Haar state, ie a function $h : C(\mathbb{G}) \to \mathbb{C}$ such that $(h \otimes id)\Delta(a) = h(a)1 = (id \otimes h)\Delta(a)$ for each $a \in A$.
Now, the Haar state will allow us to define the reduced C^*-algebra of a compact quantum group. If $\mathbb{G}$ is a compact quantum group, then we may construct the GNS representation of its Haar state h, ie there exist a $*$-homomorphism $\pi : \mathrm{Pol}(\mathbb{G}) \to B(H)$ with H a Hilbert space, such that $h(x) =< \Omega, \pi(x)\Omega >$ for all $x \in H$ and a specific $\Omega \in H$. Then, the reduced C^*-algebra $C_r(\mathbb{G})$ will be the norm completion of $\pi(\mathrm{Pol}(\mathbb{G}))$ in $B(H)$. In all the sequel of the article, we will always consider the reduced C^*-algebra instead of the universal one. The reason for this is that the Haar state is always faithful on the reduced C^*-algebra, but not on the universal one. The faithfulness of h is important to define the L^p spaces, which is done as follows. The space $L^\infty(\mathbb{G}) = C_r(\mathbb{G})''$ is the von Neumann algebra generated by $C_r(\mathbb{G})$. We then define for any $1 \leq p < \infty$, the $L^p(\mathbb{G})$ space as the completion of $L^\infty(\mathbb{G})$ for the norm $\|x\|_p = [h((x^*x)^{p/2})]^{1/p}$.
We also recall that the Haar state is a trace (ie $h(ab) = h(ba)$) whenever the compact quantum group is of Kac type, which is the case with O_N^+ and S_N^+.
Let us now say a few words about corepresentations, for more details and notations we refer to [14, 23]. For any compact quantum group, there is a notion of corepresentations, ie of matrices $v \in \mathcal{M}_k \otimes C(\mathbb{G})$ such that $(id \otimes \Delta)(v) = v_{12}v_{23}$ where the indices explain on which leg of $\mathcal{M}_k \otimes C(\mathbb{G}) \otimes C(\mathbb{G})$ v acts. Thus, v_{12} is $v \in \mathcal{M}_k \otimes C(\mathbb{G})$ seen as having its $C(\mathbb{G})$ elements coming from the middle leg of $\mathcal{M}_k \otimes C(\mathbb{G}) \otimes C(\mathbb{G})$ v. We may as well reformulate it this way:

Proposition-Definition 2.1.4 (Corepresentations). *A matrix $v \in \mathcal{M}_k \otimes C(\mathbb{G})$ is a k-dimensional corepresentation of the compact quantum group $\mathbb{G}$ if for any $1 \leq i, j \leq k$ the matricial element v_{ij} verifies:*

$$\Delta(v_{ij}) = \sum_{p=1}^{k} v_{ip} \otimes v_{pj}$$

This definition is equivalent to saying that $(id \otimes \Delta)(v) = v_{12}v_{23}$.
The matricial elements v_{ij}, for $1 \leq i, j \leq k$, are called the coefficients of the corepresentation v.

There is as well a notion of irreducible corepresentations:

Definition 2.1.5. *A corepresentation v of the compact quantum group $\mathbb{G}$ is said to be irreducible if the only matrices $T \in \mathcal{M}_k$ such that $Tv = vT$ holds are multiple of the identity.*

The set of all irreducible corepresentations is denoted $Irr(\mathbb{G})$. In the case of O_N^+ and S_N^+, the irreducible corepresentations can be indexed by $\mathbb{N}$ and we denote by $(u_{ij}^{(s)})_{1\leq i,j\leq \dim V_s}$ the coefficients of the s^{th} irreducible corepresentation, V_s being the linear span of them.

2.2 Dual Groups

The notion of dual groups was introduced by Voiculescu [47] in the 80's. We consider here the purely algebraic version. Like Hopf algebras, the idea is to generalize the notion of groups to a noncommutative setting by replacing the product by a coproduct, but we now use the free product instead of the tensor product.

Definition 2.2.1. *A* dual group $G = (\mathcal{A}, \Delta, \delta, \Sigma)$ *(in the sense of Voiculescu) is a unital $*$-algebra $\mathcal{A}$, and three unital $*$-homomorphisms $\Delta : \mathcal{A} \to \mathcal{A} \sqcup \mathcal{A}$, $\delta : \mathcal{A} \to \mathbb{C}$ and $\Sigma : \mathcal{A} \to \mathcal{A}$, such that*

- *The map Δ is a* coassociative coproduct*:* $(\mathrm{Id} \bigsqcup \Delta) \circ \Delta = (\Delta \bigsqcup \mathrm{Id}) \circ \Delta$
- *The map δ is a* counit*:* $(\delta \bigsqcup \mathrm{Id}) \circ \Delta = \mathrm{Id} = (\mathrm{Id} \bigsqcup \delta) \circ \Delta$
- *The map Σ is a* coinverse*:* $(\Sigma \sqcup \mathrm{Id}) \circ \Delta = 1_{\mathcal{A}} \circ \delta = (\mathrm{Id} \sqcup \Sigma) \circ \Delta$.

Let us give a couple of examples of such structures. We first introduce the unitary dual group$U\langle n\rangle$, first considered by Brown in [9], and which possesses naturally a structure of dual group. It has to be considered as the noncommutative analog of the classical unitary group.

Example 2.2.2. *Let $n \geq 1$. The* unitary dual group *is the dual group $U\langle n\rangle = (U_n^{\mathrm{nc}}, \Delta, \delta, \Sigma)$ where:*

- *The universal unital $*$-algebra U_n^{nc} is generated by n^2 elements $(u_{ij})_{1\leq i,j\leq n}$ with the relations*
$$\sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^{n} u_{ik} u_{jk}^*.$$
- *The coproduct is given on the generators by $\Delta(u_{ij}) = \sum_k u_{ik}^{(1)} u_{kj}^{(2)}$.*
- *The counit is given by $\delta(u_{ij}) = \delta_{ij}$.*
- *The antipode is given by $\Sigma(u_{ij}) = u_{ji}^*$.*

Let us remark that the relations defining U_n^{nc} can be summed up by saying that $u = (u_{ij})_{1\leq i,j\leq n}$ is a unitary matrix in $\mathcal{M}_n(U_n^{\mathrm{nc}})$. We do not suppose that $\bar{u} = (u_{ij}^*)_{1\leq i,j\leq n}$ is unitary. Indeed, unlike the relations $\sum_{k=1}^n u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^n u_{ik} u_{jk}^*$, the relations $\sum_{k=1}^n u_{ik}^* u_{jk} = \delta_{ij} = \sum_{k=1}^n u_{ki} u_{kj}^*$ do not pass the coproduct Δ, since we cannot simplify expressions like $\Delta(\sum_k u_{ik}^* u_{jk}) = \sum_{k,p,q} u_{pk}^{(2)*} u_{ip}^{(1)*} u_{jq}^{(1)} u_{qk}^{(2)}$ to δ_{ij}.
We will give another example which has no counterpart in the world of compact quantum groups, namely the tensor algebra:

Example 2.2.3. *Let V be a $*$-vector space. We define the tensor algebra $T(V)$ by:*

$$T(V) = \mathbb{C} \oplus \bigoplus_{k\geq 1} V^{\otimes k}$$

It has the structure of an unital $*$-algebra when we equipp it with the multiplication and the involution:

$$\begin{aligned} m \quad &: \quad T(V) \times T(V) \to T(V) \\ &\quad\ (v_1 \otimes \ldots \otimes v_k, w_1 \otimes \ldots \otimes w_l) \mapsto v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_l \\ * \quad &: \quad T(V) \to T(V) \\ &\quad\ v_1 \otimes \ldots \otimes v_k \mapsto v_k^* \otimes \ldots \otimes v_1^* \end{aligned}$$

The tensor algebra is characterized by an universal property, namely for any algebra A and any linear map $f : V \to A$, there exists a unique $*$-homomorphism $T(f)$ such that the following diagramm commutes:

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "f"'] & T(V) \arrow[dl, "T(f)"] \\ A & \end{tikzcd}$$

where $i : V \to T(V)$ is the canonical inclusion identifying V with $V^{\otimes 1}$ in $T(V)$.
We define the following $*$-homomorphisms by their values on generators:

$$\begin{aligned} \Delta \quad &: \quad T(V) \to T(V) \sqcup T(V) \\ &\quad\ v \in V \mapsto v^{(1)} + v^{(2)} \\ &\quad\ \delta = T(0) \\ \Sigma \quad &: \quad T(V) \to T(V) \\ &\quad\ v \in V \mapsto -v \end{aligned}$$

where $v^{(1)}$, resp. $v^{(2)}$, designates the element v taken from the left, resp. right, leg of $T(V) \sqcup T(V)$, and where δ is obtained from the zero linear map through the univeral property of the tensor algebra.

Following the point of view of the theory of quantum groups, we consider the $*$-algebra $\mathcal{A}$ as a set of "functions on the dual group G", and not as the dual group. This terminology of *dual group* can be ambiguous and one could prefer the term *H-algebras* used by Zhang in [52], or the term *co-group* used by Bergman and Hausknecht in [4]. However, in the following remark, we will see that the duality can be seen as the existence of some particular functor.

Remark 2.2.4. 1. Let **Alg** be the category of unital $*$-algebras. The dual category $\mathbf{Alg}^{\mathrm{op}}$ is the category **Alg** with all arrows reversed. The definition of a dual group has the immediate consequence that *an element $\mathcal{A}$ in the category* **Alg** *which defines a dual group $G = (\mathcal{A}, \Delta, \delta, \Sigma)$ has a group structure in the dual category* $\mathbf{Alg}^{\mathrm{op}}$, in the following sense of [10, Chapter 4]: we have the commutativity of all the diagrams obtained from the diagrams defining a classical group by replacing the product by Δ^{op}, the unit map by δ^{op} and the inverse map by Σ^{op}. Remark that $\mathbf{Alg}^{\mathrm{op}}$ is not a concrete category: the morphism Δ^{op} in $\mathbf{Alg}^{\mathrm{op}}$ can not be seen as an actual function from $\mathcal{A} \sqcup \mathcal{A}$ to $\mathcal{A}$. Nevertheless, as shown in [10, Chapter 4], this group structure is sufficient to endow naturally the set $\mathrm{Hom}_{\mathbf{Alg}^{\mathrm{op}}}(\mathcal{B}, \mathcal{A})$ of morphisms of $\mathbf{Alg}^{\mathrm{op}}$ from any unital $*$-algebra $\mathcal{B}$ to $\mathcal{A}$ with a classical structure of group.

2. As a consequence, for any unital $*$-algebra $\mathcal{B}$, the set $\mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A},\mathcal{B})$ of the unital $*$-homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ is a group. Moreover, one can verify that $\mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A},\cdot) : \mathcal{B} \mapsto \mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A},\mathcal{B})$ is a functor from **Alg** to the category of groups **Gr**. Conversely, if a unital $*$-algebra $\mathcal{A}$ is such that $\mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A},\cdot)$ is a functor from **Alg** to **Gr**, then $G = (\mathcal{A},\Delta,\delta,\Sigma)$ is a dual group for some particular Δ, δ and Σ (see [52] for a direct proof, or [10, Chapter 4] for a proof of the dual statement about $\mathbf{Alg}^{\mathrm{op}}$). We can summarize those considerations saying that *dual groups are in one-to-one correspondence with the representing objects of the functors from* **Alg** *to* **Gr**. As a comparison, commutative Hopf algebras are the representing objects of the functors from the category of unital commutative algebras to **Gr**.

3. Now, starting from a group G, one can ask the following question: is there a unital $*$-algebra $\mathcal{A}$ such that $\mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A},\cdot)$ is a functor and $\mathrm{Hom}_{\mathbf{Alg}}(\mathcal{A},\mathbb{C}) \simeq G$? If yes, there exist Δ, δ and Σ such that $(\mathcal{A},\Delta,\delta,\Sigma)$ is a dual group which can be called *a dual groupof* G (not unique). One of Voiculescu's motivation of [47] was to show that a dual action of a dual group of G on some operator algebra gives rise to an action of G on that operator algebra. For example, the unitary dual group$U\langle n\rangle$, the principal object of our study defined subsequently, is a dual group of the classical unitary group $U(n) = \{M \in \mathcal{M}_n(\mathbb{C}) : U^*U = I_n\}$ in the sense that $\mathrm{Hom}_{\mathbf{Alg}}(U_n^{\mathrm{nc}},\mathbb{C}) \simeq U(n)$.

Let us emphasize, in the following remark, the major differences between dual groups and compact quantum groups.

Remark 2.2.5.
1. Firstly, as for Hopf algebras, the definition is purely algebraic: we use only the idea of $*$-algebras and we do not need to consider some C^*-algebra. One possible direction of research is to consider a more analytic structure on dual groups which could lead to more powerful results.

2. The second difference is that the tensor product has here been replaced by the free product. The latter is in some way "more noncommutative" because in the case of the tensor product, the two legs of the product are still commuting. If we have gained in noncommutativity, we have lost in interpretation: while a classical (compact) group could always be seen as a (compact) quantum group via the isomorphism $C(G \times G) \simeq C(G) \otimes C(G)$, we do not have such an isomorphism any more and hence classical groups cannot be seen as special cases of dual groups.

3. Finally, let us also remark that we here impose to have $*$-homomorphisms which correspond to the idea of a neutral element and inverses, whereas in the quantum case we only imposed the quantum cancellation property. We know that this cancellation property, which in the classical case yields automatically groups, is in the quantum case somewhat weaker.

3

Quantum Probability

3.1 From Classical Probability to Quantum Probability

We refer to [35] for a comprehensive introduction to noncommutative probability. We follow it for the basic definitions.
In the axiomatization of Probability known since the time of Kolmogorov[1] all we need is to have a triple $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of a set Ω, a σ-algebra $\mathcal{F}$ on the set Ω and a probability measure $\mathbb{P}$. With these objects, one can build the whole of the theory, including its most interesting and deepest objects, as for instance the Brownian motion. But instead of taking this triple $(\Omega, \mathcal{F}, \mathbb{P})$, one can build the whole theory by starting with a couple $(\mathcal{A}, \mathbb{E})$ consisting of the (commutative) algebra $\mathcal{A}$ of all complex-valued random variables defined on Ω and an expectation $\mathbb{E}$ which replaces the probability measure.
If we investigate the properties of such a couple we see immediately that $\mathcal{A}$ is a commutative unital $*$-algebra, the involution being given by complex conjugation, and $\mathbb{E} : \mathcal{A} \to \mathbb{C}$ is an unital linear functional that is hermitian (ie, $\mathbb{E}(\overline{X}) = \overline{\mathbb{E}(X)}$) and positive (ie, $\mathbb{E}(X^*X) \geq 0$). Quantum Probability answers the question of what remains if one releases the condition that $\mathcal{A}$ be commutative.

Definition 3.1.1 ($*$-probability spaces)**.** *An (unital) $*$-noncommutative probability space $(\mathcal{A}, \phi)$ consists of an unital $*$-algebra over $\mathbb{C}$ and an unital, hermitian and positive linear functional $\phi : \mathcal{A} \to \mathbb{C}$. Therefore we have the properties:*

1. $\phi(1) = 1$
2. $\phi(x^*) = \overline{\phi(x)}$
3. $\phi(x^*x) \geq 0$

for any $x \in \mathcal{A}$.
The elements of $\mathcal{A}$ are called (noncommutative) random variables.
If we also have $\phi(xy) = \phi(yx)$ for any $x, y \in \mathcal{A}$, we say that the noncommutative probability space is tracial.
If for any $x \in \mathcal{A}$, we have the implication: $\phi(x^*x) = 0 \Rightarrow x = 0$, we say that the noncommutative probability space is faithful.

[1] Andreï Nikolaïevitch Kolmogorov, 25th April 1903, Tambov, Russia - 20th October 1987, Moscow, Soviet Union.

We have so far applied the philosophy of noncommutative mathematics explained in Section **??**. We want to stress again the fact that there is in general no more interpretation of such noncommutative spaces in "everyday life": though a (classical)probability space may serve as a model for the evolution of temperature, of the stock market, or a stack or tails game, the world of noncommutative probability space remains in the realm of abstract mathematics!

Remark 3.1.2. We could also define a (nonunital) $*$-probability space by lifting the condition that $\mathcal{A}$ needs to unital and lifting condition 1 on ϕ. We will come back to this in Section 3.2, when we will talk about independence. For brevity's sake, if we talk only about a noncommutative probability space, without further information, we will refer to an unital $*$-noncommutative probability space.

Example 3.1.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space. Then, $(L^{\infty-}(\Omega), \mathbb{E})$ is a tracial and faithful $*$-noncommutative probability space. Here, $L^{\infty-}(\Omega)$ denotes the set of all random variables on Ω with values in $\mathbb{C}$ and with finite moments of every order.*

Example 3.1.4. *Let us take $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ and $\phi = tr_n = \frac{1}{n}tr$ the reduced trace. Then, (A, ϕ) is a faithful and tracial $*$-probability space.*

Example 3.1.5. *We can build a more interesting example by merging the two preceding examples. The couple*

$$(\mathcal{M}_n(\mathbb{C}) \otimes L^{\infty-}(\Omega), tr_n \otimes \mathbb{E})$$

is a $*$-probability space. In particular, it shows that the theory of noncommutative probability spaces seems to be adapted to the study of random matrices.

Of course, we may want to give more structure to our probability space, for instance some topology:

Definition 3.1.6.

- *A C^*-probability space is a $*$-probability space $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is an unital C^*-algebra*
- *A W^*-probability space is a $*$-probability space $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a von Neumann algebra.*

In Classical Probability, the idea of distribution plays an important role. Knowing the ditribution of a random variable consists in knowing entirely its probability law. To know the distribution of a (classical) random variable X, it suffices to know the value of all $\mathbb{E}(f(X))$ for any $f : \mathbb{C} \to \mathbb{C}$ continuous bounded, for instance. In the noncommutative case, in the most general version of a $*$-algebra, without any topology, we do not have any notion of continuous bounded functions. We therefore have use of (noncommutative) polynomials.
For any set I, we denote by $K\langle (X_i)_{i\in I}\rangle$ the algebra of polynomials in the (noncommutative) indeterminates $(X_i)_{i\in I}$. The fact that the indeterminates do not commute means that $X_iX_j \neq X_jX_i$ for $i \neq j$.

Definition 3.1.7. *Let $x \in \mathcal{A}$ be an element from a $*$-probability space. The $*$-distribution of x is a map:*

$$\begin{aligned} \phi_x \quad : \quad & K\langle X, Y\rangle \to \mathbb{C} \\ & P \mapsto \phi(P(x, x^*)) \end{aligned}$$

Let $(x_i)_{i\in I}$ a set of elements from $\mathcal{A}$. The joint $*$-distribution of the $(x_i)_{i\in I}$ is a map:

$$\begin{aligned} \phi_{(x_i)_{i\in I}} \quad : \quad & K\langle (X_i, X_i^*)_{i\in I}\rangle \to \mathbb{C} \\ & P \mapsto \phi[P((x_i)_i \cup (x_i^*)_i)] \end{aligned}$$

We want to stress the fact that the polynomial must have twice as many variable than the number of elements we are considering, because we have to take into account both the element and its adjoint.
Now, in some cases, we can describe the distribution of a (noncommutative) random variable with probability laws, as in the classical case. Let $a \in \mathcal{A}$ be a normal element, ie such that $aa^* = a^*a$, then the distribution of a is entirely known if we know the value of the $\phi(a^p(a^*)^k)$ for any $p, k \in \mathbb{N}$. Let now μ be some compactly supported probability measure on $\mathbb{C}$. We say that a has the law μ if we have the relation, for any $p, k \in \mathbb{N}$:

$$\phi(a^p(a^*)^k) = \int z^p \overline{z}^k d\mu(z)$$

Remark 3.1.8. If a has a law, then this law is unique. Indeed, by the Stone-Weierstraß theorem, know $\int z^p \overline{z}^k d\mu(z)$ means that you know all the $\int f d\mu$ for any f continuous and supported on K the compact support of μ. Therefore, this determines entirely μ.

Let us also remark that, so far, nothing assures us that any normal element will have a law. Nevertheless, this will be the case in a good number of cases, in the case of C^*-algebras, see [35, Proposition 3.13].

Theorem 3.1.9. *Let $(\mathcal{A}, \phi)$ be a C^*-probability space, and let a be a normal element in $\mathcal{A}$. Then, there exists a compactly supported probability measure μ on $\mathbb{C}$ such that μ is the law of a.*

3.2 Different notions of independence

In classical probability, a very important idea is the concept of independence. When two random variables are independant, knowing the law of each is enough to know the joint distribution. We would like to have something similar in the noncommutative world, namely a way to know the joint distribution of two "independant" variables whenever we know the distribution of each one. In the noncommutative world, though, there are up to five different notions of "independence". We will introduce them below. We refer the reader to [1, 33] for a more comprehensive study of the question.
Let $(\mathcal{A}_1, \phi_1)$ and $(\mathcal{A}_2, \phi_2)$ be two (unital) $*$-noncommutative probability spaces. The free product $\mathcal{A}_1 \sqcup \mathcal{A}_2$ can be equipped with two different product states, called respectively free and tensor independent (or just tensor) product of states. We define those two constructions.

Definition 3.2.1. *Let $(\mathcal{A}_1, \phi_1)$ and $(\mathcal{A}_2, \phi_2)$ be two unital $*$-noncommutative spaces. There exist two different states $\phi_1 * \phi_2$, $\phi_1 \otimes \phi_2$, called respectively* free and tensor independent (or just tensor) product*, and defined, for all $A_1, \ldots, A_n \in \mathcal{A}_1 \sqcup \mathcal{A}_2$ such that $A_i \in \mathcal{A}_{\epsilon_i}$ and $\epsilon_1 \neq \epsilon_2 \cdots \neq \epsilon_n$, by respectively the following relations*

- $\phi_1 * \phi_2(A_1 \cdots A_n) = 0$ *whenever* $\phi_1(A_1) = \cdots = \phi_n(A_n) = 0$*;*
- $\phi_1 \otimes \phi_2(A_1 \cdots A_n) = \phi_1(\prod_{i:\epsilon_i=1} A_i)\phi_2(\prod_{i:\epsilon_i=2} A_i)$.

If we do not impose unitality on our $*$-probability space, we can get up to three more products.

Definition 3.2.2. *Let $(\mathcal{A}_1, \phi_1)$ and $(\mathcal{A}_2, \phi_2)$ be two (nonunital) $*$-noncommutative spaces. There exist five different states $\phi_1 * \phi_2$, $\phi_1 \otimes \phi_2$, $\phi_1 \diamond \phi_2$, $\phi_1 \rhd \phi_2$ and $\phi_1 \lhd \phi_2$ on $\mathcal{A}_1 \sqcup \mathcal{A}_2$, called respectively* free, tensor independent (or just tensor), boolean, monotone and anti-monotone product*, and defined, for all* $A_1, \dots, A_n \in \mathcal{A}_1 \tilde{\sqcup} \mathcal{A}_2$ *such that* $A_i \in \mathcal{A}_{\epsilon_i}$ *and* $\epsilon_1 \neq \epsilon_2 \cdots \neq \epsilon_n$*, by respectively the following relations*

- $\phi_1 * \phi_2(A_1 \cdots A_n) = 0$ *whenever* $\phi_1(A_1) = \cdots = \phi_n(A_n) = 0$*;*
- $\phi_1 \otimes \phi_2(A_1 \cdots A_n) = \phi_1(\prod_{i:\epsilon_i=1} A_i)\phi_2(\prod_{i:\epsilon_i=2} A_i)$*;*
- $\phi_1 \diamond \phi_2(A_1 \cdots A_n) = \prod_{i=1}^n \phi_{\epsilon_i}(A_i)$*;*
- $\phi_1 \rhd \phi_2(A_1 \cdots A_n) = \phi_1(\prod_{i:\epsilon_i=1} A_i) \prod_{i:\epsilon_i=2} \phi_2(A_i)$*;*
- $\phi_1 \lhd \phi_2(A_1 \cdots A_n) = \prod_{i:\epsilon_i=1} \phi_1(A_i)\phi_2(\prod_{i:\epsilon_i=2} A_i)$*.*

Now, let $(\mathcal{A}, \phi)$ be a unital (resp. nonunital) noncommutative probability space and let $\mathcal{B}$ and $\mathcal{C}$ two unital (resp. nonunital) sub-$*$-algebras of $\mathcal{A}$. We denote by $\phi_{\mathcal{B}}$, resp. $\phi_{\mathcal{C}}$, resp. $\phi_{\mathcal{B} \sqcup \mathcal{C}}$ (resp. $\phi_{\mathcal{B} \tilde{\sqcup} \mathcal{C}}$) the restriction of ϕ to $\mathcal{B}$, resp. $\mathcal{C}$, resp. $\mathcal{B} \sqcup \mathcal{C}$ (resp. $\mathcal{B} \tilde{\sqcup} \mathcal{C}$). We say that $\mathcal{B}$ and $\mathcal{C}$ are **free**, resp. **tensor independant** (resp. **boolean independant**, resp. **monotone independant**, resp. **antimonotone independant**) if $\phi_{\mathcal{B} \sqcup \mathcal{C}}$ (resp. $\phi_{\mathcal{B} \tilde{\sqcup} \mathcal{C}}$) is equal to $\phi_{\mathcal{B}} * \phi_{\mathcal{C}}$, resp. $\phi_{\mathcal{B}} \otimes \phi_{\mathcal{C}}$ (resp. $\phi_{\mathcal{B}} \diamond \phi_{\mathcal{C}}$, resp. $\phi_{\mathcal{B}} \rhd \phi_{\mathcal{C}}$, resp. $\phi_{\mathcal{B}} \lhd \phi_{\mathcal{C}}$).
Likewise, we say that two random variables are free, tensor, boolean, monotone or antimonotone independant, if the $*$-algebras they generate are free, tensor, boolean, monotone or antimonotone independant.
Let us make some observations about these definitions. First, they do exactly what we expect them to do, namely if two variables are independant in one sense or the other and if we know the distribution of each variable, then we know their joint distribution. Second, we have here various possible definitions, and we can show that they are the only ones for a suitable notion of what an independence should satisfy, see e.g. [1, Theorem 1, Theorem 3]:

Theorem 3.2.3. *The tensor independence is the only possible notion in the category of commutative unital algebras. In the category of unital algebras, the only possible independences are the free and the tensor ones.*

Let us finish this section by enlarging the scope of the boolean, monotone and antimonotone products. When we will study Haar states on dual groups, we will need to be able to compute such products on unital $*$-algebras. In order to do that, we will assume for our unital $*$-algebras the decomposition of vector spaces $\mathcal{A} = \mathbb{C}1_{\mathcal{A}} \oplus \mathcal{A}^0$, where $\mathcal{A}^0$ is a $*$-subalgebra of $\mathcal{A}$. Remark that this decomposition is not necessarily unique, and sometimes does not exist. We then set:

Definition 3.2.4. *Let* $(\mathcal{A}_1, \phi_1)$ *and* $(\mathcal{A}_2, \phi_2)$ *be two noncommutative spaces with* $\mathcal{A}_1 = \mathbb{C}1_{\mathcal{A}_1} \oplus \mathcal{A}_1^0$ *and* $\mathcal{A}_2 = \mathbb{C}1_{\mathcal{A}_1} \oplus \mathcal{A}_2^0$*. There exist five different states* $\phi_1 * \phi_2$*,* $\phi_1 \otimes \phi_2$*,* $\phi_1 \diamond \phi_2$*,* $\phi_1 \rhd \phi_2$ *and* $\phi_1 \lhd \phi_2$ *on* $\mathcal{A}_1 \sqcup \mathcal{A}_2$*, called respectively* free, tensor independent (or just tensor), boolean, monotone and anti-monotone product*, and defined, for all* $A_1, \dots, A_n \in \mathcal{A}_1 \sqcup \mathcal{A}_2$ *such that* $A_i \in \mathcal{A}_{\epsilon_i}^0$ *and* $\epsilon_1 \neq \epsilon_2 \cdots \neq \epsilon_n$*, by respectively the following relations*

- $\phi_1 * \phi_2(A_1 \cdots A_n) = 0$ *whenever* $\phi_1(A_1) = \cdots = \phi_n(A_n) = 0$*;*
- $\phi_1 \otimes \phi_2(A_1 \cdots A_n) = \phi_1(\prod_{i:\epsilon_i=1} A_i)\phi_2(\prod_{i:\epsilon_i=2} A_i)$*;*
- $\phi_1 \diamond \phi_2(A_1 \cdots A_n) = \prod_{i=1}^n \phi_{\epsilon_i}(A_i)$*;*
- $\phi_1 \rhd \phi_2(A_1 \cdots A_n) = \phi_1(\prod_{i:\epsilon_i=1} A_i) \prod_{i:\epsilon_i=2} \phi_2(A_i)$*;*

- $\phi_1 \lhd \phi_2(A_1 \cdots A_n) = \prod_{i:\epsilon_i=1} \phi_1(A_i)\phi_2(\prod_{i:\epsilon_i=2} A_i)$.

The tensor product and the free product do not depend on the choice of the decomposition $\mathcal{A}_1 = \mathbb{C}1_{\mathcal{A}_1} \oplus \mathcal{A}_1^0$ *and* $\mathcal{A}_2 = \mathbb{C}1_{\mathcal{A}_1} \oplus \mathcal{A}_2^0$, *but the other three products do.*

3.3 Free cumulants

When we consider noncommutative probability in the setting of freeness, cumulants are a tool that simplify computations and are often very useful. We shall introduce them here. Again, we refer to [35] for more details, and especially to Lecture 9.

Let S be a totally ordered set. A partition of the set S is a class $(S_1, \dots, S_k)$ of non-empty subsets of S such that the subsets are mutually disjoint and their union is the whole of S. Such a partition is said to have a crossing if there exist $i, j, k, l \in S$, with $i < j < k < l$, such that i and k belong to some block of the partition and j and l belong to another block. If a partition has no crossings, it is called non-crossing. The set of all non-crossing partitions of S is denoted by $NC(S)$. When $S = \{1, \dots, n\}$, with its natural order, we will use the notation $NC(n)$. We can endow $NC(S)$ with an order defined as follows: for all π_1 and $\pi_2 \in NC(S)$, $\pi_1 \preceq \pi_2$ if every block of π_1 is contained in a block of π_2. With this order, for any two elements ρ and σ the set $\{\tau \in NC(S) | \rho \preceq \tau \text{ and } \sigma \preceq \tau\}$ is non-empty and has a minimum, called the join and denoted $\rho \vee \sigma$, and the set $\{\tau \in NC(S) | \tau \preceq \sigma \text{ and } \tau \preceq \rho\}$ has a maximum, called the meet and denoted $\rho \wedge \sigma$. Whenever a partially ordered satisfies these two properties (existence of a join and of a meet for any two elements), it is called a lattice, which is the case for $NC(S)$.

Figure 3.1: Non-crossing partition on the left, crossing on the right

Definition 3.3.1. *The collection of free cumulants* $(\kappa_q : \mathcal{A}^q \to \mathbb{C})_{q \geq 1}$ *on some probability space* $(\mathcal{A}, \phi)$ *are defined via the following relations: for all* $A_1, \dots, A_n \in \mathcal{A}$,

$$\phi(A_1 \dots A_q) = \sum_{\sigma \in NC(q)} \prod_{\{i_1 \leq \dots \leq i_k\} \in \sigma} \kappa_k(A_{i_1}, \dots, A_{i_k})$$

where $NC(q)$ *is the set of non-crossing partitions of* $\{1, \dots, q\}$.

The importance of the free cumulants is in large part due to the following characterization of freeness.

Proposition 3.3.2. *Let* $(A_i)_{i \in I}$ *be random variables of* $(\mathcal{A}, \phi)$. *They are* $*$*-free if and only if their mixed* $*$*-cumulants vanish. That is to say: for all* $n \geq 0$, $\epsilon_1, \dots, \epsilon_n$ *be either* $\emptyset$ *or* $*$, *and all* $A_{i(1)}, \dots, A_{i(n)} \in \mathcal{A}$ *such that* $i(1), \dots i(n) \in I$, *whenever there exists some* j *and* j' *with* $i(j) \neq i(j')$, *we have* $\kappa(A_{i(1)}^{\epsilon_1}, \dots, A_{i(n)}^{\epsilon_n}) = 0$.

3.4 Lévy processes

In this section we will introduce the noncommutative counterpart to Lévy processes. To do that, we will need to recall the basic definitions in the classical case.

3.4.1 Lévy processes on Lie groups in classical Probability

Lévy[2] processes are a class of stochastic processes which generalizes nicely the idea of a Brownian motion. The natural setting in which they can be defined is the one of topological groups. We refer the reader to [30] for a comprehensive presentation on this subject in the case of Lie groups. We first recall the basic notions about topological groups.

Definition 3.4.1. *A topological group G is a group endowed with a topology τ such that the multiplication and the inverse maps are continuous.*
A Lie group G is a group endowed with a C^∞-manifold structure, such that the multiplication and the inverse maps are smooth. It is in particular a topological group.

This allows us to define Lévy processes.

Definition 3.4.2. *Let G be a topological group and $(g_t)_{t\geq 0}$ a stochastic process with values in G. The process $(g_t)_t$ is a (left) Lévy process if it verifies:*

1. *It is càdlàg (from the French "continu à droite, limite à gauche"), ie almost all its paths $t \mapsto g_t$ are right continuous on $[0,\infty)$ and have left limits on $(0,\infty)$.*
2. *It has independant right increments, ie for any $0 < t_1 < t_2 < \ldots < t_n$, the right increments $g_0, g_0^{-1}g_{t_1}, g_{t_1}^{-1}g_{t_2}, \ldots, g_{t_{n-1}}^{-1}g_{t_n}$ are independant.*
3. *It has stationary right increments, ie for $0 \leq s < t$, the right increment $g_s^{-1}g_t$ has the same distribution as $g_0^{-1}g_{t-s}$.*

The definition of a left Lévy process has been given with reference to right increments. If we considered instead left increments (ie quantities like $g_t g_s^{-1}$ for $0 \leq s < t$), we would have defined a right Lévy process. Nevertheless, right and left Lévy processes are in one-to-one correspondence because if $(g_t)_t$ is a left Lévy process, then $(g_t^{-1})_t$ is a right Lévy process, and vice-versa. In the sequel of this work we will always consider left Lévy processes, that begin at the unit element of the group ($g_0 = e$), except otherwise explicitly stated.
Let us also remark that topological groups are a nice setting to define these objects. Indeed, we needed some topology in order to be able to define what càdlàg is and also in order to be able to use the Borel σ-algebra, but we needed also a multiplication and an inverse in order to define the increments. Nevertheless, to have an inverse is not really necessary and it is possible to define Lévy processes in the case of topological semigroups, see e.g. [24, Definition 1.20]. Topological semigroups are sets endowed with a multiplication and an unit but not necessarily with an inverse map, and such that the multiplication is continuous under the topology. To give such a definition, in this most general sense, we need to take a process with two indices $(g_{st})_{0\leq s\leq t}$ where g_{st} plays the role of the increment.

[2]Paul Lévy, 15th September 1886, Paris (France) - 15th december 1971, Paris (France). Former student of the École Polytechnique he is one the founders of modern Probability Theory.

Definition 3.4.3. *A stochastic process $(g_{st})_{0\leq s\leq t}$ with values in the topological semigroup G is called a left Lévy process, if*

1. *It verifies the increment property, namely that $g_{tt} = e$ and $g_{st}g_{tu} = g_{su}$ almost surely for any $0 \leq s \leq t \leq u$.*
2. *It has independant increments, ie for any $s_1 < t_1 < s_2 < \ldots < s_n < t_n$, the right increments $g_{s_1t_1}, g_{s_2t_2}, \ldots, g_{s_nt_n}$ are independant.*
3. *It has stationary right increments, ie for $0 \leq s < t$ and for $h > 0$, the right increment $g_{s+h,t+h}$ has the same distribution as g_{st}.*
4. *It is weakly continuous, ie g_{st} converges in probability to g_{ss} when t goes to s, for any $s \geq 0$.*

Lévy processes are a large class of processes. They contain the Brownian motion B_t, which is defined in the case of the euclidean space $\mathbb{R}^d$ as a process beginning at origin, with independent and stationary increments, such that B_t has the distribution of a centered gaussian law with variance t. Brownian motion can be defined in the most general setting on Riemannian manifolds, see [30, Section 2.3], which are manifolds where one can measure distances. The definition, using an operator called the Laplace-Beltrami operator, gives a meaning to the intuitive notion of a process that "diffuses equally in every direction". Whe shall not say more on this matter, as it would go beyond the scope of this work to introduce the theory of Riemannian manifolds, but let us mention that in the case of Lie groups, who can be seen as Riemannian manifolds, Brownian motions are Lévy processes. In the euclidean space case, there are two other natural example: (deterministic) linear maps and Poisson processes, which is a process with values in $\mathbb{N}$, are Lévy processes.
When studying Lévy processes, it is important to know that they admitt several characterizations. If $(g_t)_t$ is a Lévy process, let us denote by μ_t the law of g_t. Because the increments are stationary and independant, it is clear that knowing all the $(\mu_t)_t$ is equivalent to knowing the law of the process $(g_t)_t$. The family $(\mu_t)_t$ forms a continuous convolution of semigroups:

Definition 3.4.4. *A family $(\mu_t)_{t\geq 0}$ of probability measures on a topological group G is called a continuous convolution semigroup if:*

- *For all f continuous bounded on G, $\int f d\mu_t$ converges to $f(e)$ when t goes to zero.*
- *For any $s, t \geq 0$, the relation $\mu_s \star \mu_t = \mu_{s+t}$ is verified, where $\star$ is the convolution operator, such that if X, resp. Y, is a random variable with distribution μ, resp. ν, and X is independant from Y, then $\mu \star \nu$ is the law of XY.*

There is a one-to-one correspondence between left Lévy processes and continuous convolution semigroups on the topological group G, see [24, Proposition 1.21], because the family of marginal distributions of a Lévy process is a continuous convolution semigroup, and, conversely, given a continuous convolution semigroup, one can build a corresponding Lévy process.
Now, a Lévy process $(g_t)_t$ being given, we may define its transition semigroup:

$$\begin{aligned} P_u \quad : \quad & \mathcal{B}(G) \to \mathbb{C} \\ & f \mapsto (g \in G \mapsto \int f(gg_t) d\mu_u) \end{aligned}$$

where $\mathcal{B}(G)$ is the set of nonnegative Borel functions on G, and for $u \geq 0$. Informally, the quantity $P_u f(g)$ is the expectation of the process $(g_t)_t$ after time u and by assuming that it began at point g. This semigroup of operator determines completely the law of the marginals, and therefore the distribution of the Lévy process (granted that $g_0 = e$, or more generally, that the distribution of g_0 is known). We may also define the generator of the Lévy process as being a couple (D, L) where L is defined as the following limit, when it exists:

$$Lf = \lim_{t \to 0} \frac{P_t f - f}{t}$$

with $f \in D \subset C_0(G)$ and D the subset of $C_0(G)$ of continuous functions vanishing at infinity, such that this limit exists. It is a well-known result of Probability theory that (D, L) determines entirely the semigroup of operators $(P_t)_t$, see [30, Appendix B.1]. We can actually even characterize entirely the generators of a Lévy process on a Lie group through Hunt's formula, see [24, Theorem 1.28], and in the same way we can characterize all those semigroups $(P_t)_t$ that correspond to transition semigroups for Lévy processes (namely left-invariant Feller transition semigroups, but it would be beyond the scope of this work to go into more details here, since we do not need it to understant their quantum counterparts), see [30, Proposition 1.2].
To sum it up in the Lie group case, we have a one-to-one correspondence between these four objects, knowing one of them amounting to know all the others:

$$\begin{array}{ccc}
\text{Lévy process } (g_t)_{t\geq 0} & \longleftrightarrow & \text{Continuous Convolution semigroup}(\mu_t)_{t\geq 0} \\
\updownarrow & & \updownarrow \\
\text{Transition semigroup}(P_t)_{t\geq 0} & \longleftrightarrow & \text{Generator } (D, L)
\end{array}$$

3.4.2 Noncommutative Lévy processes

Noncommutative counterparts of Lévy processes can be defined. As classical Lévy processes are defined on groups (or semigroups), their noncommutative variants must be defined on noncommutative versions of groups (or semigroups). We will define them on involutive bialgebras and on dual groups. We refer to [24, 20] for more details on this matter.

Quantum random variables

Up to this point, we have defined noncommutative random variables as being elements of a $*$-algebra. This definition gives a convenient analogue to the notion of complex-valued (classical) random variables. Nevertheless, if we want to define Lévy processes, we need the concept of noncommutative random variables with values in some abstract space, not necessarily $\mathbb{C}$ any more.

Definition 3.4.5. *Let (A, ϕ) be a $*$-probability space and B be some $*$-algebra. A quantum random variable over the quantum probability space (A, ϕ) on the $*$-algebra B is a $*$-homomorphism $j : B \to A$.*

Remark 3.4.6. Let us remark that the definition "goes in the other direction" than the classical one. A classical random variable over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on the group G is a measurable map $X : \Omega \to G$, whereas in the quantum case the map goes from B, the space wherein the variable takes values, to A the spaces whereon the variable is defined. This is due to duality. Indeed, any (classical) random variable $X : \Omega \to G$ yields a $*$-homomorphism ι_X:

$$\begin{aligned} \iota_X \quad : \quad & C(G) \to L^{\infty -}(\Omega) \\ & f \mapsto f \circ X \end{aligned}$$

Therefore, X can be seen as a quantum random variable over $L^{\infty -}(\Omega)$ on $C(G)$.

The distribution of a quantum random variable j is the map $\phi \circ j : B \to \mathbb{C}$. In a natural way, a quantum stochastic process is a family $(j_t)_{t \in I}$ of quantum random variables. The maps $\phi_t := \phi \circ j_t$ are called the marginal distribution of the quantum stochastic process, whereas its joint distribution is the map $\phi \circ \bigsqcup_{t \in I} j_t$.

Lévy processes on bialgebras

We can define Lévy processes on involutive bialgebras. In this way, we can in particular consider Lévy processes on the algebra of polynomials of a compact quantum group. We follow [24] closely for the definition.
Let in the sequel B be an involutive bialgebra. If $j_1, j_2 : B \to A$ are two linear maps with values in some algebra A, we need to be able to define the convolution of these maps:

$$j_1 \star j_2 = \mu_A \circ (j_1 \otimes j_2) \circ \Delta$$

where $\mu_A : A \otimes A \to A$ is the multiplication map.

Definition 3.4.7 (See [24], especially Definition 1.32)**.** *A quantum stochastic process $(j_{st})_{0 \leq s \leq t}$ on the involutive bialgebra B over the quantum probability space (A, ϕ) is called a Lévy process if it satisfies following properties:*

1. *(Increment property) We have:*

$$\begin{aligned} j_{rs} \star j_{st} &= j_{rt} \text{ for all } 0 \leq r \leq s \leq t \\ j_{tt} &= \epsilon.1 \text{ for all } 0 \leq t \end{aligned}$$

2. *(Indepedence of increments) For any $n \in \mathbb{N}$ and all $0 \leq s_1 \leq t_1 \leq s_2 \leq \ldots \leq t_n$, the quantum random variables $j_{s_1 t_1}, \ldots, j_{s_n t_n}$ are tensor independent.*
3. *(Stationarity of increments) The distribution $\phi_{st} = \phi \circ j_{st}$ depends only on the difference $t - s$, for any $0 \leq s \leq t$.*
4. *(Weak continuity) For any $b \in B$ and any $0 \leq s$, we have $\lim_{t \searrow s} j_{st}(b) = j_{ss}(b) = \epsilon(b)1$.*

If two Lévy processes have the same joint distribution, we shall say that they are equivalent. As in the classical case, there are different characterization of Lévy processes. They will be given here without further justification, as they are well-known in the world of quantum probability. For more informations, one can go to [24, 20, 49, 32].

Proposition-Definition 3.4.8 (Lemma 1.34 of [24])**.** *Let $(j_{st})_{0 \leq s \leq t}$ be a quantum Lévy process on a involutive bialgebra B. Then, its marginal distributions $\phi_s := \phi \circ j_{a,a+s}$ for any $0 \leq a$ (which is well-defined by the property of stationarity of the increments) form a continuous convolution semigroup of states on B, namely:*

1. *We have* $\phi_0 = \epsilon$.

2. *For all* $0 \leq s, t$*, we have:* $\phi_s \star \phi_t = \phi_{s+t}$.

3. *We have the limit* $\lim_{t \searrow 0} \phi_t(b) = \epsilon(b)$

4. *For any* $0 \leq t$*,* $\phi_t(1) = 1$ *and for any* $b \in B$*,* $\phi_t(b^*b) \geq 0$.

Linear functionals on a $*$-algebra which verify the last item are called state on the $*$-algebra.

Proposition-Definition 3.4.9 (Schoenberg correspondence)**.** *Let* $(\phi_t)_{0 \leq t}$ *be a continuous convolution semigroup fo states on an involutive bialgebra* B*. Then, the following limit exists for any* $b \in B$ *and the functional* L *is called the generator if the semigroup:*

$$L(b) = \lim_{t \searrow 0} \frac{\phi_t(b) - \epsilon(b)}{t}$$

Moreover, L *verifies the following properties:*

- *It is conditionnaly positive:* $L(b^*b) \geq 0$ *for any* $b \in Ker\epsilon$.
- *It cancells on the unit:* $L(1) = 0$.
- *It is hermitian:* $L(b^*) = \overline{L(b)}$ *for any* $b \in B$.

Moreover, for any linear functional L *that verifies the upper three conditions, there exists a continuous convolution semigroup of states* $(\phi_t)_t$ *of which* L *is the generator.*

Once a generator L is given, one can apply a variant of the GNS construction in order to get an unital $*$-representation $\rho : B \to L(D)$ on some pre-Hilbert space D and a linear map $\eta : B \to B$ such that the following two formulae hold for any $a, b \in B$:

$$\begin{aligned} \eta(ab) &= \rho(a)\eta(b) + \eta(a)\epsilon(b) \\ -\langle \eta(a^*), \eta(b) \rangle &= \epsilon(a)L(b) - L(ab) + L(a)\epsilon(b) \end{aligned}$$

This formulae can be summed up saying that η is a ρ-η-1-cocycle and L has $(a, b) \mapsto -\langle \eta(a^*), \eta(b) \rangle$ as a ϵ-ϵ-2-coboundary in the language of cohomology.
When such a triple (ρ, η, L) is given, we call it a Schürmann triple. if η is surjective, we say that the Schürmann triple is surjective.
Given such a Schürmann triple, it is clear that we do not change the description of the generator if we take an pre-Hilbert space isomorphic in some sense to the first one. Formally, if $U : D_1 \to D_2$ is an isometry, and we have the relations $\rho_2(b)U = U\rho_1(b)$ and $\eta_2(b) = U\eta_1(b)$ for any $b \in B$, we say that the Schürmann triples (ρ_1, η_1, L) and (ρ_2, η_2, L) are unitarily equivalent.
We have the correspondence between all these objects, as the following result, coming from [24, Theorem 1.39] explains:

Theorem 3.4.10. *Let* B *be an involutive bialgebra. We have a one-to-one correspondence between Lévy processes on* B *(up to equivalence), continuous convolution semigroups of states on* B*, generators on* B*, surjective Schürmann triples on* B *(up to unitary equivalence).*

Lévy processes on dual groups

The main object of study during my PhD work has been dual groups. Lévy processes are nice objects because through their study we may understand better the objects on which they take values. We shall therefore now introduce quantum Lévy processes on dual groups. To do this, we shall again follow closely [24]. Let us here remark that Lévy processes can be defined on dual semigroups in the most general sense, exactly as they were defined on semigroup in the most general (classical) sense or on bialgebra in the previous section. We shall nevertheless restrain ourselves to the dual groupcase, in order to avoid to be to cumbersome.
We must also make here an important remark, namely that Lévy processes on dual groups can be defined in five different ways, corresponding to the five different notions of independence.

Definition 3.4.11. *Let $(B, \Delta, \Sigma, \delta)$ be a dual group. A quantum stochastic process $(j_t)_0 \leq t$ on B over a quantum probability space (A, ϕ) is called a tensor (resp. free, resp. boolean, resp. monotone, resp. antimonotone) quantum Lévy process on B if it satisfies the following conditions:*

1. *For any $0 \leq t$: $j_0 = \epsilon.1$*
2. *For any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, the quantum random variables $j_{t_1}, [S \circ j_{t_1}] \star j_{t_2}, \ldots, [S \circ j_{t_{n-1}}] \star j_{t_n}$ are tensore (resp. freely, resp. boolean, resp. monotone, resp. antimonotone) independent.*
3. *The distribution $\phi_{t-s} := \phi \circ j_{t-s}$ depends only on the difference $t - s$.*
4. *For any $b \in B$ and any $0 \leq s$, we have $\lim_{t \searrow 0} j_t(b) = j_0(b) = \epsilon(b)1$*

As before, we say that two Lévy processes are equivalent if they have the same joint distribution.
The marginal distributions form again a continuous convolution semigroup of states but it is important here to pay attention to the fact that the convolution is done with respect to notion of independence used. For instance, if we study a free quantum Lévy process, than the marginals verify $(\phi_s * \phi_t) \circ \Delta = \phi_{s+t}$, or if on the contrary we study a monotone quantum Lévy process, then the marginals form a convolution semigroup with respect to following convolution: $(\phi_s \rhd \phi_t) \circ \Delta = \phi_{s+t}$. We actually begin to see why the setting of dual groups is very powerful here: the fact that the coproduct takes values in the free product instead of the tensor product means that we have tools that are adapted to the various notions of independence, not only to the tensor independence.
We may also define a generator of the process in the same way, and it is well-defined for any $b \in B$:

$$L(b) = \lim_{t \searrow 0} \frac{\phi_t(b) - \epsilon(b)}{t}$$

Again, we have a Schoenberg correspondence, namely that there is a one-to-one correspondence between continuous convolution semigroups of states and linear functionals that are conditionnaly positive, hermitian and vanishing on the unit. Once a continuous convolution semigroup of states is given, it is also possible to get back to a quantum Lévy process, one can refer e.g. to the construction given in Section 2.2 of [49].

Part II

Hypercontractivity

4

Hypercontractivity on the Orthogonal and Permutation quantum groups

In this section we study hypercontractivity properties of the free orthogonal quantum group and of the free permutation quantum group.

This chapter is taken from the article [19], which I co-authored with U. Franz, G. Hong and F. Lemeux.

4.1 Introduction

Since the 70s, when the word "hypercontractivity" was coined (see [41]), it has yielded a fruitful area of Mathematics. Stronger than the well-known and classical notion of contractivity, it has been shown that hypercontractivity is strongly linked to a class of inequalities called logarithmic Sobolev inequalities, which in turn have many applications such as in statistical mechanics (see for instance [25] for the investigation of the Ising model based on log-Sobolev inequalities). With the rise of noncommutative mathematics, the framework of hypercontractivity has also been studied in the context of noncommutative L^p spaces, for instance in [36].
The hypercontractivity for semigroups on some cocommutative compact quantum groups such as von Neumann algebras of discrete groups, e.g. free products of $\mathbb{Z}_2$ etc., has been recently studied by Junge et al., see [26] and the references therein.
The goal of this chapter is to investigate hypercontractivity for semigroups on the free orthogonal quantum group and the free permutation quantum group. Different definitions for a Brownian motion (and hence for a heat semigroup) could be considered on these quantum groups; we will be interested in the *ad*-invariant generating functionals in order to select semigroups that could pretend to the role of heat semigroups.
It is only a short introduction to this topic and it is to be hoped that much more work will be done in this direction.

4.2 Compact quantum groups and heat semigroups

4.2.1 Markov semigroups

In order to investigate hypercontractivity on heat semigroups, one must be able to define heat semigroups on the quantum groups at hand. We recall here for clarity's sake a certain number of important results, without proofs. More on this topic might be found in [14].
We can define Lévy processes on quantum groups (Definition 2.4 in [14]). If $(j_t)_{t\geq 0}$ is such a process, then we can associate to it a Markov semigroup T_t by putting $T_t = (id \otimes \phi_t) \circ \Delta$ where $\phi_t = h \circ j_t$. The Lévy process $(j_t)_t$ is also associated to a generator $L = \frac{d\phi_t}{dt}\Big|_{t=0}$ (actually, there is a one-to one correspondence between generators and Lévy processes, called the Schoenberg correspondence).
It is important to mention the domain of the Markov semigroup. The operator T_t can either be seen as $T_t : C_u(\mathbb{G}) \to C_u(\mathbb{G})$ or as $T_t : C_r(\mathbb{G}) \to C_r(\mathbb{G})$. We will in the sequel take the second definition, due to our use of the reduced C^*-algebra. The semigroup is associated to a Markovian generator $T_L : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ which is defined by $T_L = (id \otimes L) \circ \Delta = \frac{dT_t}{dt}\Big|_{t=0}$.
The two semigroups treated in this paper are KMS-symmetric (even GNS-symmetric, which means that T_L and T_t are self-adjoint on $L^2(\mathbb{G}, h)$), therefore they extend to σ-weakly continuous semigroups on the von Neumann algebra $L^\infty(\mathbb{G}) = C_r(\mathbb{G})''$, see, e.g., [13, Theorem 2.39].
Now, in the classical case, a heat semigroup is the Markov semigroup associated to a Brownian motion, which is a particular kind of Lévy process. So if we had a definition of such a Brownian motion on O_N^+ or S_N^+, we could define a heat semigroup and this semigroup should be naturally privileged in our study. Unfortunately, to define such an object is not an easy matter. In the classical case, Brownian motions are defined on Lie groups via the Laplace-Beltrami operator. On quantum groups, we do not have a differential structure which would allow us to define a quantum analogue to the Laplace-Beltrami operator. Alternative approaches must thus be found.
One way to do so is to use the notion of gaussianity first introduced by Schürmann (as has been done in Definition 5.5.1, see also [44, Section 5.3], to exhibit a Brownian motion on the unitary dual group). This approach nevertheless fails for S_N^+, as indicated by [23, Proposition 8.6], which states that there are no gaussian generators on S_N^+.
As an alternative, we will be interested in the class of *ad*-invariant generating functionals (see section 6 of [14]), ie the functionals invariant under the adjoint action. Linear functionals L that are *ad*-invariant are exactly those such that there exist numbers $(c_s)_s$ such that $L(u_{ij}^{(s)}) = c_s \delta_{ij}$. They are classified for O_N^+ in [14, Section 10] and in [23, Section 10.4] for S_N^+. This approach to the definition of a Brownian motion seems natural. Indeed, in the classical case of Lie groups, [30, Propositions 4.4, 4.5] shows that *ad*-invariant processes (or, equivalently, conjuguate-invariant processes) on simple Lie groups have a generator constituted of the Laplace-Beltrami operator plus a part due to the Lévy measure. It therefore seems reasonable to define a Brownian motion from within the class of *ad*-invariant functionals and this will be the approach which we will use in this paper.

4.2.2 Heat semigroup on the free orthogonal quantum group

We will need for this section and the next one a definition of Chebyshev polynomials of the second kind.

Definition 4.2.1. *The Chebyshev polynomials of the second kind are the polynomials U_s given by this relation*

$$U_s(X) = \sum_{p=0}^{\lfloor s/2 \rfloor} (-1)^p \binom{s-p}{p} X^{s-2p}$$

They are an orthonormal family of $\mathbb{C}[X]$ for the scalar product defined via the semicircular measure.

We recall the following proposition, found in [14, Proposition 10.3], showing that ad-invariant functionals on O_N^+ are classified with pairs (b, ν) where b is a non-negative real number and ν a finite measure with support on the interval $[-N, N]$.

Proposition 4.2.2. *The ad-invariant generating functional on $\mathrm{Pol}(O_N^+)$ with characteristic pair (b, ν) ($b \geq 0$ and ν a finite measure on $[-N, N]$) acts on the coefficients of unitary irreducible representations of O_N^+ as:*

$$L(u_{ij}^{(s)}) = \frac{\delta_{ij}}{U_s(N)} \left(-bU_s'(N) + \int_{-N}^{N} \frac{U_s(x) - U_s(N)}{N - x} \nu(dx) \right)$$

for $s \in \mathbb{N}$, where U_s denotes the s^{th} Chebyshev polynomial of the second kind (considered on the interval $[-N, N]$).

The generator of the Markov semigroup, which is defined by: $T_L = (id \otimes L) \circ \Delta$, acts as:

$$T_L(u_{ij}^{(s)}) = \frac{1}{U_s(N)} \left(-bU_s'(N) + \int_{-N}^{N} \frac{U_s(x) - U_s(N)}{N - x} \nu(dx) \right) u_{ij}^{(s)}$$

The Markov semigroup is given by $T_t = \exp(tT_L)$. We will be interested in this paper in the case $b = 1$ and $\nu = 0$, which is not only the easiest, but also it is what seems to be the most logical definition of what a Brownian motion should be. Indeed, the formula seems somewhat similar to Hunt's formula in the case of Lévy processes on Lie groups and it therefore seems natural to take ν, which seems to play a role analogous to the Lévy measure of Hunt's formula, equal to zero.
Let us now investigate further this Markovian semigroup. We have:

$$L(u_{ij}^{(s)}) = -\frac{\delta_{ij}}{U_s(N)} U_s'(N)$$

Therefore, the eigenvalues of T_L are given by:

$$\lambda_s = -\frac{U_s'(N)}{U_s(N)}$$

with eigenspace $V_s = \mathrm{span}\{u_{ij}^{(s)}, 1 \leq i, j\}$ and multiplicity $m_s = (\dim u^{(s)})^2 = U_s(N)^2$ (see [14], section 10). Now, since the leading coefficient of U_s is equal to one, we can write these polynomials with the help of their zeros:

$$U_s(x) = (x - x_1) \dots (x - x_s)$$

And therefore:

$$\frac{U_s'(x)}{U_s(x)} = \sum_{k=1}^{s} \frac{1}{x - x_k}$$

for $x \in \mathbb{R} \backslash \{x_1, \dots x_s\}$. We will need the following classical lemma about Chebyshev polynomials, which will be useful to us in this section and also in the next.

Lemma 4.2.3. *The zeros of U_s are comprised between -2 and 2.*

Proof. We will use the fact that the Chebyshev polynomials of the second kind constitute an orthonormal family with regard to Wigner's semicircle law $\frac{1}{\pi}\sqrt{4-x^2}$ on $[-2,2]$. Let $\in \mathbb{N}$. Let us denote by $S = \{y_1, \dots, y_l\}$ the set of all zeros of U_s in $(-2,2)$ that have an odd multiplicity. We set $Q = \prod_{k=1}^{l}(X - x_k)$. It is obvious that Q divides U_s. Let us now assume that $\deg Q < s = \deg U_s$. Therefore, we have:

$$\int_{-2}^{2} Q(x)U_s(x)\frac{1}{\pi}\sqrt{4-x^2}dx = 0$$

But the very definition of Q means that the zeros of U_sQ that are in $(-2,2)$ have an even multiplicity, ie U_sQ has a constant sign on this interval. For the integral to be zero, we must have $U_sQ = 0$, which is absurd. Therefore we must have $U_s = Q$ and this proves the lemma. □

We thus have the following lemma:

Lemma 4.2.4. *For $N \geq 2$,*

$$\frac{s}{N+2} \leq -\lambda_s = \frac{U_s'(N)}{U_s(N)} = \sum_{k=1}^{s} \frac{1}{N - x_s} \leq \frac{s}{N-2}$$

where, for $N = 2$, we take the convention that $1/0 = \infty$.

4.2.3 Heat semigroups on the Free Permutation quantum group

We rely on the results of [23] for S_N^+. We consider semigroups with generating functionals defined by:

$$L(u_{ij}^{(s)}) = -\frac{\delta_{ij}U_{2s}'(\sqrt{N})}{2\sqrt{N}U_{2s}(\sqrt{N})}$$

We follow the same reasoning as before. The eigenvalues are:

$$\lambda_s = -\frac{U_{2s}'(\sqrt{N})}{2\sqrt{N}U_{2s}(\sqrt{N})}$$

with eigenspace $V_s = \{u_{ij}^{(s)}, 1 \leq i,j\}$ and multiplicity $m_s = U_{2s}(\sqrt{N})^2$. We finally find the estimate:

Lemma 4.2.5. *For $N \geq 4$,*

$$\frac{s}{\sqrt{N}(\sqrt{N}+2)} \leq -\lambda_s = \frac{1}{2\sqrt{N}} \sum_{k=1}^{2s} \frac{1}{\sqrt{N} - x_k} \leq \frac{s}{\sqrt{N}(\sqrt{N}-2)}$$

where, for $N = 4$, we take the convention that $1/0 = \infty$.

4.3 Ultracontractivity and hypercontractivity

When we need to distinguish the semigroups, we will denote by T_t^O (resp, T_t^S) the semigroup we introduced on O_N^+ (resp. S_N^+)

4.3.1 Ultracontractivity

We say that a semigroup T_t is ultracontractive if it is bounded from L^2 into L^∞ for all $t > 0$. In the sequel, we will denote by $||.||_\infty = ||.||$ the operator norm and by $||x||_p^p = h((x^*x)^{p/2})$ the p-norm (h being the Haar state). We observe that this definition of the p-norm is valid only when h is a trace, which is the case here. We will prove the following result:

Theorem 4.3.1. *Let T_t be a semigroup on a compact quantum group, such that the following assumptions hold:*

- *The subspaces V_s spanned by the coefficients of the irreducible corepresentations u^s are eigenspaces for the generator T_L of the Markov semigroup, ie:*

  $$T_L x = \lambda_s x$$

  for $x \in V_s$

- *We have an estimate of the form $\lambda_s \leq -\alpha s$ for some $\alpha > 0$.*

- *We have an inequality of the form:*

  $$||x||_\infty \leq (\beta s + \gamma)||x||_2$$

  for $x \in V_s$, with $\beta, \gamma \geq 0$ and β, γ are independant of s.

Then, T_t is ultracontractive: $||T_t x||_\infty \leq \sqrt{f(t)}||x||_2$, where:

$$f(t) = \frac{\beta^2 e^{-2\alpha t}(1 + e^{-2\alpha t}) + 2\beta\gamma e^{-2\alpha t}(1 - e^{-2\alpha t}) + \gamma^2(1 - e^{-2\alpha t})^2}{(1 - e^{-2\alpha t})^3}.$$

Proof. We have for $x = \sum_s x_s$ with $x_s \in V_s$:

$$\begin{aligned}
||T_t x||_\infty &\leq \sum_{s\in\mathbb{N}} ||T_t x_s||_\infty = \sum_s e^{\lambda_s t}||x_s||_\infty \\
&\leq \sum_s e^{-\alpha s t}||x_s||_\infty \leq \sum_s e^{-\alpha s t}(\beta s + \gamma)||x_s||_2 \\
&\leq \left(\sum_s (\beta s + \gamma)^2 e^{-2\alpha s t}\right)^{1/2} \left(\sum_s ||x_s||_2^2\right)^{1/2} \\
&= \sqrt{f(t)}||x||_2
\end{aligned}$$

where we used the Cauchy-Bunjakowski-Schwarz inequality in the second to last line.
The computation of $f(t) = \sum_s (\beta^2 s^2 + 2\beta\gamma s + \gamma^2)e^{-2\alpha s t}$ is done via the classical series:

$$\begin{aligned}
\sum_{k\in\mathbb{N}} e^{-\lambda k} &= \frac{1}{1 - e^{-\lambda}} \\
\sum_{k\in\mathbb{N}} k e^{-\lambda k} &= \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2} \\
\sum_{k\in\mathbb{N}} k^2 e^{-\lambda k} &= \frac{e^{-\lambda}(1 + e^{-\lambda})}{(1 - e^{-\lambda})^3}
\end{aligned}$$

□

Let us mention the following nice consequence

Corollary 4.3.2. *We have for the heat semigroup on O_N^+ (resp. S_N^+):*

$$\|T_t x\|_\infty \leq f(t/2)\|x\|_1$$

with f the same function as in the previous Theorem.

Let us remark that, when t goes to zero, $f(t)$ is equivalent to $1/t^3$. On $\mathbb{R}^d$, the behavior when t goes to zero of a heat semigroup is in $1/t^{d/2}$, as can be seen e.g. in [45, Property R_n, section II.1], so that we have here a behavior as if we were in "dimension" 6.

Proof. We are following the reasoning of [6, Corollary 3].
The semigroup we consider are self-adjoint on $L^2(\mathbb{G}, h)$. Indeed, by [14, Remark 4.7], it follows from the fact that $L(u_{ij}^{(s)}) = L(u_{ji}^{(s)})$. By self-adjointness, we can dualize the inequality of Theorem 4.3.1 so as to obtain $\|T_t x\|_2 \leq \sqrt{f(t)}\|x\|_1$. We can then combine it to get:

$$\|T_t x\|_\infty \leq \sqrt{f(t/2)}\|T_{t/2}x\|_2 \leq f(t/2)\|x\|_1$$

$\square$

As a consequence of the Theorem, we deduce that the semigroup we considered on the free Orthogonal quantum group is ultracontractive. Indeed, [8, Proof of Theorem 2.2] shows that there exists a constant D (depending on N) such that:

$$||x||_\infty \leq D(s+1)||x||_2$$

when $x \in V_s$. Thus we can apply Theorem 4.3.1 with $\alpha = 1/(N+2)$ and $\beta = \gamma = D$.
In the same way, [7, Theorem 4.10] shows that there exists a constant C depending on N such that on S_N^+ and for $x \in V_s$, we have:

$$||x||_\infty \leq C(2s+1)||x||_2$$

This means that we can obtain ultracontractivity for our semigroup on S_N^+ by applying Theorem 4.3.1 with $\alpha = \frac{1}{\sqrt{N}(\sqrt{N}+2)}$, $\beta = 2C$ and $\gamma = C$.

4.3.2 Special cases O_2^+ and S_4^+

We can say something more in the case of O_2^+. Indeed, we have $U_s(2) = s+1$ and, by differentiating the recurrence relation, we get $U_s'(2) = s(s+1)(s+2)/6$. This therefore means that we have an exact value for the eigenvalues:

$$\lambda_s = -\frac{s(s+2)}{6}$$

If we then take up the computations from Theorem 4.3.1, we see that we have a somewhat better estimation:

$$||T_t x||_\infty \leq \sqrt{D^2 \sum_s e^{-\frac{s(s+2)}{3}t}(s+1)^2}||x||_2$$

How to compute the exact value of this series does not seem obvious. We can nevertheless find a better estimate for the "dimension" of the semigroup (cf remark following Corollary 4.3.2). Indeed, we first observe that:

$$\sum_s e^{-\frac{s(s+2)}{3}t}(s+1)^2 \leq \sum_s e^{-\frac{s^2}{3}t}(s+1)^2$$

Moreover,

$$\sum_{s=0}^{\infty} e^{-\frac{s^2}{3}t} \leq 1 + \sum_{s=1}^{\infty} s e^{-\frac{s^2}{3}t} \leq 1 + \sum_{s=1}^{\infty} s^2 e^{-\frac{s^2}{3}t}$$

This yields an inequality:

$$||T_t x||_\infty \leq \sqrt{g(t)}||x||_2 \text{ with } g(t) = 4D^2 \sum_{s=1}^{\infty} s^2 e^{-\frac{s^2}{3}t} + D^2$$

Now, we will try to estimate the series. The function $s \mapsto s^2 e^{-\frac{s^2 t}{3}}$ is decreasing on $[\sqrt{\frac{3}{t}}, +\infty[$ and increasing on $[0, \sqrt{\frac{3}{t}}]$. Let's set $s_0 = \sqrt{\frac{3}{t}}$. For fixed t, we have:

$$\begin{aligned}
\int_0^{s_0} s^2 e^{-\frac{s^2 t}{3}} ds &\leq \sum_{s=1}^{s_0} s^2 e^{-\frac{s^2 t}{3}} \leq \int_0^{s_0} s^2 e^{-\frac{s^2 t}{3}} ds + \frac{3}{et} \\
\int_{s_0}^{\infty} s^2 e^{-\frac{s^2 t}{3}} ds &\leq \sum_{s=s_0}^{\infty} s^2 e^{-\frac{s^2 t}{3}} \leq \frac{3}{et} + \int_{s_0}^{\infty} s^2 e^{-\frac{s^2 t}{3}} ds
\end{aligned}$$

We do the change of variable $u = s\sqrt{t/3}$:

$$\begin{aligned}
\left(\frac{3}{t}\right)^{3/2} \int_0^1 u^2 e^{-u^2} du &\leq \sum_{s=1}^{s_0} s^2 e^{-\frac{s^2 t}{3}} \leq \left(\frac{3}{t}\right)^{3/2} \int_0^1 u^2 e^{-u^2} du + \frac{3}{et} \\
\left(\frac{3}{t}\right)^{3/2} \int_1^{\infty} u^2 e^{-u^2} du &\leq \sum_{s=s_0}^{\infty} s^2 e^{-\frac{s^2 t}{3}} \\
&\leq \frac{3}{et} + \left(\frac{3}{t}\right)^{3/2} \int_1^{\infty} u^2 e^{-u^2} du
\end{aligned}$$

And by combining:

$$\begin{aligned}
\left(\frac{3}{t}\right)^{3/2} \int_0^{\infty} u^2 e^{-u^2} du &\leq \frac{3}{et} + \sum_{s=0}^{\infty} s^2 e^{-\frac{s^2 t}{3}} \\
&\leq 2\frac{3}{et} + \left(\frac{3}{t}\right)^{3/2} \int_0^{\infty} u^2 e^{-u^2} du
\end{aligned}$$

In other words, when t goes to zero, $g(t)$ behaves like $t^{-3/2}$ and, in the spirit of the remark following Corollary 4.3.2, this yields a "dimension" 3 for the semigroup.
The same reasoning yields for S_4^+ that the eigenvalues are:

$$\lambda_s = -\frac{s(s+1)}{6}$$

And the exact same computations can be done to find that the "dimension" of the semigroup on S_4^+ is also 3.

4.3.3 Hypercontractivity

Definition 4.3.3. *We say that a semigroup T_t is hypercontractive if for each $2 < p < \infty$, there exists a $\tau_p > 0$ such that for all $t \geq \tau_p$ we have:*

$$\|T_t x\|_p \leq \|x\|_2 \tag{4.1}$$

Let us remark that if the semigroup T_t is hypercontractive, then the inequality (4.1) is also true for $1 \leq p \leq 2$ because for such a p and for any $x \in C(\mathbb{G})$ we have $\|T_t x\|_p \leq \|x\|_p \leq \|x\|_2$. We can also notice that due to duality, we have:

$$\|T_t x\|_2 \leq \|x\|_q$$

for $t \geq \tau_p$ and q such that $1/p + 1/q = 1$. Therefore, for t big enough, T_t is also a contraction from L^q to L^2 for any $1 < q < 2$.

Theorem 4.3.4. *The semigroup T_t^O (resp. T_t^S) we consider on O_N^+ (resp. S_N^+) is hypercontractive.*

Proof. We use the following inequality shown in [37, Theorem 1], for $2 < p < \infty$. This inequality can indeed be applied in our setting, as $L^\infty(\mathbb{G}) = C_r(\mathbb{G})''$ is a von Neumann algebra and h is a faithful, finite normal trace on it. Thus,

$$\|x\|_p^2 \leq \|h(x)1\|_p^2 + (p-1)\|x - h(x)1\|_p^2$$

To use this inequality, we will write $x = h(x)1 + \sum_{s\geq 1} x_s$ with $x_s \in V_s$. We notice that $h(T_t(x))1 = T_t(h(x)1)$ because the V_s are eigenspaces for T_t. Therefore, we have:

$$\begin{aligned}
\|T_t(x)\|_p^2 &\leq \|T_t(h(x)1)\|_p^2 + (p-1)\|T_t(x - h(x)1)\|_p^2 \\
&\leq |h(x)|^2 + (p-1)\left(\sum_{s\geq 1} ||T_t(x_s)||_p\right)^2 \\
&\leq |h(x)|^2 + (p-1)\left(\sum_{s\geq 1} e^{\lambda_s t}||x_s||_p\right)^2 \\
&\leq |h(x)|^2 + (p-1)\left(\sum_{s\geq 1} e^{\lambda_s t}||x_s||_\infty\right)^2 \\
&\leq |h(x)|^2 + (p-1)\left(\sum_{s\geq 1} e^{\lambda_s t}(\beta s + \gamma)||x_s||_2\right)^2 \\
&\leq |h(x)|^2 + (p-1)\sum_{s\geq 1}\left((\beta s + \gamma)e^{\lambda_s t}\right)^2 \sum_{s\geq 1} ||x_s||_2^2 \\
&\leq ||x||_2^2
\end{aligned}$$

for $t \geq \tau_p$ and τ_p such that:

$$(p-1)\sum_{s\geq 1}(\beta s + \gamma)^2 e^{2\lambda_s \tau_p} \leq 1$$

□

Proposition 4.3.5. *Hypercontractivity is achieved for T_t^O at least from the time $\tau_p^{(O)}$ on which verifies:*

$$\tau_p^{(O)} = -\frac{(N+2)}{2}\ln X$$

$$\text{and } X \text{ is the smallest real positive root of } \frac{X^3-3X^2+4X}{(1-X)^3} = \frac{1}{(p-1)D^2}$$

Hypercontractivity is achieved for T_t^S at least from the time $\tau_p^{(S)}$ on which verifies:

$$\tau_p^{(S)} = -\frac{\sqrt{N}(\sqrt{N}+2)}{2}\ln Y$$

$$\text{and } Y \text{ is the smallest real positive root of } \frac{Y^3-2Y^2+9Y}{(1-Y)^3} = \frac{1}{(p-1)C^2}$$

Proof. We use the expression:

$$(p-1)\sum_{s\geq 1}(\beta s+\gamma)^2 e^{2\lambda_s\tau_p} = 1$$

drawn from the proof of Theorem 4.3.4. The precise value of the eigenvalues is too cumbersome to compute, therefore we use a minoration of them:

$$\begin{aligned} \lambda_s &\geq -\frac{s}{N-2} \text{ for } O_N^+ \\ \lambda_s &\geq -\frac{s}{\sqrt{N}(\sqrt{N}-2)} \text{ for } S_N^+ \end{aligned}$$

By then setting $X = \exp(-\frac{2\tau_p^{(O)}}{N-2})$ and $Y = \exp(-\frac{2\tau_p^{(S)}}{\sqrt{N}(\sqrt{N}-2)})$ and using the classical series that were already used in the proof of Theorem 4.3.1, we obtain the desired equation for X and Y. The fact that the root must be the smallest real one comes from the fact that we need to take the biggest time τ_p such that the inequalities

$$\begin{aligned} \frac{X^3-3X^2+4X}{(1-X)^3} &\leq \frac{1}{D^2(p-1)} \\ \frac{Y^3-2Y^2+9Y}{(1-Y)^3} &\leq \frac{1}{(p-1)C^2} \end{aligned}$$

are verified always for $t \geq \tau$. But X and Y diminish when the time increase. Therefore we need to take the smallest positive root. □

Nevertheless, there is no reason for $\tau_p^{(O)}$ (resp. $\tau_p^{(S)}$) to be the optimal times.

4.4 Further properties of the semigroups

We will note $\mathrm{Pol}(\mathbb{G})_+$ the subset of $\mathrm{Pol}(\mathbb{G})$ consisting of all such x such that $|x| = x$.

4.4.1 Spectral gap

Definition 4.4.1. *We say that T_t verifies a spectral gap inequality with constant $m > 0$ if we have for all $x \in \mathrm{Pol}(\mathbb{G})_+$:*

$$m||x - h(x)||_2^2 \leq -h(xT_Lx)$$

Proposition 4.4.2. *Our semigroup T_t^O on O_N^+ verifies the spectral gap inequality with constant $m = \frac{1}{N+2}$.*

Proof. The eigenvalues of the generator T_L are of the form:

$$-\frac{U_s'(N)}{U_s(N)} = -\sum_{i=1}^{s} \frac{1}{N - \lambda_i}$$

Because $-2 \leq \lambda_i \leq 2$, we get $\frac{U_s'(N)}{U_s(N)} \geq \frac{s}{N+2}$.
Let us now write $x = \sum_s x_s$. We then get:

$$h(xT_Lx) = \sum_s -\frac{U_s'(N)}{U_s(N)}||x_s||_2^2$$

Using the fact that the V_s are in orthogonal direct sum, we deduce that: $-h(xT_Lx) \geq \frac{1}{N+2}||x||_2^2$.
But, we also see that $||x - h(x)||_2 \leq ||x||_2$ and thus we finally get:

$$||x - h(x)||_2^2 \leq -(N+2)\, h(xT_Lx)$$

□

We can prove the following in the same way:

Proposition 4.4.3. *Our semigroup T_t^S on S_N^+ verifies the spectral gap inequality with constant $m = \frac{1}{2\sqrt{N}(\sqrt{N}+2)}$.*

4.4.2 Logarithmic Sobolev inequalities

Hypercontractivity is closely related to a class of inequalities called Logarithmic Sobolev inequalities, or, shorter, log-Sobolev inequalities. See, e.g., [36, Theorem 3.8]. There is nothing new in our argumentation and we will use many arguments similar to those contained in [36]; we only give it here for clarity's sake and so as to have a condensed proof.

Proposition 4.4.4. *There exist constants $c, t_0 > 0$, such that, if we denote $q(t) = 1 + e^{2t/c}$, we then have for $0 \leq t \leq t_0$:*

$$||T_t^A : L^2 \to L^{q(t)}|| \leq 1$$

where $A = O_N^+$ or S_N^+

Proof. For simplicity's sake, we will drop the exponent A of T_t^A.
We want to use Hadamard's three line lemma. Let z be a complexe number whose real part is in $[0,1]$. Let $x = \sum_{ijs} x_s$ be an element of $\mathrm{Pol}(A)$ and we define:

$$T_{t_0 z}(x) = \sum_{ijs} e^{\lambda_s t_0 z} x_s$$

where we take for t_0 the optimal time for hypercontractivity $T_t : L^2 \to L^4$. We note that this definition is a holomorphic continuation of the semigroup T_t. We now set (with $c = 2t_0/\ln 3$):

$$\phi(z) = h\left[|T_{t_0 z}(x)|^{1+e^{2t_0 z/c}}\right]$$

We must first observe that ϕ is holomorphic on its definition domain. Indeed, we may write:

$$\phi(z) = h\left[\exp\left(\frac{1+e^{2t_0 z/c}}{2}\log\left(T_{t_0 z}(x)^* T_{t_0 z}(x)\right)\right)\right]$$

We shall note $\zeta(z)$ the argument of h. Let w be a complex such that $z+w$ is still in the domain. Then, by using Taylor expansions:

$$\begin{aligned} \zeta(z+w) - \zeta(z) &= \\ e^{\frac{1+e^{2t_0 z/c}}{2}\log\left[T_{t_0 z}(x)^* T_{t_0 z}(x) + \sum_s \lambda_s t_0 \bar{w} T_{t_0 z}(x) + \sum_s \lambda_s t_0 w T_{t_0 z}(x)^* + o(\|w\|)\right]} & \\ -e^{\frac{1+e^{2t_0 z/c}}{2}\log T_{t_0 z}(x)^* T_{t_0 z}(x)} & \end{aligned}$$

By using the Taylor series expansion of the Logarithm and the Exponential we get the existence of the holomorphic derivative.
Let now y be a real. We observe:

$$\begin{aligned} |\phi(iy)| &= |h\left[|x|^{1+e^{2t_0 iy/c}}\right]| \\ &\leq h\left[|x|^2\right] \leq ||x||_2^2 \end{aligned}$$

because $\left||x|^{\exp(2ist_0/c)}\right| \leq |x|^{Re(\exp(2t_0 is/c))}$.
We also have:

$$\begin{aligned} |\phi(1+is)| &= |h\left[|T_{t_0}(x)|^{1+e^{2t_0/c}e^{2ist_0/c}}\right]| \\ &\leq h\left[|T_{t_0}(x)|^{1+\exp(2t_0/c)}\right] \leq ||x||_2^4 \end{aligned}$$

Where the last inequality was obtained thanks to the $L^2 \to L^4$ hypercontractivity (indeed, $1 + e^{2t_0/c} = 4$ since $c = 2t_0/\ln 3$).
From now on we will assume that $||x||_2 = 1$. Let t be positive. By applying Hadamard's three lines lemma, we get:

$$\begin{aligned} ||T_t(x)||_{q(t)}^{q(t)} &= |\phi(t/t_0)| \\ &\leq ||x||_2^{2(1-t/t_0)} ||T_{t_0} x||_4^{4t/t_0} \leq 1 \end{aligned}$$

□

Theorem 4.4.5. *For all $x \in A$ such that $x = |x|$ and with the same assumptions as in Proposition 4.4.4, we have the following inequality:*

$$h\left(x^2 \ln x\right) - ||x||_2^2 \ln ||x||_2 \leq -\frac{c}{2} h\left(xT_L x\right)$$

where $c = 2t_0/\ln 3$.

Proof. We define : $F(t) = ||x_t||_{q(t)}$, where we note $x_t = T_t x$. Because of Proposition 4.4.4, we know that $\log F(t) \leq \log F(0)$. Hence:

$$\frac{d}{dt} \log F(t)_{|t=0} \leq 0$$

Let us calculate this term:

$$\begin{aligned} \frac{d}{dt} \log ||x_t||_q &= \frac{d}{dt} \left(\frac{1}{q} \log ||x_t||_q^q \right) \\ &= -\frac{\dot{q}}{q} \log ||x_t||_q + \frac{1}{q||x_t||_q^q} \frac{d}{dt} ||x_t||_q^q \end{aligned}$$

We will calculate this last derivative.

$$\begin{aligned} & \frac{1}{\Delta t} \left[||x_{t+\Delta t}||_{q(t+\Delta t)}^{q(t+\Delta t)} - ||x_t||_q^q \right] \\ = \; & \underbrace{\frac{1}{\Delta t} h \left(x_{t+\Delta t}^{q(t+\Delta t)-1} \left(x_{t+\Delta t} - x_t \right) \right)}_{K_1} + \underbrace{\frac{1}{\Delta t} h \left[\left(x_{t+\Delta t}^{q(t+\Delta t)-1} - x_t^{q(t+\Delta t)-1} \right) x_t \right]}_{K_2} \\ + \; & \underbrace{\frac{1}{\Delta t} h \left[\left(x_t^{q(t+\Delta t)-1} - x_t^{q(t)-1} \right) x_t \right]}_{K_3} \end{aligned}$$

Let us then treat the three terms separately:

1. The first one K_1 converges towards $h \left[x_t^{q-1} T_L x_t \right]$.

2. The second one K_2 converges towards 0. This is proven just as in [36].

3. The third one K_3 can be written as follows We assume first that x is invertible in order to use the holomorphic functional calculus:

$$\begin{aligned} K_3 &= \frac{1}{\Delta t} \left[||x_t||_{q(t+\Delta t)}^{q(t+\Delta t)} - ||x_t||_q^q \right] \\ &= \frac{1}{\Delta t} \tau \left[\int_0^1 ds \frac{d}{ds} x_t^{q(t+s\Delta t)} \right] = \frac{1}{\Delta t} h \left[\int_0^1 ds \frac{d}{ds} e^{q(t+s\Delta t) \log x_t} \right] \\ &= \tau \left[\int_0^1 ds \log x_t \dot{q} \left(t + s\Delta t \right) e^{q(t+s\Delta t) \log x_t} \right] \end{aligned}$$

   And this converges towards $\tau \left(\dot{q} x_t^q \log x_t \right)$.
   Now, what happens if x is not invertible? We introduce $\psi_n(z) = z^q \ln(z + 1/n)$ which is defined on $[0, +\infty[$ and $\psi(z) = z^q \ln z$ which is also defined (by continuity) on $[0, +\infty[$. It is easy to see that $(\psi_n - \psi)(0) = 0$ and $\psi_n - \psi$ is increasing and pointwise converging to zero when n goes to infinity. By Dini's Theorem, this means that ψ_n converges towards ψ uniformly on the compact subsets of $[0, +\infty[$. This will allow us to intervert limits, since the spectrum is anyways compact. We replace in the same wise $z \mapsto z^q$ by $z \mapsto (z + 1/n)^q$.

Therefore we can take up the above calculation in the following way:

$$\begin{aligned}
K_3 &= \frac{1}{\Delta t}\left[||x_t||_{q(t+\Delta t)}^{q(t+\Delta t)} - ||x_t||_q^q\right] \\
&= \frac{1}{\Delta t} h\left[\int_0^1 ds \frac{d}{ds} \lim_{n\to\infty} (x_t + 1/n)^{q(t+s\Delta t)}\right] \\
&= \lim_n \frac{1}{\Delta t} h\left[\int_0^1 ds \frac{d}{ds} e^{q(t+s\Delta t)\log(x_t+1/n)}\right] \\
&= \lim_n h\left[\int_0^1 ds \log(x_t+1/n)\dot{q}\,(t+s\Delta t)\, e^{q(t+s\Delta t)\log(x_t+1/n)}\right] \\
&= h\left[\int_0^1 ds \log x_t \dot{q}\,(t+s\Delta t)\, e^{q(t+s\Delta t)\log x_t}\right]
\end{aligned}$$

Putting all of this together we obtain our inequality, because $q\,(0) = 2$, $\dot{q}\,(0) = 2/c$. $\square$

Part III

Investigating Dual Groups

5

Convergence of the (classical) Brownian motion on $U(nd)$

In this chapter we show that the Brownian motion on the (classical) unitary group converges block-wise, when the size of the matrices goes to infinity, to a quantum Lévy process on the unitary dual group. After having recalled a previous result by Biane [5] and given a new proof of it, relying on stochastic calculus and combinatorial considerations, we will introduce the main theorem of this section, Theorem 5.2.1, which generalizes Biane's result. We prove it by using the same method. In the course of a later chapter, we will see another proof of Theorem 5.2.1, as it will be seen to be the special case of Theorem 6.2.2. Nevertheless, the author thinks that the method presented here is interesting for itself and therefore deserves a chapter.

This chapter is taken from my first article [44], which I wrote alone.

5.1 Biane's result about the Brownian motion on the Unitary group

In all the following, we assume that a unital noncommutative probability space (A, ϕ) be given. Let us remind what we mean by that definition: a unital noncommutative probability space is a couple (A, ϕ) where A is a unital $*$-algebra and ϕ is a linear functional on A such that $\phi(a^*a) \geq 0$ for each $a \in A$ and $\phi(1) = 1$.
We will also write by δ_{ab} Kronecker's symbol, which is equal to 0 when $a \neq b$ and is equal to 1 when $a = b$. Let us recall following definitions and result:

Definition 5.1.1. *We denote by $(\nu_t)_{t\geq 0}$ the same family of measures on the unit circle as in [5], ie ν_t is the only probability measure such that $\xi_{\nu_t}(z) = z\exp[\frac{1}{2}\frac{1+z}{1-z}]$, where ξ_{ν_t} is the inverse function of $\frac{\psi_{\nu_t}}{1+\psi_{\nu_t}}$ and $\psi_{\nu_t} = \int \frac{z\zeta}{1-z\zeta} d\nu_t(\zeta)$ where the integration is done on the unit circle.*

Definition 5.1.2. *A free multiplicative Brownian motion is a family $(U_t)_{t\geq 0}$ such that:*

- *For every $0 \leq t_1 < t_2 < \ldots < t_n$, the family $\left(U_{t_1}, U_{t_2}U_{t_1}^{-1}, \ldots, U_{t_n}U_{t_{n-1}}\right)$ is free.*
- *For every $0 \leq s < t$ the element $U_tU_s^{-1}$ has a distribution ν_{t-s}.*

In his paper [5], Biane proved that Brownian motion on the group $U(d)$ converges, as d goes to infinity, towards a multiplicative free Brownian motion. To do this, he proves first the convergence of the marginals using representation theory arguments and secondly the freeness of the increments. We suggest here that there is an other way to prove the convergence of the marginals based on the Itô formula.
Let us first observe that the Brownian motion on the Unitary group $U(d)$ can be defined as the unique solution of:

$$\mathbf{d}U_t^{(d)} = \mathrm{i}\mathbf{d}H_t U_t^{(d)} - \frac{1}{2}U_t^{(d)}\mathbf{d}t$$

with initial condition $U_0 = I$. Note that we denote by i the complex number, so as to differentiate it from the index i. In the same way we write $\mathbf{d}$ the differential operator so as to distinguish it from the size of the matrices. In this equation, we have noted by H_t a Brownian motion on hermitian matrices defined by:

- The family $(H_{ij}(t))_{1\le i\le j\le d}$ is an independent family of random variables
- For $1 \le i \le d$, we have $H_{ii}(t)$ a gaussian variable $\mathcal{N}(0, \frac{1}{d})$
- For $1 \le k \le j \le d$, we have $H_{kj}(t) = H_{kj}^{(1)}(t) + \mathrm{i}H_{kj}^{(2)}(t)$ with $H_{kj}^{(1)}(t)$ and $H_{kj}^{(2)}(t)$ two independent gaussian variables $\mathcal{N}(0, \frac{1}{2d})$
- The matrix $H(t)$ is hermitian for each t.

In particular this means that each entry of H_t is of variance $1/d$.

Note: we shall omit the exponent (d) when there is no confusion possible.
Let us now denote by $f_{k_1,\dots,k_r}$ the following function of t:

$$f_{k_1,\dots,k_r} = \mathbb{E}\left[tr\left(U_t^{k_1}\right)\dots tr\left(U_t^{k_r}\right)\right]$$

where the trace is normalized[1] by $1/d$. We will find a differential equation involving those functions.

Lemma 5.1.3. *We have the following formula:*

$$\begin{aligned}\mathbf{d}\left(U_{i_1j_1}\dots U_{i_rj_r}\right) &= martingale - \frac{1}{2}\sum_{k=1}^{r} U_{i_1j_1}\dots U_{i_rj_r}\mathbf{d}t \\ &- \frac{\mathbf{d}t}{d}\sum_{1\le p<q\le r} U_{i_1j_1}\dots U_{i_pj_q}\dots U_{i_qj_p}\dots U_{i_rj_r}\end{aligned}$$

This means that the non-martingale part is constituted by two terms, the first one where nothing is changed in the indices and the second one where you have switched two indices: j_q replaces j_p and j_p replaces a j_q.

Proof. This is obtained by using Itô's formula and by reasoning for each element in the matrix, because:

$$\mathbf{d}\left(U_{i_1j_1}\dots U_{i_rj_r}\right) = \sum_{k=1}^{r} U_{i_1j_1}\dots\left(\mathbf{d}U_{i_kj_k}\right)\dots U_{i_rj_r} + \sum_{1\le k<l\le r}\prod_{s\ne k,l} U_{i_sj_s}\mathbf{d}\left[U_{i_kj_k}, U_{i_lj_l}\right]$$

[1] The convention we adopt in this chapter is following: whenever we mean the normalized trace, we write tr and we write Tr whenever we speak of the usual trace.

The $[.,.]$ denotes the quadratic variation. We remark that:

$$\forall i,j, \mathbf{d}U_{ij}(t) = \mathrm{i}\sum_{r=1}^{d} \mathbf{d}H_{ir}U_{rj} - \frac{1}{2}U_{ij}\mathbf{d}t$$

and

$$\mathbf{d}[H_{i_k r_k}, H_{i_l r_l}] = \mathbf{d}[H^{(1)}_{i_k r_k} + \mathrm{i}H^{(2)}_{i_k r_k}, H^{(1)}_{i_l r_l} + \mathrm{i}H^{(2)}_{i_l r_l}]$$

But we know that the quadratic variation of two processes is zero if they are independent. Thus, $\mathbf{d}[H_{i_k r_k}, H_{i_l r_l}]$ is equal to:

- If $i_k = i_l$ and $j_l = j_k$, $\mathbf{d}[H_{i_k j_k}, H_{i_l j_l}] = \frac{1}{2d} - \frac{1}{2d} = 0$
- If $i_k = j_l$ and $j_k = i_l$, $\mathbf{d}[H_{i_k j_k}, H_{i_l j_l}] = \frac{1}{2d} + \frac{1}{2d} = \frac{1}{d}$
- And it is equal to zero in all other cases.

And thus, the quadratic variation can be expressed as:

$$\begin{aligned}\mathbf{d}[U_{i_k j_k}, U_{i_l j_l}] &= \mathrm{i}\sum_{r_l, r_k=1}^{d} U_{r_k j_k}U_{r_l j_l}\mathbf{d}[H_{i_k r_k}, H_{i_l r_l}] + \text{martingale} \\ &= \mathrm{i}U_{i_l j_k}U_{i_k j_l}\end{aligned}$$

$\square$

When we take the expectation, the martingale part vanishes.
If we expand $f_{k_1,\ldots,k_r}$, we get:

$$f_{k_1,\ldots,k_r} = \frac{1}{d^r}\mathbb{E}[\sum_{\substack{i^1_1,\ldots,i^1_{k_1} \\ \ldots \\ i^r_1,\ldots,i^r_{k_r}}=1}^{d} U_{i^1_1 i^1_2}\ldots U_{i^1_{k_1} i^1_1}\ldots U_{i^r_1 i^r_2}\ldots U_{i^r_{k_r} i^r_1}]$$

To get a system of differential equations we will use the former formula that we have obtained thanks to Itô's Lemma. Especially we must see how the last term, switching p and q, can be rewritten in terms of the functions $f_{k_1,\ldots,k_r}$. There are actually two cases to study: first when p and q come from the same trace and second when they come from different traces.
When they come from the same trace: If for instance p and q both come from the m^{th} trace, the contribution of this trace is of the kind:

$$\frac{1}{d^r}\ldots U_{i^m_1 i^m_2}\ldots U_{i^m_p i^m_{p+1}}\ldots U_{i^m_q i^m_{q+1}}\ldots U_{i^m_{k_m} i^m_1}\ldots$$

So when we do the switching it yields:

$$\frac{1}{d^r}\ldots U_{i^m_1 i^m_2}\ldots U_{i^m_p i^m_{q+1}}\ldots U_{i^m_q i^m_{p+1}}U_{i^m_{q+1} i^m_{q+2}}\ldots U_{i^m_{k_m} i^m_1}$$

And when we sum over all those indices we see that we actually get: $df_{k_1,\ldots,k_m-(q-p),q-p,\ldots,k_r}$, ie the switching has produced one more trace.

When they come from two different traces: We shall here suppose that p comes from the u^{th} trace and q comes from the v^{th} trace, with $u < v$. The contribution of those two traces are:

$$\frac{1}{d^r}\dots U_{i_1^u i_2^u}\dots U_{i_p^u i_{p+1}^u}\dots U_{i_{k_u}^u i_1^u}\dots U_{i_1^v i_2^v}\dots U_{i_q^v i_{q+1}^v}\dots U_{i_{k_v}^v i_1^v}\dots$$

Switching p and q yields to:

$$\frac{1}{d^r}\dots U_{i_1^u i_2^u}\dots U_{i_p^u i_{q+1}^v}\dots U_{i_{k_u}^u i_1^u}\dots U_{i_1^v i_2^v}\dots U_{i_q^v i_{p+1}^u}U_{i_{q+1}i_{q+2}}\dots U_{i_{k_v}^v i_1^v}\dots$$

And so if we sum over all indices we see that we get $\frac{1}{d}f_{k_1,\dots k_u+k_v,\dots k_r}$, ie we have merged two traces together.

So, if we put it all together we see by using Lemma 5.1.3 that the system of differential equations we get is:

$$\begin{aligned} f'_{k_1,\dots,k_r} &= -\frac{k_1+\dots k_r}{2}f_{k_1,\dots,k_r} - \sum_{\kappa=1}^{r}\sum_{l=1}^{k_\kappa}(k_\kappa - l)\, f_{k_1,\dots,k_\kappa-l,l,\dots,k_r} \\ &- \frac{1}{d^2}\sum_{1\le\kappa<\lambda\le r}\sum_{p=1}^{k_\kappa}\sum_{q=1}^{k_\lambda} f_{k_1,\dots,k_\kappa+k_\lambda,\dots} \end{aligned}$$

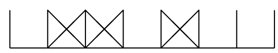
Let us observe here that we have a nice combinatorial structure for these equations. Indeed, we can interpret $(k_1,\dots,k_r)$ as an integer partition for the integer $k_1+\dots+k_r$. By doing so, we see that the equation only involves partitions for the same integer because we either split an integer into two parts or we merge two integers into one. These equations thus have the same structure as the equations in Proposition 2.3 in [28] via the identification between a permutation and the length of the cycles of its canonical decomposition.
Let us also note that an integer l has only finitely many partitions.[2] So that means that each function is involved in a system of finitely many linear differential equations with fixed initial conditions.
What can we say about the convergence of this family of functions? We actually have that for each $r \ge 1$ and every $k_1,\dots,k_r \ge 1$, the function $f^{(d)}_{k_1,\dots,k_r}$ converges, as d goes to infinity, towards a function $f_{k_1,\dots,k_r}$ verifying:

$$f'_{k_1,\dots,k_r} = -\frac{k_1+\dots+k_r}{2}f_{k_1,\dots,k_r} - \sum_{\kappa=1}^{r}\sum_{l=1}^{k_\kappa}(k_\kappa - l)\, f_{k_1,\dots,k_\kappa-l,l,\dots,k_r}$$

Indeed, let us fix such a partition $k_1+\dots+k_r = k$. If we note

$$P(k) := \{(k_1,\dots,k_r) | r \ge 0, k_1+\dots+k_r = k\}$$

the set of partitions of the integer k, we have just shown that this set if finite. The function $f^{(d)}_{k_1,\dots,k_r}$ thus only shows up in a finite number of linear differential equations with constant

[2]Without going into the details of the theory of integer partitions, we may find a gross upper bound for this number in the following way: A partition of l cannot have more than l parts. So let's consider a line consisting of $l+l-1 = 2l-1$ boxes. We then put crosses in $l-1$ boxes. Each such cross helps separate two parts of the partition. For instance:  represents the partition $(1,1,2)$ of the integer 4. Hence we see that the number of such partitions is bounded by $\binom{2l-1}{l-1}$, which is finite.

coefficients. This finite number of differential equations can be rewritten in a matricial form: let $\Phi_t^{(d)}$ be a vector in $\mathbb{C}^{\sharp P(k)}$ consisting of all functions $f^{(d)}_{p_1,\ldots,p_l}$ where $p_1,\ldots,p_l$ is a partition of the same integer k. Then $\Phi^{(d)}$ is solution of a differential equation of the form:

$$(\Phi^{(d)})' = A^{(d)}\Phi^{(d)}$$

where $A^{(d)}$ is a (constant) matrix formed with the coefficients of our differential equations. It is well-known that $\Phi^{(d)}$ is thus of the form $\Phi^{(d)} = \Phi_0^{(d)}e^{A^{(d)}t}$. But the coefficients of the equations for $f^{(d)}$, namely $A^{(d)}$, converge towards the coefficients for the equation of f, namely A, and thus $\Phi^{(d)}$ converges towards Φ, or in other words, $f^{(d)}_{k_1,\ldots,k_r}$ converges towards $f_{k_1,\ldots,k_r}$.

We will now denote by $F_{k_1,\ldots,k_r}$ the function $\phi\left(u_t^{k_1}\right)\ldots\phi\left(u_t^{k_r}\right)$ where u is here a free multiplicative Brownian motion. To prove the convergence of the marginals it will be enough to prove that the family of functions F verifies the differential equations system:

$$F'_{k_1,\ldots,k_r} = -\frac{k_1+\ldots k_r}{2}F_{k_1,\ldots,k_r} - \sum_{\kappa=1}^{r}\sum_{l=1}^{k_\kappa}(k_\kappa - l)\,F_{k_1,\ldots,k_\kappa-l,l,\ldots,k_r}$$

Indeed, if we have proven it, then it implies that for all $r \geq 1$ and all $0 \leq t_1 \leq \ldots \leq t_r$ the function $f^{(d)}_{t_1,\ldots,t_r}$ converges towards $F_{t_1\ldots t_r}$ when d goes to infinity. In particular, if we take $r=1$, we see that we have the convergence of the marginals (in moments).
In order to prove that formula we must remark that a free multiplicative Brownian motion is given by a free stochastic equation with initial conditions $u_0 = 1$ (1 is the unit element of A):

$$\mathbf{d}u_t = \mathrm{i}\mathbf{d}X_t u_t - \frac{1}{2}u_t\mathbf{d}t$$

where X_t is a free additive Brownian motion. This result is stated in [5, Theorem 2]. We will simplify the calculations by putting $V_t := e^{t/2}u_t$. Using the free analogue of Itô's Lemma (see e.g. [27, Theorem 5]), Biane demonstrated following formula

$$\mathbf{d}V_t^n = \mathrm{i}\sum_{k=0}^{n}V_t^k\mathbf{d}X_tV_t^{n-k} - \sum_{k=1}^{n-1}kV_t^k\phi\left(V_t^{n-k}\right)\mathbf{d}t$$

In other words this means:

$$\mathbf{d}u_t^n = \mathrm{i}\sum_{k=0}^{n}u_t^k\mathbf{d}X_tu_t^{n-k} - \sum_{k=1}^{n-1}ku_t^k\phi\left(u_t^{n-k}\right)\mathbf{d}t - \frac{n}{2}u_t^n\mathbf{d}t$$

Taking the trace of it we obtain:

$$\phi\left(u_t^n\right)' = -\sum_{k=1}^{n-1}k\phi\left(u_t^k\right)\phi\left(u_t^{n-k}\right) - \frac{n}{2}\phi\left(u_t^n\right)$$

And so it finally yields the following system of differential equations:

$$F'_{k_1,\ldots k_r} = -\frac{k_1+\ldots+k_r}{2}F_{k_1\ldots k_r} - \sum_{\kappa=1}^{r}\sum_{p=1}^{k_\kappa-1}pF_{k_1,\ldots,p,k_\kappa-p,\ldots,k_r}$$

And this is exactly the system we wanted because $F_{k_1,\ldots,p,k_\kappa-p,\ldots} = F_{k_1,\ldots,k_\kappa-p,p,\ldots}$.
To put it in a nutshell: we were able to reprove Biane's result by using a different method (by comparing systems of differential equations) to prove the convergence of marginals. The freeness of the increments can still be proven as did Biane but it will also follow from the results of section 4. We will now try to use that alternative method to generalize Biane's result. To do that we will need the concept of dual groups.

5.2 The main Theorem

How can express this result in the language of quantum Lévy processes? Can we relate it to dual groups? We could indeed generalize Biane's question by taking $U_t^{\langle d \rangle}$ a Brownian motion on the Unitary Group $U(nd)$, where n is a fixed integer. The matrix $U_t^{\langle d \rangle}$ can be decomposed in n^2 blocks of size $d \times d$. In the sequel of the article we will denote by $[U_t^{\langle d \rangle}]_{ij}$ the $(i,j)^{\text{th}}$ block of our Brownian motion. For each d we thus get a quantum stochastic process on the Dual Unitary Group by setting for $0 \leq s \leq t$:

$$\begin{aligned} j_{st}^{\langle d \rangle)} \quad : \quad & U_n^{\text{nc}} \to (A, \phi) \\ & u_{ij} \mapsto [U_t^{\langle d \rangle}]_{ij} \end{aligned}$$

We will in the sequel of the article omit the exponent $\langle d \rangle$ whenever no confusion can arise.
The question that is natural to ask and that generalizes Biane's result is whether or not j_{st} converges to a Lévy process on $U\langle n \rangle$ in the limit when d goes to infinity.
We will show that we have following result

Theorem 5.2.1 (Main Theorem)**.** *We assume that ϕ is tracial.*
Let $X = (X_{ij})_{1 \leq i,j \leq n}$ be a matrix whose entries are free stochastic variables verifying that:

- *For each i, X_{ii} is an additive free Brownian motion.*
- *For every $i \neq j$, $X_{ij} = X_{ij}^{(1)} + \mathrm{i} X_{ij}^{(2)}$ with $\sqrt{2} X_{ij}^{(1)}$ and $\sqrt{2} X_{ij}^{(2)}$ who are two additive free Brownian motions who are free one with another.*
- *For each i, j we have $X_{ij} = X_{ji}^*$.*
- *The family $(X_{ij})_{1 \leq i \leq j \leq n}$ is free.*

Let also $\Psi = (\Psi_{ij})$ be a free stochastic process defined by the free stochastic equation with initial condition $\Psi_0 = I$:

$$d\Psi_t = \frac{\mathrm{i}}{\sqrt{n}} dX_t \Psi_t - \frac{1}{2} \Psi_t dt$$

Through Ψ we may define a free Lévy process J through[3]*:*

$$\begin{aligned} J_{st} \quad : \quad & U_n^{\text{nc}} \to (A, \phi) \\ & u_{ij} \mapsto \Psi_{ij} \end{aligned}$$

Then, $(j_{st}^{\langle d \rangle)})$ converges towards (J_{st}) as d goes to infinity.

In the rest of this chapter we will give a proof of Theorem 5.2.1 by combinatorial and stochastic calculus arguments. In Section 6.2, we give another proof by showing that Theorem 5.2.1 is actually a special case of a more general result given by Theorem 6.2.2.

[3]By calculating $d(\sum_k \Psi_{ki}^* \Psi_{kj})$ we find zero. Moreover, when we calculate $d(\sum_k \Psi_{ik} \Psi_{jk}^*)$ we find a free stochastic diffenrential equation that is verified by the constant δ_{ij}. By unicity of the solution (see e.g. [27][Theorem 4]), we have that $\sum_k \Psi_{ik} \Psi_{jk}^* = \delta_{ij}$. Thus J_{st} respects the defining relations of $U\langle n \rangle$

5.3 Convergence of the marginals

We will first study the convergence of the marginals. Hence we will fix in this section a $t \geq 0$. To prove such a convergence we must study the moments of the type $\phi \circ j_{0t}(u^{\epsilon_1}_{i_1 j_1} \dots u^{\epsilon_r}_{i_r j_r})$, where $\epsilon_1, \dots, \epsilon_r \in \{\varnothing, *\}$. For convenience, we will identify $\varnothing$ with 0 and $*$ with 1. We will use exactly the same method as in the first section but, because there are n^2 variables, we will have many more indices.

5.3.1 Notations

We consider the dual group $U\langle n\rangle$ which is generated by n^2 variables. We will need to introduce some notations to describe all the indices that will be involved.
From now on and until the end of the paper, when we have a matrix $M \in \mathcal{M}_{nd}(\mathbb{C})$, we will denote:

- by M_{ij} the (i,j)-matrix entry of M.
- by $[M]_{ij}$ the (i,j)-block of size $d \times d$ of the matrix M

We denote by $[\mathcal{I}]$ the set $[\mathcal{I}] = \{1, \dots, n\}^2 \times \{0,1\}$. For such a triple $\alpha = (i,j,\epsilon)$, we will denote $[U]_\alpha$ the $d \times d$ block $[U]^\epsilon_{ij}$ where we identity $\epsilon = 1$ with $*$ and $\epsilon = 0$ with $\varnothing$.
We denote by $\mathcal{I}$ the set $\mathcal{I} = \{1, \dots, nd\}^2 \times \{0,1\}$. For such a triple $\rho = (\mu, \nu, \omega)$, we will denote U_ρ the coefficient $U_{\mu\nu}$ if $\omega = 0$ and the coefficient $\bar{U}_{\mu\nu}$ if $\omega = 1$.
When Ψ is in $\mathcal{M}_n(\mathcal{A})$, with $\mathcal{A}$ a $*$-algebra, we denote by Ψ_α the element Ψ^ϵ_{ij}.

5.3.2 A system of differential equations for the Brownian motion on $U(nd)$

To achieve our purpose we need to consider the family of functions (as always, we will omit the exponents everytime we may do so without risk):

$$\begin{aligned}
&\gamma^{\langle d\rangle}_{\alpha_{11},\dots,\alpha_{k_1 1};\dots;\alpha_{1r},\dots,\alpha_{k_r r}} \\
=\ &\mathbb{E}[tr([U]_{\alpha_{11}} \dots [U]_{\alpha_{k_1 1}}) \dots tr(\dots [U]_{\alpha_{k_r r}})]
\end{aligned}$$

where $r \geq 1; k_1, \dots, k_r \in \mathbb{N}, \alpha_{kl} \in [\mathcal{I}]$.
In other words, we take functions very similar to what we had before in the simpler case of the convergence to Biane's result. They still are the product of traces[4]. The difficulty arises here from the fact that we consider blocks and that we thus have to consider all possible products of the blocks and their adjoints. The indices we use specify which U_{ij} appear and if they have a $*$ or not and the semicolumns separate two traces. We will, as previously, try to find a system of differential equations. Let us fix the indices $\alpha_{11} \dots \alpha_{k_r r}$.
Again, we apply Lemma 5.1.3 in order to calculate the differential equation. For the sake of simplicity let us first observe what happens if we suppose that there are no $*$ in our function and we will later explain how to get the general case. As previously we treat separately the case where the switch occurs inside a same trace and the case where it affects two distinct traces.
The switch occurs in the same trace: Let's say that the switch is between p and q inside

[4]The renormalization is here done with a coefficient $1/d$.

the κ^{th} trace. Then, when we develop the traces, we see that the contribution of this trace, after the switch, is of the type:

$$\mathbb{E}[\sum_{s_{11}\dots s_{k_r r}} \dots U_{(i_{p\kappa}-1)d+s_{p\kappa},(j_{q\kappa}-1)d+s_{q\kappa}} \dots U_{(i_{q\kappa}-1)d+s_{q\kappa},(j_{p\kappa}-1)d+s_{p\kappa}} \dots]$$

As we could have expected the κ^{th} trace will be divided into two distinct traces: we get $d\gamma_{\dots;i_{1\kappa}j_{1\kappa},\dots,i_{p\kappa}j_{q\kappa},i_{q+1,\kappa}j_{q+1,\kappa},\dots;i_{p+1,\kappa}j_{p+1,\kappa},\dots,i_{q\kappa}j_{p\kappa};\dots}$ (we recall that the normalization constant we now use for the trace is $1/d$).

The switch concerns two distinct traces: If we do the calculations, we see that we reunite these two traces and that we get a multiplicative factor $1/d$.

So, if we put it all together (in the case we have no $*$ at all), the equation we will have is:

$$\begin{aligned}
& \gamma'_{\alpha_{11},\dots,\alpha_{k_1 1};\dots;\dots,\alpha_{k_r r}} \\
= & -\frac{k_1+\dots+k_r}{2}\gamma_{\alpha_{11},\dots,\alpha_{k_1 1};\dots;\dots,\alpha_{k_r r}} \\
- & \sum_{\kappa=1}^{r}\sum_{1\le p<q\le k_\kappa}\frac{1}{n}\gamma_{\dots;\alpha_{1\kappa},\dots,(i_{p\kappa}j_{q\kappa}0),\alpha_{q+1,\kappa},\dots;\alpha_{p+1,\kappa},\dots,(i_{q\kappa},j_{p\kappa},0);\dots} \\
+ & \mathcal{O}(\frac{1}{d^2})
\end{aligned}$$

Now, in the general case. We can remark that $[U^*]_{ij} = [U]^*_{ji}$. We also have:

$$\begin{aligned}
\mathbf{d}U_{\mu\nu} &= \mathrm{i}\sum_{\tau=1}^{d}\mathbf{d}H_{\mu\tau}U_{\tau\nu} - \frac{1}{2}U_{\mu\nu}\mathbf{d}t \\
\mathbf{d}\bar{U}_{\mu\nu} &= -\mathrm{i}\sum_{\tau=1}^{d}\bar{U}_{\tau\nu}\mathbf{d}H_{\tau\mu} - \frac{1}{2}\bar{U}_{\mu\nu}\mathbf{d}t
\end{aligned}$$

In turn this yields the more general Lemma:

Lemma 5.3.1. *We have, for $\rho_1,\dots,\rho_r \in \mathcal{I}$:*

$$\begin{aligned}
\mathbf{d}(U_{\rho_1}\dots U_{\rho_r}) &= -\frac{r}{2}U_{\rho_1}\dots U_{\rho_r}\mathbf{d}t \\
&+ \quad \textit{martingale part} - \frac{\mathbf{d}t}{nd}\sum_{1\le p<q\le r}(-1)^{\omega_p+\omega_q}\zeta^{\langle d\rangle)}_{pq}
\end{aligned}$$

where:

$$\zeta^{\langle d\rangle)}_{pq} = \begin{cases} U_{\rho_1}\dots U_{\mu_p\nu_q}\dots U_{\mu_q\nu_p}\dots U_{\rho_r} & \textit{if } \omega_p=\omega_q=0 \\ U_{\rho_1}\dots \bar{U}_{\mu_p\nu_q}\dots \bar{U}_{\mu_q\nu_p}\dots U_{\rho_r} & \textit{if } \omega_p=\omega_q=1 \\ \sum_{\tau=1}^{nd}\delta_{\mu_p\nu_q}U_{\rho_1}\dots\bar{U}_{\tau\nu_p}\dots U_{\tau\nu_q}\dots U_{\rho_r} & \textit{if } \omega_p=1,\omega_q=0 \\ \sum_{\tau=1}^{nd}\delta_{\mu_p\mu_q}U_{\rho_1}\dots U_{\tau\nu_p}\dots\bar{U}_{\tau\nu_q}\dots U_{\rho_r} & \textit{if } \omega_p=0,\omega_q=1 \end{cases} \tag{5.1}$$

Proof. It is an application of Itô's Lemma along with the observation that:

$$\mathbf{d}[U_{\mu\nu},U_{\theta\eta}] = -\frac{\mathbf{d}t}{nd}U_{\theta\nu}U_{\mu\eta} \text{ and } \mathbf{d}[\bar{U}_{\mu\nu},U_{\theta\eta}] = \sum_{\tau=1}^{nd}\frac{\mathbf{d}t}{nd}B_{\tau\nu}B_{\tau\eta}\delta_{\mu\theta}$$

□

So, taking up the same calculations as before, we get the following system of differential equations:

$$\begin{aligned}\gamma'_{\alpha_{11},\ldots} &= -\frac{k_1+\ldots+k_r}{2}\gamma_{\alpha_{11},\ldots}\\ &- \sum_{\kappa=1}^{r}\sum_{1\le p<q\le k_\kappa}(-1)^{\epsilon_{p\kappa}+\epsilon_{q\kappa}}\gamma_{(p,q,\kappa)}\\ &+ \mathcal{O}(\frac{1}{d^2})\end{aligned}$$

where we note:
If $\epsilon_{p\kappa}=\epsilon_{q\kappa}=0$:

$$\gamma_{(p,q,\kappa)} = \gamma_{\ldots;\alpha_{1\kappa},\ldots,(i_{p\kappa}j_{q\kappa}\epsilon_{q\kappa}),\alpha_{q+1,\kappa},\ldots;\alpha_{p+1,\kappa},\ldots,(i_{q\kappa}j_{p\kappa}\epsilon_{p\kappa});\ldots}$$

That is, we have a switch exactly as before.
If $\epsilon_{p\kappa}=\epsilon_{q\kappa}=1$:

$$\gamma_{(p,q,\kappa)} = \gamma_{\ldots;\alpha_{1\kappa},\ldots,\alpha_{p-1,\kappa},(i_{q\kappa}j_{p\kappa}\epsilon_{p\kappa}),\ldots;(i_{p\kappa}j_{q\kappa}\epsilon_{q\kappa}),\ldots;\ldots}$$

That is, we also have here a switch as we have already seen.
If $\epsilon_{p\kappa}=1,\epsilon_{q\kappa}=0$:

$$\gamma_{(p,q,\kappa)} = \sum_{t=1}^{n}\delta_{i_{p\kappa}i_{q\kappa}}\gamma_{\ldots;\alpha_{1\kappa},\ldots,(tj_{p\kappa}\epsilon_{p\kappa}),(tj_{q\kappa}\epsilon_{q\kappa}),\ldots;\alpha_{p+1,\kappa}\ldots\alpha_{q-1,\kappa};\ldots}$$

The structure is here a little more complicated, with a sum over t and t replacing the indices i_p and i_q and everything situated between the places p and q gets located in a new trace.
If $\epsilon_{p\kappa}=0,\epsilon_{q\kappa}=1$:

$$\gamma_{(p,q,\kappa)} = \sum_{t=1}^{n}\delta_{i_{p\kappa}i_{q\kappa}}\gamma_{\ldots;\alpha_{1\kappa},\ldots,\alpha_{p-1,\kappa},\alpha_{q+1,\kappa},\ldots;(tj_{p\kappa}\epsilon_{p\kappa}),\ldots,(tj_{q\kappa}\epsilon_{q\kappa});\ldots}$$

the structure is almost the same as in the previous case, with the only difference that the places p and q and everything in between gets into a new trace.

5.3.3 A system of differential equations for the free stochastic process

We will now introduce:

$$\Gamma_{\alpha_{11},\ldots;\ldots;\alpha_{1r},\ldots,\alpha_{k_r r}} = \phi(\Psi_{\alpha_{11}}\ldots)\ldots\phi(\Psi_{\alpha_{1r}}\ldots\Psi_{\alpha_{k_r r}})$$

To prove the convergence of the marginals, we will show that Γ verifies the system of differential equations that we have just found, in the limit where d goes to infinity.
By using free stochastic calculus we can see that the quadratic variation is $\mathbf{d}X_{ij}dX_{kl}=\delta_{il}\delta_{jk}\mathbf{d}t$. Moreover, the free stochastic differential equation yields, coefficient by coefficient:

$$\mathbf{d}\Psi_{uv} = \frac{\mathrm{i}}{\sqrt{n}}\sum_{k=1}^{n}\mathbf{d}X_{uk}\Psi_{kv} - \frac{1}{2}\Psi_{uv}\mathbf{d}t$$

and

$$\mathbf{d}\Psi^*_{uv} = -\frac{\mathrm{i}}{\sqrt{n}}\sum_{k=1}^{n}\Psi^*_{kv}\mathbf{d}X_{ku} - \frac{1}{2}\Psi^*_{uv}\mathbf{d}t$$

This allows us to prove following technical Lemma:

Lemma 5.3.2. *For each $r \geq 2$ and all indices we have:*

$$\begin{aligned}
\mathbf{d}(\Psi_{\alpha_1}\dots\Psi_{\alpha_r}) &= -\frac{r\mathbf{d}t}{2}\Psi_{\alpha_1}\dots\Psi_{\alpha_r} \\
&+ \frac{\mathrm{i}}{\sqrt{n}}\sum_{l=1}^{r}\sum_{k=1}^{n}(-1)^{\epsilon_l}\Psi_{\alpha_1}\dots\left\{\begin{array}{ll}\mathbf{d}X_{i_lk}\Psi_{kj_l} & \textit{if } \epsilon_l = 0\\ \Psi^*_{kj_l}\mathbf{d}X_{ki_l} & \textit{if } \epsilon_l = 1\end{array}\right\}\dots\Psi_{\alpha_r} \\
&- \frac{\mathbf{d}t}{n}\sum_{1\leq p<q\leq r}(-1)^{\epsilon_p+\epsilon_q}\zeta_{pq}
\end{aligned}$$

where

$$\zeta_{pq} = \begin{cases}
\Psi_{\alpha_1}\dots\Psi_{\alpha_{p-1}}\phi(\Psi^{\epsilon_p}_{i_qj_p}\dots\Psi_{\alpha_{q-1}})\Psi^{\epsilon_q}_{i_pj_q}\dots & \textit{if } \epsilon_p=\epsilon_q=0\\
\Psi_{\alpha_1}\dots\Psi_{\alpha_{p-1}}\Psi^{\epsilon_p}_{i_qj_p}\phi(\Psi_{\alpha_{p+1}}\dots\Psi_{\alpha_{q-1}}\Psi^{\epsilon_q}_{i_pj_q})\Psi_{\alpha_{q+1}}\dots & \textit{if } \epsilon_p=\epsilon_q=1\\
\sum_{k=1}^{n}\delta_{i_pi_q}\Psi_{\alpha_1}\dots\Psi_{\alpha_{p-1}}\phi(\Psi^{\epsilon_p}_{kj_p}\dots\Psi_{\alpha_{q-1}}\Psi^{\epsilon_q}_{kj_q})\dots & \textit{if } \epsilon_p=0,\epsilon_q=1\\
\sum_{k=1}^{n}\delta_{i_pi_q}\Psi_{\alpha_1}\dots\Psi^{\epsilon_p}_{kj_p}\phi(\Psi_{\alpha_{p+1}}\dots\Psi_{\alpha_{q-1}})\Psi^{\epsilon_q}_{kj_q}\dots & \textit{if } \epsilon_p=1,\epsilon_q=0
\end{cases}$$

Proof. The proof is done by recurrence and by using Itô's formula. For simplicity's sake we will do it only in the case where all ϵ are put equal to zero.
For $r = 2$ we get:

$$\mathbf{d}(\Psi_{ij}\Psi_{kl}) = \frac{\mathrm{i}}{\sqrt{n}}\sum_{s=1}^{n}\Psi_{ij}\mathbf{d}X_{ks}\Psi_{sl} + \frac{\mathrm{i}}{\sqrt{n}}\sum_{s=1}^{n}\mathbf{d}X_{is}\Psi_{sj}\Psi_{kl} - \Psi_{ij}\Psi_{kl}\mathbf{d}t - \frac{\mathbf{d}t}{n}\phi(\Psi_{kj})\psi_{il}$$

Hence we have the desired result for $r = 2$. Let us now assume that the Lemma is right until a certain r. Then, by Itô's Lemma:

$$\begin{aligned}
\mathbf{d}(\Psi_{u_1v_1}\dots\Psi_{u_{r+1}v_{r+1}}) &= -\frac{r+1}{2}\psi_{u_1v_1}\dots\Psi_{u_{r+1}v_{r+1}}\mathbf{d}t \\
&+ \frac{\mathrm{i}}{\sqrt{n}}\sum_{k=1}^{n}\psi_{u_1v_1}\dots\Psi_{u_rv_r}\mathbf{d}X_{u_{r+1}k}\Psi_{kv_{r+1}} \\
&+ \frac{\mathrm{i}}{\sqrt{n}}\sum_{k=1}^{n}\sum_{l=1}^{r}\Psi_{u_1v_1}\dots\mathbf{d}X_{u_lk}\Psi_{kv_l}\dots\Psi_{u_{r+1}v_{r+1}} \\
&- \frac{\mathbf{d}t}{n}\sum_{1\leq p<q\leq r}\Psi_{u_1v_1}\dots\phi(\Psi_{u_qv_p}\dots)\Psi_{u_pv_q}\dots\Psi_{u_{r+1}v_{r+1}} \\
&- \frac{\mathbf{d}t}{n}\sum_{l=1}^{r}\Psi_{u_1v_1}\dots\Psi_{u_{l-1}v_{l-1}}\phi(\Psi_{u_{r+1}v_l}\dots\Psi_{u_rv_r})\Psi_{u_lv_{r+1}}
\end{aligned}$$

And so we see that the result is also right for $r + 1$. □

We now introduce, as expected, the family of functions:

$$\Gamma_{\alpha_{11},\dots;\dots;\alpha_{1r}} = \phi(\Psi_{\alpha_{11}}\dots)\dots\phi(\Psi_{\alpha_{1r}}\dots)$$

By applying Lemma 5.3.2 we get:

$$\begin{aligned}
\Gamma'_{\alpha_{11},\dots;\dots;\alpha_{1r},\dots} &= -\frac{k_1+\dots+k_r}{2}\Gamma_{\alpha_{11},\dots;\dots;\alpha_{1r},\dots} \\
&- \frac{1}{n}\sum_{\kappa=1}^{r}\sum_{1\leq p<q\leq k_\kappa}(-1)^{\epsilon_p+\epsilon_q}\Gamma_{(p,q,\kappa)}
\end{aligned}$$

where we defined:

$$\Gamma_{(p,q,\kappa)} = \begin{cases} \Gamma_{\dots;\alpha_{1\kappa},\dots,i_{p\kappa}j_{q\kappa}\epsilon_{q\kappa},\dots,\alpha_{k_\kappa\kappa};i_{q\kappa}j_{p\kappa}\epsilon_{p\kappa},\dots,\alpha_{q-1,\kappa};\dots} & \text{if } \epsilon_{p\kappa} = \epsilon_{q\kappa} = 0 \\ \Gamma_{\dots;\alpha_{1\kappa},\dots,i_{q\kappa}j_{p\kappa}\epsilon_{p\kappa},\alpha_{q+1,\kappa},\dots;\alpha_{p+1,\kappa},\dots,i_{p\kappa}j_{q\kappa}\epsilon_{q\kappa};\dots} & \text{if } \epsilon_{p\kappa} = \epsilon_{q\kappa} = 1 \\ \sum_{l=1}^{n} \delta_{i_{p\kappa}i_{q\kappa}} \Gamma_{\dots;\alpha_{1\kappa},\dots,\alpha_{p-1\kappa},\alpha_{q+1,\kappa}\dots;lj_{p\kappa}\epsilon_{p\kappa},\dots,lj_{q\kappa}\epsilon_{q\kappa};\dots} & \text{if } \epsilon_{p\kappa} = 0, \epsilon_{q\kappa} = 1 \\ \sum_{l=1}^{n} \delta_{i_{p\kappa}i_{q\kappa}} \Gamma_{\dots;\alpha_{1\kappa},\dots,lj_{p\kappa}\epsilon_{p\kappa},lj_{q\kappa}\epsilon_{q\kappa},\dots;\alpha_{p+1,\kappa}\dots,\alpha_{q-1,\kappa};\dots} & \text{if } \epsilon_{p\kappa} = 1, \epsilon_{q\kappa} = 0 \end{cases}$$

Hence we see that the family of functions γ truly converges towards the family of functions Γ. In particular, taking $r = 1$, we see that the $*$-moments of the family $(U_{ij}^{\langle d \rangle)})_{1 \leq i,j \leq n}$ converges towards the $*$-moments of $(\Psi_{ij})_{1 \leq i,j \leq n}$. This proves the convergence of the marginals.

5.4 Conditional expectation

In order to prove Theorem 5.2.1 we must prove the convergence of all mixed moments of the kind: $\mathbb{E} \circ tr(U_{i_1 j_1}^{\epsilon_1}(t_1) \dots U_{i_r j_r}^{\epsilon_r}(t_r))$ towards $\phi(\Psi_{i_1 j_1}^{\epsilon_1}(t_1) \dots \Psi_{i_r j_r}^{\epsilon_r}(t_r))$. In the previous section we have already proven that this is indeed the case when $\sharp\{t_1, \dots, t_r\} = 1$. In order to prove the general case we will use a method consisting of computing the joint moments by taking recursively conditional expectations.

5.4.1 Notations

In order to use this method, we must generalize somewhat our notations. In the sequel, we fix $s \geq 0$ and our time variable t will always verify $t \geq s$. We note:

1. by $[\mathcal{I}]$ the set $\{1, \dots, n\}^2 \times \{0, 1\} \times \mathcal{M}_d^{(s)}$, where $\mathcal{M}_d^{(s)}$ is the set of $d \times d$ matrices whose entries are $\mathcal{F}_s$-measurable random variables. Of course, we have $\mathcal{F}_s = \sigma(j_u, u \leq s)$.
2. by $\mathcal{I}$ the set $\{1, \dots, nd\}^2 \times \{0, 1\} \times V^{(s)}$, where $V^{(s)}$ designates the set of $\mathcal{F}_s$-measurable random variables.
3. by $\mathcal{I}^f$ the set $\{1, \dots, n\}^2 \times \{0, 1\} \times \mathcal{A}_s$, where $\mathcal{A}_s$ is the $*$-algebra generated by all $\Psi_{pq}(u), u \leq s$.

We use these sets as sets of indices in the following way:

1. If $\alpha = (i, j, \epsilon, m) \in [\mathcal{I}]$, we note $[U]_\alpha = m[U]_{ij}^\epsilon$
2. If $\rho = (\mu, \nu, \omega, \pi) \in \mathcal{I}$, we note $U_\rho = \pi U_{\mu\nu}^\omega$
3. If $\alpha = (i, j, \epsilon, m) \in \mathcal{I}^f$, we note $\Psi_\alpha = m\Psi_{ij}^\epsilon$.

5.4.2 A system of differential equations for the Brownian motion on $U(nd)$

We are interested in the family of functions:

$$\begin{aligned} & \gamma_{\alpha_{11},\dots,\alpha_{k_1 1};\dots;\dots,\alpha_{k_r r}}(t) \\ = \ & \mathbb{E}[tr([U]_{\alpha_{11}}(t) \dots [U]_{\alpha_{k_1 1}}(t)) \dots tr(\dots [U]_{\alpha_{k_r r}}(t))] \end{aligned}$$

In other words, we use the same family as before but we put $\mathcal{F}_s$-measurable elements between the blocks of the Brownian motion.
We want to use the same method as before. We will need following Lemma:

Lemma 5.4.1. *We have for any choice of indices in $\mathcal{I}$ and for $t \geq s$:*

$$\begin{aligned} \mathbf{d}(U_{\rho_1} \dots U_{\rho_k}) &= -\frac{k}{2} U_{\rho_1} \dots U_{\rho_k} \mathbf{d}t \\ &- \frac{1}{nd} \sum_{1 \leq p < q \leq k} (-1)^{\omega_p + \omega_q} \zeta_{pq}^{\langle d \rangle >} \mathbf{d}t \\ &+ \quad \textit{martingale part} \end{aligned}$$

where:

$$\zeta_{pq}^{\langle d \rangle} = \begin{cases} U_{\rho_1} \dots \pi_p U_{\mu_p \nu_q} \dots \pi_q U_{\mu_q \nu_p} \dots U_{\rho_k} & \textit{if } \omega_p = \omega_q = 0 \\ U_{\rho_1} \dots \pi_p U^*_{\mu_p \nu_q} \dots \pi_q U^*_{\mu_q \nu_p} \dots U_{\rho_k} & \textit{if } \omega_p = \omega_q = 1 \\ \sum_{\tau=1}^{nd} \delta_{\mu_p \mu_q} U_{\rho_1} \dots \pi_p U^*_{\tau \nu_p} \dots \pi_q U_{\tau \nu_q} \dots U_{\rho_k} & \textit{if } \omega_p = 1, \omega_q = 0 \\ \sum_{\tau=1}^{nd} \delta_{\mu_p \mu_q} U_{\rho_1} \dots \pi_p U_{\tau \nu_p} \dots \pi_q U^*_{\tau \nu_q} \dots U_{\rho_k} & \textit{if } \omega_p = 0, \omega_q = 1 \end{cases} \tag{5.2}$$

Proof. As always, this is proven using Itô's Lemma. □

Applying this Lemma, we get:

Lemma 5.4.2. *The system of differential equations is:*

$$\begin{aligned} & \gamma'_{\alpha_{11},\dots,\alpha_{k_1 1};\dots;\dots,\alpha_{k_r r}} \\ = \quad & -\frac{k_1 + \dots + k_r}{2} \gamma_{\alpha_{11},\dots,\alpha_{k_1 1};\dots;\dots,\alpha_{k_r r}} \\ - \quad & \frac{1}{n} \sum_{\kappa=1}^{r} \sum_{1 \leq p < q \leq k_\kappa} (-1)^{\epsilon_{p\kappa} + \epsilon_{q\kappa}} \gamma_{(p,q,\kappa)} \\ + \quad & \mathcal{O}(\frac{1}{d^2}) \end{aligned}$$

where:
If** $\epsilon_{p\kappa} = \epsilon_{q\kappa} = 0$**:

$$\gamma_{(p,q,\kappa)} = \gamma_{\dots;\dots,(m_{p\kappa}, i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa}),\alpha_{q+1,\kappa}\dots;\alpha_{p+1,\kappa},\dots,(m_{q\kappa}, i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa});\dots}$$

If** $\epsilon_{p\kappa} = \epsilon_{q\kappa} = 1$**:

$$\gamma_{(p,q,\kappa)} = \gamma_{\dots;\dots,(m_{p\kappa}, i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa}),\alpha_{q+1,\kappa},\dots;\alpha_{p+1,\kappa},\dots,(1, i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa});\dots}$$

If** $\epsilon_{p\kappa} = 1, \epsilon_{q\kappa} = 0$**:

$$\gamma_{(p,q,\kappa)} = \sum_{t=1}^{n} \delta_{i_{p\kappa} i_{q\kappa}} \gamma_{\dots;\dots,(m_{p\kappa},t,j_{p\kappa}\epsilon_{p\kappa}),(t,j_{q\kappa},\epsilon_{q\kappa},1),\dots;(i_{p+1,\kappa},j_{p+1,\kappa},\epsilon_{p+1,\kappa},m_{q\kappa} m_{p+1,\kappa}),\dots;\dots}$$

If** $\epsilon_{p\kappa} = 0, \epsilon_{q\kappa} = 1$**:

$$\gamma_{(p,q,\kappa)} = \sum_{t=1}^{n} \delta_{i_{p\kappa} i_{q\kappa}} \gamma_{\dots;\dots,(i_{q+1,\kappa},j_{q+1,\kappa},\epsilon_{q+1,\kappa},m_{p\kappa} m_{q+1,\kappa},)\dots;(t,j_{p\kappa},\epsilon_{p\kappa},1),\dots,(t,j_{q\kappa},\epsilon_{q\kappa},m_{q\kappa});\dots}$$

The structure is very similar to what we had proved in the previous section. We just have to be careful to what happens with the m's.
When we proved Biane's result we saw that the system of differential equations had a combinatorial structure related to the idea of integer partitions. I do not see any obvious combinatorial structure in this generalized formula but it is a question that is worth being asked.

5.4.3 A system of differential equations for the free stochastic process

Of course, we will be interested in the behavior of the family of functions:

$$\Gamma_{\alpha_{11},\dots,\alpha_{k_11};\dots} = \phi(\Psi_{\alpha_{11}}(t)\dots)\dots\phi(\dots)$$

Lemma 5.4.3. *For any choice of indices in $\mathcal{I}^f$ and for $t \geq s$, we have:*

$$\begin{aligned}
\mathbf{d}(\Psi_{\alpha_1}\dots\Psi_{\alpha_k}) &= -\frac{k}{2}\Psi_{\alpha_1}\dots\Psi_{\alpha_k}\mathbf{d}t \\
&+ \frac{\mathrm{i}}{\sqrt{n}}\sum_{r=1}^{n}\sum_{l=1}^{k}\Psi_{\alpha_1}\dots\alpha_l\left\{\begin{array}{ll}\mathbf{d}X_{i_lr}\Psi_{rj_l} & \text{if } \epsilon_l=0\\ \Psi_{rj_l}\mathbf{d}X_{ri_l} & \text{if } \epsilon_l=1\end{array}\right\}\dots\Psi_{\alpha_k}\\
&- \frac{\mathbf{d}t}{n}\sum_{1\leq p<q\leq k}(-1)^{\epsilon_p+\epsilon_q}\zeta_{pq}
\end{aligned}$$

where

$$\zeta_{pq} = \begin{cases}
\Psi_{\alpha_1}\dots\phi(\Psi^{\epsilon_p}_{i_qj_p}\dots\Psi_{\alpha_{q-1}}m_q)\Psi^{\epsilon_q}_{i_pj_q}\dots & \text{if } \epsilon_p=\epsilon_q=1\\
\Psi_{\alpha_1}\dots\alpha_p\Psi^{\epsilon_p}_{i_qj_p}\phi(\Psi_{\alpha_{p+1}}\dots\Psi^{\epsilon_q}_{i_pj_q})\Psi_{\alpha_{q+1}}\dots & \text{if } \epsilon_p=\epsilon_q=1\\
\sum_{t=1}^{k}\delta_{i_pi_q}\Psi_{\alpha_1}\dots\alpha_p\phi(\Psi^{\epsilon_p}_{tj_p}\dots\Psi^{\epsilon_q}_{tj_q})\Psi_{\alpha_{q+1}}\dots & \text{if } \epsilon_p=0,\epsilon_q=1\\
\sum_{t=1}^{k}\delta_{i_pi_q}\Psi_{\alpha_1}\dots\Psi^{\epsilon_p}_{tj_p}\phi(\Psi_{\alpha_{p+1}}\dots\alpha_q)\Psi^{\epsilon_q}_{tj_q}\dots & \text{if } \epsilon_p=1,\epsilon_q=0
\end{cases}$$

Proof. It is the same proof as before, based on Itô's formula. □

Applying this Lemma, we get:

Lemma 5.4.4. *The system of differential equations for the free stochastic process is:*

$$\begin{aligned}
&\Gamma'_{\alpha_{11},\dots;\dots}\\
&= -\frac{k_1+\dots+k_r}{2}\Gamma_{\alpha_{11},\dots;\dots}\\
&- \sum_{\kappa=1}^{r}\sum_{1\leq p<q\leq k_\kappa}(-1)^{\epsilon_{p\kappa}+\epsilon_{q\kappa}}\Gamma_{(p,q,\kappa)}
\end{aligned}$$

where:
If $\epsilon_{p\kappa}=\epsilon_{q\kappa}=0$***:***

$$\Gamma_{(p,q,\kappa)} = \Gamma_{\dots;\dots,(i_{p\kappa}j_{q\kappa}\epsilon_{q\kappa}m_{p\kappa}),\dots;(i_{q\kappa}j_{p\kappa}\epsilon_{p\kappa}m_{q\kappa}),\dots,\alpha_{q-1,\kappa};\dots}$$

If $(\epsilon_{p\kappa},\epsilon_{q\kappa})=(1,1)$***:***

$$\Gamma_{(p,q,\kappa)} = \Gamma_{\dots;\dots,(i_{q\kappa}j_{p\kappa}\epsilon_{p\kappa}m_{p\kappa}),\alpha_{q+1,\kappa},\dots;\alpha_{p+1,\kappa},\dots,(i_{p\kappa}j_{q\kappa}\epsilon_{q\kappa}1);\dots}$$

If $\epsilon_{p\kappa} = 0, \epsilon_{q\kappa} = 1$***:***

$$\Gamma_{(p,q,\kappa)} = \sum_{t=1}^{n} \delta_{i_{p\kappa} i_{q\kappa}} \Gamma_{\ldots;\ldots,(i_{q+1,\kappa},j_{q+1,\kappa},\epsilon_{q+1,\kappa},m_{p\kappa}m_{q+1,\kappa}),\ldots;(tj_{p\kappa}\epsilon_{p\kappa}1),\ldots,(tj_{q\kappa}\epsilon_{q\kappa}m_{q\kappa});\ldots}$$

If $\epsilon_{p\kappa} = 1, \epsilon_{q\kappa} = 0$***:***

$$\Gamma_{(p,q,\kappa)} = \sum_{t=1}^{n} \delta_{i_{p\kappa} i_{q\kappa}} \Gamma_{\ldots;\ldots,(tj_{p\kappa}\epsilon_{p\kappa}m_{p\kappa}),(tj_{q\kappa}\epsilon_{q\kappa}1),\ldots;(i_{p+1,\kappa},j_{p+1,\kappa},\epsilon_{p+1,\kappa},m_{q\kappa}m_{p+1,\kappa}),\ldots,\ldots}$$

5.4.4 Recurrence

We are now able to finish the proof of Theorem 5.2.1. We want to show that the moments $\mathbb{E} \circ tr([U]_{i_1 j_1}(t_1)^{\epsilon_1} \ldots [U]_{i_k j_k}^{\epsilon_k}(t_k))$ converge towards $\phi(\Psi_{i_1 j_1}^{\epsilon_1}(t_1) \ldots \Psi_{i_k j_k}^{\epsilon_k}(t_k))$. Let us denote $\sigma = \sharp\{t_1, \ldots, t_k\}$ the number of different times showing up in our moment. We are going to prove that result through recurrence on σ.

1. If $\sigma = 1$ the result has already been shown because it is just the convergence of the marginals.

2. Let us suppose that the result is true until a certain σ. We will now consider a moment using $\sigma + 1$ different times. We can order those times in increasing order: $t_1 \leq t_1 \leq \ldots \leq t_{\sigma+1}$.The recurrence hypothesis tells us that:

   $$(U_{p,q}(t_i))_{\substack{1\leq i\leq\sigma \\ 1\leq p,q\leq n}} \underset{\text{in }*\text{-moments}}{\longrightarrow} (\Psi_{p,q}(t_i))_{\substack{1\leq i\leq\sigma \\ 1\leq p,q\leq n}}$$

   We can write the moment under consideration as:

   $$\gamma_{(i_1 j_1 \epsilon_1 m_1^{(d)}),,\ldots,(i_k j_k \epsilon_k m_k^{(d)})}(t_{\sigma+1})$$

   where the $m_i^{(d)}$ are $\mathcal{F}_{t_\sigma}$-measurable. Now, let us remark that the family of functions $(\gamma_{\alpha_{11},\ldots,\alpha_{k_1 1};\alpha_{12},\ldots})$ is entirely characterized by the system of differential equations from Lemma 5.4.2 along with all the relationships between the $\{m_{ij}^{(d)}, 1 \leq j \leq r, 1 \leq i \leq k_j\}$. In the same way, the family $\Gamma_{\ldots}$ is entirely defined by the system from Lemma 5.4.4 along with the relationships between the $\{m_{ij}, 1 \leq j \leq r, 1 \leq i \leq k_j\}$
   Now, the recurrence hypothesis allows us to say that the $m_i^{(d)}, 1 \leq i \leq k$ converges towards some $m_i, 1 \leq i \leq k$. This tells us that the relationships between the $\{m_i^{(d)}\}$ "converges" towards the relationships between the $\{m_i\}$. Moreover, the system of differential equations from Lemma 5.4.2 converges towards that of Lemma 5.4.4. To put it in a nutshell, this means:

   $$\gamma_{\alpha_1^{(d)},\ldots,\alpha_1^{(d)}}(t_{\sigma+1}) \underset{d\to\infty}{\longrightarrow} \Gamma_{\alpha_1,\ldots,\alpha_k}(t_{\sigma+1})$$

   Or, in other words, we have the convergence of our moment.

Thus, we have proven that all $*$-moments converge and this means that Theorem 5.2.1 is proven.

5.5 Some examples of calculations and gaussianity

We will now use the differential equations that we obtained to calculate some simple moments of our process. We will then be able to draw a consequence about the gaussianity of the free process. In the sequel, we denote by ϕ_t the function defined on U_n^{nc} by $\phi_t = \phi \circ J_{0t}$ where J_t is the limit (free) process.

5.5.1 The first moments

Let us take now $1 \leq i \neq j \leq n$. We have the following differential equations:

$$\begin{aligned} \frac{d}{dt}\phi_t(u_{ii}) &= -\frac{1}{2}\phi_t(u_{ii}) \\ \frac{d}{dt}\phi_t(u_{ij}) &= -\frac{1}{2}\phi_t(u_{ij}) \end{aligned}$$

with initial conditions: $\phi_0(u_{ii}) = 1$ and $\phi_0(u_{ij}) = 0$. It thus yields:

$$\begin{aligned} \phi_t(u_{ii}) &= e^{-\frac{1}{2}t} \\ \phi_t(u_{ij}) &= 0 \end{aligned}$$

We find the same expression for $\phi_t(u_{ii}^*)$ and $\phi_t(u_{ij}^*)$ because they obey the same differential equation with the same initial conditions.

5.5.2 The second moments

Let us take $1 \leq i, j, k, l \leq n$. We have following equation:

$$\begin{aligned} \frac{d}{dt}\phi_t(u_{ij}u_{kl}) &= -\phi_t(u_{ij}u_{kl}) - \phi_t(u_{il})\phi_t(u_{kj})\frac{1}{n} \\ &= -\phi_t(u_{ij}u_{kl}) - \frac{1}{n}\delta_{il}\delta_{kj}e^{-t} \end{aligned}$$

with initial conditions $\phi_0(u_{ij}u_{kl}) = \delta_{ij}\delta_{kl}$ because $\Psi_0 = I$. This equation is a linear differential equation of order 1 and the well-known method allows us to say:

$$\phi_t(u_{ij}u_{kl}) = \frac{\delta_{ij}\delta_{kl}}{n}e^{-t} - t\delta_{il}\delta_{kj}e^{-t}$$

The moments $\phi_t(u_{ij}^*u_{kl}^*)$ also obey the same equation with the same initial condition and they therefore have the same expression. If we are interested in $\phi_t(u_{ij}u_{kl}^*)$ we get the equation:

$$\frac{d}{dt}\phi_t(u_{ij}u_{kl}^*) = -\phi_t(u_{ij}u_{kl}^*) + \frac{1}{n}\sum_{p=1}^{n}\phi_t(u_{pj}u_{pl}^*)$$

with initial conditions $\phi_0(u_{ij}u_{kl}^*) = \delta_{ij}\delta_{kl}$. This can be put in the form of a system of linear differential equations by puting $\Phi_t = (\phi_t(u_{ij}u_{kl}))_{1\leq i,j,k,l\leq n}$ seen as a vector of $\mathbb{C}^{n^4}$ and $A = (a_{(r_1,r_2,r_3,r_4),(s_1,s_2,s_3,s_4)})$ as a matrix acting on $\mathbb{C}^{n^4}$, with:

$$\begin{cases} a_{rs} = 0 & \text{if } s_1 = s_3 \text{ and } r = s \\ a_{rs} = 1/n & \text{if } s_1 = s_3 \text{ and } r \neq s \\ a_{rs} = -1 & \text{if } r = s \text{ and } r_1 \neq r_3 \end{cases}$$

The equation then is:

$$\Phi' \;=\; A\Phi$$

The solution of such an equation is of the form $\Phi_t = Ce^{At}$ with C a constant.

5.5.3 Gaussianity

We would like to define a Brownian motion on $U\langle n\rangle$ as a free stochastic process having the same law (the same $*$-moments) as Ψ_t. This would seem natural because it is just the limit of the Brownian motion on $U(nd)$. To know if this definition makes sense, we would like Ψ_t to verify some properties, and especially the gaussian property as defined in [20], Proposition 1.12 and in [39], Proposition 5.1.1.

Definition 5.5.1 (Proposition 5.1.1 from [39])**.** *We say that a Lévy process on $U\langle n\rangle$ is gaussian if one of the following equivalent properties are verified:*

- *For each $a, b, c \in Ker\delta$, we have $L(abc) = 0$.*
- *For each $a, b \in Ker\delta$, we have $L(b^*a^*ab) = 0$.*
- *For all $a, b, c \in U\langle n\rangle$ we have the following formula:*

$$\begin{aligned} L(abc) &= L(ab)\delta(c) + L(ac)\delta(b) + \delta(a)L(bc) - \delta(a)\delta(b)L(c) \\ &- \delta(a)\delta(c)L(b) - L(a)\delta(b)\delta(c) \end{aligned}$$

- *The representation π is zero on $Ker\delta$: $\pi_{|Ker\delta} = 0$.*
- *We have for each $a \in U\langle n\rangle$: $\pi(a) = \delta(a)Id$.*
- *For each a, b in $Ker\delta$, we have: $\eta(ab) = 0$.*
- *We have for all a, b in $U\langle n\rangle$: $\eta(ab) = \delta(a)\eta(b) + \eta(a)\delta(b)$.*

Theorem 5.5.2. *Let us take $D = \mathcal{M}_n(\mathbb{C})$. We then define a Schürmann triple by setting:*

$$\eta(u_{jk}) = \epsilon_{jk}/\sqrt{n}, \eta(u^*_{jk}) = -\epsilon_{kj}\sqrt{n}$$

$$\pi(u_{jk}) = \delta_{jk}Id$$

$$L(u_{jk}) = -\frac{1}{2}\sum_{r=1}^{n}\langle\eta(u^*_{rj}), \eta(u_{rk})\rangle$$

where ϵ_{jk} describe the elementary matrices.
Then, the Schürmann triple (η, π, L) is associated to the Lévy process on $U\langle n\rangle$ we are interested in.

Proof. We prove it by recurrence on the length of the words:
For the length 1: we have:

$$L(u_{jk}) \;=\; -\frac{1}{2}\sum_{r=1}^{n}\langle\epsilon_{rj}, \epsilon_{rk}\rangle = -\frac{1}{2n}\sum_{r=1}^{n}Tr(\epsilon_{jr}\epsilon_{rk}) = -\delta_{jk}/2$$

Let us suppose the result is true for words of length up to k: We must first find an expression for η. The cocycle property for η allows us to find through an easy recurrence that:

$$\eta(u_{i_1j_1}^{\epsilon_1}\dots u_{i_kj_k}^{\epsilon_k}) = \sum_{p=1}^{k}\delta_{i_1j_1}\dots\begin{cases}\epsilon_{i_pj_p} & \text{if } \epsilon_p=0\\ -\epsilon_{j_pi_p} & \text{if } \epsilon_p=1\end{cases}\dots\delta_{i_kj_k}$$

We can now use the coboundary property to write:

$$\begin{aligned}L(u_{i_1j_1}^{\epsilon_1}\dots u_{i_{k+1}j_{k+1}}^{\epsilon_{k+1}}) &= \epsilon(u_{i_2j_2}^{\epsilon_1}\dots u_{i_{k+1}j_{k+1}}^{\epsilon_{k+1}})L(u_{i_1j_1}^{\epsilon_{k+1}})+L(u_{i_2j_2}^{\epsilon_2}\dots u_{i_{k+1}j_{k+1}}^{\epsilon_{k+1}})\epsilon(u_{i_1j_1}^{\epsilon_1})\\ &+ \langle\eta(u_{i_1j_1}^{1-\epsilon_1}),\eta(u_{i_2j_2}^{\epsilon_2}\dots u_{i_kj_k}^{\epsilon_k}u_{i_{k+1}j_{k+1}}^{\epsilon_{k+1}})\rangle\\ &= -\frac{k}{2}\delta_{i_1j_1}\dots\delta_{i_kj_k}\delta_{i_{k+1}j_{k+1}}\\ &- \sum_{2\le p<q\le k+1}(-1)^{\epsilon_p+\epsilon_q}\Gamma_{(p,q,1)}\delta_{i_1j_1}\\ &- \delta_{i_1j_1}\delta_{i_2j_2}\dots\delta_{i_{k+1}j_{k+1}}/2\\ &+ \clubsuit\end{aligned}$$

where we have used the fact that the Brownian motion on $U(nd)$ at time $t=0$ is just Id. So we only have to compute the value of $\clubsuit$, which is the term arising from $\langle\eta(u_{i_1j_1}^{1-\epsilon_1}),\eta(u_{i_2j_2}^{\epsilon_2}\dots u_{i_kj_k}^{\epsilon_k}u_{i_{k+1}j_{k+1}}^{\epsilon_{k+1}})\rangle$. We also remark that to finish our recurrence, it suffices to show that this $\clubsuit$ is equal to

$$-\sum_{2\le p\le k+1}(-1)^{\epsilon_p+\epsilon_1}\Gamma_{(1,p,1)}$$

So we may now write:

$$\begin{aligned}&\langle\eta(u_{i_1j_1}^{1-\epsilon_1}),\eta(u_{i_2j_2}^{\epsilon_2}\dots u_{i_kj_k}^{\epsilon_k}u_{i_{k+1}j_{k+1}}^{\epsilon_{k+1}})\rangle\\ =\ &\frac{1}{n}\langle\begin{cases}-\epsilon_{j_1i_1} & \text{if } \epsilon_1=0\\ \epsilon_{i_1j_1} & \text{if } \epsilon_1=1\end{cases},\sum_{p=2}^{k+1}\delta_{i_2j_2}\dots\begin{cases}\epsilon_{i_pj_p} & \text{if } \epsilon_p=0\\ -\epsilon_{j_pi_p} & \text{if } \epsilon_p=1\end{cases}\dots\delta_{i_{k+1}j_{k+1}}\rangle\\ =\ &\sum_{p=2}^{k+1}\spadesuit_p\end{aligned}$$

We may now study the four cases:
Case where $\epsilon_1=\epsilon_p=0$: we have

$$\spadesuit_p = -\frac{1}{n}\delta_{i_1j_p}\delta_{i_2j_2}\dots\delta_{i_pj_1}\dots = -(-1)^{\epsilon_1+\epsilon_p}\Gamma_{(1,p,1)}$$

Case where $\epsilon_1=\epsilon_p=1$: we have

$$\spadesuit_p = -\frac{1}{n}\delta_{i_1j_p}\delta_{i_2j_2}\dots\delta_{i_pj_1}\dots = -(-1)^{\epsilon_1+\epsilon_p}\Gamma_{(1,p,1)}$$

Case where $\epsilon_1=0,=\epsilon_p=1$: we have

$$\spadesuit_p = \frac{1}{n}\delta_{i_1i_p}\delta_{i_2j_2}\dots\delta_{j_pj_1}\dots = -(-1)^{\epsilon_1+\epsilon_p}\Gamma_{(1,p,1)}$$

Case where $\epsilon_1=1,\epsilon_p=0$: we have

$$\spadesuit_p = \frac{1}{n}\delta_{i_1i_p}\delta_{i_2j_2}\dots\delta_{j_pj_1}\dots = -(-1)^{\epsilon_1+\epsilon_p}\Gamma_{(1,p,1)}$$

Thus, we have proven the result by recurrence. □

Theorem 5.5.3. *The Lévy process from Theorem 5.2.1 is gaussian.*

Proof. It is immediate by using the fifth characterization from Definition 5.5.1.

□

Because it satisfies the gaussianity property, the quantum Lévy process under consideration is a good candidate to define what we would like to call a Brownian motion on $U\langle n\rangle$.

6

Free Lévy processes on the unitary dual group

In this section, we study free Lévy processes on the unitary dual group. We recall their definition and the correspondence between Lévy processes, generators, and Schürmann triples. We describe a class of free Lévy processes which appears as limit of Lévy processes on the classical unitary group, and compute their generators thanks to a representation theorem which was still missing in the free case.

This chapter is taken from section 4 *of the article [11], which I co-authored with G. Cébron.*

6.1 Free Lévy processes

Definition 6.1.1. *A* free unitary Lévy process *is a family* $(U_t)_{t\geq 0}$ *of unitary elements of a noncommutative probability space* $(\mathcal{A},\phi)$ *such that:*

- $U_0 = 1_{\mathcal{A}}$.
- *For all* $0 \leq s \leq t$*, the distribution of* $U_s^{-1}U_t$ *depends only on* $t-s$.
- *For all* $0 \leq t_1 \leq \ldots \leq t_k$*, the random variables* $U_{t_1}, U_{t_1}^{-1}U_{t_2}, \ldots, U_{t_{n-1}}^{-1}U_{t_n}$ *are free.*
- *The distribution of* U_t *converges weakly to* δ_1 *as* t *goes to* 0.

One can generalize this definition by considering a process $(U_t)_{t\geq 0}$ of matrices of elements of $\mathcal{A}$ which are unitary, instead of considering only one element. In other words, we want to consider a process $(j_t)_{t\geq 0}$ of quantum random variables on $U\langle n\rangle$ over $(\mathcal{A},\phi)$ (for all times $t\geq 0$, $j_t : U_n^{\mathrm{nc}} \to \mathcal{A}$ is a $*$-homomorphism, which is equivalent with requiring that the matrix $(j_t(u_{ij}))_{i,j=1}^n$ is unitary).

Definition 6.1.2. *A* free Lévy process *on* $U\langle n\rangle$ *over* $(\mathcal{A},\phi)$ *is a family of quantum random variables* $(j_t)_{t\geq 0}$ *on* $U\langle n\rangle$ *over* $\mathcal{A}$ *such that:*

- $j_0 = \delta 1_{\mathcal{A}}$.
- *For all* $0 \leq s \leq t$, $\phi \circ ((j_s \circ \Sigma) \star j_t) = \phi \circ j_{t-s}$ *(stationarity of the distributions).*
- *For all* $0 \leq t_1 \leq \ldots \leq t_k$, *the homomorphisms* $j_{t_1}, (j_{t_1} \circ \Sigma) \star j_{t_2}, \ldots, (j_{t_n} \circ \Sigma) \star j_{t_{n-1}}$ *are freely independent in the sense that the image* $*$*-algebras of* U_n^{nc} *are freely independent in* $(\mathcal{A}, \phi)$.
- *For all* $b \in U_n^{\mathrm{nc}}$, $\phi \circ j_s(b)$ *converges towards* $\delta(b)$ *when* s *tends to* 0.

Some authors find more convenient to make the following assumptions on the family of increments $(j_{st})_{0 \leq s \leq t}$ linked with $(j_t)_{t \geq 0}$ by the relation $j_{st} = (j_s \circ \Sigma) \star j_t$ (for all $0 \leq s \leq t$):

- For all $0 \leq t$, $j_{tt} = \delta 1_{\mathcal{A}}$.
- For all $0 \leq r \leq s \leq t$, $j_{rs} \star j_{st} = j_{rt}$.
- For all $0 \leq s \leq t$, $\phi \circ j_{st} = \phi \circ j_{0,t-s}$.
- For all $0 \leq t_1 \leq \ldots \leq t_k$, the homomorphisms $j_{0t_1}, \ldots, j_{t_{n-1}t_n}$ are freely independent in the sense that the image algebras are freely independent.
- For all $b \in U_n^{\mathrm{nc}}$, $\phi \circ j_{0s}(b)$ converges towards $\delta(b)$ when s tends to 0.

Of course, the two points of view are equivalent. Let us observe that a free unitary Lévy process $(U_t)_{t \geq 0}$ in $(\mathcal{A}, \phi)$ is also a free Lévy process $(u \mapsto U_t)_{t \geq 0}$ on $U\langle 1 \rangle$ over $(\mathcal{A}, \phi)$.

6.2 Free Lévy processes as limit of random matrices

Let us present here an example of a free Lévy process constructed thanks to the homomorphism j_U described in Section 7.1, and which is the limit of random matrices in the sense of Theorem 6.2.2. We will see that it generalizes results from Chapter 5.

Proposition 6.2.1. *Let* $(U_t)_{t \geq 0}$ *be a free unitary Lévy process. Let us consider the family of quantum random variables* $(j_t)_{t \geq 0}$ *on* $U\langle n \rangle$ *over* $E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$ *defined by* $j_t := j_{U_t}$, *or, in other words, for all* $1 \leq i, j \leq n$, *we have* $j_t(u_{ij}) = E_{1i} U_t E_{j1}$.

Then, $(j_t)_{t \geq 0}$ *is a free Lévy process on* $U\langle n \rangle$ *over the non-commutative probability space* $\Big(E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}, n(\phi * \mathrm{tr}_n)\Big)$.

Proof. The fact that $(j_t)_{0 \leq t}$ is indeed a free Lévy process on $U\langle n \rangle$ follows from Proposition 7.2.2, and from the definition of a free unitary Lévy process $(U_t)_{t \geq 0}$. $\square$

Theorem 6.2.2. *Let* $(U_t)_{t \geq 0}$ *be a free unitary Lévy process in* $(\mathcal{A}, \phi)$ *and let* $(j_t)_{t \geq 0}$ *be the Lévy process over* $U\langle n \rangle$ *defined by Proposition 6.2.1. For each* $N \in \mathbb{N}$, *let us consider a process* $(U_t^{(N)})_{t \geq 0}$ *on the classical unitary group* $U(N)$.

Assume that the family $\{U_t^{(N)}\}_{t \geq 0}$ *converges almost surely in* $*$*-distribution to the family* $\{U_t\}_{t \geq 0}$ *as* N *tends to* ∞. *Then, the block matrices* $\Big([U_t^{(nN)}]_{ij}\Big)_{\substack{1 \leq i,j \leq n \\ t \geq 0}}$ *converge almost surely in* $*$*-distribution to* $\Big(j_t(u_{ij})\Big)_{\substack{1 \leq i,j \leq n \\ t \geq 0}}$ *as* N *tends to* ∞.

In the particular case where $(U_t)_{t\geq 0}$ is a free unitary Brownian motion (see the last section of the paper), this theorem above is the result stated in Theorem 5.2.1 that has been proved in Chapter 5 via stochastic calculus and combinatorial arguments.

Proof. Setting $\{A_k^{(N)}\}_{k\in K} = \{U_t^{(N)}\}_{t\geq 0}$ (with $K = \{t \geq 0\}$), it is a direct consequence of Corollary 7.4.2. ☐

In [12], one of the authors defined a matrix model for every unitary free Lévy process $(U_t)_{t\geq 0}$. More precisely, for each $N \in \mathbb{N}$, there exists a Lévy process $(U_t^{(N)})_{t\geq 0}$ on the classical unitary group $U(N)$ such that the family $\{U_t^{(N)}\}_{t\geq 0}$ converges almost surely in $*$-distribution to the family $\{U_t\}_{t\geq 0}$. As a consequence, every free Lévy process defined according to Proposition 6.2.1 from a one-dimensional free Lévy process is indeed the limit of a family of random matrices when the dimension tends to ∞.

The rest of this chapter is devoted to compute the generator of such free Lévy processes, whose expression is given in Theorem 6.3.3.

6.3 Generator and Schürmann triple

In this section, we define two different objects which characterize Lévy processes on $U\langle n\rangle$.

Definition 6.3.1. *The* generator *of a free Lévy process $(j_t)_{t\geq 0}$ on $(\mathcal{A}, \phi)$ over $U\langle n\rangle$ is the linear form $L : U_n^{\mathrm{nc}} \to \mathbb{C}$ defined, for all $u \in U_n^{\mathrm{nc}}$, by*

$$L(u) = \frac{\mathrm{d}}{\mathrm{d}t}\phi \circ j_t(u) = \lim_{t\to 0} \frac{1}{t}\left(\phi(j_t(u)) - \delta(u)\right).$$

In [2], it is proved that L is well-defined and determines completely the family of law $(\phi \circ j_t)_{t\geq 0}$. The generator satisfies $L(1) = 0$, is hermitian and is conditionally positive, in the sense that

- $L(u^*) = \overline{L(u)}$ for all $u \in U_n^{\mathrm{nc}}$,
- $L(u^*u) \geq 0$ for all $u \in U_n^{\mathrm{nc}}$ such that $\delta(u) = 0$.

Conversely, the recent article [40] proves that, for any hermitian and conditionally positive $L : U_n^{\mathrm{nc}} \to \mathbb{C}$ such that $L(1) = 0$, there exists a free Lévy process on $U\langle n\rangle$ whose generator is L. We will call such a linear functional *a generator*, without mentioning any Lévy process. The description of the generators is made easier by the following notion of Schürmann triple.

Definition 6.3.2. *A* Schürmann triple *(ρ, η, L) on $U\langle n\rangle$ over a Hilbert space H consists of*

- *a generator L,*
- *a linear map $\eta : U_n^{\mathrm{nc}} \to H$ such that, for all $a, b \in U_n^{\mathrm{nc}}$, we have*

$$L(ab) = \delta(a)L(b) + \langle \eta(a^*), \eta(b)\rangle + L(a)\delta(b),$$

- *a unital $*$-representation ρ of U_n^{nc} on H such that, for all $a, b \in U_n^{\mathrm{nc}}$, we have*

$$\eta(ab) = \rho(a)\eta(b) + \eta(a)\delta(b).$$

It simplifies the data of L because the three maps ρ, η and L of a Schürmann triple are uniquely determined by their values on the generators $\{u_{ij}, u_{ij}^*\}_{1\leq i,j\leq n}$ of U_n^{nc}. A sort of GNS-construction (see [38]) allows conversely to construct a Schürmann triple (ρ, η, L) for every generator L.

In the next section, we will prove the following theorem, which computes the Schürmann triple of the Lévy process over $U\langle n\rangle$ defined by Proposition 6.2.1.

Theorem 6.3.3. *Let $(U_t)_{t\geq 0}$ be a free unitary Lévy process in $(\mathcal{A}, \phi)$ and let (ρ, η, L) be its Schürmann triple on $U\langle 1\rangle = \mathbb{C}[u, u^{-1}]$ over a Hilbert space H. Let $j_t : U\langle n\rangle \to E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$ be the Lévy process defined by setting, for all $1 \leq i, j \leq n$, $j_t(u_{ij}) = E_{1i}U_tE_{j1}$.*

The Schürmann triple (ρ_n, η_n, L_n) of $(j_t)_{t\geq 0}$ over $H \otimes \mathcal{M}_n(\mathbb{C})$ is given, for all $1 \leq i, j \leq n$, by

$$\rho_n(u_{ij}) = \frac{1}{n}(\rho(u) - \mathrm{Id}_H) \otimes E_{ij} + \delta_{ij}\mathrm{Id}_H \otimes I_N,$$
$$\eta_n(u_{ij}) = \eta(u) \otimes E_{ij}, \quad \eta_n(u_{ij}^*) = \eta(u^{-1}) \otimes E_{ij}, \quad L_n(u_{ij}) = \delta_{ij}L(u). \tag{6.1}$$

As a corollary, we have a sufficient characterization for the existence of a random matrix model in terms of the generator (we believe that this condition is not necessary).

Corollary 6.3.4. *Let $(j_t)_{t\geq 0}$ be a free Lévy process on $U\langle n\rangle$. Let H be a Hilbert space such that the Schürmann triple (ρ_n, η_n, L_n) of $(j_t)_{t\geq 0}$ is given over $H \otimes \mathcal{M}_n(\mathbb{C})$ by*

$$\rho_n(u_{ij}) = \frac{1}{n}(W - \mathrm{Id}_H) \otimes E_{ij} + \delta_{ij}\mathrm{Id}_H \otimes I_N,$$
$$\eta_n(u_{ij}) = h \otimes E_{ij}, \quad \eta_n(u_{ij}^*) = -W^*h \otimes E_{ij}, \quad L_n(u_{ij}) = (iR - \frac{1}{2}\|h\|_H^2)\delta_{ij},$$

where W is a unitary operator of $\mathcal{B}(H)$, $h \in H$ and $R \in \mathbb{R}$. Then, for each $N \in \mathbb{N}$, there exists a process $(U_t^{(N)})_{t\geq 0}$ on the classical unitary group $U(N)$ such that the family of $N \times N$-blocks $\Big([U_t^{(nN)}]_{ij}\Big)_{\substack{1\leq i,j\leq n\\ t\geq 0}}$ converges almost surely in $*$-distribution to $\Big(j_t(u_{ij})\Big)_{\substack{1\leq i,j\leq n\\ t\geq 0}}$ as N tends to ∞.

We give the proof of Corollary 6.3.4 right now, and postpone the proof of Theorem 6.3.3 to the next section.

Proof. Let us show that we are indeed in the situation of Theorem 6.3.3, and that W, h and R can be read as the Schürmann triple of some Lévy process over $U\langle 1\rangle$. This is a consequence of the following general description of the generators on $U\langle n\rangle$.

Proposition 6.3.5 (Proposition 4.4.7 of [39]). *Let H be a Hilbert space, $(h_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_n(H)$ be elements of H, $(W_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_n(\mathcal{B}(H))$ unitary and $(R_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_n(\mathbb{C})$ self-adjoint. Then there exists a unique Schürmann triple (ρ, η, L) over H such that*

$$\rho(u_{ij}) = W_{ij}; \quad \eta(u_{ij}) = h_{ij}; \quad \eta(u_{ij}^*) = -\sum_{k=1}^n W_{ki}^* h_{kj}; \quad L(u_{ij}) = iR_{ij} - \frac{1}{2}\sum_{k=1}^n \langle h_{ki}, h_{kj}\rangle_H. \tag{6.2}$$

Conversely, each generator L appears in a Schürmann triple (ρ, η, L) over a Hilbert space H as (6.2) *for some $(h_{ij})_{1\leq i,j\leq n}$, $(W_{ij})_{1\leq i,j\leq n}$ unitary, and $(R_{ij})_{1\leq i,j\leq n}$ selfadjoint given by*

$$h_{ij} = \eta(u_{ij}); \quad W_{ij} = \rho(u_{ij}); \quad R_{ij} = -i\left(L(u_{ij}) + \frac{1}{2}\sum_{k=1}^n \langle \eta(u_{ki}), \eta(u_{kj})\rangle_H\right). \tag{6.3}$$

Using this proposition for W, h and R shows that the generator (ρ_n, η_n, L_n) can be written in the form (6.1) for some Schürmann triple (ρ, η, L) over H. But let us consider a free unitary Lévy process $(U_t)_{t\geq 0}$ with Schürmann triple (ρ, η, L), and the Lévy process $(j_{U_t})_{t\geq 0}$ of Theorem 6.3.3 defined by setting, for all $1 \leq i, j \leq n$, $j_{U_t}(u_{ij}) = E_{1i}U_tE_{j1}$. Using the result [12, Theorem 3], there exists a random matrix model $(U_t^{(N)})_{t\geq 0}$ on the unitary group for the Lévy process $(U_t)_{t\geq 0}$, and Theorem 6.2.2 allows us to conclude that $\left([U_t^{(nN)}]_{ij}\right)_{\substack{1\leq i,j\leq n\\ t\geq 0}}$ is a random matrix model for $(j_{U_t})_{t\geq 0}$. Theorem 6.3.3 shows that $(j_{U_t})_{t\geq 0}$ has the same Schürmann triple that our Lévy process $(j_t)_{t\geq 0}$. Thus their distributions are equal, and $\left([U_t^{(nN)}]_{ij}\right)_{\substack{1\leq i,j\leq n\\ t\geq 0}}$ is also a random matrix model for $(j_t)_{t\geq 0}$. □

6.4 Proof of Theorem 6.3.3

In the three next steps, we will

1. establish a concrete realization of any free Lévy process $(j_t)_{t\geq 0}$ on $U\langle n\rangle$ on a full Fock space, starting from any Schürmann triple;
2. show that, considering a one dimensional free Lévy process $(U_t)_{t\geq 0}$, this concrete realization behaves nicely when applying the boosting $j_{U_t}(u_{ij}) = E_{1i}U_tE_{j1}$ to define a free Lévy process $(j_{U_t})_{t\geq 0}$ on $U\langle n\rangle$;
3. conclude the proof by reading the Schürmann triple directly from the stochastic equation of $(j_{U_t})_{t\geq 0}$.

Step 1

In this step, we give a direct construction of a free Lévy process starting from a Schürmann triple on $U\langle n\rangle$. To achieve this purpose, we will use the free quantum stochastic calculus. We do not recall the definition of the free stochastic equations on the full Fock space, but we define now the objects involved, and we refer the reader to [27] and [42] for further details.

Let us consider a Hilbert space H. We denote by K the Hilbert space $L^2(\mathbb{R}, H) \simeq L^2(\mathbb{R})\otimes H$, and consider the full Fock space

$$\Gamma(K) = \mathbb{C}\Omega \oplus \bigoplus_{n\geq 1} K^{\otimes n}.$$

We turn $\mathcal{B}(\Gamma(K))$, the $*$-algebra of bounded operator on $\Gamma(K)$, into a noncommutative probability space by endowing it with the state $\tau(\cdot) = \langle \Omega, (\cdot)\Omega\rangle$. Let $h \in H$ and $t \geq 0$. The creation operator $c_t(h) \in \mathcal{B}(\Gamma(K))$ is defined by setting, for all $n \geq 0$,

$$c_t(h)(k_1 \otimes \cdots \otimes k_n) = (h1_{[0,t[}) \otimes k_1 \otimes \cdots \otimes k_n,$$

and the annihilation operator $c_t^*(h) \in \mathcal{B}(\Gamma(K))$ is its adjoint operator. Let W a bounded operator on H and $t \geq 0$. The conservation operator $\Lambda_t(W) \in \mathcal{B}(\Gamma(K))$ is defined by setting, for all $n \geq 1$,

$$\Lambda_t(W)((h1_{[r,s[}) \otimes \cdots \otimes k_n) = ((W(h)1_{[0,t[\cap[r,s[}) \otimes \cdots \otimes k_n)$$

and $\Lambda_t(W)(\Omega) = 0$ otherwise.

The following general result is the free counterpart of the general results of Schürmann (see Section 4.4. of [39] for the tensor case). The free case turns out to be the only case which has not yet been written down.

Theorem 6.4.1. *Let H be a Hilbert space, and let (ρ, η, L) be a Schürmann triple on $U\langle n\rangle$ over the Hilbert space H. Then the coupled free stochastic equations*

$$\mathrm{d}j_t(u_{ij}) = \sum_{1\le k\le n} j_t(u_{ik})\Big(\mathrm{d}c_t^*(\eta(u_{kj})) + \mathrm{d}c_t(\eta(u_{kj}^*)) + \mathrm{d}\Lambda_t((\rho-\delta)(u_{kj})) + L(u_{kj})\mathrm{d}t\Big) \tag{6.4}$$

for $1 \le i,j \le n$, with initial conditions $j_0(u_{ij}) = \delta_{ij}Id$, has a unique solution $\big(j_t(u_{ij})_{i,j=1}^n\big)_{t\ge 0}$ which extends to a free Lévy process $(j_t)_{t\ge0}$ on $U\langle n\rangle$ with value in $(\mathcal{B}(\Gamma(L^2(\mathbb{R},H))),\tau)$, and with generator L.

Proof. The existence and uniqueness of the solution of (6.4) is a consequence of a very general theorem in [42], from which we can also deduce the extension of the solution to a free Lévy process. On the contrary, proving that L is indeed the generator of this solution is not a direct consequence of [42], and requires some computations very similar to those of [38].

The existence theorem which we will use is [42, Theorem 10.1]. In order to use Theorem 10.1 of [42], we must write the n^2 stochastic equations (6.4) as one stochastic equation involving only one variable. This is routine using the explanations of Chapter 13 of [42]. For the convenience of the reader, we sketch the ideas: we consider the full Fock $\mathcal{M}_N(\mathbb{C})$-module $\mathcal{M}_N(\mathcal{B}(\Gamma(K))) \simeq \mathcal{B}(\mathbb{C}^n \otimes \Gamma(K))$. The stochastic equations (6.4) can be summed up into the following stochastic equation in $\mathcal{M}_N(\mathcal{B}(\Gamma(K)))$ (where c_t, c_t^* and Λ_t are defined accordingly)

$$\begin{aligned}&\mathrm{d}\Big(j_t(u_{ij})_{i,j=1}^n\Big)\\ &= \Big(j_t(u_{ij})_{i,j=1}^n\Big)\cdot\Big(\mathrm{d}c_t^*(\eta(u_{ij})_{i,j=1}^n) + \mathrm{d}c_t(\eta(u_{ij}^*)_{i,j=1}^n) + \mathrm{d}\Lambda_t(\rho(u_{ij})_{i,j=1}^n - \mathrm{Id}) + L(u_{ij})_{i,j=1}^n\mathrm{d}t\Big)\end{aligned} \tag{6.5}$$

with initial condition $(j_t(u_{ij}))_{i,j=1}^n = \mathrm{Id}$. Let us define $h = (h_{ij})_{1\le i,j\le n}$, $W = (W_{ij})_{1\le i,j\le n}$ unitary, and $R = (R_{ij})_{1\le i,j\le n}$ selfadjoint by the relation (6.3). The stochastic equation (6.5) can be rewritten

$$\mathrm{d}\Big(j_t(u_{ij})_{i,j=1}^n\Big) = \Big(j_t(u_{ij})_{i,j=1}^n\Big)\cdot\Big(\mathrm{d}c_t^*(h) - \mathrm{d}c_t(W^{-1}h) + \mathrm{d}\Lambda_t(W-\mathrm{Id}) + \big(iR - \frac{1}{2}\sum_{k=1}^n \langle h_{ki}, h_{kj}\rangle_H\big)\mathrm{d}t\Big). \tag{6.6}$$

According to Theorem 10.1 of [42] (see the end of [42, Chapter 10] to make the link with this particular case), there exists a unique solution to (6.6) whenever $W = (W_{ij})_{1\le i,j\le n}$ is unitary and $R = (R_{ij})_{1\le i,j\le n}$ is selfadjoint, which is indeed true thanks to Proposition 6.3.5. Finally, there exists a unique solution $(j_t(u_{ij}))_{i,j=1}^n$ to the coupled stochastic equations (6.4), and another consequence of [42, Theorem 10.1] is that $(j_t(u_{ij}))_{i,j=1}^n$ is unitary. This is sufficient to extend $(j_t(u_{ij}))_{i,j=1}^n$ as a process $(j_t)_{t\ge0}$ of quantum random variables. The stationarity of the distribution is a consequence of the stationarity of the underlying driven process and the freeness of the increments is a consequence of the particular underlying filtration for which $(j_t(u_{ij}))_{i,j=1}^n$ is adapted (see Chapter 11 of [42] for the statements of those two facts).

It remains to prove that L is indeed the generator of $(j_t)_{t\geq 0}$. Let us denote by $\mathcal{L}$ the generator of $(j_t)_{t\geq 0}$, defined for all $a \in U_n^{\mathrm{nc}}$ by

$$\mathcal{L}(a) = \frac{\mathrm{d}}{\mathrm{d}t}\phi \circ j_t(a). \tag{6.7}$$

In order to prove that $\mathcal{L} = L$, it suffices to prove first that, for all $b, c \in U_n^{\mathrm{nc}}$, we have

$$\mathcal{L}(b^*c) = \mathcal{L}(b^*)\delta(c) + \delta(b^*)\mathcal{L}(c) + \langle \eta(b), \eta(c)\rangle, \tag{6.8}$$

which implies that $(\rho, \eta, \mathcal{L})$ is a Schürmann triple, and to prove secondly that $(\mathcal{L}(u_{i,j}))_{i,j=1}^n = (L(u_{i,j}))_{i,j=1}^n$, which implies that the Schürmann triples $(\rho, \eta, \mathcal{L})$ and (ρ, η, L) are equal.

The quantum stochastic calculus allows us to write the quantum stochastic differential equation of $j_t(b^*c)$, thanks to the following result.

Theorem 6.4.2 (Corollary 9.2. of [42]). *Let $h, h' \in H$ and $W, W' \in \mathcal{B}(H)$. Let I_t be one of the following four processes $t \mapsto t$, $c_t(h)$, $c_t^*(h)$ or $\Lambda_t(W)$, and I_t' one of the following four processes $t \mapsto t$, $c_t(h')$, $c_t^*(h')$ or $\Lambda_t(W')$. Let F, G, F' and G' be adapted and bounded. For all M_t and M_t' such that $\mathrm{d}M_t = F_t\mathrm{d}I_tG_t$ and $\mathrm{d}M_t' = F_t'\mathrm{d}I_t'G_t'$, we have*

$$\mathrm{d}(M_tM_t') = F_t\mathrm{d}I_t(G_tM_t') + (M_tF_t')\mathrm{d}I_t'G_t' + \tau(G_tF_t')F_t\mathrm{d}I_t''G_t',$$

where the integrator $\mathrm{d}I_t''$ has to be chosen according to Itô's table (see Table 6.1).

$\mathrm{d}I_t \backslash \mathrm{d}I_t'$	$\mathrm{d}t$	$\mathrm{d}c_t(h')$	$\mathrm{d}c_t^*(h')$	$\mathrm{d}\Lambda_t(W')$
$\mathrm{d}t$	0	0	0	0
$\mathrm{d}c_t(h)$	0	0	0	0
$\mathrm{d}c_t^*(h)$	0	$\langle h, h'\rangle\mathrm{d}t$	0	$\mathrm{d}c_t^*(W'^*h)$
$\mathrm{d}\Lambda_t(W)$	0	$\mathrm{d}c_t(Wh')$	0	$\mathrm{d}\Lambda_t(WW')$

Table 6.1: Itô's table

By induction, it follows that $j_t(b)$ and $j_t(c)$, which can be written as a polynomial in the operators $\{j_t(u_{i,j}), j_t(u_{i,j})^*\}_{1\leq i,j\leq n}$, satisfies a quantum stochastic differential equation. Moreover, by the previous theorem,

$$\mathrm{d}j_t(b^*c) = \mathrm{d}j_t(b^*)j_t(c) + j_t(b^*)\mathrm{d}j_t(c) + \mathrm{d}j_t(b^*) \cdot \mathrm{d}j_t(c) \tag{6.9}$$

where the third term is computed thanks to the quantum Itô table. But in the definition (6.7) of $\mathcal{L}$, we are only dealing with expectations in the vacuum state τ, and the $\mathrm{d}c_t$-part, the $\mathrm{d}c_t^*$-part and the $\mathrm{d}\Lambda_t$-part are martingales under the vacuum state τ. Thus we need only to compute the integrand of the $\mathrm{d}t$-part of $\mathrm{d}j_t(b^*c)$. This coefficient is a complex valued function in $t \in \mathbb{R}_+$ and its value at $t = 0$ gives us $\mathcal{L}(b^*c)$. Using the initial condition, one checks that the first two terms on the right hand side of (6.9) give rise, under the vacuum state, to the first two terms on the right hand side of (6.8). We are left with the computation of the coefficient of the $\mathrm{d}t$-part of $\mathrm{d}j_t(b^*) \cdot \mathrm{d}j_t(c)$ at $t = 0$. Because of the Itô table, this $\mathrm{d}t$-part is coming from the $\mathrm{d}c_t$-parts of $\mathrm{d}j_t(b)$ and $\mathrm{d}j_t(c)$ by the formula

$$\mathrm{d}c_t^*(h) \cdot \mathrm{d}c_t(h') = \langle h, h'\rangle\mathrm{d}t. \tag{6.10}$$

Thus we are left to compute the $\mathrm{d}c_t$-parts of $\mathrm{d}j_t(b)$ and $\mathrm{d}j_t(c)$. Of course, we can assume that both b and c are monomials in u_{ij} and u_{ij}^*. Assuming $b = u_{i_1,j_1}^{\epsilon_1}\cdots u_{i_r,j_r}^{\epsilon_r}$, we can compute from the differential equation of j_t and the quantum Itô table the exact expression for the $\mathrm{d}c_t$-part of $\mathrm{d}j_t(b)$. For simplicity, we give here the expression of the $\mathrm{d}c_t$-part of $\mathrm{d}j_t(b)$ where we have already put the integrand at time $t = 0$, as this will not affect the final result (notice that it allows us to replace $j_0(u_{ij})$ by $\delta(u_{ij})$, and $\sum_{k=1}^n j_0(u_{ik})\mathrm{d}\Lambda_t((\rho-\delta)(u_{kj}))$ by $\mathrm{d}\Lambda_t((\rho-\delta)(u_{ij}))$):

$$\begin{aligned}
&\sum_{\substack{1\le l\le r\\ 1\le m(1)\le\ldots\le m(l)\le r}} \delta(u_{i_1,j_1}^{\epsilon_1}\cdots \hat{u}_{i_{m(1)}j_{m(1)}}^{\epsilon_{m(1)}}\cdots \hat{u}_{i_{m(l-1)}j_{m(l-1)}}^{\epsilon_{m(l-1)}}\cdots \hat{u}_{i_{m(l)}j_{m(l)}}^{\epsilon_{m(l)}}\cdots u_{i_r,j_r}^{\epsilon_r})\\
&\qquad\cdot \mathrm{d}\Lambda_t((\rho-\delta)(u_{i_{m(1)}j_{m(1)}}^{\epsilon_{m(1)}}))\cdots \mathrm{d}\Lambda_t((\rho-\delta)(u_{i_{m(l-1)}j_{m(l-1)}}^{\epsilon_{m(l-1)}}))\mathrm{d}c_t(\eta(u_{i_{m(l)}j_{m(l)}}^{\epsilon_{m(l)}})\\
&=\sum_{1\le l\le r}\mathrm{d}\Lambda_t(\rho(u_{i_1j_1}^{\epsilon_1}))\cdots \mathrm{d}\Lambda_t(\rho(u_{i_{l-1}j_{l-1}}^{\epsilon_{l-1}}))\mathrm{d}c_t(\eta(u_{i_lj_l}^{\epsilon_l})\delta(u_{i_{l+1},j_{l+1}}^{\epsilon_{l+1}}\cdots u_{i_r,j_r}^{\epsilon_r})\\
&=\mathrm{d}c_t(\eta(u_{i_1,j_1}^{\epsilon_1}\cdots u_{i_r,j_r}^{\epsilon_r})) = \mathrm{d}c_t(\eta(b)),
\end{aligned}$$

where the hats mean that we omit the terms in the product. Finally, using (6.10), the integrand of the $\mathrm{d}t$-part of $(\mathrm{d}j_t(b^*))\cdot(\mathrm{d}j_t(c))$ at time $t=0$ is equal to $\langle\eta(b),\eta(c)\rangle$, which completes the equality (6.8).

Now, for $1\le i,j\le n$, $\mathcal{L}(u_{ij})$ is given by the integrand of the $\mathrm{d}t$-part of $\mathrm{d}j_t(u_{ij})$ at time $t=0$. Indeed, the three other parts are martingales. This integrand is given by (6.4):

$$\mathcal{L}(u_{ij}) = \sum_{k=1}^n \tau(j_0(u_{ik})L(u_{kj})) = \sum_{k=1}^n \delta(u_{ik})L(u_{kj}) = L(u_{ij});$$

and it concludes the proof. □

Using Proposition 6.3.5, it is possible to rewrite Theorem 6.4.1 without mentioning any Schürmann triple.

Corollary 6.4.3. *Let H be a Hilbert space, $(h_{ij})_{1\le i,j\le n}\in\mathcal{M}_n(H)$ be elements of H, $(W_{ij})_{1\le i,j\le n}\in\mathcal{M}_n(\mathcal{B}(H))$ unitary and $(R_{ij})_{1\le i,j\le n}\in\mathcal{M}_n(\mathbb{C})$ self-adjoint. Then the coupled free stochastic equations*

$$\begin{aligned}
\mathrm{d}j_t(u_{ij}) = &\sum_{1\le k\le n} j_t(u_{ik})\\
&\cdot\Big(\mathrm{d}c_t^*(h_{kj}) - \mathrm{d}c_t(\sum_{l=1}^n W_{lk}^*h_{lj}) + \mathrm{d}\Lambda_t(W_{kj}-\delta_{kj}Id_H) + (iR_{kj} - \frac{1}{2}\sum_{l=1}^n\langle h_{lk},h_{lj}\rangle_H)\mathrm{d}t\Big)
\end{aligned}$$

for $1\le i,j\le n$, with initial conditions $j_0(u_{ij}) = \delta_{ij}Id$ has a unique solution $\big(j_t(u_{ij})_{i,j=1}^n\big)_{t\ge0}$ which extends to a free Lévy process $(j_t)_{t\ge0}$ on $U\langle n\rangle$ over $(\mathcal{B}(\Gamma(L^2(\mathbb{R},H))),\tau)$.

Step 2

Let H be a Hilbert space, (ρ,η,L) be a Schürmann triple on $U\langle1\rangle$ over H, and $K = L^2(\mathbb{R}_+,H)$. From Theorem 6.4.1, we know that

$$\mathrm{d}U_t = U_t\Big(\mathrm{d}c_t^*(\eta(u)) + \mathrm{d}c_t(\eta(u)^*) + \mathrm{d}\Lambda_t((\rho-\delta)(u)) + L(u)\mathrm{d}t\Big) \tag{6.11}$$

with initial conditions $U_0 = 1$, has a unique solution $(U_t)_{t\geq 0}$ in $(\mathcal{B}(\Gamma(K)), \tau)$ which is a free Lévy process with Schürmann triple (ρ, η, L). We consider

$$j_t : U\langle n\rangle \to E_{11}(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}))E_{11} \tag{6.12}$$

the free Lévy process defined by setting $j_t(u_{ij}) = E_{1i}U_tE_{j1}$ as in Proposition 6.2.1. The following theorem gives a stochastic equation on $\mathcal{B}(\Gamma(L^2(\mathbb{R}, H) \otimes \mathcal{M}_n(\mathbb{C})))$ whose solution has the same distribution under the vacuum state than j_t.

Let us first remark that $L^2(\mathbb{R}, H) \otimes \mathcal{M}_n(\mathbb{C}) \simeq L^2(\mathbb{R}, H \otimes \mathcal{M}_n(\mathbb{C}))$. Thus, for all $h \otimes M \in H \otimes \mathcal{M}_n(\mathbb{C})$, the process $c_t^*(h \otimes M), c_t(h \otimes M) \in \mathcal{B}(\Gamma(K \otimes \mathcal{M}_n(\mathbb{C})))$ are defined as previously. Furthermore, for all $W \in \mathcal{B}(H)$ and $M \in \mathcal{M}_n(\mathbb{C})$, the conservation operator $\Lambda_t(W \otimes M)$ is defined as previously, with M acting on $\mathcal{M}_n(\mathbb{C})$ by the left multiplication.

Proposition 6.4.4. *Let H be a Hilbert space and (ρ, η, L) be a Schürmann triple on $U\langle 1\rangle = \mathbb{C}[u, u^{-1}]$ over H. Let $(U_t)_{t\geq 0}$ defined by (6.11) and $(j_t)_{t\geq 0}$ defined by (6.12). There exists a homomorphism of noncommutative probability spaces*

$$\rho : \Big(E_{11}(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}, n(\phi * \mathrm{tr}_n)\Big) \to \Big(\mathcal{B}(\Gamma(K \otimes \mathcal{M}_n(\mathbb{C}))), \langle \Omega, (\cdot)\Omega\rangle\Big)$$

such that the free Lévy process $(J_t)_{t\geq 0} = (\rho \circ j_t)_{t\geq 0}$ is the solution of the following differential equation, starting at $J_0(u_{ij}) = \delta_{ij}\mathrm{Id}$:

$$\begin{aligned} \mathrm{d}J_t(u_{ij}) = \sum_{1\leq k\leq n} & J_t(u_{ik}) \\ & \cdot \Big(\mathrm{d}c_t^*(\eta(u) \otimes E_{kj}) + \mathrm{d}c_t(\eta(u^*) \otimes E_{kj}) + \frac{1}{n}\mathrm{d}\Lambda_t(((\rho - \delta)(u)) \otimes E_{kj}) + \delta_{kj}L(u)\mathrm{d}t\Big). \end{aligned}$$

Proof. Let us first describe the free product representation of $\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C})$ given in [46]. We consider $\mathcal{M}_n(\mathbb{C})$ acting on itself by the left multiplication. We denote by $\Gamma(K)^\circ$ the Hilbert space $\bigoplus_{n\geq 1} K^{\otimes n}$ and by $\mathcal{M}_n(\mathbb{C})^\circ$ the Hilbert space $\mathcal{M}_n(\mathbb{C}) \ominus \mathbb{C}I_n$, in such a way that

$$\Gamma(K) = \Gamma(K)^\circ \oplus \mathbb{C}\Omega \quad \text{and} \quad \mathcal{M}_n(\mathbb{C}) = \mathcal{M}_n(\mathbb{C})^\circ \oplus \mathbb{C}I_n.$$

We denote by $k \mapsto k^\circ$ and $M \mapsto M^\circ$ the respective orthogonal projection of $\Gamma(K)$ onto $\Gamma(K)^\circ$ and of $\mathcal{M}_n(\mathbb{C})$ onto $\mathcal{M}_n(\mathbb{C})^\circ$. We consider the Hilbert space $\Gamma(K) * \mathcal{M}_n(\mathbb{C})$ given by

$$\Gamma(K) * \mathcal{M}_n(\mathbb{C}) = \mathbb{C}\Omega \oplus \bigoplus_{m\geq 1} \left(\bigoplus_{\substack{H_1,\ldots,H_n = \Gamma(K)^\circ \text{ or } \mathcal{M}_n(\mathbb{C})^\circ \\ H_i \neq H_{i+1}}} H_1 \otimes \cdots \otimes H_m \right).$$

The algebra $\mathcal{B}(\Gamma(K))$ acts on $\Gamma(K) * \mathcal{M}_n(\mathbb{C})$ as follows: for $A \in \mathcal{B}(\Gamma(K)), k \in \Gamma(K)^\circ$ and $M \in \mathcal{M}_n(\mathbb{C})^\circ$, we have

$$\begin{aligned} (\pi(A))(\Omega) &= (A\Omega)^\circ + \langle \Omega, A\Omega\rangle\Omega, \\ (\pi(A))(k \otimes (\cdots)) &= (Ak)^\circ \otimes (\cdots) + \langle \Omega, Ak\rangle \cdot (\cdots), \\ (\pi(A))(M \otimes (\cdots)) &= (A\Omega)^\circ \otimes M \otimes (\cdots) + \langle \Omega, A\Omega\rangle \cdot M \otimes (\cdots). \end{aligned}$$

Similarly, the algebra $\mathcal{M}_n(\mathbb{C})$ acts on $\Gamma(K) * \mathcal{M}_n(\mathbb{C})$ as follows: for $A \in \mathcal{M}_n(\mathbb{C}), k \in \Gamma(K)^\circ$ and $M \in \mathcal{M}_n(\mathbb{C})^\circ$, we have

$$\begin{aligned}
(\lambda(A))(\Omega) &= A^\circ + \langle I_N, A\rangle\Omega, \\
(\lambda(A))(k \otimes (\cdots)) &= A^\circ \otimes k \otimes (\cdots) + \langle I_N, A\rangle \cdot k \otimes (\cdots), \\
(\lambda(A))(M \otimes (\cdots)) &= (AM)^\circ \otimes (\cdots) + \langle I_N, AM\rangle \cdot (\cdots).
\end{aligned}$$

According to [46, Section 1.5], the $*$-homomorphism $\pi \sqcup \lambda : \Big(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}), \phi * \mathrm{tr}_n\Big) \to \Big(\mathcal{B}(\Gamma(K) * \mathcal{M}_n(\mathbb{C})), \langle \Omega, (\cdot)\Omega\rangle\Big)$ is a $*$-homomorphism of noncommutative probability spaces.

Lemma 6.4.5. *There exists a Hilbert space isomorphism*

$$\Gamma(K) * \mathcal{M}_n(\mathbb{C}) \to \Gamma(K \otimes \mathcal{M}_n(\mathbb{C})) \otimes \mathcal{M}_n(\mathbb{C})$$

which induces a $*$-algebra isomorphism

$$f : \mathcal{B}\Big(\Gamma(K) * \mathcal{M}_n(\mathbb{C})\Big) \to \mathcal{B}\Big(\Gamma(K \otimes \mathcal{M}_n(\mathbb{C})) \otimes \mathcal{M}_n(\mathbb{C})\Big).$$

Proof. We will use the three well-known isomorphisms

$$K \simeq K \otimes \mathbb{C},\ (K \otimes \mathbb{C}) \oplus (K \otimes \mathcal{M}_n(\mathbb{C})^\circ) \simeq K \otimes \mathcal{M}_n(\mathbb{C}) \ \text{ and } \ \mathcal{M}_n(\mathbb{C}) \simeq \mathbb{C}^n \otimes \mathbb{C}^n.$$

It suffices to write

$$\begin{aligned}
\Gamma(K) * \mathcal{M}_n(\mathbb{C}) &\simeq \mathbb{C}\Omega \oplus \bigoplus_{m \geq 1} \left(\bigoplus_{\substack{H_1,\dots,H_m = \Gamma(K)^\circ \text{ or } \mathcal{M}_n(\mathbb{C})^\circ \\ H_i \neq H_{i+1}}} H_1 \otimes \cdots \otimes H_m \right) \\
&\simeq \mathbb{C}\Omega \oplus \bigoplus_{m \geq 1} \left(\bigoplus_{\substack{k_1+1+k_2+\ldots+1+k_N = m \\ k_2,\dots,k_{N-1} \geq 1}} K^{\otimes k_1} \otimes \mathcal{M}_n(\mathbb{C})^\circ \otimes K^{\otimes k_2} \cdots \otimes \mathcal{M}_n(\mathbb{C})^\circ \otimes K^{\otimes k_N} \right) \\
&\simeq \bigoplus_{m' \geq 1} \left(\bigoplus_{H_1,\dots,H_{m'} = \mathbb{C} \text{ or } \mathcal{M}_n(\mathbb{C})^\circ} H_1 \otimes K \otimes H_2 \otimes \cdots \otimes K \otimes H_{m'} \right) \\
&\simeq \bigoplus_{m' \geq 1} \underbrace{\mathcal{M}_n(\mathbb{C}) \otimes K \otimes \mathcal{M}_n(\mathbb{C}) \otimes \cdots \otimes K \otimes \mathcal{M}_n(\mathbb{C})}_{\text{where } \mathcal{M}_n(\mathbb{C}) \text{ appears } m' \text{ times}} \\
&\simeq \bigoplus_{m' \geq 1} \underbrace{(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes K \otimes (\mathbb{C}^n \otimes \mathbb{C}^n) \otimes \cdots \otimes K \otimes (\mathbb{C}^n \otimes \mathbb{C}^n)}_{\text{where } \mathbb{C}^n \otimes \mathbb{C}^n \text{ appears } m' \text{ times}} \\
&\simeq \mathbb{C}^n \otimes \left(\bigoplus_{m \geq 0} \underbrace{(\mathbb{C}^n \otimes K \otimes \mathbb{C}^n) \otimes \cdots \otimes (\mathbb{C}^n \otimes K \otimes \mathbb{C}^n)}_{\text{where } \mathbb{C}^n \otimes K \otimes \mathbb{C}^n \text{ appears } m \text{ times}} \right) \otimes \mathbb{C}^n \\
&\simeq \mathbb{C}^n \otimes \Gamma(\mathbb{C}^n \otimes K \otimes \mathbb{C}^n) \otimes \mathbb{C}^n \\
&\simeq \Gamma(K \otimes \mathcal{M}_n(\mathbb{C})) \otimes \mathcal{M}_n(\mathbb{C})
\end{aligned}$$

and to define the Hilbert space isomorphism $\Gamma(K) * \mathcal{M}_n(\mathbb{C}) \to \Gamma(K \otimes \mathcal{M}_n(\mathbb{C})) \otimes \mathcal{M}_n(\mathbb{C})$ accordingly. ☐

Unfortunately, we do not see any way of writing f directly, and for computing it, we will always follow the different steps of the proof of Lemma 6.4.5.

We are interested in the $*$-subalgebra $E_{11}(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$, and it is important to remark here that its image by $f \circ (\pi \sqcup \lambda)$ is an algebra of operators which leaves the space $\Gamma(K \otimes \mathcal{M}_n(\mathbb{C})) \otimes \mathbb{C}E_{11}$ invariant (it suffices to follow each step of the proof of Lemma 6.4.5). Consequently, when restricted to $E_{11}(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$, the $*$-homomorphism $f \circ (\pi \sqcup \lambda)$ can be seen as a $*$-homomorphism

$$\rho : E_{11}(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}))E_{11} \to \mathcal{B}\left(\Gamma(K \otimes \mathcal{M}_n(\mathbb{C}))\right)$$

using the trivial isomorphism $\Gamma(K \otimes \mathcal{M}_n(\mathbb{C})) \otimes \mathbb{C}E_{11} \simeq \Gamma(K \otimes \mathcal{M}_n(\mathbb{C}))$. It is now a routine, following the steps of Lemma 6.4.5, to verify that

- $n(\phi * \mathrm{tr}_n)(A) = \langle \Omega, \rho(A)\Omega \rangle$ for all $A \in E_{11}(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$;
- $\rho(E_{1i}c_t^*(h)E_{j1}) = c_t^*(h \otimes E_{ij})$ for all $h \in H$,
- $\rho(E_{1i}c_t(h)E_{j1}) = c_t(h \otimes E_{ij})$ for all $h \in H$,
- and $\rho(E_{1i}\Lambda_t(W)E_{j1}) = \frac{1}{n}\Lambda_t(W \otimes E_{ij})$ for all $W \in \mathcal{B}(H)$.

To conclude, let us write

$$\begin{aligned}
\mathrm{d}j_t(u_{ij}) =& E_{1i}\mathrm{d}U_t E_{j1} \\
=& E_{1i}U_t \cdot \left(\mathrm{d}c_t^*(\eta(u)) + \mathrm{d}c_t(\eta(u)^*) + \mathrm{d}\Lambda_t((\rho-\delta)(u)) + L(u)\mathrm{d}t\right) E_{j1} \\
=& \sum_{1 \le k \le n} E_{1i}U_t E_{k1} \cdot E_{1k}\left(\mathrm{d}c_t^*(\eta(u)) + \mathrm{d}c_t(\eta(u)^*) + \mathrm{d}\Lambda_t((\rho-\delta)(u)) + L(u)\mathrm{d}t\right) E_{j1} \\
=& \sum_{1 \le k \le n} j_t(u_{ik}) \cdot E_{1k}\left(\mathrm{d}c_t^*(\eta(u)) + \mathrm{d}c_t(\eta(u)^*) + \mathrm{d}\Lambda_t((\rho-\delta)(u)) + L(u)\mathrm{d}t\right) E_{j1}
\end{aligned}$$

and then apply the homomorphism ρ. $\square$

Step 3

We conclude the proof of Theorem 6.3.3. Recall that we start from a free unitary Lévy process $(U_t)_{t\geq 0}$ with Schürmann triple (ρ, η, L). Because Theorem 6.3.3 uniquely depends on the distribution of our random variables, we can without loss of generality represent $(U_t)_{t\geq 0}$ as the solution of the stochastic equation (6.11). Let $j_t : U\langle n \rangle \to E_{11}(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$ be the Lévy process defined by setting, for all $1 \le i, j \le n$, $j_t(u_{ij}) = E_{1i}U_t E_{j1}$ as in Proposition 6.2.1.

We want to prove that (ρ_n, η_n, L_n) defined by setting, for all $1 \le i, j \le n$,

$$\begin{aligned}
\rho_n(u_{ij}) = \frac{1}{n}(\rho(u) - \mathrm{Id}_H) \otimes E_{ij} + \delta_{ij}\mathrm{Id}_H \otimes I_N,& \\
\eta_n(u_{ij}) = \eta(u) \otimes E_{ij}, \quad \eta_n(u_{ij}^*) = \eta(u^*) \otimes E_{ij}, \quad L_n(u_{ij}) = \delta_{ij}L(u),& \qquad (6.13)
\end{aligned}$$

is the Schürmann triple of $(j_t)_{t\geq 0}$.

First of all, (ρ_n, η_n, L_n) given by (6.13) is a well-defined Schürmann triple. Indeed, defining $(h_{ij})_{1\leq i,j\leq n}$, $(W_{ij})_{1\leq i,j\leq n}$ unitary, and $(R_{ij})_{1\leq i,j\leq n}$ selfadjoint by

$$W_{ij} = \frac{1}{n}(\rho(u) - \mathrm{Id}_H) \otimes E_{ij} + \delta_{ij}\mathrm{Id}_H \otimes I_N,$$
$$h_{ij} = \eta(u) \otimes E_{ij}, \quad R_{ij} = -i\left(\delta_{ij}L(u) + \frac{1}{2}\sum_{k=1}^{n}\langle h_{ki}, h_{kj}\rangle_{H\otimes\mathcal{M}_n(\mathbb{C})}\right),$$

we can apply Proposition 6.3.5 and conclude that (ρ_n, η_n, L_n) is a Schürmann triple whenever $\eta(u^*)\otimes E_{ij} = -\sum_{k=1}^n W_{ki}^* h_{kj}$ (because in that case the relations (6.2) and (6.13) are the same). Let us verify this fact:

$$-\sum_{k=1}^{n} W_{ki}^* h_{kj} = -\frac{1}{n}\sum_{k=1}^{n}(\rho(u)^*\eta(u)\otimes E_{ik}E_{kj} - \eta(u)\otimes E_{ik}E_{kj}) - \eta(u)\otimes E_{ij}$$
$$= -\rho(u)^*\eta(u)\otimes E_{ij} = \eta(u^*)\otimes E_{ij}.$$

Proposition 6.4.4 gives us the stochastic equation which drives the process $(j_t)_{t\geq 0}$ (or at least a process which has the same distribution):

$$\mathrm{d}j_t(u_{ij}) = \sum_{1\leq k\leq n} j_t(u_{ik}) \cdot \left(\mathrm{d}c_t^*(\eta(u)\otimes E_{kj}) + \mathrm{d}c_t(\eta(u^*)\otimes E_{kj}) + \frac{1}{n}\mathrm{d}\Lambda_t(((\rho-\delta)(u))\otimes E_{kj}) + \delta_{kj}(L(u))\mathrm{d}t\right),$$

or equivalently,

$$\mathrm{d}j_t(u_{ij}) = \sum_{1\leq k\leq n} j_t(u_{ik})\Big(\mathrm{d}c_t^*(\eta_n(u_{kj})) + \mathrm{d}c_t(\eta_n(u_{kj}^*)) + \mathrm{d}\Lambda_t((\rho_n-\delta)(u_{kj})) + L_n(u_{kj})\mathrm{d}t\Big). \quad (6.14)$$

Theorem 6.4.1 allows us to conclude that (ρ_n, η_n, L_n) is the Schürmann triple of $(j_t)_{t\geq 0}$, which concludes the proof of Theorem 6.3.3.

6.5 An example: the free unitary Brownian motion

The free unitary Brownian motion introduced in [5] is the unique solution $(U_t)_{t\geq 0}$ in $\mathcal{B}(\Gamma(L^2(\mathbb{R},\mathbb{C})))$, starting at $U_0 = \mathrm{Id}$, of the free stochastic equation

$$\mathrm{d}U_t = iU_t(\mathrm{d}c_t^*(1) + \mathrm{d}c_t(1)) - \frac{1}{2}U_t\mathrm{d}t,$$

or equivalently, of the equation $\mathrm{d}U_t = U_t\cdot(\mathrm{d}c_t^*(-i) + \mathrm{d}c_t(i) - \frac{1}{2}\mathrm{d}t)$. It corresponds to a Lévy process over $U\langle 1\rangle$ given by $(u\mapsto U_t)_{t\geq 0}$, and from Theorem 6.4.1, we know that the Schürmann triple (ρ,η,L) over $\mathbb{C}$ of this process is given by $\rho(u) = \mathrm{Id}_{\mathbb{C}}$, $\eta(u) = -\eta(u^*) = -i$ and $L(u) = -\frac{1}{2}$. Concretely, using the definition of a Schürmann triple, it means that, for all polynomials $P\in\mathbb{C}[X]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau(P(U_t)) = L(P(u)) = -\frac{1}{2}P'(1) - P''(1).$$

The free Lévy process $j_t : U\langle n\rangle \to E_{11}(\mathcal{B}(\Gamma(K)) \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$ defined by $j_t(u_{ij}) = E_{1i}U_tE_{j1}$ is then (thanks to Proposition 6.4.4), equal in distribution to the solution $(J_t)_{t\geq 0}$ of

$$\begin{aligned}
\mathrm{d}J_t(u_{ij}) &= \sum_{1\leq k\leq n} J_t(u_{ik}) \cdot \left(\mathrm{d}c_t^*(-iE_{kj}) + \mathrm{d}c_t(iE_{kj}) - \frac{1}{2}\delta_{kj}\mathrm{d}t\right) \\
&= i \sum_{1\leq k\leq n} J_t(u_{ik}) \cdot (\mathrm{d}c_t^*(E_{kj}) + \mathrm{d}c_t(E_{kj})) - \frac{1}{2}J_t(u_{ij})\mathrm{d}t, \qquad (6.15)
\end{aligned}$$

the Lévy process on $U\langle n\rangle$ under study in [44]. Theorem 6.2.2 gives the same conclusion as in [44]: because the Brownian motion $(U_t^{(N)})_{t\geq 0}$ on the unitary group $U(N)$ defined and studied in [5] converges in $*$-distribution to the free unitary Brownian motion $(U_t)_{t\geq 0}$ as N tends to ∞, the $N \times N$-block matrices $\left([U_t^{(nN)}]_{ij}\right)_{\substack{1\leq i,j\leq n \\ t\geq 0}}$ converge almost surely in $*$-distribution to $\left(J_t(u_{ij})\right)_{\substack{1\leq i,j\leq n \\ t\geq 0}}$ as N tends to ∞.

7

Haar states and Haar traces on dual groups

This section is devoted to define a convenient counterpart to the notion of Haar measures on Lie groups or Haar states on quantum groups. We will focus on the dual groups $U\langle n\rangle$, as this is the one that has been most studied during my thesis. This work has been done in collaboration with Guillaume Cébron[1].

This chapter is adapted from the article [11], which I co-authored with G. Cébron.

7.1 How to build states on $U\langle n\rangle$?

We expose now a general method for defining quantum random variables on $U\langle n\rangle$. Consider the noncommutative probability space $\mathcal{M}_n(\mathbb{C})$ composed of matrices of dimension n equipped with its normalized trace $\mathrm{tr}_n := \frac{1}{n}\mathrm{Tr}$. Let us denote by E_{ij} the usual matricial units (ie, the matrix whose entries are zero, except for the (i,j)-th coefficient which is 1).

Let A be a random variable in a noncommutative space $(\mathcal{A},\phi)$. One way to consider A as a matrix is to count A as an element of $\mathcal{M}_n(\mathcal{A})\simeq\mathcal{A}\otimes\mathcal{M}_n(\mathbb{C})$. In this way, the (i,j)-th block of A is just $\delta_{ij}A$. The starting point of our reflexion is the following: there is another way to consider A as a matrix. Let us denote by $E_{11}(\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C}))E_{11}$ the $*$-subalgebra $\{E_{11}XE_{11} : X\in\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C})\}\subset\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C})$. We have the $*$-isomorphism

$$\begin{array}{rcl}\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C}) & \simeq & \Big(E_{11}(\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C}))E_{11}\Big)\otimes\mathcal{M}_n(\mathbb{C})\\ X & \mapsto & \displaystyle\sum_{1\le i,j\le n} E_{1i}XE_{j1}\otimes E_{ij}\\ \displaystyle\sum_{1\le i,j\le n} E_{i1}A_{ij}E_{1j} & \leftarrow\!\shortmid & \displaystyle\sum_{1\le i,j\le n} A_{ij}\otimes E_{ij}.\end{array}$$

It tells us that the (i,j)-th blocks of A viewed as an element of $\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C})$ can be defined as $E_{1i}AE_{j1}\in E_{11}(\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C}))E_{11}$. We endow the $*$-algebra $E_{11}(\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C}))E_{11}$ with the state $n(\phi * \mathrm{tr}_n)$, where we recall that tr_n is the normalized trace on $\mathcal{M}_n(\mathbb{C})$.

[1] Cébron is postdoc at the Universität des Saarlandes (Germany).

Proposition-Definition 7.1.1. *For any unitary random variable $U \in \mathcal{A}$, there exists a unique quantum random variable $j_U : U_n^{\mathrm{nc}} \to E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$ determined by $j_U(u_{ij}) = E_{1i}UE_{j1}$, which induces a state $(n\ \phi * \mathrm{tr}_n) \circ j_U$ on $U\langle n\rangle$.*

Proof. It follows from the unitarity of $(E_{1i}UE_{j1})_{1\leq i,j\leq n}$. Indeed, we have

$$\sum_{k=1}^{n} E_{1i}U^*E_{k1}E_{1k}UE_{j1} = E_{1i}U^*I_nUE_{j1} = \delta_{ij}$$

and the same for the other relation. □

The elements $j_U(u_{ij}) = E_{1i}UE_{j1}$ have to be considered as the (i,j)-th blocks of U, and, when there is no confusion, we will denote them by U_{ij}. The matrix $(U_{ij})_{1\leq i,j\leq n}$ seen as an element of $E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11} \otimes \mathcal{M}_n(\mathbb{C}) \simeq \mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C})$ is exactly $U \in \mathcal{A}$ seen as an element of $\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C})$, which justifies this notation.

Remark that we have $(U_{ij})^* = (U^*)_{ji}$, and that the notation U_{ij}^* is ambiguous.

7.2 Free cumulants

The compression of random variables by a family of matrix units has been considered in different situations, and it is possible to write explicitly the free cumulants of U_{ij} in terms of those of U. For the definition of free cumulants, we refer to Section 3.3.

Let us express the free cumulants of $U_{ij} = j_U(u_{ij}) = E_{1i}UE_{j1}$ as defined in Proposition-Definition 7.1.1 in terms of the free cumulants of $U \in \mathcal{A}$.

Proposition 7.2.1 (Theorem 14.18 of [35]). *Let $U^{(1)},\dots,U^{(m)}$ be unitary random variables of $(\mathcal{A},\phi)$. The free cumulants of $(U^{(k)})_{ij} = j_{U^{(k)}}(u_{ij})$ in the noncommutative probability space $\Big(E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}, n(\phi * \mathrm{tr}_n)\Big)$ are given as follows. Let $1 \leq i_1, j_1, \dots, i_q, j_q \leq n$ and $1 \leq m_1,\dots,m_q \leq m$. If the indices are cyclic, i.e. if $i_l = j_{l-1}$ for $2 \leq l \leq q$ and $i_1 = j_q$, we have*

$$\kappa_q\Big((U^{(m_1)})_{i_1j_1},\dots,(U^{(m_q)})_{i_qj_q}\Big) \quad = \quad n\ \kappa_q\Big(\frac{1}{n}U^{(m_1)},\dots,\frac{1}{n}U^{(m_q)}\Big).$$

If the indices are not cyclic, the left handside is equal to zero.

Let us mention two basic properties about the quantum random variables $j_U : U\langle n\rangle \to E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$ defined in Definition 7.1.1.

Proposition 7.2.2. *Let $U, V \in \mathcal{A}$ be two unitary variables of $(\mathcal{A},\phi)$.*

1. *We have $j_{U^{-1}} = j_U \circ \Sigma$ and $j_{UV} = j_U \star j_V$.*
2. *If U and V are $*$-free, then, the image $*$-algebras of j_U and j_V are $*$-free in the noncommutative probability space $(E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}, n(\phi * \mathrm{tr}_n))$.*

Proof. The first property follows from the relation $j_{UV}(u_{ij}) = \sum_k j_U(u_{ik})j_V(u_{kj})$. The second one follows from Proposition 7.2.1 and the characterization of freeness of Proposition 3.3.2. □

7.3 Haar state on the unitary dual group

In this section, we will investigate the existence of the Haar state on $U\langle n\rangle$ for the five different convolutions. Unfortunately, the definition of a Haar state on $U\langle n\rangle$ is too strong, and we need to define a weaker notion of Haar state, namely the notion of Haar trace, to have some existence results.

Definition 7.3.1. *The free (resp. tensor independent, boolean, monotone, anti-monotone)* Haar state *on $U\langle n\rangle$, if it exists, is the unique state h on U_n^{nc} such that, for all other states ϕ on U_n^{nc}, we have $\phi\star_F h = h = h\star_F \phi$ (resp. the same relation for $\star_T$, $\star_B$, $\star_M$ or $\star_{AM}$).*

Theorem 7.3.2.
1. *The Haar measure on $\{z\in\mathbb{C} : |z| = 1\}$ is the Haar state for the free, tensor independent, boolean, monotone and anti-monotone convolution on $U\langle 1\rangle$.*
2. *For all $n\geq 2$, there exists no Haar state on $U\langle n\rangle$ for the free, tensor independent, boolean, monotone or anti-monotone convolution.*

Proof. In Section 7.3.1, we prove the first item. In Section 7.3.2, we prove the second item for the free and the tensor convolution. In Section 7.3.3, we prove the second item for the boolean convolution, and finally, in Section 7.3.4, we prove the second item for the monotone and anti-monotone convolution. □

Let us define a weaker notion of Haar state. A state ϕ on U_n^{nc} is called a *tracial state*, or a *trace*, if, for all $a,b\in U_n^{\mathrm{nc}}$, we have $\phi(ab)=\phi(ba)$.

Definition 7.3.3. *The free (resp. tensor independent, boolean, monotone, anti-monotone)* Haar trace *on $U\langle n\rangle$, if it exists, is the unique tracial state h on U_n^{nc} such that, for all other tracial states ϕ on U_n^{nc}, we have $\phi\star_F h = h = h\star_F \phi$ (resp. the same relation for $\star_T$, $\star_B$, $\star_M$ or $\star_{AM}$).*

Remark that a Haar state which is tracial is automatically a Haar trace.

Theorem 7.3.4.
1. *For all $n\geq 2$, there exist no Haar trace on $U\langle n\rangle$ for the boolean, monotone or anti-monotone convolution.*
2. *For all $n\geq 1$, there exist a Haar trace on $U\langle n\rangle$ for the free convolution, which is faithful, and a Haar trace on $U\langle n\rangle$ for the tensor convolution, which is not faithful whenever $n\neq 1$.*

Remark 7.3.5. As nicely communicated by Moritz Weber, a careful examination of the proof of Theorem 7.3.4 allows us to conclude a more general result: the free Haar trace h on $U\langle n\rangle$ is such that $\phi\star_F h = h = h\star_F\phi$ for all states ϕ on U_n^{nc} such that $\phi(\sum_{k=1}^n u_{ki}u_{kj}^*) = \phi(\sum_{k=1}^n u_{ik}^* u_{jk}) = \delta_{ij}$ $(1\leq i,j\leq n)$, a case which includes the tracial states but not only. For example, a state which factorizes on the unitary quantum group, where $\sum_{k=1}^n u_{ki}u_{kj}^* = \sum_{k=1}^n u_{ik}^* u_{jk} = \delta_{ij}$, fulfills this condition and so is absorbed by the free Haar trace.

Proof. In Section 7.3.3, we prove the first item for the boolean convolution. In Section 7.3.4, we prove the first item for the monotone and anti monotone convolution. In Section 7.3.5, we prove the second item for the free convolution, and give a more explicit description of the free Haar trace. In Section 7.3.6, we prove the second item for the tensor convolution, and give a more explicit description of the tensor Haar trace. □

Let us remark that one could also choose a side and ask about a *right (resp. left) Haar state* for each of these independences. It would be a state h such that for each state ϕ, it holds that $h \star \phi = h$ (resp. $\phi \star h = h$). We define similarly a *right (resp. left) Haar trace*. Nevertheless, the following result shows that this notion does not introduce any more generality.

Proposition 7.3.6. *Let us consider one of the five notions of independence. If h is a right (resp. left) Haar state on $U\langle n\rangle$ then it is also a left (resp. right) Haar state. As well, if h is a right (resp. left) Haar trace on $U\langle n\rangle$ then it is also a left (resp. right) Haar trace.*

Proof. Let h be a right Haar state. We define the flip τ on $U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$ as the $*$-homomorphism such that $\tau(u_{ij}^{(1)}) = u_{ij}^{(2)}$ and $\tau(u_{ij}^{(2)}) = u_{ij}^{(1)}$, where the exponent (1) and (2) indicate if the element is in the first leg of $U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$ or in the second leg. A simple computation on the generators u_{ij} shows that $\tau \circ (\Sigma \sqcup \Sigma) \circ \Delta = \Delta \circ \Sigma$. Therefore, by denoting the notion of independence at hand by $\odot$, we have for all states ϕ:

$$h \circ \Sigma = (h \odot \phi) \circ \Delta \circ \Sigma = (h \odot \phi) \circ \tau \circ (\Sigma \sqcup \Sigma) \circ \Delta = [(\phi \circ \Sigma) \odot (h \circ \Sigma)] \circ \Delta.$$

Because Σ is invertible, this says exactly that $h \circ \Sigma$ is a left Haar state. But then we have:

$$h = h \star (h \circ \Sigma) = h \circ \Sigma$$

by using the right (resp. left) Haar state property of h (resp. $h \circ \Sigma$). Therefore, $h = h \circ \Sigma$ is a right and left Haar state. The argument is valid when replacing h and ϕ by tracial states since it implies that $h \circ \Sigma$ and $\phi \circ \Sigma$ are also tracial. $\square$

7.3.1 The Haar state in the one-dimensional case

Let us emphasize first that we identify the states on $U\langle 1\rangle$ with the probability measures on $\{z \in \mathbb{C} : |z| = 1\}$ via $\mu(u^k) = \int_{\mathbb{U}} z^k d\mu(z)$ and $\mu(u^{*k}) = \int_{\mathbb{U}} \bar{z}^k d\mu(z)$ for $k \in \mathbb{N}$. The Haar measure is the uniform measure on the unit circle and is given by $h(u^k) = h(u^{*k}) = \delta_{k0}$ for $k \in \mathbb{N}$.

The free, tensor independent, boolean, monotone and anti-monotone convolutions on $U\langle 1\rangle$ correspond to five different multiplicative convolutions on probability measures on $\mathbb{U}$ which have been already studied in the literature. In each of those cases, it is straightforward to prove that h is absorbing.

For the free multiplicative convolution, we refer to [46], or to Section 7.3.5. For the tensor independent convolution, one has just to observe that $\phi \star_T h(u^k) = \phi(u^k)h(u^k) = \delta_{k0}$.

For the Boolean, the monotone, and the anti-monotone convolutions, our references are [3, 21, 22]. Let μ be a probability measure on $\mathbb{U}$. We define the K-transform of μ for $|z| < 1$ by

$$K_\mu(z) = \left(\int_{\mathbb{U}} \frac{zx}{1 - zx} d\mu(x)\right) \Big/ \left(\int_{\mathbb{U}} \frac{1}{1 - zx} d\mu(x)\right).$$

Let us remark that $K_h(z) = 0$. The K-transform of the multiplicative Boolean convolution of μ and ν is given by $\frac{1}{z} K_\mu(z) \cdot K_\nu(z)$, and consequently, h is absorbing for the Boolean convolution. The K-transform of the multiplicative monotone (resp. anti-monotone) convolution of μ and ν is given by $K_\mu \circ K_\nu$ (resp. $K_\nu \circ K_\mu$), and consequently, h is absorbing for the monotone and anti-monotone convolutions.

7.3.2 The non existence of Haar state in the free and tensor cases

In this section, we prove that there exists no free Haar state, nor tensor Haar state, for $n \geq 2$.

Let us take $n \geq 2$ and assume that h is a free Haar state. We take $1 \leq k \leq n-1$ and we consider the unitary matrix of size $2n \times 2n$ (which is a version of [50, Non-example 4.1], attributed to Woronowicz):

$$M_k = \left(\begin{array}{c|cc|cc|c} I_{2k-2} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_{2n-2k-2} \end{array}\right).$$

For all $1 \leq i, j \leq n$, we set $j_k(u_{ij})$ the (i,j)-th block of M_k of size 2×2. Because M_k is unitary, j_k extends to a quantum random variable $j : U_n^{\mathrm{nc}} \to \mathcal{M}_2(\mathbb{C})$. We define the state ϕ_k for all $a \in U_n^{\mathrm{nc}}$ as $\phi_k(a) = \langle e_2, j_k(a)e_2 \rangle$, or equivalently, as the $(2,2)$-th coefficient of $j_k(a)$. Then, for every $1 \leq i, j \leq n$, we have $\phi_k(u_{ik}u_{jk}^*) = 0$. Let us remark that h being a free Haar state, we also have

$$h(u_{ik}u_{ik}^*) = \sum_{p,q=1}^{n} (h \star_F \phi_k)(u_{ip}^{(1)} u_{pk}^{(2)} u_{qk}^{(2)*} u_{iq}^{(1)*}) = \sum_{p,q=1}^{n} h(u_{ip}u_{iq}^*)\phi_k(u_{pk}u_{qk}^*) = 0$$

This reasoning can be done for any $1 \leq k \leq n-1$. For $k = n$ we take the matrix M_k in the which we have exchanged the last two columns of blocks. We therefore also have $\phi_k(u_{in}u_{jn}^*) = 0$ and thus $h(u_{in}u_{in}^*) = 0$. Therefore we should have:

$$\sum_{k=1}^{n} h(u_{ik}u_{ik}^*) = \sum_{k=1}^{n} 0 = 0$$

which contradicts the unitarity relation $\sum_{k=1}^{n} u_{ik}u_{ik}^* = 1$.

The same proof can be done for the tensor case as well. Indeed, the tensor independence also verifies that, for any $1 \leq i, j, p, q, k \leq n$,

$$(h \star_T \phi)(u_{ip}^{(1)} u_{pk}^{(2)} u_{qk}^{(2)*} u_{iq}^{(1)*}) = h(u_{ip}u_{iq}^*)\phi(u_{pk}u_{qk}^*).$$

7.3.3 The boolean case

In this section, we prove that for $n \geq 2$, there exist no boolean Haar state and no boolean Haar trace on $U\langle n \rangle$.

First of all, we remark the following general result: if ϕ and ψ are two states on U_n^{nc} and if a, c come from the left leg and b from the right leg of $U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$, then we have

$$(\phi \star_B \psi)(abc) = (\phi \star_B \psi)\Big(a(b - \delta(b))c\Big) + \delta(b)\phi(ac) = \phi(a)\psi(b - \delta(b))\phi(c) + \delta(b)\phi(ac).$$

For all states ϕ, let us introduce the following matrices:

$$\begin{aligned} N_\phi &= (\phi(u_{ij}))_{ij} \in \mathcal{M}_n(\mathbb{C}), \\ \bar{N}_\phi &= (\phi(u_{ij}^*))_{ij} \in \mathcal{M}_n(\mathbb{C}), \\ M_\phi &= (\phi(u_{ij}^* u_{kl}))_{(i,k)(j,l)} \in \mathcal{M}_{n^2}(\mathbb{C}). \end{aligned}$$

Suppose that there exists a boolean Haar state h. Then, for any state ϕ,

$$\begin{aligned} h(u_{ij}^* u_{kl}) &= \sum_{\alpha,\beta=1}^{n} (h \star_B \phi)(u_{\alpha j}^{(2)*} u_{i\alpha}^{(1)*} u_{k\beta}^{(1)} u_{\beta l}^{(2)}) \\ &= \sum_{\alpha,\beta=1}^{n} [\phi(u_{\alpha j}^*) h(u_{i\alpha}^* u_{k\beta} - \delta(u_{i\alpha}^* u_{k\beta}))\phi(u_{\beta l}) + \delta(u_{i\alpha}^* u_{k\beta})\phi(u_{\alpha j}^* u_{\beta l})] \\ &= \sum_{\alpha,\beta=1}^{n} \phi(u_{\alpha j}^*) h(u_{i\alpha}^* u_{k\beta})\phi(u_{\beta l}) - \phi(u_{ij}^*)\phi(u_{kl}) + \phi(u_{ij}^* u_{kl}), \end{aligned}$$

which can be written

$$\begin{aligned} M_h &= M_h(\bar{N}_\phi \otimes N_\phi) - (\bar{N}_\phi \otimes N_\phi) + M_\phi \\ &= (M_h - I_{n^2})(\bar{N}_\phi \otimes N_\phi) + M_\phi \end{aligned} \tag{7.1}$$

where $\otimes$ denotes here the tensor product (or Kronecker product) of matrices.

A measure μ on the unitary group $U(n) = \{M \in \mathcal{M}_n(\mathbb{C}) : U^*U = I_N\}$ can be seen as a unique state on U_n^{nc} via the integration map

$$\mu(u_{i_1 j_1}^{\epsilon_1} \dots u_{i_q j_q}^{\epsilon_q}) = \int_{U(n)} U_{i_1 j_1}^{\epsilon_1} \dots U_{i_q j_q}^{\epsilon_q} \mathrm{d}\mu(U).$$

Let us set $\phi_1 = (1/2)(\delta_{I_n} + \delta_{-I_n})$. For all $1 \leq i,j,k,l \leq n$, we have $\phi_1(u_{ij}) = \phi_1(u_{ij}^*) = 0$ and $\phi_1(u_{ij}^* u_{kl}) = \frac{1}{2}(\delta_{ij}\delta_{kl} + \delta_{ij}\delta_{kl}) = \delta_{ij}\delta_{kl}$, or equivalently $N_{\phi_1} = 0$ and $M_{\phi_1} = I_{n^2}$. By replacing it into (7.1), we get $M_h = I_{n^2}$.

Consider now another state ϕ_2 defined by $(1/2)(\delta_A + \delta_{\bar{A}})$ where $A = \mathrm{Diag}(i, 1, \dots, 1)$. We see that $M_{\phi_2} \neq I_{n^2}$ because $\phi_2(u_{22}^* u_{11}) = 0$. Replacing M_h by I_{n^2} and M_ϕ by $M_{\phi_2} \neq I_{n^2}$ in (7.1) yields to a contradiction.

Now, let us remark that ϕ_1 and ϕ_2 are both tracial, and consequently the proof allows also to conclude that there exists no Haar trace for the boolean convolution.

7.3.4 The monotone and the antimonotone case

In the proof of the nonexistence of a boolean Haar state, the only property of the boolean independence that we needed was

$$(h \star_B \phi)(abc) = \phi(a)h(b - \delta(b))\phi(c) + \delta(b)\phi(ac)$$

for a, c in the right leg and b in the left leg of $U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$. The monotone independence verifies this same property and we can thus deduce that there exists no monotone Haar state. On the contrary, the antimonotone case verifies $(h \star_{AM} \phi)(abc) = h(b)\phi(ac)$. Nevertheless, for x, z in the left leg and y in the right leg of $U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$, we have

$$(\phi \star_{AM} h)(xyz) = \phi(x)h(y - \delta(y))\phi(z) + \delta(y)\phi(xz).$$

We can then do the computation of the relation $h(u_{ij}u_{kl}^*) = (\phi \star_{AM} h)\Delta(u_{ij}u_{kl}^*)$ in the exact same way as before and we find that $M_h = (N_\phi \otimes \bar{N}_\phi)(M_h - I_{n^2}) + M_\phi$. We again find a contradiction by looking on the particular states ϕ_1 and ϕ_2 . To sum it up, for $n \geq 2$, there exists no monotone (resp. antimonotone) Haar state on $U\langle n\rangle$.

The same remark, about the traciality of the states used, allows us to conclude about the non-existence of a Haar trace.

7.3.5 The free Haar trace

In this section, we define the free Haar trace and prove that is is indeed an absorbing state for the free convolution on U_n^{nc} with other tracial states.

Let us first interpret the existence result of the free Haar trace on $U\langle n\rangle$ in a very concrete way as follows. Let us denote by h the Haar trace of $U\langle n\rangle$ for the free convolution, and by $u = (u_{i,j})_{1\leq i,j\leq n}$ the collection of generators of U_n^{nc}. Let $A = (a_{ij})_{1\leq i,j\leq n} \in \mathcal{M}_n(\mathcal{A})$ be a collection of random variables in $(\mathcal{A}, \phi)$ (ϕ tracial) such that $(a_{ij})_{1\leq i,j\leq n}$ is unitary. Setting $(b_{ij})_{1\leq i,j\leq n} = uA \in \mathcal{M}_n(\mathcal{A} \sqcup U_n^{\text{nc}})$ and $(c_{ij})_{1\leq i,j\leq n} = Au \in \mathcal{M}_n(\mathcal{A} \sqcup U_n^{\text{nc}})$, the collection $\{b_{ij}\}_{1\leq i,j\leq n}$ and $\{c_{ij}\}_{1\leq i,j\leq n}$ have both the same distribution as $\{u_{ij}\}_{1\leq i,j\leq n}$ in the noncommutative probability space $(\mathcal{A} \sqcup U_n^{\text{nc}}, \phi * h)$.

In order to define the state which will play the role of the Haar trace, we have to define a Haar unitary variable. A noncommutative variable U of a noncommutative probability space $(\mathcal{A}, \phi)$ is called *Haar unitary* if it is a unitary variable, and $\phi(U^k) = 0$ for all $k \geq 0$. Here is a description of its free cumulants.

Proposition 7.3.7 (Remark 3.4.3. of [43])**.** *Let U be a Haar unitary element on some noncommutative probability space. Then, for all $r \geq 1$ and $\epsilon_1, \ldots, \epsilon_r \in \{1, *\}$, we have:*

$$\kappa_r(U^{\epsilon_1}, \ldots, U^{\epsilon_r}) = \begin{cases} (-1)^{r/2-1}C_{r/2-1} & \text{if } r \text{ is even and the } \epsilon_i \text{ are alternating } (\epsilon_i \neq \epsilon_{i+1}) \\ 0 & \text{else,} \end{cases}$$

where $C_i = (2i)!/(i+1)!i!$ designate the Catalan numbers.

Let us consider a Haar unitary random variable U in $(\mathcal{A}, \phi)$ and construct from there a quantum variable $j_U : U_n^{\text{nc}} \to E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$ determined by $j_U(u_{ij}) = E_{1i}UE_{j1}$ for all $1 \leq i, j \leq n$ as indicated in Proposition-Definition 7.1.1. We will study the state $h = [n(\phi * \text{tr}_n)] \circ j_U$ on U_n^{nc}. We compute first the free cumulants of our variables u_{ij} and u_{ij}^*. In fact, for all $1 \leq i, j \leq n$, we denote by $(u^*)_{ij}$ the generator u_{ji}^*. The free cumulants of u_{ij} and $(u^*)_{ij}$ turn out to be more convenient than the free cumulants of u_{ij} and u_{ij}^*.

Corollary 7.3.8. *The free cumulants of $(u_{ij})_{1\leq i,j\leq n}$ and $((u^*)_{ij})_{1\leq i,j\leq n} = (u_{ji}^*)_{1\leq i,j\leq n}$ in the noncommutative probability space (U_n^{nc}, h) are given as follows.*

Let $1 \leq i_1, j_1, \ldots, i_r, j_r \leq n$ and $\epsilon_1, \ldots, \epsilon_r$ be either $\emptyset$ or $*$. If the indices are cyclic (i.e. if $j_{l-1} = i_l$ for $2 \leq l \leq q$ and $i_1 = j_r$), r is even and the ϵ_i are alternating, we have

$$\kappa_r\Big((u^{\epsilon_1})_{i_1j_1}, \ldots, (u^{\epsilon_r})_{i_rj_r}\Big) = n^{1-r}(-1)^{r/2-1}C_{r/2-1}.$$

If not, the left handside is equal to zero.

Proof. It suffices to apply Theorem 14.18 of [35], reminded in Proposition 7.2.1, to $U^{(1)} = U$ and $U^{(2)} = U^*$ in order to get the free cumulants of $j_U(u_{ij}) = U_{ij}$ and $j_U((u^*)_{ij}) = (U^*)_{ij}$. □

We will need another property of free cumulants. Let us first introduce new notation. For all $r \in \mathbb{N}$, $S \subset \{1, \ldots, r\}$, $\sigma \in NC(S)$, and $A_1, \ldots, A_r \in \mathcal{A}$, set

$$\begin{aligned}\phi_\sigma(A_1, \ldots, A_r) &= \prod_{\{i_1 \leq \ldots \leq i_k\} \in \sigma} \phi(A_{i_1} \cdots A_{i_k}),\\ \kappa_\sigma(A_1, \ldots, A_r) &= \prod_{\{i_1 \leq \ldots \leq i_k\} \in \sigma} \kappa_k(A_{i_1}, \ldots, A_{i_k}).\end{aligned} \tag{7.2}$$

Remark that, even if we write r variables on the left side, the right side only involves the variables which correspond to indices which are in $S \subset \{1, \ldots, r\}$.

Proposition 7.3.9. *Let $\{1, \ldots, r\} = E \cup F$ be a disjoint union of two subsets. We suppose that σ is a non-crossing partition on E. Then, for all $A_1, \ldots, A_n \in \mathcal{A}$, we have*

$$\sum_{\substack{\mu \in NC(F) s.t.\\ \mu \cup \sigma \in NC(r)}} \kappa_\mu(A_1, \ldots, A_r) = \phi_{K(\sigma)}(A_1, \ldots, A_r)$$

where $K(\sigma)$ is the biggest partition on F such that $\sigma \cup K(\sigma)$ is non-crossing.

Proof. Let us compute

$$\phi_{K(\sigma)}(A_1, \ldots, A_r) = \sum_{\substack{\mu \in NC(F)\\ \mu \preceq K(\sigma)}} \kappa_\mu(A_1, \ldots, A_r) = \sum_{\substack{\mu \in NC(F)\\ \mu \cup \sigma \in NC(r)}} \kappa_\mu(A_1, \ldots, A_r)$$

because, by definition of $K(\sigma)$, the set $\{\mu \in NC(F) : \mu \preceq K(\sigma)\}$ is in one-to-one correspondence with the set $\{\mu \in NC(F) : \mu \cup \sigma \in NC(r)\}$. □

We are now ready to prove that $h = [n(\phi * \mathrm{tr}_n)] \circ j_U$ is indeed a Haar trace for the free convolution.

Proof of Theorem 7.3.4 in the free case. Let ϕ be a tracial state on U_n^{nc}. Let $1 \leq i_1, j_1, \ldots, i_r, j_r \leq n$, let $\epsilon_1, \ldots, \epsilon_r$ be either $\emptyset$ or $*$ and set

$$m = (u^{\epsilon_1})_{i_1 j_1} \ldots (u^{\epsilon_r})_{i_r j_r}$$

where we recall that $((u)_{ij})_{1 \leq i,j \leq n} = (u_{ij})_{1 \leq i,j \leq n}$ and $((u^*)_{ij})_{1 \leq i,j \leq n} = (u^*_{ji})_{1 \leq i,j \leq n}$ by convention. Remark that we prefer to work with the word m instead of the word $u^{\epsilon_1}_{i_1 j_1} \ldots u^{\epsilon_r}_{i_r j_r}$, since the computations are easier.

Let us compute $(h \star_F \phi)(m)$. We have $(h \star_F \phi)(m) = (h * \phi) \circ \Delta(m)$ and

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u^{(1)}_{ik} u^{(2)}_{kj}, \text{ and } \Delta((u^*)_{ij}) = \Delta(u^*_{ji}) = \sum_{k=1}^{n} (u^*)^{(2)}_{ik} (u^*)^{(1)}_{kj},$$

where the exponent (1) and (2) indicate if the element is in the first leg of $U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$ or in the second leg. So, when computing $\Delta(m)$, we obtain something of the form $\sum_{k_1, \ldots, k_r} m_{k_1, \ldots, k_r}$ where $m_{k_1, \ldots, k_r}$ are words of length $2r$ of the form $(u^{\epsilon_1})_{i_1 k_1} (u^{\epsilon_1})_{k_1 j_1} \cdots (u^{\epsilon_r})_{i_r k_r} (u^{\epsilon_r})_{k_r j_r}$ with the generators coming from both legs of $U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$. More precisely, let us decompose $\{1,\dots,2r\} = S \cup T$ where S contains the positions of the generators which are in the first leg and T contains the positions of the generators which are in the second leg, according to

$$\begin{aligned} S &= \{2i-1 : 1 \le i \le n, \epsilon(i) = \emptyset\} \cup \{2i : 1 \le i \le n, \epsilon(i) = *\}, \\ T &= \{1,\dots,2r\} \setminus S. \end{aligned}$$

We can develop the computation using the freeness of the legs:

$$\begin{aligned} (h * \phi) \circ \Delta(m) &= (h * \phi)\left(\sum_{k_1,\dots,k_r} m_{k_1,\dots,k_r}\right) \\ &= \sum_{k_1,\dots,k_r} \sum_{\substack{\sigma \in NC(S),\ \mu \in NC(T) \\ \text{s.t. } \sigma \cup \mu \in NC(2r)}} \kappa_\sigma^h\Big((u^{\epsilon_1})_{i_1k_1}, (u^{\epsilon_1})_{k_1j_1}, \dots, (u^{\epsilon_r})_{i_rk_r}, (u^{\epsilon_r})_{k_rj_r}\Big) \\ &\qquad \cdot \kappa_\mu^\phi\Big((u^{\epsilon_1})_{i_1k_1}, (u^{\epsilon_1})_{k_1j_1}, \dots, (u^{\epsilon_r})_{i_rk_r}, (u^{\epsilon_r})_{k_rj_r}\Big), \end{aligned}$$

where we recall that, according to (7.2), the free cumulant $\kappa_\sigma^h(\cdots)$ only involves the variables which correspond to indices in S and $\kappa_\mu^\phi(\cdots)$ only involves the variables which correspond to indices in T.

Using Corollary 7.3.8, we know that, whenever the ϵ_i are alternating and the indices are cyclic within the blocks of $\sigma \in NC(S)$, the quantity $\kappa_\sigma^h((u^{\epsilon_1})_{i_1k_1}, (u^{\epsilon_1})_{k_1j_1}, \dots, (u^{\epsilon_r})_{i_rk_r}, (u^{\epsilon_r})_{k_rj_r})$ does not depend on the indices $k_1,\dots,k_r$. We denote it by κ_σ^h, and compute

$$\begin{aligned} &(h * \phi) \circ \Delta(m) \\ &= \sum_{\substack{\sigma \in NC(S),\ \mu \in NC(T) \\ \text{s.t. } \sigma \text{ alternates the } \epsilon_i \\ \text{and } \sigma \cup \mu \in NC(2r)}} \sum_{\substack{k_1,\dots,k_r \\ \text{s.t. the indices are cyclic} \\ \text{within each block of } \sigma}} \kappa_\sigma^h \cdot \kappa_\mu^\phi\Big((u^{\epsilon_1})_{i_1k_1}, (u^{\epsilon_1})_{k_1j_1}, \dots, (u^{\epsilon_r})_{i_rk_r}, (u^{\epsilon_r})_{k_rj_r}\Big) \end{aligned}$$

Thanks to Proposition 7.3.9, we can sum over μ and we obtain

$$\begin{aligned} &(h * \phi) \circ \Delta(m) \\ &= \sum_{\substack{\sigma \in NC(S) \\ \text{s.t. } \sigma \text{ alternates the } \epsilon_i}} \kappa_\sigma^h \cdot \sum_{\substack{k_1,\dots,k_r \\ \text{s.t. the indices are cyclic} \\ \text{within each block of } \sigma}} \phi_{K(\sigma)}\Big((u^{\epsilon_1})_{i_1k_1}, (u^{\epsilon_1})_{k_1j_1}, \dots, (u^{\epsilon_r})_{i_rk_r}, (u^{\epsilon_r})_{k_rj_r}\Big). \end{aligned} \tag{7.3}$$

So let us now examine equation (7.3) in greater details. Because the blocks of σ alternate the ϵ_i, the blocks of $K(\sigma)$ must also alternate the ϵ_i. One can convince himself on a few examples, but also find a full proof in Proposition 7.7. of [34]. Now one has to understand how the cyclicity of the indices $i_1k_1, k_1j_1, \dots, i_rk_r, k_rj_r$ in the blocks of σ is translated in terms of the blocks of $K(\sigma)$. For every block $B = \{r(1) \le \dots \le r(q)\}$, we say that $r(1)$ and $r(q)$ are opposites in B.

A condition $k_q = k_{q'}$ for some $1 \le q < q' \le r$ appears twice. Once in the case where $2q$ and $2q'-1$ are opposites in the same block of σ, which is equivalent to the fact that $2q-1$ and $2q'$ are consecutive in the same block of $K(\sigma)$ (it corresponds to the case $\epsilon_q = *$ and $\epsilon_{q'} = \emptyset$, see the Figure 7.1). The other case is when $2q-1$ and $2q'$ are consecutive in the same block of σ, which is equivalent to the fact that $2q$ and $2q'-1$ are opposites in the same block of $K(\sigma)$ (it corresponds to the case $\epsilon_q = \emptyset$ and $\epsilon_{q'} = *$, see Figure 7.2).

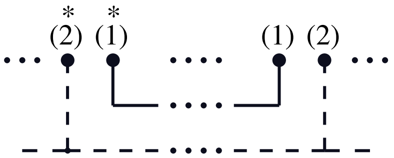


Figure 7.1: The case $k_q = k_{q'}$ in $\cdots (u^*)^{(2)}_{i_q k_q} (u^*)^{(1)}_{k_q j_q} \cdots u^{(1)}_{i_{q'} k_{q'}} u^{(2)}_{k_{q'} j_{q'}} \cdots$ (the continuous line represents σ while the dashed line represents $K(\sigma)$)

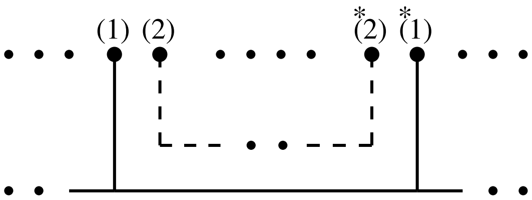


Figure 7.2: The case $k_q = k_{q'}$ in $\cdots u^{(1)}_{i_q k_q} u^{(2)}_{k_q j_q} \cdots (u^*)^{(2)}_{i_{q'} k_{q'}} (u^*)^{(1)}_{k_{q'} j_{q'}} \cdots$ (the continuous line represents σ while the dashed line represents $K(\sigma)$)

Now, let us consider one block $B = \{r(1) \leq \ldots \leq r(q)\} \in K(\sigma)$. If $\epsilon(r(1)) = *$, a case illustrated in Figure 7.3, we have

$$\begin{aligned}
&\phi_B\Big((u^{\epsilon_1})_{i_1 k_1}, (u^{\epsilon_1})_{k_1 j_1}, \ldots, (u^{\epsilon_r})_{i_r k_r}, (u^{\epsilon_r})_{k_r j_r}\Big) \\
&= \phi\Big((u^*)_{i_{r(1)} k_{r(1)}} u_{k_{r(2)} j_{r(2)}} \cdots (u^*)_{i_{r(q-1)} k_{r(q-1)}} u_{k_{r(q)} j_{r(q)}}\Big) \\
&= \phi\Big(u^*_{k_{r(1)} i_{r(1)}} u_{k_{r(2)} j_{r(2)}} \cdots u^*_{k_{r(q-1)} i_{r(q-1)}} u_{k_{r(q)} j_{r(q)}}\Big)
\end{aligned}$$

and summing over the indices $k_{r(1)} = k_{r(2)}, k_{r(3)} = k_{r(4)}, \cdots, k_{r(q-1)} = k_{r(q)}$ yields to

$$\delta_{i_{r(1)} j_{r(2)}} \ldots \delta_{i_{r(q-1)} j_{r(q)}}.$$

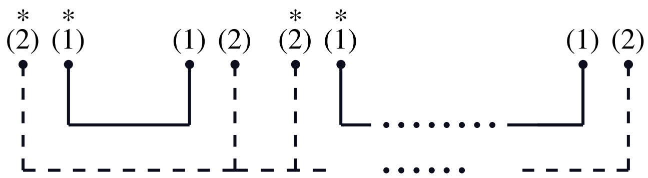


Figure 7.3: The case $k_{r(1)} = k_{r(2)}, k_{r(3)} = k_{r(4)}, \cdots, k_{r(q-1)} = k_{r(q)}$ in B (the continuous line represents σ while the dashed line represents $K(\sigma)$)

As well, if $\epsilon(r(1)) = \emptyset$, a case illustrated in Figure 7.4, we have

$$\begin{aligned}
&\phi_B\Big((u^{\epsilon_1})_{i_1 k_1}, (u^{\epsilon_1})_{k_1 j_1}, \ldots, (u^{\epsilon_r})_{i_r k_r}, (u^{\epsilon_r})_{k_r j_r}\Big) \\
&= \phi\Big(u_{k_{r(1)} j_{r(1)}} (u^*)_{i_{r(2)} k_{r(2)}} \cdots u_{k_{r(q-1)} j_{r(q-1)}} (u^*)_{i_{r(q)} k_{r(q)}}\Big) \\
&= \phi\Big(u_{k_{r(1)} j_{r(1)}} u^*_{k_{r(2)} i_{r(2)}} \cdots u_{k_{r(q-1)} j_{r(q-1)}} u^*_{k_{r(q)} i_{r(q)}}\Big)
\end{aligned}$$

and summing over the indices $k_{r(2)} = k_{r(3)}, \cdots, k_{r(q-2)} = k_{r(q-1)}$ yields to

$$\delta_{i_{r(q)}j_{r(1)}} \ldots \delta_{i_{r(q-2)}j_{r(q-1)}} \phi\Big(u_{k_{r(1)}j_{r(1)}} u^*_{k_{r(q)}i_{r(q)}}\Big).$$

Using the fact that ϕ is tracial, $\phi\Big(u_{k_{r(1)}j_{r(1)}} u^*_{k_{r(q)}i_{r(q)}}\Big) = \phi\Big(u^*_{k_{r(q)}i_{r(q)}} u_{k_{r(1)}j_{r(1)}}\Big)$. Thus we can also sum over $k_{r(1)}$ and get $\delta_{i_{r(q)}j_{r(1)}} \ldots \delta_{i_{r(q-2)}j_{r(q-1)}}$.

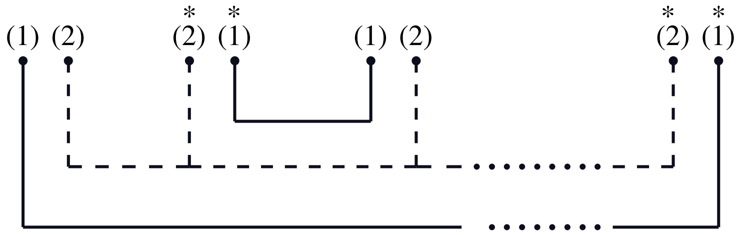


Figure 7.4: The case $k_{r(q)} = k_{r(1)}, k_{r(2)} = k_{r(3)}, \cdots, k_{r(q-2)} = k_{r(q-1)}$ in B (the continuous line represents σ while the dashed line represents $K(\sigma)$)

Those computations shows that the quantity $(h * \phi) \circ \Delta(m)$ expressed as (7.3) does not depend on the choice of ϕ, and in particular, we can replace ϕ by δ and obtain $(h * \phi) \circ \Delta(m) = (h * \delta) \circ \Delta(m)$. Since m is arbitrary, we have $(h * \phi) \circ \Delta = (h * \delta) \circ \Delta$. Now, let us remark that $h * \delta$ and $h \circ (\mathrm{Id} \sqcup \delta)$ are two unital linear functionals which vanish on products in $U_n^{\mathrm{nc}} \sqcup U_n^{\mathrm{nc}}$ which alternate elements from $\ker(h)$ in the first leg and elements from $\ker(\delta)$ in the second leg. As a consequence, we have $h * \delta = h \circ (\mathrm{Id} \sqcup \delta)$, and we can write $(h * \phi) \circ \Delta = (h * \delta) \circ \Delta = h \circ (\mathrm{Id} \sqcup \delta) \circ \Delta = h$. This prove that h is a Haar trace, thanks to Proposition 7.3.6. □

The free Haar state can be computed with the help of the following proposition, which is just a reformulation of Corollary 7.3.8.

Proposition 7.3.10. *When U_n^{nc} is endowed with its Haar trace for the free convolution, the free cumulants of $\{u_{ij}\}_{1 \le i,j \le n}$ are given as follows.*

Let $1 \le i_1, j_1, \ldots, i_r, j_r \le n$. We have

$$\begin{aligned} \kappa_r\Big(u_{i_1j_1}, u^*_{i_2j_1}, u_{i_2j_2}, u^*_{i_3j_2}, \ldots, u_{i_rj_r}, u^*_{i_1j_r}\Big) &= n^{1-2r}(-1)^{r-1}C_{r-1} \\ \text{and} \quad \kappa_r\Big(u^*_{i_1j_1}, u_{i_1j_2}, u^*_{i_2j_2}, u_{i_2j_3}, \ldots, u^*_{i_rj_r}, u_{i_rj_1}\Big) &= n^{1-2r}(-1)^{r-1}C_{r-1} \end{aligned}$$

where $C_r = (2r)!/(r+1)!r!$ designate the Catalan numbers. Moreover, the free cumulants which are not given in such a way are equal to 0.

In [31], Mc Clanahan defines a state on U_n^{nc} which is in fact equal to our free Haar trace. More precisely, let us denote by $C(\mathbb{U})$ the algebra of continuous functions on the unit complex circle $\mathbb{U}$ and by $\mathcal{M}_n(\mathbb{C})'$ the relative commutant of $\mathcal{M}_n(\mathbb{C})$ in $C(\mathbb{U}) \sqcup \mathcal{M}_n(\mathbb{C})$. It is straightforward to verify that there exists a unique $*$-homomorphism $\varphi : U_n^{\mathrm{nc}} \to \mathcal{M}_n(\mathbb{C})'$ such that

$$\varphi(u_{ij}) = \sum_{1 \le k \le n} E_{ki} \mathrm{Id}_{\mathbb{U}} E_{jk}.$$

Endowing $C(\mathbb{U})$ with the uniform measure λ on the unit circle gives us a state $(\lambda * \mathrm{tr}_n)_{|\mathcal{M}_n(\mathbb{C})'} \circ \varphi$ on U_n^{nc}.

Proposition 7.3.11. *The state $(\lambda * \mathrm{tr}_n)_{|\mathcal{M}_n(\mathbb{C})'} \circ \varphi$ of Mc Clanahan is the Haar trace for the free convolution on U_n^{nc}.*

Proof. Let us first observe the $*$-homomorphism of noncommutative probability spaces (where $\mathcal{A} = C(\mathbb{U})$ equipped with Haar measure):

$$\begin{array}{rcl}\tilde{\varphi} : \Big(E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}, n(\lambda * \mathrm{tr}_n)\Big) & \to & \Big(\mathcal{M}_n(\mathbb{C})', (\lambda * \mathrm{tr}_n)_{|\mathcal{M}_n(\mathbb{C})'}\Big) \\ A & \mapsto & \displaystyle\sum_{1 \le k \le n} E_{k1}AE_{1k}\end{array}$$

which follows from the equality $\lambda * \mathrm{tr}_n(\sum_k E_{k1}AE_{1k}) = n(\lambda * \mathrm{tr}_n(A))$ for all elements A of $E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$. Observe also that $\mathrm{Id}_{\mathbb{U}}$ is a Haar unitary element U of $(C(\mathbb{U}), \lambda)$. The result follows from the equality $\varphi = \tilde{\varphi} \circ j_U$ which shows that the state of Mc Clanahan $(\lambda * \mathrm{tr}_n)_{|\mathcal{M}_n(\mathbb{C})'} \circ \varphi$ is exactly the Haar trace $[n(\lambda * \mathrm{tr}_n)] \circ j_U = (\lambda * \mathrm{tr}_n)_{|\mathcal{M}_n(\mathbb{C})'} \circ \tilde{\varphi} \circ j_U$. ☐

Proposition 7.3.12. *The free Haar trace h is faithful for $n \geq 1$.*

Proof. Let $\mathcal{A}$ be the dual groups U_1^{nc} generated by one element U (also known as the space of Laurent polynomials $\mathbb{C}[U, U^{-1}]$) endowed with the uniform measure λ on the circle given by Section 7.3.1 (for all $k \in \mathbb{N}$, $\lambda(U^k) = \lambda(U^{*k}) = \delta_{k0}$), which is faithul. As a consequence of [35, Proposition 6.14], which says that the free product of faithful states is faithful, we know that $n(\phi * tr_n)$ is faithful on the $*$-algebra $\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C})$, thus also on $E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$.

We have $h = [n(\lambda * tr_n)] \circ j_U$ because U is a Haar unitary random variable in $(\mathcal{A}, \lambda)$. The faithfulness of h follows then from the injectivity of the map j_U, which can be seen in the following way. Let us define the homomorphism of $*$-algebras k_1 by

$$\begin{array}{rcl} k_1 : \mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}) & \to & U_n^{nc} \otimes \mathcal{M}_n(\mathbb{C}) \\ U & \mapsto & \displaystyle\sum_{ij} u_{ij} \otimes E_{ij} \\ E_{ij} & \mapsto & 1 \otimes E_{ij} \end{array}$$

and the $*$-linear map k_2 by

$$\begin{array}{rcl} k_2 : U_n^{nc} \otimes \mathcal{M}_n(\mathbb{C}) & \to & U_n^{nc} \\ w \otimes M & \mapsto & M(1,1) \cdot w. \end{array}$$

Let us prove that $k_2 \circ k_1 \circ j_U = \mathrm{Id}$. First, $k_1(E_{11}) = 1 \otimes E_{11}$, and, as a consequence,

$$k_1\Big(E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}\Big) = U_n^{nc} \otimes (\mathbb{C} \cdot E_{11}).$$

Of course, k_2 is a homomorphism of $*$-algebras on $U_n^{nc} \otimes (\mathbb{C} \cdot E_{11})$, which means that $k_2 \circ k_1$ is a homomorphism of $*$-algebras on $E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}$ and that $k_2 \circ k_1 \circ j_U$ is a homomorphism of $*$-algebras from U_n^{nc} to itself. Finally, for all $1 \leq i, j \leq n$, we compute

$$\begin{aligned} &k_2 \circ k_1 \circ j_U(u_{ij}) \\ &\quad = k_2 \circ k_1(E_{1i}UE_{j1}) = k_2\left[(1 \otimes E_{1i})(\sum_{pq} u_{pq} \otimes E_{pq})(1 \otimes E_{j1})\right] = k_2(u_{ij} \otimes E_{11}) = u_{ij}. \end{aligned}$$

Therefore, $k_2 \circ k_1 \circ j_U = \mathrm{Id}$, and consequently j_U is injective, which concludes the proof. ☐

7.3.6 The tensor Haar trace

In this section, we prove that there exists a tensor Haar trace.

Let us define the state which will be the tensor Haar trace. It is constructed via a very different method than the free Haar trace. We consider the Hilbert space $H = \ell^2(\mathbb{Z}) \otimes \bigotimes_{k\in\mathbb{Z}} \mathcal{M}_n(\mathbb{C})$, where $\ell^2(\mathbb{Z})$ is Hilbert space of square-summable families of complex numbers indexed by $\mathbb{Z}$ and $\bigotimes_{k\in\mathbb{Z}} \mathcal{M}_n(\mathbb{C})$ is the infinite tensor product of copies of the Hilbert space $\mathcal{M}_n(\mathbb{C})$, where the number of matrices different from I_n is finite and the scalar product on $\mathcal{M}_n(\mathbb{C})$ is given by $\mathrm{tr}_n(A^*B) = \mathrm{Tr}(A^*B)/n$.

For all $1 \leq i,j \leq n$, we define the following bounded operator on H by setting, for all $\delta_k \otimes \bigotimes_{l\in\mathbb{Z}} M_l \in H$,

$$U_{ij}(\delta_k \otimes (\ldots \otimes M_{k-1} \otimes M_k \otimes M_{k+1} \otimes \ldots)) = \delta_{k+1} \otimes (\ldots \otimes M_{k-1} \otimes E_{ji}M_k \otimes M_{k+1} \otimes \ldots)$$

and therefore its adjoint, given by

$$U_{ij}^*(\delta_k \otimes (\ldots \otimes M_{k-1} \otimes M_k \otimes M_{k+1} \otimes \ldots)) = \delta_{k-1} \otimes (\ldots \otimes E_{ij}M_{k-1} \otimes M_k \otimes M_{k+1} \otimes \ldots).$$

We introduce $\Omega = \delta_0 \otimes \bigotimes_{k\in\mathbb{Z}} I_n$ and the state on the algebra $B(H)$ of bounded operators on H given by $A \mapsto \langle \Omega, A\Omega\rangle$. The operators U_{ij} verify that $\sum_{k=1}^n U_{ki}^* U_{kj} = \delta_{ij} = \sum_{k=1}^n U_{ik}U_{jk}^*$ and so the quantum random variable over $j : U_n^{\mathrm{nc}} \ni u_{ij} \mapsto U_{ij} \in B(H)$ is well-defined. It induces a state h on $U\langle n\rangle$, given for all $a \in U_n^{\mathrm{nc}}$ by

$$h(a) = \langle \Omega, j(a)\Omega\rangle.$$

Let us compute first the value of h, thanks to the following lemmas.

Lemma 7.3.13. *For all* $1 \leq i_1, j_1, \ldots, i_r, j_r \leq n$, *we have*

$$\begin{aligned} h(u_{i_1j_1}u_{i_2j_2}^* \ldots u_{i_{r-1}j_{r-1}}u_{i_rj_r}^*) &= \frac{1}{n}\delta_{i_1i_2}\delta_{j_2j_3}\delta_{i_3i_4}\ldots\delta_{i_{r-1}i_r}\delta_{j_rj_1}, \\ h(u_{i_1j_1}^*u_{i_2j_2} \ldots u_{i_{r-1}j_{r-1}}^*u_{i_rj_r}) &= \frac{1}{n}\delta_{j_1j_2}\delta_{i_2i_3}\delta_{j_3j_4}\ldots\delta_{j_{r-1}j_r}\delta_{i_ri_1}. \end{aligned}$$

Proof. We have

$$U_{i_1j_1}U_{i_2j_2}^* \ldots U_{i_{r-1}j_{r-1}}U_{i_rj_r}^*(\Omega) = \delta_0 \otimes (\cdots \otimes I_n \otimes \underbrace{E_{j_1i_1}E_{i_2j_2}\ldots E_{j_{r-1}i_{r-1}}E_{i_rj_r}}_{\text{at the level } -1} \otimes I_n \otimes \cdots),$$

$$U_{i_1j_1}^*U_{i_2j_2} \ldots U_{i_{r-1}j_{r-1}}^*U_{i_rj_r}(\Omega) = \delta_0 \otimes (\cdots \otimes I_n \otimes \underbrace{E_{i_1j_1}E_{j_2i_2}\ldots E_{i_{r-1}j_{r-1}}E_{j_ri_r}}_{\text{at the level } 0} \otimes I_n \otimes \cdots),$$

which yields the first and the second result. ☐

For more general words, it is possible to reduce them and fit into the previous case. Fix $1 \leq i_1, j_1, \ldots, i_r, j_r \leq n$, $\epsilon_1, \ldots, \epsilon_r \in \{\emptyset, *\}$, and consider the word $u_{i_1j_1}^{\epsilon_1} \ldots u_{i_rj_r}^{\epsilon_r}$. We can decompose $\{1, \ldots, r\}$ into $\bigcup_{k=-r}^r S_k$, where

$$S_k = \Big\{ l \in \{1, \ldots, r\} : k = \sharp\{m > l : \epsilon_m = \emptyset\} - \sharp\{m \geq l : \epsilon_m = *\}\Big\}. \tag{7.4}$$

If we assume that $\emptyset$ corresponds to a North step, $*$ to a South step, and consider the path given by $\epsilon_r, \ldots, \epsilon_1$, the set S_k contains the positions where the path goes from the level k to the level $k+1$, or from the level $k+1$ to the level k. Consequently, the S_k form a partition of $\{1, \ldots, r\}$, and the ϵ_m are alternating inside each S_k.

Lemma 7.3.14. *Let $1 \leq i_1, j_1, \dots, i_r, j_r \leq n$ and $\epsilon_1, \dots, \epsilon_r$ be either $\emptyset$ or $*$.*

If $\sharp\{m : \epsilon_m = *\} \neq \sharp\{m : \epsilon_m = 1\}$, then $h(u^{\epsilon_1}_{i_1 j_1} \dots u^{\epsilon_r}_{i_r j_r}) = 0$.

If $\sharp\{m : \epsilon_m = *\} = \sharp\{m : \epsilon_m = 1\}$, then

$$h(u^{\epsilon_1}_{i_1 j_1} \dots u^{\epsilon_r}_{i_r j_r}) = \prod_{k=-r}^{r} h\left(\overrightarrow{\prod_{l \in S_k}} u^{\epsilon_l}_{i_l j_l}\right).$$

This lemma combined with Lemma 7.3.13 describes entirely the state h.

Proof. Let us prove by decreasing induction on l that, for all $1 \leq l \leq r$, and $l \in S_k$, we have

$$\begin{aligned} U^{\epsilon_l}_{i_l j_l} \dots U^{\epsilon_r}_{i_r j_r}(\Omega) = &\quad \delta_{k+1} \otimes \bigotimes_{p \in \mathbb{Z}} \left(\overrightarrow{\prod_{q \in S_p \cap \{l, \dots, r\}}} E^{\epsilon_q}_{j_q i_q} \right) \quad \text{if } \epsilon_l = 1, \\ \text{and } U^{\epsilon_l}_{i_l j_l} \dots U^{\epsilon_r}_{i_r j_r}(\Omega) = &\quad \delta_{k} \otimes \bigotimes_{p \in \mathbb{Z}} \left(\overrightarrow{\prod_{q \in S_p \cap \{l, \dots, r\}}} E^{\epsilon_q}_{j_q i_q} \right) \quad \text{if } \epsilon_l = *. \end{aligned}$$

First of all, we have $U_{i_r j_r}(\Omega) = \delta_1 \otimes (\dots \otimes E_{j_r i_r} \otimes \dots)$ with the non-identity matrix at level 0 and $U^*_{i_r j_r}(\Omega) = \delta_{-1} \otimes (\dots \otimes E_{i_r j_r} \otimes \dots)$ where the non-identity matrix is at level -1. Thus the property is true for $l = r$.

Fix now $1 \leq l < r$ and assume that the property is true for $l + 1$. Suppose first that $\epsilon_l = \epsilon_{l+1} = *$, and denote by k the integer such that $l + 1 \in S_k$, then $l \in S_{k-1}$ and:

$$\begin{aligned} U^{\epsilon_l}_{i_l j_l} \dots U^{\epsilon_r}_{i_r j_r}(\Omega) &= U^{\epsilon_l}_{i_l j_l} \left[\delta_k \otimes \bigotimes_{p \in \mathbb{Z}} \left(\overrightarrow{\prod_{q \in S_p \cap \{l, \dots, r\}}} E^{\epsilon_q}_{j_q i_q} \right) \right] \\ &= \delta_{k-1} \otimes \bigotimes_{p \in \mathbb{Z}} \left(\overrightarrow{\prod_{q \in S_p \cap \{l+1, \dots, r\}}} E^{\epsilon_q}_{j_q i_q} \right). \end{aligned}$$

The other cases (i.e., $\epsilon_l = \epsilon_{l+1} = \emptyset$, $\epsilon_l = *, \epsilon_{l+1} = \emptyset$ and $\epsilon_{l+1} = *, \epsilon_l = \emptyset$) are treated in the exact same way. Therefore, the property is true for every $l \in \{1, \dots, r\}$.

Finally, $h(u^{\epsilon_1}_{i_1 j_1} \dots u^{\epsilon_r}_{i_r j_r}) = \langle \Omega, U^{\epsilon_1}_{i_1 j_1} \dots U^{\epsilon_r}_{i_r j_r}(\Omega) \rangle$ is exactly as expected. □

We are now ready to prove that h is indeed the Haar trace for the tensor convolution. Thanks to Proposition 7.3.6, it is a consequence of the following proposition.

Proposition 7.3.15. *The state h is tracial, and for all other tracial states ϕ, we have $h \star_T \phi = h$.*

Proof. Firstly, h is tracial. Indeed, let us fix $1 \leq i_1, j_1, \dots, i_q, j_q \leq n$, $\epsilon_1, \dots, \epsilon_q \in \{\emptyset, *\}$ and compare $h(u^{\epsilon_1}_{i_1 j_1} \dots u^{\epsilon_r}_{i_r j_r})$ with $h(u^{\epsilon_r}_{i_r j_r} u^{\epsilon_1}_{i_1 j_1} \dots u^{\epsilon_{r-1}}_{i_{r-1} j_{r-1}})$. Thanks to Lemma 7.3.13, if the ϵ_i are alternating, we are done. If not, remark that acting by a cyclic permutation just shifts the S_k's. Thus, up to a cyclic permutation, the decomposition in the S_k's is the same for $u^{\epsilon_1}_{i_1 j_1} \dots u^{\epsilon_r}_{i_r j_r}$ and $u^{\epsilon_r}_{i_r j_r} u^{\epsilon_1}_{i_1 j_1} \dots u^{\epsilon_{r-1}}_{i_{r-1} j_{r-1}}$. Consequently, by Lemma 7.3.14, the full traciality is a consequence of the traciality for words alternating the ϵ_i's.

Now, let us prove that $h\star_T\phi = h$. Equivalently, we will prove that, for all $1 \le i_1, j_1, \dots, i_r, j_r \le n$ and $\epsilon_1, \dots, \epsilon_r \in \{\emptyset, *\}$,

$$h \star_T \phi(u_{i_1j_1}^{\epsilon_1} \dots u_{i_rj_r}^{\epsilon_r}) = \sum_{k_1,\dots,k_r=1}^{n} h(u_{i_1k_1}^{\epsilon_1} \dots u_{i_rk_r}^{\epsilon_r})\phi(u_{k_1j_1}^{\epsilon_1} \dots u_{k_rj_r}^{\epsilon_r}) \tag{7.5}$$

is equal to $h(u_{i_1j_1}^{\epsilon_1} \dots u_{i_rj_r}^{\epsilon_r})$. If $\sharp\{m : \epsilon_m = *\} \neq \sharp\{m : \epsilon_m = 1\}$, this is a direct consequence of Lemma 7.3.14. If not, let us prove the result by induction on the even length $r = 2q$ of the word.

Remark that $h\star_T \phi(1) = h(1) = 1$. Fix $q > 0$ and suppose that the result is true for words of length less than $2q$. Let us prove that the result is true for words $u_{i_1j_1}^{\epsilon_1} \dots u_{i_rj_r}^{\epsilon_r}$ of length $r = 2q$ such that $\sharp\{m : \epsilon_m = *\} = \sharp\{m : \epsilon_m = 1\}$.

Fix $1 \le i_1, j_1, \dots, i_r, j_r \le n$ and $\epsilon_1, \dots, \epsilon_r \in \{\emptyset, *\}$. Consider $k_0 = \min\{k : S_k \neq \emptyset\}$. For all $q \in S_{k_0} \setminus \{1\}$ such that $\epsilon_q = \emptyset$, we must have $\epsilon_{q-1} = *$ and consequently $q-1 \in S_{k_0}$ (indeed, if $\epsilon_{q-1} = \emptyset$, then $q-1 \in S_{k_0+1}$ and k_0 is not minimal). Moreover, if $1 \in S_{k_0}$ and $\epsilon_1 = \emptyset$, we must have $\epsilon_r = *$ and consequently $r \in S_{k_0}$ (indeed, in this case, $\epsilon_1 = \emptyset$ implies that $1 \in S_{-1}$ and $k_0 = -1$, and if $\epsilon_r = \emptyset$, then $r \in S_0$ and -1 is not minimal). To sum up, S_{k_0} can be written in the form

$$\{r(1) \le r(1)+1 \le r(2) \le r(2)+1 \le \dots \le r(q) \le r(q)+1\}$$

if the first element is labelled by $*$, and in the form

$$\{1 \le r(1) \le r(1)+1 \le \dots \le r(q) \le r(q)+1, r\}$$

if the first element is labelled by $\emptyset$.

In the case where the first element is labelled by $*$, let us decompose $S_{k_0} = \{r(1) \le r(1)+1 \le \dots \le r(q) \le r(q)+1\}$. Set $r(q+1) = r(1)$, and compute, thanks to Lemmas 7.3.13 and 7.3.14,

$$\begin{aligned} h \star_T \phi(u_{i_1j_1}^{\epsilon_1} \dots u_{i_rj_r}^{\epsilon_r}) &= \sum_{k_1,\dots,k_r=1}^{n} h(u_{i_1k_1}^{\epsilon_1} \dots u_{i_rk_r}^{\epsilon_r})\phi(u_{k_1j_1}^{\epsilon_1} \dots u_{k_rj_r}^{\epsilon_r}) \\ = \sum_{k_1,\dots,k_r=1}^{n} h\left(\overrightarrow{\prod_{l\notin S_{k_0}}} u_{i_lk_l}^{\epsilon_l}\right) &\frac{1}{n}\prod_{l=1}^{q}(\delta_{k_{r(l)}k_{r(l)+1}}\delta_{i_{r(l)+1}i_{r(l+1)}}) \cdot \phi(u_{k_1j_1}^{\epsilon_1} \dots u_{k_rj_r}^{\epsilon_r}). \end{aligned}$$

Summing over the indices $k_{r(1)} = k_{r(1)+1}, k_{r(2)} = k_{r(2)+1}, \cdots, k_{r(q)} = k_{r(q)+1}$ in ϕ and use the induction hypothesis yields to

$$\begin{aligned} &\sum_{\substack{l\notin S_{k_0} \\ 1\le k_l\le n}} h\left(\overrightarrow{\prod_{l\notin S_{k_0}}} u_{i_lk_l}^{\epsilon_l}\right)\phi\left(\overrightarrow{\prod_{l\notin S_{k_0}}} u_{k_lj_l}^{\epsilon_l}\right)\frac{1}{n}\prod_{l=1}^{q}(\delta_{j_{r(l)}j_{r(l)+1}}\delta_{i_{r(l)+1}i_{r(l+1)}}) \\ &= h\star_T\phi\left(\overrightarrow{\prod_{l\notin S_{k_0}}} u_{i_lj_l}^{\epsilon_l}\right)\frac{1}{n}\prod_{l=1}^{q}(\delta_{j_{r(l)}j_{r(l)+1}}\delta_{i_{r(l)+1}i_{r(l+1)}}) \\ &= h\left(\overrightarrow{\prod_{l\notin S_{k_0}}} u_{i_lj_l}^{\epsilon_l}\right)\frac{1}{n}\prod_{l=1}^{q}(\delta_{j_{r(l)}j_{r(l)+1}}\delta_{i_{r(l)+1}i_{r(l+1)}}) \\ &= h(u_{i_1j_1}^{\epsilon_1} \dots u_{i_rj_r}^{\epsilon_r}). \end{aligned}$$

In the case where the first element is labelled by $\emptyset$, we decompose $S_{k_0} = \{1 \le r(1) \le r(1)+1 \le \cdots \le r(q) \le r(q)+1 \le r\}$. The previous computation can be written as well, with a needed shift which has to be done in order to sum over the index $k_1 = k_r$:

$$\sum_{k_1=1}^{n} \phi(u_{k_1j_1}u_{k_2j_2}^{\epsilon_2}\dots u_{k_{r-1}j_{r-1}}^{\epsilon_{r-1}}u_{k_rj_r}^{*})$$
$$= \sum_{k_1=1}^{n} \phi(u_{k_rj_r}^{*}u_{k_1j_1}u_{k_2j_2}^{\epsilon_2}\dots u_{k_{r-1}j_{r-1}}^{\epsilon_{r-1}}) = \delta_{j_1,j_r}\phi(u_{k_2j_2}^{\epsilon_2}\dots u_{k_{r-1}j_{r-1}}^{\epsilon_{r-1}}).$$

Finally, we always have $h \star_T \phi(u_{i_1j_1}^{\epsilon_1}\dots u_{i_rj_r}^{\epsilon_r}) = h(u_{i_1j_1}^{\epsilon_1}\dots u_{i_rj_r}^{\epsilon_r})$ and the proof is done. ☐

Proposition 7.3.16. *The tensor Haar trace h is not faithful for $n \ge 2$.*

Proof. Set $X = u_{21}^*u_{11}$. Then, $h\left(X^2(X^2)^*\right) = 0$ thanks to Lemma 7.3.13. But, on the other hand, X^2 is nonzero. This can be seen for instance via the representation $\pi : U_n^{nc} \to \mathbb{C}$ given by

$$\Big(\pi(u_{ij})\Big)_{1\le i,j\le n} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix},$$

which is a well-defined representation of $U\langle n\rangle$ because the matrix $(\pi(u_{ij}))_{i,j=0}^n$ is unitary. We have $\pi(X^2) = 1/4 \neq 0$, which implies that X^2 is nonzero. ☐

7.4 Random matrix and Haar traces

In this section we investigate the relationship between Haar traces and matrices.

7.4.1 Matrix models

In this section, we define a model of random matrices which converges to the free Haar trace defined in Section 7.3.

Let us fix an arbitrary set I of indices. Let $(M_i)_{i\in I}$ be a family of random variables in some non-commutative space $(\mathcal{A}, \phi)$. For each $N \in \mathbb{N}$, let $(M_i^{(N)})_{i\in I}$ be a family of random $N \times N$ matrices. We will say that $(M_i^{(N)})_{i\in I}$ *converges almost surely in $*$-distribution* to $(M_i)_{i\in I}$ as N tends to ∞ if for all noncommutative polynomials $P \in \mathbb{C}\langle X_i, X_i^* : i \in I\rangle$ we have almost surely the following convergence:

$$\lim_{N\to\infty} \mathrm{tr}_N\left(P(M_i^{(N)})\right) = \phi\left(P(M_i)\right),$$

where we recall that tr_N is the normalized trace.

The following theorem, whose first version is due to Voiculescu [48], is a well-known phenomenon which makes freeness appear from independence and invariance by unitary conjugation. (see also [15, 29, 35, 46]).

Theorem 7.4.1 (Theorem 23.14 of [35])**.** *Let I and J be two arbitrary set of indices. Let $(A_i)_{i\in I}$ be a family of random variables in $(\mathcal{A}, \phi)$ and $(B_j)_{i\in J}$ be a family of random variables in $(\mathcal{B}, \tau)$. We suppose that*

1. *For each $N \in \mathbb{N}$, $\{A_i^{(N)}\}_{i\in I}$ is a family of random $N \times N$ matrices which converges almost surely in $*$-distribution to $\{A_i\}_{i\in I}$ as N tends to ∞.*

2. *For each $N \in \mathbb{N}$, $\{B_j^{(N)}\}_{i\in J}$ is a family of constant $N \times N$ matrices which converges almost surely in $*$-distribution to $\{B_j\}_{j\in J}$ as N tends to ∞.*

3. *The law of $\{A_i^{(N)}\}_{i\in I}$ is invariant by unitary conjugation, i.e. it is equal to the law of $\{UM_i^{(N)}U^*\}_{i\in I}$ for all $U \in \mathcal{M}_N(\mathbb{C})$ which is unitary.*

Then the matrices $\{A_i^{(N)}\}_{i\in I} \cup \{B_j^{(N)}\}_{i\in J}$ converge almost surely in $*$-distribution together to $\{A_i\}_{i\in I} \cup \{B_j\}_{i\in J}$ seen as elements of $(\mathcal{A} \sqcup \mathcal{B}, \phi * \tau)$ as N tends to ∞

For a matrix $M \in \mathcal{M}_{nN}(\mathbb{C})$ and $1 \leq i, j \leq n$, we denote by $[M]_{ij}$ the (i, j)-block of M when it is divided in n^2 matrices of size $N \times N$.

Corollary 7.4.2. *Let I be an arbitrary set of indices. Let $(A_k)_{k\in K}$ be a family of random variables in $(\mathcal{A}, \phi)$. For each $N \in \mathbb{N}$, let $\{A_k^{(N)}\}_{k\in K}$ be a family of random $N \times N$ matrices which converges almost surely in $*$-distribution to $\{A_k\}_{k\in K}$ as N tends to ∞ and whose law is invariant by unitary conjugation.*

Then, the family of block matrices $\{[A_k^{(nN)}]_{ij}\}_{k\in K,1\leq i,j\leq n}$ converges almost surely in $*$-distribution to $\{E_{1i}A_kE_{j1}\}_{k\in K,1\leq i,j\leq n}$ seen as an element of $(E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}, n\ \phi * \mathrm{tr}_n)$ when N tends to ∞.

Let us remark that the invariance of the law by unitary conjugation is not very restrictive. Indeed, if the law of $\{A_k^{(N)}\}_{k\in K}$ is not invariant by unitary conjugation, we can replace the family $\{A_k^{(N)}\}_{k\in K}$ by the family $\{UA_k^{(N)}U^*\}_{k\in K}$, where U is a uniform unitary random matrix of $\mathcal{M}_N(\mathbb{C})$ independent from $\{A_k^{(N)}\}_{k\in K}$.

Proof. First, remark that the family of constant $nN \times nN$ matrices $\{P_{ij}^{(N)}\}_{1\leq i,j\leq n}$ defined by the block matrices $[P_{ij}^{(N)}]_{lm} = \delta_{il}\delta_{jm}I_N$ (the block of $P_{ij}^{(N)}$ are zero except the (i, j)-th block which is I_N) converges to $\{E_{ij}\}_{1\leq i,j\leq n} \subset \mathcal{M}_n(\mathbb{C})$ as N tends to ∞. Using Theorem 7.4.1, the family $\{P_{1i}^{(N)}A_k^{(nN)}P_{j1}^{(N)}\}_{k\in K,1\leq i,j\leq n}$ converges to $\{E_{1i}A_kE_{j1}\}_{k\in K,1\leq i,j\leq n}$ seen as an element of $(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}), \phi * \mathrm{tr}_n)$ when N tends to ∞.

But let us remark that $P_{1i}^{(N)}A_k^{(nN)}P_{j1}^{(N)} = \begin{pmatrix} [A_k^{(nN)}]_{ij} & 0 \\ 0 & 0 \end{pmatrix}$. Consequently, the morphism of algebra from $\mathcal{M}_N(\mathbb{C})$ to $\mathcal{M}_{nN}(\mathbb{C})$ given by $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$ and the previous convergence implies the convergence of $\{[A_k^{(nN)}]_{ij}\}_{k\in K,1\leq i,j\leq n}$ as N tends to ∞. However, one has to be careful that the trace tr_N is transformed via this map into the linear functional $n\,\mathrm{tr}_{nN}$, and that consequently the family $\{[A_k^{(nN)}]_{ij}\}_{k\in K,1\leq i,j\leq n}$ converges to $\{E_{1i}A_kE_{j1}\}_{k\in K,1\leq i,j\leq n}$ seen as elements of $\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C})$ endowed with the linear functional $n(\phi * \mathrm{tr}_n)$, or equivalently, seen as elements of the noncommutative probability space $(E_{11}(\mathcal{A} \sqcup \mathcal{M}_n(\mathbb{C}))E_{11}, n(\phi * \mathrm{tr}_n))$. ☐

A *Haar unitary matrix* on the unitary group $U(N) = \{M \in \mathcal{M}_N(\mathbb{C}) : U^*U = I_N\}$ is a uniformly distributed unitary matrix $U^{(N)}$, or equivalently a random unitary matrix $U^{(N)}$ which is equal in law to $VU^{(N)}$ and $U^{(N)}V$ for every unitary matrix V.

Theorem 7.4.3. *Let us consider* $(u_{ij})_{1\le i,j\le n}$*, the generators of the non-commutative space* $U\langle n\rangle$ *endowed with its free Haar trace. For all* $N\ge 1$*, let* $U^{(N)}$ *be a Haar unitary matrix on the classical unitary group* $U(N)$*.*

Then, the matrices $\left([U^{(nN)}]_{ij}\right)_{1\le i,j\le n}$ *converge almost surely in* $*$*-distribution to* $(u_{ij})_{1\le i,j\le n}$ *when* N *tends to* ∞*.*

Proof. Setting $\{U_k^{(N)}\}_{k\in K}=\{U^{(N)},U^{(N)*}\}$ (with $K=\{1,2\}$), it is a direct consequence of Corollary 7.4.2. Indeed, $U^{(N)}$ converge almost surely to a Haar unitary random variable U, and the Haar trace is given by $[n(\phi * \mathrm{tr}_n)]\circ j_U)$. $\square$

7.4.2 Back to the Brownian motion on $U\langle n\rangle$

In Chapter 5, we have shown that the free quantum Lévy process that was considered there satisfied the gaussianity property and was therefore a nice candidate to be called a Brownian motion on $U\langle n\rangle$. We show in this section that it converges, when times goes to infinity, toward the free Haar trace, thus providing one more argument to call it a Brownian motion.

Proposition 7.4.4. *Let* $(J_t)_{t\ge 0}$ *be the Lévy process on* $U\langle n\rangle$ *defined in Chapter 5. Then, when* t *goes to infinity, the distribution of* $(J_t)_{t\ge 0}$ *converges towards the free Haar trace.*

Proof. Let $(U_t)_{t\ge 0}$ be a free multiplicative Brownian motion in a non-commutative probability space $(\mathcal{A},\Phi)$. Then, $(J_t)_{t\ge 0}$ is equal in distribution to $j_t: U\langle n\rangle\to E_{11}(\mathcal{A}\sqcup\mathcal{M}_n(\mathbb{C}))E_{11}$ defined by setting, for all $1\le i,j\le n$, $j_t(u_{ij})=E_{1i}U_tE_{j1}$.

It is well-known that $(U_t)_{t\ge 0}$ converges in $*$-distribution to a Haar unitary variable U as t tends to ∞. Indeed, there is an explicit description of the moments of U_t in [5], namely

$$\tau(U_t^k)=e^{-\frac{kt}{2}}\sum_{i=0}^{k-1}(-1)^i\frac{t^i}{i!}k^{i-1}\binom{k}{i+1},\quad k\ge 1,$$

and they converge to zero, which are the moments of a Haar unitary variable U. As a consequence, $(j_t(u_{ij}))_{1\le i,j\le n}$ converges in $*$-distribution to $(E_{1i}UE_{j1})_{1\le i,j\le n}$ as t tends to ∞, where $(E_{ij})_{1\le i,j\le n}$ are free from U. But $u_{ij}\mapsto E_{1i}UE_{j1}$ is a quantum random variable whose distribution is the free Haar trace (see Section 7.3.5). Consequently, $(j_t)_{t\ge 0}$ converge in distribution to the free Haar trace, and so do $(J_t)_{t\ge 0}$. $\square$

Erklärung

Hiermit erkläre ich, dass diese Arbeit – **außer am Laboratoire de Mathématiques de Besançon (Université de Franche-Comté) im Rahmen der Cotutelle** – bisher von mir weder an der Mathematisch-Natur-wissenschaftlichen Fakultät der Ernst-Moritz-Arndt-Universität Greifswald noch einer anderen wissenschaftlichen Einrichtung zum Zwecke der Promotion eingereicht wurde.
Ferner erkläre ich, dass ich diese Arbeit selbständig verfasst und keine anderen als die darin angegebenen Hilfsmittel und Hilfen benutzt und keine Textabschnitte eines Dritten ohne Kennzeichnung übernommen habe.

Résumé

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Publications

They are referenced in the bibliography under [44, 11, 19]. They are also given here for completeness' sake:

- M. Ulrich, Construction of a Free Lévy Process as high-dimensional limit of a Brownian Motion on the Unitary Group, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, Vol. 18, No. 3 (2015)
- G. Cébron and M. Ulrich, Haar states and Lévy processes on the unitary dual group, *Journal of Functional Analysis*, 2016
- U. Franz, G. Hong, F. Lemeux and M. Ulrich, Hypercontractivity of heat semigroups on free quantum groups, *Preprint. Arxiv: 1511.02753*, 2015

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2015	*How to define Haar states on Dual Groups*, Conference "Quantum Probability, Groups and Geometry", Warsaw

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