DOCTORAL THESIS

Study of compact quantum groups with probabilistic methods: characterization of ergodic actions and quantum analogue of Noether's isomorphisms theorems

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submitted by

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Résumé

Cette thèse étudie des problèmes liés aux treillis des sous-groupes quantiques et la caractérisation des actions ergodiques et des états idempotents d'un groupe quantique compact. Elle consiste en 3 parties. La première partie présente des résultats préliminaires sur les groupes quantiques localement compacts, les sous-groupes quantiques normaux ainsi que les actions ergodiques et les états idempotents. La seconde partie étudie l'analogue quantique de la règle de modularité de Dedekind et de l'analogue quantique des théorèmes d'isomorphisme de Noether ainsi que leur conséquences comme le théorème de raffinement de Schreier, et le théorème Jordan-Hölder. Cette partie s'inspire du travail de recherche de Shuzhou Wang sur l'analogue quantique du troisième théorème d'isomorphisme de Noether pour les groupes quantiques compacts ainsi que le travail récent de Kasprzak, Khosravi et Soltan sur l'analogue quantique du premier théorème d'isomorphisme de Noether pour les groupes quantiques localement compacts. Dans la troisième partie, nous caractérisons les états idempotents du groupe quantique compact $O_{-1}(2)$ en s'appuyant sur la caractérisation de ses actions ergodiques plongeables. Cette troisième partie est dans la ligne des travaux fait par Franz, Skalski et Tomatsu pour les groupes quantiques compacts $\mathrm{U}_q(2)$, $\mathrm{SU}_q(2)$ et $\mathrm{SO}_q(3)$. Nous classifions au préalable les actions ergodiques et les actions ergodiques plongeables du groupe quantique compact $O_{-1}(2)$.

Les travaux présentés dans cette thèse se basent sur deux articles de l'auteur et al. Le premier s'intitule "Fundamental isomorphism theorems for quantum groups" et a été accepté pour publication dans *Expositionae Mathematicae* et le second est intitulé "Ergodic actions and idempotent states of $O_{-1}(2)$" et est en cours de finalisation pour être soumis.

Mots-clefs

Groupe quantique localement compact, groupe quantique discret, groupe quantique linéairement reductif, lemme de Zassenhaus, théorème de raffinement de Schreier, théorème de Jordan-Hölder, groupe quantique compact, action ergodique, état idempotent, règle de modularité de Dedekind, théorèmes d'isomorphismes de Noether.

Abstract

This thesis studies problems linked to the lattice of quantum subgroups and characterization of ergodic actions and idempotent states of a compact quantum group. It consists of three parts. The first part present some preliminary results about locally compact quantum groups, normal quantum subgroups, ergodic actions and idempotent states. The second part studies the quantum analog of Dedekind's modularity law, Noether's isomorphism theorem and their consequences as the Schreier refinement theorem and the Jordan-Hölder theorem. This part completes the work of Shuzhou WANG on the quantum analog of the third isomorphism theorem for compact quantum group and the recent work of Kasprzak, Khosravi and Soltan on the quantum analog of the first Noether isomorphism theorem for locally compact quantum groups. In the third part, we characterize idempotent states of the compact quantum group $O_{-1}(2)$ relying on the characterization of embeddable ergodic actions. This third part is in the sequence of the seminal works of Franz, Skalski and Tomatsu for the compact quantum groups $\mathrm{U}_q(2)$, $\mathrm{SU}_q(2)$ and $\mathrm{SO}_q(3)$. We classify in advance the ergodic actions and embeddable ergodic actions of the compact quantum group $O_{-1}(2)$.

This thesis is based on two papers of the author and al. The first one is entitled "Fundamental isomorphism theorems for quantum groups" which have been accepted for publication in *Expositionae Mathematicae* and the second one is entitled "Ergodic actions and idempotent states of $O_{-1}(2)$" and is being finalized for submission.

Keywords

Locally compact quantum group, discrete quantum group, linearly reductive quantum group, Zassenhaus lemma, Schreier refinement theorem, Jordan-Hölder theorem, compact quantum group, ergodic action, idempotent state, Dedekind's modularity law, Noether's isomorphism theorem

Contents

Introduction

Cette thèse étudie les treillis des sous-groupes normaux, la règle de modularité de Dedekind, les théorèmes d'isomorphismes de Noether et ses conséquences ainsi que la caractérisation des actions ergodiques et des états idempotents du groupe quantique compact $O_{-1}(2)$. La théorie des groupes quantiques commence par les travaux de Drinfeld [33], Jimbo [43][44] , Reshetikhin, Takhtadzhyan et Faddeev [37] ainsi que Woronowicz [118]. Pour plus de détails et d'autres aspects sur les groupes quantiques compacts, voir par exemple [15][61][57] [68]. La théorie des groupes quantiques localement compact a été introduite quant à elle dans les années 2000 par Kustermans et Vaes [66]. L'étude. Ce travail fût poursuivi par Kasprzak-Khosravi-Soltan[53].

L'autre but de cette thèse sera de prouver l'analogue quantiques des théorèmes d'isomorphismes et de leurs conséquences pour les groupes quantiques. Cette partie de la thèse s'inspire du travail de S.Natale[76] sur le théorème de Jordan-Hölder pour les algèbres de dimension finie.des états idempotents des groupes quantiques compacts et leur caractérisation a débuté avec les travaux de Franz-Skalski[39] et Franz-Skalski-Tomatsu[40]. Les actions ergodiques sur des algèbres d'opérateurs des groupes classiques ont été introduites dans les travaux fondateurs de Wassermann [110][108][109].

Le but de cette thèse est de poursuivre les travaux sur la caractérisation des états idempotents des groupes quantiques compacts ainsi que la caractérisation des actions ergodiques des groupes classiques. Un autre aspect d'étude des groupes quantiques compacts est de généraliser les résultats des groupes classiques aux groupes quantiques. Par exemple, Wang[107] établie l'analogue quantique du troisième théorème d'isomorphisme de Noether

Cette thèse est composée de quatre chapitres. Elle repose sur les travaux[22, 23] de l'auteur et *al*, intitulés "Fundamental isomorphism theorems for quantum groups" et "Ergodic actions and idempotent states of $O_{-1}(2)$" qui sont respectivement accepté pour publication dans *Expositionae Mathematicae* et en préparation. Dans cette introduction, nous rappellerons l'assise théorique de notre thèse puis nous énoncerons les résultats centraux de cette dernière.

0.1 Brève esquisse historique

Treillis de sous-groupes classiques et Théorèmes d'isomorphismes de Noether et conséquences

En mathématiques, en particulier l'algèbre abstraite, les théorèmes d'isomorphisme sont trois théorèmes qui décrivent la relation entre quotients, homomorphismes et sous-objets. Des versions de ces théorèmes existent pour les groupes, les anneaux, les espaces vectoriels, les modules, les algèbres de Lie et diverses autres structures algébriques. Dans l'algèbre universelle, les théorèmes d'isomorphisme peuvent être généralisés au contexte des algèbres et des congruences.

Les théorèmes d'isomorphisme ont été formulés dans une certaine généralité pour les homomorphismes de modules par Emmy Noether dans son papier *Abstrakter Aufbau der Idealtheorie* dans *algebraischen Zahl- und Funktionenkörpern* qui a été publié en 1927 dans *Mathematische Annalen*. Des versions moins générales de ces théorèmes peuvent être trouvées dans le travail de Richard Dedekind et des articles précédents de Noether.

Trois ans plus tard, B.L. Van der Waerden a publié son influent livre *Algebra*, le premier manuel d'algèbre abstrait qui a introduit l'aspect théorique des groupes et anneaux . Van der Waerden s'est appuyé, comme principales références pour son manuel, sur des conférences de Noether sur la théorie des groupes et d'Emil Artin sur l'algèbre, ainsi que d'un séminaire dirigé par Artin, Wilhelm Blaschke, Otto Schreier et van der Waerden lui-même sur les idéaux. Les trois théorèmes d'isomorphisme, appelés théorème d'homomorphisme, et deux lois d'isomorphisme appliquées à des groupes, apparaissent explicitement.

Cette thèse s'intéresse à l'analogue quantique des trois théorèmes d'isomorphisme dans le contexte des groupes. Notez que certaines sources commutent la numérotation des deuxième et troisième théorèmes. Une autre variante rencontrée dans la littérature, en particulier dans l'Algèbre de Van der Waerden, est d'appeler le premier théorème d'isomorphisme le Théorème d'Homomorphisme Fondamental et par conséquent de décrémenter la numérotation des théorèmes d'isomorphisme restants par un. Enfin, dans le schéma de numérotage le plus étendu, le théorème de correspondance est parfois appelé le quatrième théorème d'isomorphisme.

L'étude des treillis de sous-groupes a une longue histoire, commençant par le travail de Richard Dedekind [31] en 1877, y compris le papier d'Ada Rottlaender [82] de 1928. Et plus tard de nombreuses contributions importantes de Reinhold Baer, Øystein Ore, Kenkichi Iwasawa, Léonid Efimovitch Sadovskii, Michio Suzuki, Giovanni Zacher, Mario Curzio, Federico Menegazzo, Roland Schmidt, Stewart Stonehewer, Giorgio Busetto et beaucoup d'autres.

Zassenhaus étudia son doctorat sous la supervision d'Emil Artin. Pendant ce temps il a prouvé le lemme de Zassenhaus (papillon), un résultat magnifique sur des sous-groupes qui peuvent être employés pour donner une preuve simple, et très belle, du théorème de Jordan-Hölder. Il a publié ceci dans un article de 3 pages [119].

Dans la théorie des groupes, certains systèmes ordonnés de sous-groupes d'un groupe classique donné jouent un rôle important: les sous-groupes sont inclus les uns dans les autres et le système obéit à certaines conditions additionnelles. Dans cette sous-section, nous allons étudier les propriétés de ces systèmes ordonnés qu'on appellera désormais *série* de sous-groupes.

Une série de composition fournit un moyen de décomposer une structure algébrique, comme un groupe ou un module, en morceaux simples. La nécessité de considérer les séries de compositions dans le contexte des modules résulte du fait que de nombreux modules naturels ne sont pas semi-simples et ne peuvent donc pas être décomposés en une somme directe de modules simples.

Une série de composition peut ne pas exister, et quand elle le fait, elle n'a pas besoin d'être unique. Néanmoins, un groupe de résultats connus sous le nom de théorème de Jordan-Hölder affirme que chaque fois que la série de composition existe, les classes d'isomorphisme de pièces simples (mais peut-être pas leur emplacement dans la série de composition en question) et leurs multiplicités sont déterminées de façon unique.

Mesure idempotente sur un groupe localement compact

Soit G un groupe compact. La classe des ensembles de Borel dans G noté $\mathcal{B}$ est la plus petite σ-algèbre des sous-ensembles de G qui contient chaque sous-ensemble ouvert de G.

Une *mesure de probabilité* μ sur G est une mesure additive réelle positive vérifiant $\mu(G) = 1$. Si μ et ν sont deux mesures de probabilités sur G alors leur convolution $\mu * \nu$ est aussi une mesure de probabilité sur G. En effet si X et Y sont deux variables aléatoires indépendantes sur un espace probabilisé quelconque prenant leurs valeurs dans G et si μ et ν sont leurs distributions respectives alors $\mu * \nu$ est la distribution du produit point par point XY. L'ensemble des mesures de probabilité sur G sera noté $\mathcal{P}(G)$.

Soit H un groupe compact séparé , il existe une unique mesure $\mu_H \in \mathcal{P}(H)$ tel que:

$$\mu_H(E) = \mu_H(xE) = \mu_H(Ex) = \mu_H(E^{-1})$$

pour tout ensemble de Borel $E \subset \mathbb{H}$ et $x \in H$. Cette mesure μ_H est appelée *mesure de Haar* sur H. Si $H \leq G$ est un sous-groupe fermé compact de G alors on appellera $\omega_H \in \mathcal{P}(G)$ définie par: $\omega_H(E) = \mu_H(E \cap H) \quad \forall E \in \mathcal{B}$. On notera désormais ω_H la *mesure de Haar* de H.

La mesure de Haar ω_G du groupe G est l'unique mesure de $\mathcal{P}(G)$ vérifiant:

$$\omega_G * \mu = \mu * \omega_G = \omega_G \qquad \forall \mu \in \mathcal{P}(G).$$

Vorob'yov[104] a considéré le cas où G est un groupe commutatif fini. Hewitt et Zuckerman [41] ont étudié une classe de semi-groupes commutatifs finis incluant tous les groupes commutatifs finis. Kakehashi[50] a étudié le cas où G est le tore. Kawada et Ito[58] ont montré que les mesures idempotentes d'un groupe compact séparé découlent toutes de la mesure de Haar d'un sous-groupe fermé. Wendel[111] identifia toutes les mesures idempotentes quand G est un groupe compact séparé. Kloss[60] et Urbanik[96] ont obtenu des résultats dans le cas où G est un groupe compact séparé. Ces premières investigations ont été poursuivis par Rudin[83] et complétés par le travail de Cohen[24] qui caractérisa toutes les mesures idempotentes sur un groupe abélien localement compact.

Nous allons tout d'abord donner la définition d'une mesure idempotente avant de rappeler le résultat de Kawada-Ito.

Une mesure μ sur un groupe G est dite *idempotente* si $\mu * \mu = \mu$.

Par exemple, la mesure de Haar d'un groupe G est une mesure idempotente.

Kawada et Ito ont montré que:
Une mesure de probabilité sur un groupe compact séparé G est idempotente si et seulement si c'est la mesure de Haar d'un sous-groupe fermé $H \leq G$.

Un *groupe localement compact* est un groupe topologique G tel que tout élément $g \in G$ possède un voisinage compact. On supposera toujours que G possède une base dénombrable d'ouverts.

Soit G un groupe localement compact et $(X, \mathcal{B}, \mu)$ un espace mesuré où

$$\mu : \mathcal{B} \to \mathbb{R}_+ \cup \{+\infty\}$$

est une mesure positive que nous supposerons σ-finie.

Une *action mesurable* de G sur X notée $(X,\mu) \overset{\alpha}{\curvearrowleft} G$ est une action telle que:

$$\begin{cases} G \times X & \to X \\ (g,x) & \mapsto x \end{cases}$$

est mesurable.

L'action $(X,\mathcal{B},\mu) \overset{\alpha}{\curvearrowleft} G$ est *ergodique* si pour tout $A \in \mathcal{B}$ tel que $gA = A$ pour tout $g \in G$, on a soit $\mu(A) = 0$ soit $\mu(X\ \ A) = 0$.

Groupes quantiques: Les débuts

La notion de groupe quantique fait référence à divers objets qui sont des déformations de groupes, mais qui ont encore des propriétés très similaires aux groupes, et en particulier à des groupes de Lie semi-simples. Les plus importantes sont les algèbres de Hopf déformant les algèbres de fonction sur les groupes de Lie semi-simples ou les algèbres enveloppantes des algèbres de Lie de Kac-Moody.

Le thème populaire actuel des groupes quantiques peut être abordé à partir de deux directions essentiellement différentes. La première approche, la plus répandue, est de nature algébrique. Les premiers succès de cette approche remontent à Drinfel'd (voir [33]) et Jimbo (voir [43]), qui ont défini les déformations à un paramètre des algèbres enveloppantes universelles d'algèbres de Lie complexes semisimples en 1985. Beaucoup d'autres classes de Hopf Algèbres ont été étudiées depuis 1985 et beaucoup ont reçu le label «groupe quantique». La deuxième approche est analytique dans la nature: la motivation de base dans le développement précoce de la théorie a été la généralisation de la dualité de Pontryagin pour les groupes abéliens localement compacts. Parce que le dual d'un groupe non-abélien ne peut plus être un groupe, on cherchait une catégorie plus grande contenant aussi le dual. Ces objets généralisés seraient à nouveau appelés groupes quantiques.

Cette thèse traitera aussi bien de l'approche algébrique que de l'approche analytique.

La première fois que le terme « groupe quantique » est apparu à la fois de manière algébrique et de manière analytique était lors des travaux [45, 46] de G.I.Kac sur les *groupes d'anneaux*. Kac essayait d'étendre la dualité de Pontryagin aux groupes classiques non-abéliens puisqu'en général, le dual d'un groupe classique non-abélien n'est pas forcément un groupe. Doù la nécéssité d'introduire une catégorie incluant à la fois les groupes classiques localement compacts et leurs duaux.

Dans cette optique, T.Tannaka obtint un théorème de dualité pour les groupes compacts en 1938 dans [95]. Il a réussi à recouvrir un groupe compact à partir de ses représentations irréductibles. Cependant ces derniers ne possedaient pas encore une structure de groupe. M.G.Krein, de son côté, définit le dual d'un groupe compact à partir de représentations irréductibles en les modélant sous forme d'algèbre de matrices en block.

W.F.Stinespring démontra un théorème de dualité pour les groupes localement compacts unimodulaires. Il réussit à recouvrir ce dernier à partir de l'algèbre de von Neumann de groupe en 1959.

En 1964, G.I.Kac&V.G.Paljutkin dans [47] donnerent le premier exemple d'un groupe quantique localement compact de dimension infinie. Ils étudièrent aussi les groupes quantiques de dimension finie avec leur célèbre exemple: le groupe quantique Kac-Paljutkin de dimension 8 [48].

Indépendamment dans les années 70, G.I.Kac et L.Vainermann [49, 100] d'un coté et M.Enock et J.M.Schwartz[36] de l'autre, définirent une catégorie complète englobant les groupes localements compacts et leurs duaux. Cet objet qu'ils définirent portent de nos jours le nom d' *algèbres de Kac*. S'ensuivirent plusieurs exemples d' *algèbres de Kac*. Dans les années 1980, les groupes quantiques apparurent sous une forme différente par déformation quantique d'algèbres de Lie. Ces déformations sont connues sous le nom d'*algèbres de Hopf*. La théorie d'*algèbres de Hopf* est algébrique alors que l'approche des *algèbres de Kac* est quant-à-elle analytique.

En 1987, S.L.Woronowicz developpa la théorie des *pseudo-groupes de matrices compacts*[113]. Il donna comme exemple d'illustration le groupe quantique $SU_q(2)$ dans [114]. En 1998, S.L.Woronowicz défini les *groupes quantiques compacts* dans [118]. Le point essentiel de sa théorie fut la démonstration de l'existence et l'unicité d'une *mesure de Haar* sur ces *groupes quantiques compacts*.

De leur part, E.Effros et Z.-J.Ruan[35] et A.Van Daele [102] développerent une approche duale en définissant *les groupes quantiques discrets*.

En 1998, A.Van Daele [103] définit une classe de groupes quantiques, appellé *groupes quantiques algébriques* incluant à la fois les groupes quantiques discrets et compacts. En 2000, Johan Kustermans & Stefaan Vaes définirent les *groupes quantiques localement compacts* dans [66]. Il existe deux approches C*- algébrique et Von Neumann algébrique d'un groupe quantique localement compact et ceux-ci généralisent les approches de l'algèbre de Kac, du groupe quantique compact et de l'algèbre de Hopf. D'autres tentatives d'unification de toutes ces définitions notamment avec les *unitaires multiplicatives* de S.Baaj et G.Skandalis [5], ont eu peu de succès à cause de leurs difficultés techniques. L'une des principales caractéristiques de cette nouvelle approche par rapport à ses prédécesseurs est l'existence axiomatique des poids invariants à gauche et à droite. Ceci donne un analogue non commutatif des mesures de Haar à gauche et à droite sur un groupe séparé localement compact.

En 1996, A.Pal [78] montra que pour un groupe quantique, les états idempotents ne découlaient pas forcément de la mesure de Haar d'un sous-groupe quantique comme dans le cas classique.

En 2009, U. Franz et A.Skalski[39] démontrèrent que les états idempotents d'un groupe quantique compact comoyennable sont en correspondance bijective avec les sous-algèbres coidéales avec espérance conditionnelle. Un résultat analogue pour les groupes quantiques localement compacts fut prouvé par P.Salmi et A.Skalski en 2011 dans [85].

0.2 Treillis de sous-groupes quantiques

Soit $\mathbb{G}$ un groupe quantique localement compact avec la comultiplication $\Delta_{\mathbb{G}}$ et l'unitaire multiplicatif $\mathrm{W}^{\mathbb{G}}$. Une sous-algèbre de von Neumann $N \subset L^\infty(\mathbb{G})$ est appelée:

- *coidéale à gauche* si $\Delta_{\mathbb{G}}(\mathsf{N}) \subset \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathsf{N}$;
- *sous-algèbre invariante* si $\Delta_{\mathbb{G}}(\mathsf{N}) \subset \mathsf{N} \bar{\otimes} \mathsf{N}$;

- *sous-algèbre de Baaj-Vaes* si N est une sous-algèbre invariante de $\mathrm{L}^\infty(\mathbb{G})$ qui est préservée par antipode unitaire R et le groupe dilatant $(\tau_t)_{t\in\mathbb{R}}$ de $\mathbb{G}$;
- *normale* si $\mathrm{W}^{\mathbb{G}}(\mathbb{1}\otimes\mathsf{N})\mathrm{W}^{\mathbb{G}*}\subset\mathrm{L}^\infty(\widehat{\mathbb{G}})\bar{\otimes}\mathsf{N}$;
- *intégrable* si l'ensemble des éléments intégrables par rapport aux poids de Haar à droite $\psi_{\mathbb{G}}$ est dense dans N^+; En d'autres termes, la restriction de $\psi_{\mathbb{G}}$ à N est semifinie.

Dans la suite, une sous-algèbre de von Neumann de $\mathrm{L}^\infty(\mathbb{G})$ qui est une sous-algèbre coidéale à gauche sera nommée une $\mathbb{G}$-coidéale ou tout simplement un coidéal.

Définition 0.2.1 (Definition 1.1.9 page 49)**.** Soit $\mathbb{G}$ un groupe quantique localement compact. L'ensemble de $\mathbb{G}$-coidéaux sera noté $\mathcal{CI}(\mathbb{G})$. On équipe $\mathcal{CI}(\mathbb{G})$ avec la structure suivante: pour $\mathsf{N},\mathsf{M}\in\mathcal{CI}(\mathbb{G})$, on écrit $\mathsf{N}\leq\mathsf{M}$ si $\mathsf{N}\subset\mathsf{M}$ et $\mathsf{N}\trianglelefteq\mathsf{M}$ si $\mathsf{N}\subset\mathsf{M}$ et N normal. L'ensemble $(\mathcal{CI}(\mathbb{G}),\leq)$ admet deux opérations $\wedge,\vee$

- $\mathsf{N}\wedge\mathsf{M}=\mathsf{N}\cap\mathsf{M}$,
- $\mathsf{N}\vee\mathsf{M}=\{xy : x\in\mathsf{N},y\in\mathsf{M}\}''$.

$(\mathcal{CI}(\mathbb{G}),\leq,\wedge,\vee)$ forme un treillis qu'on appellera le *treillis de coidéaux* de $\mathbb{G}$.

Le sous-ensemble de $\mathcal{CI}(\mathbb{G})$ des $\mathbb{G}$-coidéaux normaux sera noté $\mathcal{NCI}(\mathbb{G})$. Le sous-ensemble de $\mathcal{CI}(\mathbb{G})$ des sous-algèbres de Baaj-Vaes de $\mathrm{L}^\infty(\mathbb{G})$ sera noté $\mathcal{BV}(\mathbb{G})$.

On peut vérifier que: $\mathcal{NCI}(\mathbb{G})$ and $\mathcal{BV}(\mathbb{G})$ forment des sous-treillis de $\mathcal{CI}(\mathbb{G})$. De manière similaire, $\mathcal{NCI}(\mathbb{G})\cap\mathcal{BV}(\mathbb{G})$ forme un sous-treillis de $\mathcal{CI}(\mathbb{G})$.

Un groupe quantique localement compact admet un objet dual $\widehat{\mathbb{G}}=(\mathrm{L}^\infty(\widehat{\mathbb{G}}),\Delta_{\widehat{\mathbb{G}}},\varphi_{\widehat{\mathbb{G}}},\psi_{\widehat{\mathbb{G}}})$.

On a définit ensuite un treillis de sous-groupes quantiques comme suit:

Définition 0.2.2 (Definition 1.3.1 page 54)**.** Soit $\mathbb{G}$ un groupe quantique localement compact. Le treillis $\mathcal{BV}(\widehat{\mathbb{G}})$ sera noté $\mathcal{QS}(\mathbb{G})$ et appellé le *treillis de sous-groupes quantiques* de $\mathbb{G}$.

Le treillis $\mathcal{NCI}(\widehat{\mathbb{G}})\cap\mathcal{BV}(\widehat{\mathbb{G}})$ sera noté $\mathcal{NQS}(\mathbb{G})$ et appellé le *treillis de sous-groupes quantiques normaux* de $\mathbb{G}$.

On a explicité le plus grand sous-groupe quantique fermé de $\widehat{\mathbb{G}}$ ou en d'autres termes, la plus grande sous-algèbre de Baaj-Vaes commutative.

Proposition 0.2.3 (Proposition 1.3.3 page 55)**.** *Soit $\mathbb{G}$ un groupe quantique localement compact et considérons*

$$\mathsf{M}=\{x\in\mathrm{L}^\infty(\mathbb{G}) : (\mathrm{id}\otimes\Delta_{\mathbb{G}}^{op})(\Delta_{\mathbb{G}}(x))=(\mathrm{id}\otimes\Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x))\}. \tag{0.2.1}$$

Alors M est une sous-algèbre de Baaj-Vaes de $\mathrm{L}^\infty(\mathbb{G})$. Le groupe quantique $\mathbb{H}$ tel que $\mathsf{M}=\mathrm{L}^\infty(\mathbb{H})$ est abélien. Soit N une autre sous-algèbre de Baaj-Vaes et $\mathbb{L}$ le groupe quantique localement compact assigné à N. Si $\mathbb{L}$ est abélien alors $\mathsf{N}\subset\mathsf{M}$.

En s'appuyant sur [53, Theorem 6.2, Corollary 6.5], on a formulé le Premier Théorème d'Isomorphisme de Noether pour les groupes quantiques localement compact.

Théorème 0.2.4 (Theorem 1.3.11 page 59)**.** *Soit $\mathbb{H}$ et $\mathbb{G}$ deux groupes quantiques localement compacts, $\Pi:\mathbb{H}\to\mathbb{G}$ un homomorphisme et soit $\Pi_{\mathbb{H}/\ker\Pi\to\overline{im\Pi}}$, $\Pi_{\mathbb{H}/\ker\Pi\to\mathbb{G}}$, $\widehat{\Pi}_{\widehat{\overline{im\Pi}}\to\widehat{\mathbb{H}}}$ les homormorphismes induits par Π comme formulé sur la page 59. Alors les conditions suivantes sont équivalentes:*

(i) $\Pi_{\mathbb{H}/\ker\Pi \to \overline{im\Pi}}$ *est un isomorphisme;*

(ii) *L'action* $\alpha : \mathrm{L}^\infty(\overline{im\Pi}) \to \mathrm{L}^\infty(\overline{im\Pi}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H}/\ker\Pi)$ *est intégrable;*

(iii) $\Pi_{\mathbb{H}/\ker\Pi \to \mathbb{G}}$ *identifie* $\mathbb{H}/\ker\Pi$ *avec un sous-groupe quantique fermé de* $\mathbb{G}$*;*

(iv) $\widehat{\Pi}_{\widehat{\overline{im\Pi}} \to \widehat{\mathbb{H}}}$ *identifie* $\widehat{\overline{im\Pi}}$ *avec un sous-groupe quantique fermé de* $\widehat{\mathbb{H}}$*.*

Définition 0.2.5 (Definition 1.3.17 page 62)**.** Soit $\mathbb{G}$ un groupe quantique localement compact et $\mathbb{H}, \mathbb{M} \in \mathcal{QS}(\mathbb{G})$. On dit que $\mathbb{H}$ est *normalisé par* $\mathbb{M}$ si $\mathbb{H} \in \mathcal{NQS}(\mathbb{H} \vee \mathbb{M})$.

Dans [51, Definition 2.2], Kalantar-Kasprzak-Skalski ont défini un sous-groupe quantique ouvert. Le sous-ensemble de $\mathcal{QS}(\mathbb{G})$ qui consiste des sous-groupes quantiques ouverts de $\mathbb{G}$ sera noté par $\mathcal{OQS}(\mathbb{G})$. On a montré en particulier que $\mathcal{OQS}(\mathbb{G})$ forme un treillis.

Proposition 0.2.6 (Proposition 1.3.26 page 64)**.** *Soit* $\mathbb{G}$ *un groupe quantique localement compact,* $\mathbb{H} \in \mathcal{OQS}(\mathbb{G})$ *et* $\mathbb{M} \in \mathcal{QS}(\mathbb{G})$*. Alors* $\mathbb{H} \wedge \mathbb{M} \in \mathcal{OQS}(\mathbb{M})$*.*

On a aussi prouvé une transitivité forte de l'ouverture.

Proposition 0.2.7 (Proposition 1.3.28 page 65)**.** *Soit* $\mathbb{H} \leq \mathbb{M} \leq \mathbb{G}$ *une chaîne d'inclusion fermée de groupes quantiques localement compacts. Alors,* $\mathbb{H}$ *est ouvert dans* $\mathbb{G}$ *si et seulement si*

$$\mathbb{H} \leq \mathbb{M} \quad \textit{et} \quad \mathbb{M} \leq \mathbb{G}$$

sont tous deux ouverts.

Ensuite on a défini les sous-groupes quantiques bien positionnés.

Pour des sous-groupes $\mathbb{H} \leq \mathbb{G}$, nous travaillerons avec les espaces quantiques homogènes (voir Remark 1.3.2 page 54):

$$A_{\mathbb{H}} = \mathrm{L}^\infty(\mathbb{G}/\mathbb{H}) = \mathrm{cd}(\mathrm{L}^\infty(\widehat{\mathbb{H}})) \subseteq \mathrm{L}^\infty(\mathbb{G}).$$

Définition 0.2.8 (Definition 1.3.31 page 66)**.** Soit $\mathbb{H}$ et $\mathbb{M}$ deux sous-groupes quantiques d'un groupe quantique localement compact $\mathbb{G}$. On dit que $\mathbb{H}$ et $\mathbb{M}$ sont *(relativement) bien positionnés* si nous avons l'égalité

$$A_{\mathbb{H}} \vee A_{\mathbb{M}} = \{A_{\mathbb{H}} A_{\mathbb{M}}\}^{\sigma-\mathrm{cls}} \tag{0.2.2}$$

En effet, comme nous le verrons dans le Théorème 0.4.8, la propriété d'être bien positionnés est nécessaire pour la règle de modularité des sous-groupes quantiques d'un groupe quantique localement compact. On donnera maintenant quelques conditions suffisantes assurant le bien positionnement. Remarquons dans le cas algébrique, cette propriété existe toujours comme stipulé dans le Corollary 1.2.10.

Proposition 0.2.9 (Proposition 1.3.32 page 66)**.** *Les sous-groupes quantiques fermés* $\mathbb{H}, \mathbb{M} \leq \mathbb{G}$ *sont relativement bien positionnés si:*

(a) $\mathbb{G}$ *est un groupe classique;*

(b) *l'un de* $\mathbb{H}$ *et* $\mathbb{M}$ *est compact;*

(c) *l'un de* $\mathbb{H}$ *et* $\mathbb{M}$ *est normal;*

(d) $\mathbb{G}$ *est le dual d'un groupe classique.*

0.3 Théorèmes fondamentaux d'isomorphismes de Noether

Nous allons énoncer maintenant une version quantique du Second Théorème d'Isomorphisme de Noether pour les groupes quantiques linéairement réductifs et les groupes quantiques discrets.

Théorème 0.3.1 (Theorem 2.1.4 page 75)**.** *Soit $\mathbb{H} \leq \mathbb{G}$ et $\mathbb{K} \trianglelefteq \mathbb{G}$ des sous-groupes quantiques linéairement réductifs d'un groupe quantique linéairement réductif. Si $\mathbb{H}$ et $\mathbb{K}$ génère $\mathbb{G}$, alors l'homomorphisme canonique $\mathbb{H}/\mathbb{H} \wedge \mathbb{K} \to \mathbb{G}/\mathbb{K}$ est un isomorphisme.*

Remarque 0.3.2. *Notons qu'une version triviale du Premier Théorème d'Isomorphisme de Noether est implicite dans la preuve du Théorème 0.3.1. Pour un homomorphisme $\Pi : \mathbb{H} \to \mathbb{G}$ de groupe quantique localement compact, $\mathbb{H}/\ker \Pi$ est essentiellement le plus petit "quotient de GQLC" $\mathbb{H} \to ?$ pour lequel Π se factorise de cette manière*

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\quad\Pi\quad} & \mathbb{G} \\ & \searrow\!\!\!\!\twoheadrightarrow \; ? \; \nearrow & \end{array} \tag{0.3.1}$$

(voir par exemple Definition 1.3.8 page 58) D'une manière similaire, $\overline{\mathrm{im}\Pi}$ est le plus petit $? \leq \mathbb{G}$ tel que Π se factorise d'une manière similaire à (0.3.1) comme ceci

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\quad\Pi\quad} & \mathbb{G} \\ & \searrow \; ? \; \hookrightarrow & \end{array} \tag{0.3.2}$$

Dans le cas algébrique, l'image d'un homomorphisme d'algèbre de Hopf $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$ admet clairement les deux factorisations universelles comme propriétés, et ainsi ces deux concepts coïncident par défaut. Pour cette raison, on n'a pas établit un Premier Théorème d'Isomorphisme pour les groupes quantiques linéairement réductifs ainsi que pour les groupes quantiques discrets.

On a aussi montré une version du Théorème 0.3.1 pour les groupes quantiques discrets.

Théorème 0.3.3 (Theorem 2.1.6 page 75)**.** *Si le sous-groupe $\mathbb{H}$ et le sous-groupe normal $\mathbb{K}$ génèrent le groupe quantique discret $\mathbb{G}$, alors l'homomorphisme canonique $\mathbb{H}/\mathbb{H} \wedge \mathbb{K} \to \mathbb{G}/\mathbb{K}$ est un isomorphisme.*

Après avoir reformulé, l'analogue quantique du Premier Théorème d'Isomorphisme de Noether (qui s'obtient trivialement dans le cas algébrique) établi dans [53, Theorem 6.2, Corollary 6.5] dans le contexte localement compact avec une condition supplémentaire d'intégrabilité, nous avons prouvé l'analogue quantique du Second Théorème d'Isomorphisme de Noether pour les groupes quantiques localement compacts toujours avec la condition supplémentaire d'intégrabilité.

Théorème 0.3.4 (Theorem 3.1.1 page 83)**.** *Soit $\mathbb{G}$ un groupe quantique localement compact, $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$ et $\mathbb{N} \in \mathcal{NQS}(\mathbb{G})$. Notons $\Pi : \mathbb{H} \to \mathbb{G}/\mathbb{N}$ l'homomorphisme induit. Alors $\mathbb{H} \wedge \mathbb{N} = \ker \Pi$. Si de plus $\mathbb{G} = \mathbb{H} \vee \mathbb{N}$ alors $\overline{\mathrm{im}\Pi} = \mathbb{G}/\mathbb{N}$.*

En utilisant Théorème 0.2.4 et Théorème 0.3.4, on obtient ainsi:

Corollaire 0.3.5 (Corollary 3.1.2 page 84)**.** *L'homomorphisme $\mathbb{H}/\mathbb{H} \wedge \mathbb{N} \to \mathbb{G}/\mathbb{N}$ est un isomorphisme si et seulement si l'action correspondante de $\mathbb{H}/\mathbb{H} \wedge \mathbb{N}$ sur $\mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ est intégrable.*

Remarque 0.3.6. *Notons que les résultats dans le corollaire précédent (et donc ceux du second théorème d'isomorphisme aussi) ne sont pas vrais en général pour les groupes abéliens localement compacts classiques, comme le montre l'exemple 3.1.4.*

La caractéristique fondamentale de l'exemple 3.1.4 est que le produit naïf $\mathbb{H}\mathbb{N}$ n'est pas fermé dans $\mathbb{G}$, et ainsi $\mathbb{H} \vee \mathbb{N}$ est "plus large qu'espéré". Même, classiquement, c'est cet échec que $\mathbb{H}\mathbb{N}$ soit fermé qui empêche que les conditions du Corollaire 0.3.5 s'obtiennent. Ceci est résumé dans le résultat suivant.

Proposition 0.3.7 (Proposition 3.1.5 page 84)**.** *Soit G un groupe classique localement compact et $H \leq G$ et $N \trianglelefteq G$ des sous-groupes fermés.*

Alors, $H/H \wedge N$ agit de manière intégrable sur G/N si et seulement si pour tout sous-ensemble fermé F, $(H \wedge N)$-invariant de H le produit FN est fermé.

En 2013, Shuzhou Wang démontra pour la première fois un analogue quantique de l'un des trois théorèmes d'isomorphismes de Noether, en l'occurrence l'analogue quantique du Troisième Théorème d'Isomorphisme de Noether pour les groupes quantiques compacts dans [107, Theorem 4.1]. Nous avons établi son équivalent pour les groupes quantiques localement compacts.

Théorème 0.3.8 (Theorem 3.2.6 page 89)**.** *Soit $\mathbb{N} \leq \mathbb{H} \trianglelefteq \mathbb{G}$ des inclusions fermées de groupes quantiques localement compacts, et supposons de plus que $\mathbb{N}$ est normal dans $\mathbb{G}$. Alors, on a*

$$\mathbb{H}/\mathbb{N} \trianglelefteq \mathbb{G}/\mathbb{N} \quad \textit{and} \quad (\mathbb{G}/\mathbb{N})/(\mathbb{H}/\mathbb{N}) \cong \mathbb{G}/\mathbb{H}.$$

Comme le fait

$$\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$$

est une inclusion fermée ne requiert pas la normalité de $\mathbb{H}$, **??** 0.3.8 se généralise de la manière suivante:

Théorème 0.3.9 (Theorem 3.2.7 page 89)**.** *Soit $\mathbb{N} \leq \mathbb{H} \leq \mathbb{G}$ des inclusions fermées de groupes quantiques localement compacts, avec $\mathbb{N}$ normal dans $\mathbb{G}$. Alors l'homomorphisme canonique*

$$\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$$

est une inclusion fermée, et

$$\mathrm{L}^\infty((\mathbb{G}/\mathbb{N})/(\mathbb{H}/\mathbb{N})) = \mathrm{L}^\infty(\mathbb{G}/\mathbb{H})$$

0.4 La règle de modularité de Dedekind et lemme de Zassenhaus

Comme dans le cas classique, on a obtenu un analogue quantique du lemme de Zassenhaus pour les groupes quantiques. Cependant, il a été nécessaire tout d'abord de montrer un analogue quantique de la règle de modularité de Dedekind pour les groupes quantiques.

Proposition 0.4.1 (Proposition 2.2.1 page 76)**.** *Soit $\mathbb{N} \leq \mathbb{H}$ et $\mathbb{M}$ des sous-groupes normaux d'un groupe quantique linéairement réductif $\mathbb{G}$. Alors, on a*

$$\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{N}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{N}. \tag{0.4.1}$$

La règle de modularité de Dedekind s'obtient aussi pour les groupes quantiques compacts. Cependant on aura besoin de définir avant dans quel cas un sous-groupe quantique normalise un autre sous-groupe quantique.

Définition 0.4.2 (Definition 2.2.3 page 77)**.** Un sous-groupe quantique $\mathbb{L} \leq \mathbb{G}$ *normalise* un autre $\mathbb{M} \leq \mathbb{G}$ si ce dernier est normal dans le sous-groupe quantique $\mathbb{M} \vee \mathbb{L}$.

On est maintenant prêt à énoncer.

Proposition 0.4.3 (Proposition 2.2.4 page 77)**.** *Soit $\mathbb{G}$ un groupe quantique compact, avec des sous-groupes quantiques $\mathbb{L} \leq \mathbb{H} \leq \mathbb{G}$ et $\mathbb{M} \leq \mathbb{G}$ tel que $\mathbb{L}$ normalise $\mathbb{M}$. Alors, l'égalité $\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L}$ s'obtient.*

On a ensuite montré une version quantique de l'argument classique suivant: Étant donné une fonction continue f sur un groupe classique $\mathbb{G}$ et $\mathbb{L} \leq \mathbb{G}$. Alors l'expression de l'espérance conditionnelle $E_{\mathbb{L}} : L^{\infty}(\mathbb{G}) \to A_{\mathbb{L}}$ est donnée par:

$$(E_{\mathbb{L}} f)(\mathbb{L}_g) = \int_{\mathbb{L}} f(l_g)\ \mathrm{d}l.$$

Lemme 0.4.4 (Lemma 2.2.5 page 78)**.** *Soit $\mathbb{M}$ et $\mathbb{L}$ deux sous-groupes quantiques d'un groupe quantique compact $\mathbb{G}$ tel que $\mathbb{L}$ normalise $\mathbb{M}$. Alors:*

$$E_{\mathbb{L}}(A_{\mathbb{M}}) \subseteq A_{\mathbb{M}} \wedge A_{\mathbb{L}}.$$

De la même manière que pour les groupes quantiques compacts, on démontre une version quantique de la règle de modularité de Dedekind pour les groupes quantiques algébriques discrets en définissant au préalable un sous-groupe quantique discret normalisant un autre sous-groupe quantique discret.

On a montré une version duale de la Proposition 0.4.3, pour les groupes quantiques discrets dans le sens de Definition 1.2.1.

Les sous-groupes quantiques $\mathbb{M} \leq \mathbb{G}$ correspondent donc à des sous-algèbres de Hopf

$$k\mathbb{M} \subseteq k\mathbb{G}.$$

Définition 0.4.5 (Definition 2.2.9 page 80)**.** Soit $\mathbb{G}$ un groupe quantique algébrique discret. Un sous-groupe quantique $\mathbb{L}$ *normalise* un autre $\mathbb{M} \leq \mathbb{G}$ si l'algèbre de groupe $k\mathbb{M}$ de ce dernier est invariante par l'action adjointe de $k\mathbb{L}$ sur $k\mathbb{G}$.

Proposition 0.4.6 (Proposition 2.2.10 page 80)**.** *Soit $\mathbb{G}$ un groupe quantique algébrique discret ayant les sous-groupes quantiques $\mathbb{L} \leq \mathbb{H} \leq \mathbb{G}$ et $\mathbb{M} \leq \mathbb{G}$ tels que $\mathbb{L}$ normalise $\mathbb{M}$. Alors, l'égalité* (0.4.1) *s'obtient.*

Les propositions précédentes nous ont permis de démontrer une version quantique du lemme papillon (de Zassenhaus) ([62, Vol. 1, p. 77] or [81, Chapter 2, Lemma 5.10]) à la fois pour les groupes quantiques compacts et les groupes quantiques discrets.

Proposition 0.4.7 (Proposition 2.2.11 page 80)**.** *Soit $\mathbb{A}' \trianglelefteq \mathbb{A}$ et $\mathbb{B}' \trianglelefteq \mathbb{B}$ deux sous-groupes quantiques d'un groupe quantique soit compact soit algébriquement discret $\mathbb{G}$. Alors, on a un isomorphisme*

$$\frac{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B}')} \cong \frac{\mathbb{B}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{B}' \vee (\mathbb{A}' \wedge \mathbb{B})}.$$

Un résultat analogue s'obtient aussi pour les groupes quantiques linéairement réductifs $\mathbb{G}$ dés lors que $\mathbb{A}$, $\mathbb{A}'$, etc. sont tous normaux dans $\mathbb{G}$.

Pour les groupes quantiques localement compacts, l'analogue quantique de la règle de modularité de Dedekind et du lemme de Zassenhaus ne sont vrais que dans certains cas. Le théorème et la proposition suivants résument la situation.

Théorème 0.4.8 (Theorem 3.3.4 page 91)**.** *Soit $\mathbb{L} \leq \mathbb{H}$ et $\mathbb{M}$ des sous-groupes quantiques fermés d'un groupe quantique localement compact $\mathbb{G}$ tel que $\mathbb{L}$ normalise $\mathbb{M}$. Alors, on a*

$$\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L}. \tag{0.4.2}$$

si en plus

(a) $\mathbb{L}$ est compact, ou

(b) $\mathbb{H}$ est ouvert dans $\mathbb{G}$.

Remarque 0.4.9. *Le lecteur doit noter que comme $\mathbb{L} = \mathbb{H} \wedge \mathbb{L}$, la règle de modularité de Dedekind est réellement une forme de la loi de distribution $\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = (\mathbb{H} \wedge \mathbb{M}) \vee (\mathbb{H} \wedge \mathbb{L})$. Cette dernière, est néanmoins, fausse en générale.*

Proposition 0.4.10 (Proposition 3.3.9 page 93)**.** *Soit $\mathbb{A}' \trianglelefteq \mathbb{A}$ et $\mathbb{B}' \trianglelefteq \mathbb{B}$ des sous-groupes quantiques soit*

(a) compact ou

(b) ouvert

d'un groupe quantique localement compact $\mathbb{G}$. Alors, on a un isomorphisme

$$\frac{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B}')} \cong \frac{\mathbb{B}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{B}' \vee (\mathbb{A}' \wedge \mathbb{B})}.$$

0.5 Le théorème de raffinement de Schreier et le théorème de Jordan-Hölder pour les groupes quantiques

Dans cette section, nous prouvons l'analogue quantique du théorème de raffinement de Schreier et le théorème de Jordan-Hölder pour les groupes quantiques compacts et discrets (voir par exemple [81, Chapter 5, Theorem 5.11] et [81, Chapter 5, Theorem 5.12]) pour l'analogue classique pour les groupes discrets ordinaires). Dans cet objectif, on a tout d'abord défini l'analogue quantique de la notion de série sous-normale et d'une série de composition.

Définition 0.5.1 (Definition 2.3.1 page 81)**.** Soit $\mathbb{G}$ un groupe quantique soit compact soit algébrique discret soit linéairement réductif. Un système fini

$$\mathbb{G} = \mathbb{G}_0 \geq \mathbb{G}_1 \geq \mathbb{G}_2 \geq \mathbb{G}_3 \geq \cdots \geq \mathbb{G}_k = 1 \tag{0.5.1}$$

de sous-groupes quantiques fermés de $\mathbb{G}$ est appelé une *série sous-normale* de $\mathbb{G}$ si tout sous-groupe $\mathbb{G}_i$ est un sous-groupe quantique normal propre de $\mathbb{G}_{i-1}$, $i \in \{1, 2, \cdots, k\}$. En particulier, $\mathbb{G}_1$ est un sous-groupe quantique fermé normal de $\mathbb{G}$, $\mathbb{G}_2$ est un sous-groupe quantique fermé normal de $\mathbb{G}_1$, mais pas nécessairement de $\mathbb{G}$, et ainsi de suite.

Une série sous-normale est *normale* si chaque $\mathbb{G}_i$ est normal dans le groupe ambiant $\mathbb{G}$.

Les groupes quantiques sous-quotients correspondants

$$\mathbb{G}_1 \backslash \mathbb{G}, \ \mathbb{G}_2 \backslash \mathbb{G}_1, \ \cdots, \ \mathbb{G}_k \backslash \mathbb{G}_{k-1}$$

de $\mathbb{G}$ sont les *facteurs* de la série sous-normale (0.5.1).

L'entier k est la *longueur* de la série (0.5.1).

Définition 0.5.2 (Definition 2.3.2 page 81)**.** Une série sous-normale

$$\mathbb{G} = \mathbb{H}_0 \geq \mathbb{H}_1 \geq \mathbb{H}_2 \geq \mathbb{H}_3 \geq \cdots \geq \mathbb{H}_l = 1 \tag{0.5.2}$$

est appelée un *raffinement* de la série sous-normale (0.5.1) si chaque sous-groupe quantique $\mathbb{G}_i$ de (0.5.1) coïncide avec un des sous-groupes quantiques $\mathbb{H}_j$, i.e. si chaque sous-groupe quantique apparaissant dans (0.5.1) apparaît aussi dans (0.5.2).

En particulier, toute série normale est son propre raffinement. Les longueurs de la série normale (0.5.1) et de son raffinement (0.5.2) satisfont bien-sur l'inégalité $k \leq l$.

Deux séries sous-normales d'un groupe quantique compact sont dites *équivalentes* si leurs longueurs sont égales et leurs sous-quotients respectifs sont isomorphes à permutation près.

On est maintenant prêt à énoncer l'analogue quantique des théorèmes de raffinement de Schreier et de Jordan-Hölder. Leurs preuves découlent respectivement du lemme de Zassenhaus et du théorème de raffinement de Schreier.

Théorème 0.5.3 (Theorem 2.3.3 page 82)**.** *Deux séries sous-normales d'un groupe quantique soit compact soit discret soit linéairement réductif* $\mathbb{G}$ *admettent des raffinements équivalents.*

Définition 0.5.4 (Definition 2.4.1 page 82)**.** Une série sous-normale comme (0.5.1) est appelée *série de composition* de $\mathbb{G}$ si $\mathbb{G}_i$ est un sous-groupe quantique normal maximal propre de $\mathbb{G}_{i-1}$ pour $1 \leq i \leq k$.

Remarque 0.5.5. *En d'autres termes, une série de composition est une série sous-normale ne pouvant plus être raffinée une autre fois.*

Théorème 0.5.6 (Theorem 2.4.3 page 82)**.** *Deux séries de composition d'un groupe quantique soit compact soit discret soit linéairement réductif* $\mathbb{G}$ *sont équivalents.*

Remarque 0.5.7. *Ceci est tout simplement une adaptation au cas quantique des preuves usuelles du théorème de raffinement de Schreier et du théorème de Jordan-Hölder (voir par exemple les preuves de* [81, Chapter 5, Theorem 5.11] *et* [81, Chapter 5, Theorem 5.12]*). Comme mentionné ci-dessus, dès lors qu'on a obtenu une version quantique du lemme de Zassenhaus, les mêmes arguments standard marchent mécaniquement.*

Pour les groupes quantiques localement compacts, l'analogue quantique du théorème de raffinement de Schreier et du théorème de Jordan-Hölder ne sont vrais aussi que dans certains cas. Notons au préalable que les Définition 0.5.1, Définition 0.5.2 et Définition 0.5.4 restent valable aussi pour les groupes quantiques localement compacts.

Les deux théorèmes suivants résument la situation.

On écrira $\{\mathbb{G}_\ell\}_{\ell \geq 0}$ pour la série (sous)-normale générique

$$\mathbb{G} = \mathbb{G}_0 \geq \mathbb{G}_1 \geq \mathbb{G}_2 \geq \mathbb{G}_3 \geq \cdots \geq \mathbb{G}_k = 1. \tag{0.5.3}$$

de sous-groupes quantiques fermés d'un groupe quantique localement compact $\mathbb{G}$.

Théorème 0.5.8 (Theorem 3.4.1 page 94)**.** *Soit* $\mathbb{G}$ *un groupe quantique localement compact. Alors, deux séries sous-normales* $\{\mathbb{G}_\ell\}$ *et* $\{\mathbb{G}'_t\}$ *de* $\mathbb{G}$ *admettent des raffinements équivalents, dès lors que*

$$\mathbb{G}_\ell,\ \ell \geq 1 \quad et \quad \mathbb{G}'_t,\ t \geq 1$$

sont soit

(a) compacts soit

(b) ouverts.

Théorème 0.5.9 (Theorem 3.4.2 page 94)**.** *Soit $\mathbb{G}$ un groupe quantique localement compact. Alors, toutes les séries de composition de $\mathbb{G}$ consistant en des sous-groupes quantiques soit*

(a) compacts soit

(b) ouverts

sont équivalentes.

Les versions compacts des Théorème 0.5.8 et Théorème 0.5.9 se réfèrent aux séries sous-normales (0.5.3) dans lesquelles tous les $\mathbb{G}_\ell$, $\ell \geq 1$ sont compacts, mais $\mathbb{G} = \mathbb{G}_0$ n'a pas besoin de l'être. Notons que ceci est équivalent au fait que le sous-quotient $\mathbb{G}/\mathbb{G}_1$ est non-compact. Cependant, on a:

Proposition 0.5.10 (Proposition 3.4.3 page 95)**.** *Un groupe quantique localement compact $\mathbb{G}$ est compact si et seulement s'il admet une série sous-normale comme* (0.5.3) *avec des sous-quotients $\mathbb{G}_i/\mathbb{G}_{i+1}$ compacts.*

0.6 Caractérisation des actions ergodiques et états idempotents du groupe quantique compact $O_{-1}(2)$

Une mesure idempotente sur un groupe compact séparé classique est la mesure de Haar d'un sous-groupe fermé. Pal a montré dans [78] qu'un résultat analogue n'est en aucun cas possible dans le cas quantique en donnant l'exemple d'une mesure idempotente sur le groupe quantique compact Kac-Paljutkin de dimension 8 ne découlant pas de la mesure de Haar d'un sous-groupe quantique. Dans cette sous-section, nous allons tout d'abord définir un groupe quantique compact avant d'énoncer le résultat de Franz-Skalski [39] qui établirent en premier une caractérisation des états idempotents d'un groupe quantique compact. Ce résultat fut généralisé pour les groupes quantiques localement compact par Salmi-Skalski[85]. Dans cette section, nous allons donner la liste complète des actions ergodiques et des actions ergodiques plongeables du groupe quantique compact $O_{-1}(2)$ et nous caractériserons les états idempotents de ce dernier en nous appuyant sur le résultat de Franz-Skalski [39, Theorem 4.1].

Tout d'abord, notons que lister les actions ergodiques du groupe quantique compact $O_{-1}(2)$ revient à lister les actions ergodiques du groupe compact classique $O(2)$. En effet, un résultat fondamental de Banica-Bichon-Collins plus précisement [6, Theorem 4.3] stipule que la catégorie de coreprésentations du groupe quantique compact $O_{-1}(n)$ est tenseur équivalent à la catégorie de représentation du groupe compact classique $O(n)$. On en déduit donc que les groupes quantiques compact $O_{-1}(2)$ et $O(2)$ (qui est aussi un groupe quantique compact avec $q = 1$) sont monoidalement équivalents. Un autre résultat fondamental de De Rijt-Vander Vennet plus précisement [30, Theorem 7.3] stipule que les actions ergodiques de deux groupes quantiques compacts monoidalement équivalents sont en correspondance bijective. On en déduit donc que les coactions ergodiques du groupe quantique compact $O_{-1}(2)$ sont en correspondance bijective avec les actions ergodiques du groupe compact classique $O(2)$. La première difficulté fut donc de lister les actions ergodiques du groupe compact classique $O(2)$. Rappelons tout d'abord que si G est un groupe compact classique, $H \leq G$ un sous-groupe fermé et $H \overset{\alpha}{\curvearrowright} N$ une action ergodique de $\mathbb{H}$ sur une algèbre de von Neumann N, on a l'action ergodique induite de G sur

$$\begin{aligned} \mathrm{Ind}_H^{\mathbb{G}}(N) &= \{f \in L^\infty(\mathbb{G}, N) \mid \forall g \in \mathbb{G}, h \in \mathbb{H}, \alpha_h(f(gh)) = f(g)\} \\ &\subseteq L^\infty(\mathbb{G}) \bar{\otimes} N, \end{aligned}$$

donnée par

$$(\alpha_g(f))(g') = f(g^{-1}g').$$

En particulier, si π est une représentation projective irreductible de H sur un espace de Hilbert V_π, on a une action ergodique de G sur une algèbre de von Neumann $\mathrm{Ind}_H^G(B(V_\pi))$ où $B(V_\pi)$ est équipé avec la $\mathbb{H}$-action ergodique

$$\alpha_h(x) = \pi(h) x \pi(h)^*.$$

Par exemple, quand π est la représentation triviale (ou une représentation de dimension une), on a $\mathrm{Ind}_H^G(\mathbb{C}) = L^\infty(G/H)$ avec l'action de translation à gauche.

Dans ce qui suit, on va identifier $O(2) \cong C_2 \ltimes T$, avec le groupe cercle $T = \{z \in \mathbb{C} \mid |z| = 1\}$, et avec le groupe cyclique $C_2 = \{1, \sigma\}$ agissant sur T par $\sigma(z) = \bar{z}$. Le résultat

suivant est une communication privée de Kenny De Commer.

Théorème 0.6.1 (Theorem 4.1.2 page 98). *Soit $O(2) \underset{\alpha}{\curvearrowright} M$ une action ergodique. Alors $M \cong \mathrm{Ind}_{\mathbb{H}}^{O(2)}(B(V_\pi))$ pour un sous-groupe fermé quelconque $\mathbb{H} \subseteq O(2)$ et π une représentation irréductible de $\mathbb{H}$.*

Après avoir déterminé les isomorphismes entres les actions induites, on a obtenu la liste complète des actions ergodiques non-équivalentes du groupe compact classique $O(2)$.

Les représentations projectives irréductibles de $O(2)$ donnent soit l'action triviale $\beta_0^{(\infty)}$ sur $\mathbb{C}$ (pour les caractères) soit l'action ergodique $\beta_{l/2}^{(\infty)} = \alpha$ sur $M_2(\mathbb{C})$ définie par

$$\alpha_z \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z^l b \\ z^{-l} c & d \end{pmatrix}, \quad \alpha_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

pour $l \in \mathbb{N} \setminus \{0\}$ (avec l'indice pair l venant des représentations et les indices impairs l venant des représentations projectives non-triviales).

Les représentations irréductibles du tore T donnent toutes la même action ergodique $\alpha = \alpha^{(\infty)}$ sur $\mathbb{C}^2 = L^\infty(O(2)/T)$, précisément

$$\alpha_z((x, y)) = (x, y), \quad \alpha_\sigma(x, y) = (y, x).$$

Les représentations irréductibles du groupe cyclique C_k pour un certain k fixé donnent toutes la même action ergodique $\alpha = \alpha^{(k)}$ on $L^\infty(O(2)/C_k) = L^\infty(T/C_k) \oplus L^\infty(T/C_k)$, précisément

$$\alpha_z(f, g) = (f_z, g_z), \quad \alpha_\sigma(f, g) = (g, f),$$

où f_z désigne le z-translaté de f.

Finalement, pour le groupe diédral D_k on a l'action ergodique $\alpha = \beta_0^{(k)}$ découlant des caractères de D_k, donnant l'action sur $L^\infty(O(2)/D_k)$, ou les actions ergodiques $\beta_{l/2}^{(k)}$

où $0 < l < k$ un entier naturel découlant des représentations projectives n'étant pas des caractères et $\beta^{(k)}_{l/2}$ l'action ergodique induite venant de la D_k-représentation projective

$$\alpha_z \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z^l b \\ z^{-l} c & d \end{pmatrix}, \quad \alpha_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

On a obtenu ainsi:

Proposition 0.6.2 (Proposition 4.1.3 page 100)**.** *La liste complète des actions ergodiques non-équivalentes de $O(2)$:*

$$\{\beta^{(k)}_{l/2}, \beta^{(\infty)}_{l'/2}, \alpha^{(\infty)}, \alpha^{(k')} \mid k, k' \in \mathbb{N}_0, l' \in \mathbb{N}, 0 \leq l < k\}.$$

On veut en déduire la liste des actions ergodiques plongeables de $O_{-1}(2)$. Pour cela, nous procédons comme suit.

Notons tout d'abord que par définition, une action ergodique plongeable est une $*$-algèbre de comodule de l'algèbre de groupe quantique compact $\mathcal{A}_{-1}$ associée à $O_{-1}(2)$ qui s'injecte dans $\mathcal{A}_{-1}$ comme telle (c-à-d par une inclusion qui préserve toutes les structures d'algèbre, de comodule, etc).

Puisque l'équivalence twistant $\lambda\triangleright$ (voir la notation de Section 4.4) qui implémente Theorem 1.4.19 implémente aussi une équivalence entre les catégories de $*$-algèbres coidéales sur $\mathcal{A}_{-1}$ et la version non-twistée $\mathcal{A}$ (algèbre des fonctions représentatives sur le groupe classique $O(2)$), il sera suffisant d'identifier l'$O(2)$-action ergodique $\mathcal{B}$ dans la liste de Proposition 0.6.2 pour laquelle

$$\lambda \triangleright \mathcal{B} \cong \mathcal{A}_{-1}$$

comme $*$-algèbres de comodule $\mathcal{A}_{-1}$, et d'identifier aussi alors les membres de cette liste s'injectant dans $\mathcal{B}$.

On a montré qu'il n'y a qu'un candidat pour $\mathcal{B}$ (précisement $\beta^{(2)}_{1/2}$) en utilisant le théorème de Peter-Weyl pour déterminer le type de représentation des actions ergodiques identifiées dans la Proposition 0.6.2 (où par représentation type, on sous-entend les multiplicités des différentes $O(2)$-représentations irréductibles). Ceci est en l'occurrence la substance de notre résultat suivant:

Proposition 0.6.3 (Proposition 4.2.1 page 100)**.** *Les seules algèbres de comodule parmi celles listées dans la Proposition 0.6.2 qui sont isomorphiques à $\mathcal{A}$ comme $O(2)$-représentations sont $\alpha^{(1)} \cong \mathcal{A}$ elle-même et $\beta^{(2)}_{1/2}$.*

On a ainsi obtenu comme conséquence la liste complète des actions ergodiques plongeables du groupe quantique compact $O_{-1}(2)$:

Corollaire 0.6.4 (Corollary 4.2.2 page 101)**.** *L'équivalence twistant $\lambda\triangleright$ induit une bijection entre*

$$\{\alpha^{(k)},\ \beta^{(k)}_{l/2} \mid k = \infty \text{ ou pair },\ l = 0 \text{ ou impair }\}$$

de Proposition 0.6.2 et l'isomorphisme de classe des actions ergodiques plongeables de $O_{-1}(2)$.

La liste complète des coactions ergodiques plongeables non-équivalentes de $O_{-1}(2)$ est donc:

$$\{\alpha^{(k)},\ \beta^{(k)}_{l/2} \mid k = \infty \text{ ou pair },\ l = 0 \text{ ou impair }\}$$

Pour finir cette section, on s'est posé la question suivante motivée par Theorem 1.4.19 page 71:

Question 0.6.5 (Question 1.4.1 page 71). *Soit $\mathbb{G}_1$ et $\mathbb{G}_2$ deux groupes quantiques compacts monoidalement équivalents. A-t-on une correspondance bijective entre leurs actions ergodiques plongeables?*

Remarque 0.6.6. *Une réponse affirmative à la question précédente élargira le résultat fondamental* [30, Theorem 7.3] *de De Rijt-Vander Vennet aux actions ergodiques plongeables. Mais la réponse est négative en général (Voir Question 0.6.7 ci-dessous) sauf dans le cas où nous avons une condition supplémentaire (Voir Proposition 4.4.5 page 106).*

Spécifiquement, supposons que $\mathbb{G}_1$ et $\mathbb{G}_2$ sont deux groupes quantiques compacts monoidalement équivalents. Supposons de plus que l'équivalence monoidale est implémenté par un twist avec un cocycle

$$\lambda : C(\mathbb{G}_1)^{\otimes 2} \to \mathbb{C}.$$

Alors, d'après la preuve de Theorem 1.4.19, on a une opération de twist $\lambda \triangleright$ qui implémente une équivalence entre la catégorie $\mathcal{E}rg(\mathbb{G}_1)$ des $\mathbb{G}_1$-actions ergodiques et la catégorie analogue $\mathcal{E}rg(\mathbb{G}_2)$. Dans cet esprit, la formulation de la Question 0.6.5 à laquelle nous avons

répondu est:

Question 0.6.7 (Question 4.4.1 page 104). *Peut-on restreindre*

$$\lambda \triangleright : \mathcal{E}rg(\mathbb{G}_1) \to \mathcal{E}rg(\mathbb{G}_2)$$

à une équivalence entre les sous-catégories des coactions ergodiques plongeables?

Malheureusement, la réponse à la Question 0.6.7 est négative en générale, comme en l'a montré dans le contre-exemple ci-dessous.

Prenons le groupe quantique compact $\mathbb{G}_1$ comme étant le groupe de Heisenberg d'ordre 64

$$H_4 := \langle \varepsilon_1, \varepsilon_2, \delta \mid \varepsilon_1^4 = \varepsilon_2^4 = \delta^4 = 1,\ \delta\varepsilon_1 = \varepsilon_1\delta,\ \varepsilon_2\delta = \delta\varepsilon_2,\ \varepsilon_1\varepsilon_2 = \delta\varepsilon_2\varepsilon_1 \rangle.$$

Sa déformation $\mathbb{G}_2$ est celle utilisée dans [16] et décrite en détail dans la section 6 de [16]. Cependant, on n'écrira pas explicitement ici le cocycle

$$\lambda : C(H_4)^{\otimes} \to \mathbb{C},$$

rappelons juste qu'il est obtenu comme la composition

$$\begin{array}{ccc} C(H_4)^{\otimes 2} & \xrightarrow{\quad\lambda\quad} & \mathbb{C} \\ & \searrow\ C(\Gamma)^{\otimes 2}\ \nearrow & \end{array}$$

ou la flèche du membre de gauche est la restriction de l'inclusion

$$(\mathbb{Z}/2)^2 \cong \Gamma := \langle \varepsilon_1^2, \varepsilon_2^2 \rangle \subset H_4$$

et la flèche du membre de droite est un certain cocycle induisant la classe de cohomologie non-nulle dans

$$H^2(\widehat{\Gamma}, \mathbb{C}^\times) \cong \mathbb{Z}/2.$$

Il est plus habile de dualiser et de travailler avec l'algèbre de groupe $\mathbb{C}H_4$ et sa version déformée H^*, qui coïncide avec $\mathbb{C}H_4$ comme algèbre mais ayant une comultiplication twistée.

On a montré que:

Proposition 0.6.8 (Proposition 4.4.3 page 106)**.** *Il n'existe pas de bijection préservant l'inclusion, la dimension, et l'action de G entre les sous-algèbres coidéales de $\mathbb{C}H_4$ et ceux H^*.*

Comme dernière observation, on a montré que s'il arrive que $\mathbb{G}_1$ admet un ensemble de 2-cohomologie $H^2(\widehat{\mathbb{G}_1}, \mathbb{S}^1)$ trivial, la réponse à la Question 0.6.7 est affirmative:

Proposition 0.6.9 (Proposition 4.4.5 page 106)**.** *Soit $\mathbb{G}_1$ et $\mathbb{G}_2$ deux groupes quantiques compacts monoidalement équivalents, et supposons que $H^2(\widehat{\mathbb{G}_2}, \mathbb{S}^1)$ est trivial. Alors,*

$$\lambda \triangleright : \mathcal{E}rg(\mathbb{G}_1) \to \mathcal{E}rg(\mathbb{G}_2)$$

se restreint à une équivalence entre les sous-catégories des coactions ergodiques plongeables.

États idempotents du groupe quantique compact $O_{-1}(2)$

Ayant réussi à faire la liste des actions ergodiques plongeables de $O_{-1}(2)$, vu que par définition, une action ergodique plongeable s'identifie à une sous-algèbre coidéale et vu que d'après le résultat fondamental [39, Theorem 4.1] de Franz-Skalski, il y a une correspondance bijective entre les sous-algèbres coidéaux avec espérance conditionnelle et les états idempotents d'un groupe quantique compact; notre projet s'est heurté à un dernier souci: *Comment exprimer par une formule concrète un état idempotent à partir d'une action d'un groupe quantique compact?*

En fait, bien que nous les ayons caractérisées grâce à [39], il nous reste à lister les états idempotents de $O_{-1}(2)$ à partir de la liste de ses actions ergodiques plongeables.

Vu que:

1. Rappelons que, si on prend la comultiplication elle-même comme l'action (c-à-d on prend le groupe quantique lui-même comme la sous-algèbre coidéale), ce n'est pas immédiat de produire le correspondent état de Haar depuis la comultiplication.

2. Aussi si on considère la sous-algèbre $\mathbb{C}$, alors on voit qu'on ne peut pas produire l'état de Haar depuis l'action qui est simplement $\mathbb{C} \to \mathbb{C} \otimes \mathbb{C}, 1 \mapsto 1 \otimes 1$.

Le problème est que partant de la preuve de [39, Theorem 4.1], il y a plusieurs éléments que nous avons abstraitement mais de manière non-explicite, et ils dépendent les uns des autres. En effet, pour obtenir l'état idempotent de la sous-algèbre coidéale, on a besoin de l'espérance conditionnelle E; Et pour obtenir E, on a malheureusement besoin de l'état dans la preuve de [39, Theorem 4.1]. C'est donc circulaire et ne conduit pas à quelque chose d' explicite (juste au théorème de l'article Franz-Skalski, c-à-d le fait qu'il y a cette bijection). Nous pensons que s'il y a une chance d'obtenir une formule concrète de l'état idempotent à partir d'une action, ce serait en exprimant l'espérance conditionnelle E seulement en fonction de l'action ergodique plongeable α en ne faisant pas intervenir l'état lui-même, puis de composer E ainsi obtenue avec la counité ε.

La question suivante résume notre point de vue:

Question 0.6.10 (Question 4.3.1 page 104)**.** *Est-il possible d'obtenir une telle formule concrète en généralité pour un groupe quantique compact au moins dans le cas Kac?*

Introduction

This thesis studies the lattice of subgroups, Dedekind's modular law, Noether's isomorphism theorems and its consequences, as well as the characterization of the ergodic actions and idempotent states of the compact quantum group $O_{-1}(2)$. The theory of quantum groups begins with the work of Drinfeld [33], Jimbo [43] [44], Reshetikhin, Takhtadzhyan and Faddeev [37] and Woronowicz [118]. For more details and other aspects on compact quantum groups, see for example [15] [61] [57] [68]. The theory of locally compact quantum groups was introduced in the 2000s by Kustermans and Vaes [66]. The study of the idempotent states of the compact quantum groups and their characterization began with the work of Franz-Skalski [39] and Franz-Skalski-Tomatsu [40]. The ergodic actions on algebras of operators of classical groups have been introduced in the founding works of Wassermann [110] [108] [109].

The aim of this thesis is to continue the work on the characterization of the idempotent states of the compact quantum groups as well as the characterization of the ergodic actions of the classical groups.

Another aspect of the study of compact quantum groups is to generalize the results of the classical groups to the quantum groups. For example, Wang [107] establishes the quantum analogue of Noether's third isomorphism theorem. This work was pursued by Kasprzak-Khosravi-Soltan [53].

The other goal of this thesis will be to prove the quantum analogue of isomorphism theorems and their consequences for quantum groups. This part of the thesis is inspired by the work of S.Natale [76] on the theorem of Jordan-Hölder for finite-dimensional algebras.

This thesis is composed of four chapters. It is based on the author's [22, 23] works and *al*, entitled "Fundamental isomorphism theorems for quantum groups " and "Ergodic actions and idempotent states of $O_1(2)$ " which are respectively accepted for publication in *Expositionae Mathematicae* and in preparation. In this introduction, we will recall the theoretical basis of our thesis and then we will state the central results of the latter.

0.1 Brief historical outline

Lattices of classical subgroups and Noether's Isomorphisms Theorems and consequences.

In mathematics, in particular abstract algebra, isomorphism theorems are three theorems which describe the relation between quotients, homomorphisms and subobjects. Versions of these theorems exist for groups, rings, vector spaces, modules, Lie algebras and various other algebraic structures. In universal algebra, the isomorphic theorems can be generalized to the context of algebras and congruences.

The isomorphims theorems have been formulated in a certain generality for the homomorphisms of modules by Emmy Noether in her paper *Abstrakter Aufbau der Idealtheorie* in *algebraischen Zahl- und Funktionenkörpern* which was published in 1927 in *Mathematische Annalen*. Less general versions of these theorems can be found in the work of Richard Dedekind and previous papers by Noether.

Three years later, B. Van der Waerden published his influential book *Algebra*, the first abstract algebra manual that introduced the theoretical aspect of groups and rings. Van der Waerden relied, as the main references for his textbook, on Noether's lectures on group theory and Emil Artin lectures on algebra, as well as a seminar led by Artin, Wilhelm Blaschke, Otto Schreier and Van der Waerden himself on ideals. The three isomorphism theorems, called the homomorphism theorem, and two isomorphism laws applied to groups, appear explicitly.

This thesis deals with the quantum analogue of the three isomorphism theorems in the context of groups. Note that some sources switch the numbering of the second and third theorems. Another variant encountered in the literature, particularly in Van der Waerden's Algebra, is to call the first isomorphism theorem the Theorem of Fundamental Homomorphism and consequently to decrement the numbering of the remaining isomorphism theorems by one. Finally, in the most extended numbering scheme, the correspondence theorem is sometimes called the fourth isomorphism theorem.

The study of subgroup lattices has a long history, beginning with the work of Richard Dedekind [31] in 1877, including the 1928 Ada Rottlaender paper [82]. And later many important contributions of Reinhold Baer, Øystein Ore, Kenkichi Iwasawa, Leonid Efimovich Sadovskii, Michio Suzuki, Giovanni Zacher, Mario Curzio, Federico Menegazzo, Roland Schmidt, Stewart Stonehewer, Giorgio Busetto and many others.

Zassenhaus studied his thesis under the supervision of Emil Artin. Meanwhile he proved the Zassenhaus lemma (butterfly), a magnificent result on subgroups that can be used to give a simple, and very beautiful proof of the Jordan-Hölder theorem. He has published this in a 3 pages paper [119].

In group theory, some ordered systems of subgroups of a given classical group play an important role: subgroups are included in one another and the system obeys certain additional conditions. In this subsection, we will study the properties of these ordered systems, which will now be called *series* of subgroups.

A composition serie provides a means of decomposing an algebraic structure, such as a group or a module, into simple pieces. The need to consider the series of compositions in the context of modules results from the fact that many natural modules are not semi-simple and therefore can not be decomposed into a direct sum of simple modules.

A composition serie may not exist, and when it does, it does not need to be unique. Nevertheless, a group of results known as the Jordan-Hölder theorem asserts that whenever the composition series exists, the isomorphism classes of simple pieces (but perhaps not their location in the composition series) and their multiplicities are determined in a unique way.

Idempotent measure on a locally compact group

Let G be a compact group. The class of Borel sets in G denoted by $\mathcal{B}$ is the smallest σ-algebra of the subsets of G which contains each open subset of G.

A *probability measure* μ on G is a positive real additive measure satisfying $\mu(G) = 1$. If μ and ν are two probability measures on G then their convolution $\mu * \nu$ is also a probability measure on G.

In fact if X and Y are two independent random variables on any probability space taking their values in G and if μ and ν are their respective distributions then $\mu * \nu$ is the distribution of the point by point product XY. The set of probability measures on G will be denoted $\mathcal{P}(G)$.

Let H be a Hausdorff compact group, there exists a unique measure $\mu_H \in \mathcal{P}(H)$ such that:

$$\mu_H(E) = \mu_H(xE) = \mu_H(Ex) = \mu_H(E^{-1})$$

for all Borel set $E \subset H$ and $x \in H$. This measure μ_H is called *Haar measure* on H. If $H \leq G$ is a compact closed subgroup of G then $\omega_H \in \mathcal{P}(G)$ is defined by: $\omega_H(E) = \mu_G(E \cap H)$ $\forall E \in \mathcal{B}$. We will henceforth note ω_H the *Haar measure* of H.

The Haar measure ω_G of the group G is the unique measure of $\mathcal{P}(G)$ verifying:

$$\omega_G * \mu = \mu * \omega_G = \omega_G \qquad \forall \mu \in \mathcal{P}(G).$$

Vorob'yov [104] considered the case where G is a finite commutative group. Hewitt and Zuckerman [41] studied a class of finite commutative semigroups including all finite commutative groups. Kakehashi [50] studied the case where G is the circle group. Kawada and Ito [58] have shown that all idempotent measures of a Hausdorff compact group arises from the Haar measure of a closed subgroup. Wendel [111] identified all the idempotent measures when G is a separable compact group. Kloss [60] and Urbanik [96] have obtained some results in the case where G is a Hausdorff compact group. These first investigations were continued by Rudin [83] and completed by the work of Cohen [24] who characterized all idempotent measures on a locally compact abelian group.

We will first define an idempotent measure before recalling Kawada-Ito result.

A measure μ on a group G is called *idempotent* if $\mu * \mu = \mu$.

For example, the Haar measure of a group G is an idempotent measure.
Kawada and Ito proved that: A probability measure on a Hausdorff compact group G is idempotent if and only if it is the Haar measure of a closed subgroup $H \leq G$.

A *locally compact group* is a topological group G such that every element $g \in G$ has a compact neighborhood. We always assume that G has a countable base of open sets.

Let G be a locally compact group and $(X, \mathcal{B}, \mu)$ a measured space where

$$\mu : \mathcal{B} \to \mathbb{R}_+ \cup \{+\infty\}$$

is a positive measure that we assume σ-finite.

A *measurable action* of G on X denoted by $(X, \mu) \overset{\alpha}{\curvearrowleft} G$ is an action such that:

$$\begin{cases} G \times X & \to X \\ (g, x) & \mapsto x \end{cases}$$

is μ-measurable.

The action $(X, \mathcal{B}, \mu) \overset{\alpha}{\curvearrowleft} G$ is *ergodic* if for any $A \in \mathcal{B}$ such that $gA = A$ for all $g \in G$, we have either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Quantum groups: The early days

The notion of quantum group refers to various objects which are deformations of groups but which still have very similar properties to groups, and in particular to semi-simple Lie groups. The most important are the Hopf algebras which deform function algebras on semi-simple Lie groups or enveloping algebras of Kac-Moody Lie algebras.

The current popular theme of quantum groups can be approached from two essentially different directions. The first approach, the most widespread, is algebraic in nature. The first successes of this approach date back to Drinfel'd (see [33]) and Jimbo (see [43]), who defined the deformations with one parameter of a universal enveloping algebra of semisimple complex Lie algebras in 1985. Many other classes of Hopf Algebras have been studied since 1985 and many have received the label "quantum group". The second approach is analytical in nature: the basic motivation in the early development of the theory has been the generalization of Pontryagin duality for locally compact abelian groups. Because the dual of a non-abelian group can no longer be a group, we are looking for a larger category that also contains the dual. These generalized objects would again be called quantum groups.

This thesis will deal with both the algebraic approach and the analytical approach.

The first time that the term "quantum group" appeared both algebraically and analytically was during G.I.Kac [45,46] work on the *ring groups*. Kac tried to extend the duality of Pontryagin to non-abelian classical groups, since in general the dual of a classical non-abelian group is not necessarily a group. Hence the need to introduce a category including both locally compact classical groups and their duals.

From this perspective, T. Tannaka obtained a duality theorem for compact groups in 1938 in [95]. He managed to recover a compact group from its irreducible representations. However, the latter did not yet have a group structure. M.G. Krein, for his part, defined the dual of a compact group from irreducible representations by modeling them in the form of an algebra of block matrix.

W.F.Stinespring proved a duality theorem for locally compact unimodular groups. He succeeded to recover the latter from the group von Neumann algebra in 1959.

In 1964, G.I.Kac & V.G.Paljutkin in [47] gave the first example of a locally compact quantum group of infinite dimension. They also studied quantum groups of finite dimension with their famous example: the Kac-Paljutkin quantum group of dimension 8 [48].

Independently in the 1970s, G.I.Kac and L.Vainermann [49,100] on the one hand and M.Enock and J.M.Schwartz [36] on the other, defined a complete category encompassing locally compact groups and their duals. This object, which they define, now bears the name of *Kac algebras*. Several examples of *Kac algebras* followed. In the 1980s, quantum groups appeared in a different form by quantum deformation of Lie algebras. These deformations are known as *Hopf algebras*. The theory of *Hopf algebras* is algebraic whereas the approach of *Kac algebras* is analytical.

In 1987, S.L.Woronowicz developed the theory of *compact matrix pseudo-groups* [113]. He gave as an illustration example the quantum group $SU_q(2)$ in [114]. In 1998, S.L.Woronowicz defined the *compact quantum groups* in [118]. An essential point of his theory was the demonstration of the existence and the uniqueness of a *Haar measure* on these *compact quantum groups*.

On the other hand, E.Effros and Z.-J.Ruan [35] and A.Van Daele [102] developed a dual approach by defining *the discrete quantum groups*.

In 1998, A.Van Daele [103] defines a class of quantum groups, called *algebraic quantum groups* including both discrete and compact quantum groups. In 2000, Johan Kustermans & Stefaan Vaes defined *locally compact quantum groups* in [66]. There is two approaches to locally compact quantum groups: C^*-algebraic approache and Von Neumann algebraic approache, and these generalize the approaches to the Kac algebras, the compact quantum group, and the Hopf algebras. Other attempts to unify all these definitions, particularly with the *multiplicatives unitaries* of S.Baaj and G.Skandalis [5], have had little success because of their technical difficulties. One of the main characteristics of this new approach compared to its predecessors is the axiomatic existence of invariant weights on the left and right. This gives a non-commutative analogue of the left and right Haar measure on a Hausdorff locally compact group.

In 1996, A.Pal [78] showed that for a quantum group the idempotent states did not necessarily result from the Haar measure of a quantum subgroup as in the classical case.

In 2009, U. Franz and A.Skalski [39] proved that the idempotent states of a compact quantum group are in bijective correspondence with the coideal subalgebras with conditional expectation. A similar result for locally compact quantum groups was proved by P.Salmi and A.Skalski in 2011 in [85].

0.2 Lattices of quantum subgroups

Let $\mathbb{G}$ be a locally compact quantum group with comultiplication $\Delta_{\mathbb{G}}$ and multiplicative unitary $\mathrm{W}^{\mathbb{G}}$. A von Neumann subalgebra $\mathsf{N} \subset \mathrm{L}^\infty(\mathbb{G})$ is called

- *Left coideal* if $\Delta_{\mathbb{G}}(\mathsf{N}) \subset \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathsf{N}$;
- *Invariant subalgebra* if $\Delta_{\mathbb{G}}(\mathsf{N}) \subset \mathsf{N} \bar{\otimes} \mathsf{N}$;
- *Baaj-Vaes subalgebra* if N is an invariant subalgebra of $\mathrm{L}^\infty(\mathbb{G})$ which is preserved by unitary antipode R and the scaling group $(\tau_t)_{t\in\mathbb{R}}$ of $\mathbb{G}$;
- *Normal* if $\mathrm{W}^{\mathbb{G}}(\mathbb{1} \otimes \mathsf{N})\mathrm{W}^{\mathbb{G}^*} \subset \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathsf{N}$;
- *Integrable* if the set of integrable elements with respect to the right Haar weight $\psi_{\mathbb{G}}$ is dense in N^+; in other words, the restriction of $\psi_{\mathbb{G}}$ to N is semifinite.

In the sequel a von Neumann subalgebra of $\mathrm{L}^\infty(\mathbb{G})$ which is a left coideal will be called a $\mathbb{G}$-coideal or simply a coideal.

Definition 0.2.1 (Definition 1.3.1 page 54)**.** Let $\mathbb{G}$ be a locally compact quantum group. The set of $\mathbb{G}$-coideals will be denoted by $\mathcal{CI}(\mathbb{G})$. We equip $\mathcal{CI}(\mathbb{G})$ with the poset structure: for $\mathsf{N}, \mathsf{M} \in \mathcal{CI}(\mathbb{G})$ we write $\mathsf{N} \leq \mathsf{M}$ if $\mathsf{N} \subset \mathsf{M}$. The poset $(\mathcal{CI}(\mathbb{G}), \leq)$ admits two operations $\wedge, \vee$

- $\mathsf{N} \wedge \mathsf{M} = \mathsf{N} \cap \mathsf{M}$,
- $\mathsf{N} \vee \mathsf{M} = \{xy : x \in \mathsf{N}, y \in \mathsf{M}\}''$.

$(\mathcal{CI}(\mathbb{G}), \leq, \wedge, \vee)$ forms a lattice which will be called the *lattice of coideals* of $\mathbb{G}$.

The subset of $\mathcal{CI}(\mathbb{G})$ of normal $\mathbb{G}$-coideals will be denoted $\mathcal{NCI}(\mathbb{G})$. The subset of $\mathcal{CI}(\mathbb{G})$ of Baaj-Vaes subalgebras of $\mathrm{L}^\infty(\mathbb{G})$ will be denoted $\mathcal{BV}(\mathbb{G})$.

It is easy to check that $\mathcal{NCI}(\mathbb{G})$ and $\mathcal{BV}(\mathbb{G})$ form sublattices of $\mathcal{CI}(\mathbb{G})$. Similarly $\mathcal{NCI}(\mathbb{G}) \cap \mathcal{BV}(\mathbb{G})$ forms a sublattice of $\mathcal{CI}(\mathbb{G})$.

A locally compact quantum group admits a dual object $\widehat{\mathbb{G}} = (\mathrm{L}^\infty(\widehat{\mathbb{G}}), \Delta_{\widehat{\mathbb{G}}}, \varphi_{\widehat{\mathbb{G}}}, \psi_{\widehat{\mathbb{G}}})$.

A lattice of quantum subgroups was then defined as follows:

Definition 0.2.2. Let $\mathbb{G}$ be a locally compact quantum group. The lattice $\mathcal{BV}(\widehat{\mathbb{G}})$ will be denoted $\mathcal{QS}(\mathbb{G})$ and called a *lattice of quantum subgroups* of $\mathbb{G}$.

The lattice $\mathcal{NCI}(\widehat{\mathbb{G}}) \cap \mathcal{BV}(\widehat{\mathbb{G}})$ will be denoted $\mathcal{NQS}(\mathbb{G})$ and called a *lattice of normal quantum subgroups* of $\mathbb{G}$.

We make explicit the largest closed quantum subgroup of $\widehat{\mathbb{G}}$ or in other terms, the largest commutative Baaj-Vaes subalgebra.

Proposition 0.2.3 (Proposition 1.3.3 page 55)**.** *Let $\mathbb{G}$ be a locally compact quantum group and let us consider*

$$\mathsf{M} = \{x \in \mathrm{L}^\infty(\mathbb{G}) : (\mathrm{id} \otimes \Delta_{\mathbb{G}}^{op})(\Delta_{\mathbb{G}}(x)) = (\mathrm{id} \otimes \Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x))\}. \tag{0.2.1}$$

Then M is a Baaj-Vaes subalgebra of $\mathrm{L}^\infty(\mathbb{G})$. The quantum group $\mathbb{H}$ such that $\mathsf{M} = \mathrm{L}^\infty(\mathbb{H})$ is abelian. Let N be another Baaj-Vaes subalgebra and $\mathbb{L}$ be the locally compact quantum group assigned to N. If $\mathbb{L}$ is abelian then $\mathsf{N} \subset \mathsf{M}$.

Relying on [53, Theorem 6.2, Corollary 6.5], we formulate the First Noether's Isomorphism Theorem for locally compact quantum group.

Theorem 0.2.4 (Theorem 1.3.11 page 59)**.** *Let $\mathbb{H}$ and $\mathbb{G}$ be locally compact quantum groups, $\Pi : \mathbb{H} \to \mathbb{G}$ a morphism and let $\Pi_{\mathbb{H}/\ker\Pi \to \overline{im\Pi}}$, $\Pi_{\mathbb{H}/\ker\Pi \to \mathbb{G}}$, $\widehat{\Pi}_{\widehat{\overline{im\Pi}} \to \widehat{\mathbb{H}}}$ be the morphisms induced by Π as described above. Then the following conditions are equivalent:*

(i) $\Pi_{\mathbb{H}/\ker\Pi \to \overline{im\Pi}}$ *is an isomorphism;*

(ii) the action $\alpha : \mathrm{L}^\infty(\overline{im\Pi}) \to \mathrm{L}^\infty(\overline{im\Pi}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H}/\ker\Pi)$ *is integrable;*

(iii) $\Pi_{\mathbb{H}/\ker\Pi \to \mathbb{G}}$ *identifies* $\mathbb{H}/\ker\Pi$ *with a closed quantum subgroup of* $\mathbb{G}$*;*

(iv) $\widehat{\Pi}_{\widehat{\overline{im\Pi}} \to \widehat{\mathbb{H}}}$ *identifies* $\widehat{\overline{im\Pi}}$ *with a closed quantum subgroup of* $\widehat{\mathbb{H}}$*.*

Definition 0.2.5 (Definition 1.3.17 page 62)**.** Let $\mathbb{G}$ be a locally compact quantum group and $\mathbb{H}, \mathbb{M} \in \mathcal{QS}(\mathbb{G})$. We say that $\mathbb{H}$ is normalized by $\mathbb{M}$ if $\mathbb{H} \in \mathcal{NQS}(\mathbb{H} \vee \mathbb{M})$.

In [51, Definition 2.2], Kalantar-Kasprzak-Skalski defined an open quantum subgroup. The subset of $\mathcal{QS}(\mathbb{G})$ that consists of open quantum subgroups of $\mathbb{G}$ will be denoted by $\mathcal{OQS}(\mathbb{G})$. We showed in particular that $\mathcal{OQS}(\mathbb{G})$ forms a lattice.

Proposition 0.2.6 (Proposition 1.3.26 page 64)**.** *Let $\mathbb{G}$ be a locally compact quantum group, $\mathbb{H} \in \mathcal{OQS}(\mathbb{G})$ and $\mathbb{M} \in \mathcal{QS}(\mathbb{G})$. Then $\mathbb{H} \wedge \mathbb{M} \in \mathcal{OQS}(\mathbb{M})$.*

We proved also a strong transitivity of openness.

Proposition 0.2.7 (Proposition 1.3.28 page 65)**.** *Let $\mathbb{H} \leq \mathbb{M} \leq \mathbb{G}$ be a chain of closed embeddings of locally compact quantum groups. Then, $\mathbb{H}$ is open in $\mathbb{G}$ if and only if*

$$\mathbb{H} \leq \mathbb{M} \quad \textit{and} \quad \mathbb{M} \leq \mathbb{G}$$

are both open.

After that, we define the well-positioned quantum subgroup.

For subgroups $\mathbb{H} \leq \mathbb{G}$, we will be working with the quantum homogeneous spaces (see Remark 1.3.2 page 54)

$$A_{\mathbb{H}} = \mathrm{L}^{\infty}(\mathbb{G}/\mathbb{H}) = \mathrm{cd}(\mathrm{L}^{\infty}(\widehat{\mathbb{H}})) \subseteq \mathrm{L}^{\infty}(\mathbb{G}).$$

Definition 0.2.8. Let $\mathbb{H}$ and $\mathbb{M}$ be two closed quantum subgroups of a locally compact quantum group $\mathbb{G}$. We say that $\mathbb{H}$ and $\mathbb{M}$ are *(relatively) well positioned* if we have the equality

$$A_{\mathbb{H}} \vee A_{\mathbb{M}} = \{A_{\mathbb{H}} A_{\mathbb{M}}\}^{\sigma-\mathrm{cls}} \tag{0.2.2}$$

(or equivalently its analogue with $\mathbb{H}$ and $\mathbb{M}$ reversed).

As we will see in Theorem 3.3.4, the well positioning property is relevant to the modular law for quantum subgroups of a locally compact quantum group. Here, we discuss sufficient conditions that ensure well positioning. Let us also note that in the algebraic context the counterpart of well positioning always holds as noted in Corollary 1.2.10.

Proposition 0.2.9 (Proposition 1.3.32 page 66)**.** *The closed quantum subgroups $\mathbb{H}, \mathbb{M} \leq \mathbb{G}$ are relatively well positioned if*

(a) $\mathbb{G}$ *is classical;*

(b) one of $\mathbb{H}$ *and* $\mathbb{M}$ *is compact;*

(c) one of $\mathbb{H}$ *and* $\mathbb{M}$ *is normal;*

(d) $\mathbb{G}$ *is dual-classical.*

0.3 Noether's fundamental isomorphism theorems

We shall now state a quantum version of Noether's Second Isomorphism Theorem for linearly reductive quantum groups and discrete quantum groups.

Theorem 0.3.1 (Theorem 2.1.4 page 75)**.** *Let $\mathbb{H} \leq \mathbb{G}$ and $\mathbb{K} \trianglelefteq \mathbb{G}$ be linearly reductive quantum subgroups of a linearly reductive quantum group. If $\mathbb{H}$ and $\mathbb{K}$ generate $\mathbb{G}$, then the canonical morphism $\mathbb{H}/\mathbb{H} \wedge \mathbb{K} \to \mathbb{G}/\mathbb{K}$ is an isomorphism.*

Remark 0.3.2. *Note incidentally that a trivial version of the first isomorphism theorem is implicit in the proof of Theorem 2.1.4. For a morphism $\Pi : \mathbb{H} \to \mathbb{G}$ of locally compact quantum groups, $\mathbb{H}/\ker \Pi$ is essentially the smallest "quotient LCQG" $\mathbb{H} \to ?$ for which Π factors as*

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\quad\Pi\quad} & \mathbb{G} \\ & \searrow \quad ? \quad \nearrow & \end{array} \tag{0.3.1}$$

(see e.g. Definition 1.3.8 page 58) Similarly, $\overline{\mathrm{im}\Pi}$ *is the smallest* $? \leq \mathbb{G}$ *such that* Π *factors similarly to (2.1.3) as*

$$\mathbb{H} \xrightarrow{\Pi} \mathbb{G}, \quad \mathbb{H} \twoheadrightarrow \; ? \hookrightarrow \mathbb{G} \tag{0.3.2}$$

In the algebraic case, the image of a Hopf algebra morphism $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$ *clearly has both factorization universality properties, and hence by default the two concepts coincide. For this reason, we do not state a First Isomorphism Theorem in the present section.*

We have also shown a version of the Theorem 0.3.1 for discrete quantum groups.

Theorem 0.3.3 (Theorem 2.1.6 page 75)**.** *If the subgroup* $\mathbb{H}$ *and the normal subgroup* $\mathbb{K}$ *generate the discrete quantum group* $\mathbb{G}$*, then the canonical morphism* $\mathbb{H}/\mathbb{H} \wedge \mathbb{K} \to \mathbb{G}/\mathbb{K}$ *is an isomorphism.*

After we reformulated the quantum analogue of the Noether First Isomorphism Theorem (which is obtained trivially in the algebraic case) established in [53, Theorem 6.2, Corollary 6.5] in the locally compact context with an additional condition of integrability, we proved the quantum analogue of the Noether's Second Isomorphism Theorem for locally compact quantum groups always with the additional condition of integrability.

Theorem 0.3.4 (Theorem 3.1.1 page 83)**.** *Let* $\mathbb{G}$ *be a locally compact quantum group,* $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$ *and* $\mathbb{N} \in \mathcal{NQS}(\mathbb{G})$*. Let us denote by* $\Pi : \mathbb{H} \to \mathbb{G}/\mathbb{N}$ *the induced morphism. Then* $\mathbb{H} \wedge \mathbb{N} = \ker \Pi$*. If moreover* $\mathbb{G} = \mathbb{H} \vee \mathbb{N}$ *then* $\overline{im\Pi} = \mathbb{G}/\mathbb{N}$*.*

Using Theorem 1.3.11 and Proposition 3.1.1 we get

Corollary 0.3.5 (Corollary 3.1.2 page 84)**.** *The homomorphism* $\mathbb{H}/\mathbb{H} \wedge \mathbb{N} \to \mathbb{G}/\mathbb{N}$ *is an isomorphism if and only if the corresponding action of* $\mathbb{H}/\mathbb{H} \wedge \mathbb{N}$ *on* $\mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ *is integrable.*

Remark 0.3.6. *Let us also note that equivalent statements in Corollary 0.3.5 fails (and hence so does the second isomorphism theorem) in general even classically, for locally compact abelian groups, as the Example 3.1.4 page 84 shows.*

The fundamental characteristic of Example 3.1.4 page 84 is that the naive product $\mathbb{HN}$ *is not closed in* $\mathbb{G}$*, and hence* $\mathbb{H} \vee \mathbb{N}$ *is "larger than expected". Indeed, classically, it is this failure of* $\mathbb{HN}$ *to be closed that prevents the conditions of Corollary 0.3.5 from holding. This is summarized in the following result.*

Proposition 0.3.7 (Proposition 3.1.5 page 84)**.** *Let* $\mathbb{G}$ *be a classical locally compact group, and* $\mathbb{H} \leq \mathbb{G}$ *and* $\mathbb{N} \trianglelefteq \mathbb{G}$ *closed subgroups.*

Then, $\mathbb{H}/\mathbb{H} \wedge \mathbb{N}$ *acts integrably on* $\mathbb{G}/\mathbb{N}$ *if and only if for every* $(\mathbb{H} \wedge \mathbb{N})$*-invariant closed subset* $\mathbb{F}$ *of* $\mathbb{H}$ *the product* $\mathbb{FN}$ *is closed.*

In 2013, Shuzhou Wang demonstrated for the first time a quantum analogue of one of Noether's three isomorphism theorems, namely the quantum analogue of the Noether Third Isomorphism Theorem for compact quantum groups in [107, Theorem 4.1]. We have established its equivalent for locally compact quantum groups.

Theorem 0.3.8 (Theorem 3.2.6 page 89)**.** *Let* $\mathbb{N} \leq \mathbb{H} \trianglelefteq \mathbb{G}$ *be inclusions of closed locally compact quantum subgroups, and assume furthermore that* $\mathbb{N}$ *is normal in* $\mathbb{G}$*. Then, we have*

$$\mathbb{H}/\mathbb{N} \trianglelefteq \mathbb{G}/\mathbb{N} \quad \textit{and} \quad (\mathbb{G}/\mathbb{N})/(\mathbb{H}/\mathbb{N}) \cong \mathbb{G}/\mathbb{H}.$$

Since the conclusion that

$$\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$$

is a closed embedding does not actually require the normality of $\mathbb{H}$, Theorem 0.3.8 is generalized in the following way:

Theorem 0.3.9 (Theorem 3.2.7 page 89). *Let $\mathbb{N} \leq \mathbb{H} \leq \mathbb{G}$ be closed embeddings of locally compact quantum groups, with $\mathbb{N}$ normal in $\mathbb{G}$. Then the canonical morphism*

$$\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$$

is a closed embedding, and

$$\mathrm{L}^\infty((\mathbb{G}/\mathbb{N})/(\mathbb{H}/\mathbb{N})) = \mathrm{L}^\infty(\mathbb{G}/\mathbb{H})$$

0.4 Dedekind's modular law and Zassenhaus lemma

As in the classical case, a quantum analogue of the Zassenhaus lemma has been obtained for quantum groups. However, it was first necessary to show a quantum analogue of Dedekind's modular law for quantum groups.

Proposition 0.4.1 (Proposition 2.2.1 page 76). *Let $\mathbb{N} \leq \mathbb{H}$ and $\mathbb{M}$ be normal subgroups of the linearly reductive quantum group $\mathbb{G}$. Then, we have*

$$\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{N}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{N}. \tag{0.4.1}$$

Dedekind's modular law is also obtained for compact quantum groups. However, it was necessary to define before when a quantum subgroup normalizes another quantum subgroup.

Definition 0.4.2 (Definition 2.2.3 page 77). A quantum subgroup $\mathbb{L} \leq \mathbb{G}$ *normalizes* another $\mathbb{M} \leq \mathbb{G}$ if the latter is normal in the quantum subgroup $\mathbb{M} \vee \mathbb{L}$.

We are now ready to state

Proposition 0.4.3 (Proposition 2.2.4 page 77). *Let $\mathbb{G}$ be a compact quantum group, with quantum subgroups $\mathbb{L} \leq \mathbb{H} \leq \mathbb{G}$ and $\mathbb{M} \leq \mathbb{G}$ such that $\mathbb{L}$ normalizes $\mathbb{M}$. Then, the equality $\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L}$ holds.*

After this, we prove a quantum version of the following classical argument:

Given a continuous function f on $\mathbb{G}$, the expression for its expectation $E_{\mathbb{L}} : L^\infty(\mathbb{G}) \to A_{\mathbb{L}}$ is

$$(E_{\mathbb{L}} f)(\mathbb{L}_g) = \int_{\mathbb{L}} f(l_g)\ \mathrm{d}l.$$

Lemma 0.4.4 (Lemma 2.2.5 page 78). *Let $\mathbb{M}$ and $\mathbb{L}$ be quantum subgroups of a compact quantum group $\mathbb{G}$ such that $\mathbb{L}$ normalizes $\mathbb{M}$. Then, we have*

$$E_{\mathbb{L}}(A_{\mathbb{M}}) \subseteq A_{\mathbb{M}} \wedge A_{\mathbb{L}}.$$

In the same way as for compact quantum groups, we demonstrate a quantum version of Dedekind's modular law for discrete algebraic quantum groups by defining a discrete quantum subgroup normalizing another discrete quantum subgroup.

We have shown a dual version of the previous Proposition 0.4.3, for discrete quantum groups in the sense of Definition 1.2.1.

Quantum subgroups $\mathbb{M} \leq \mathbb{G}$ then correspond to Hopf subalgebras

$$k\mathbb{M} \subseteq k\mathbb{G}.$$

Definition 0.4.5 (Definition 2.2.9 page 80). Let $\mathbb{G}$ be an algebraic discrete quantum group. A quantum subgroup $\mathbb{L}$ *normalizes* another $\mathbb{M} \leq \mathbb{G}$ if the group algebra $k\mathbb{M}$ of the latter is invariant under the adjoint action of $k\mathbb{L}$ on $k\mathbb{G}$.

Proposition 0.4.6 (Proposition 2.2.10 page 80). *Let $\mathbb{G}$ be an algebraic discrete quantum group, with quantum subgroups $\mathbb{L} \leq \mathbb{H} \leq \mathbb{G}$ and $\mathbb{M} \leq \mathbb{G}$ such that $\mathbb{L}$ normalizes $\mathbb{M}$. Then, the equality (2.2.1) holds.*

The previous propositions will allow us to prove the following version of the butterfly (or Zassenhaus) lemma ([62, Vol. 1, p. 77] or [81, Chapter 5, Lemma 5.10]) for compact and discrete quantum groups.

Proposition 0.4.7 (Proposition 2.2.11 page 80). *Let $\mathbb{A}' \trianglelefteq \mathbb{A}$ and $\mathbb{B}' \trianglelefteq \mathbb{B}$ be quantum subgroups of either a compact or an algebraic discrete quantum group $\mathbb{G}$. Then, we have an isomorphism*

$$\frac{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B}')} \cong \frac{\mathbb{B}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{B}' \vee (\mathbb{A}' \wedge \mathbb{B})}.$$

The analogous statement holds for linearly reductive $\mathbb{G}$ provided $\mathbb{A}$, $\mathbb{A}'$, etc. are all normal in $\mathbb{G}$.

For locally compact quantum groups, the quantum analog of Dedekind's modular law and the Zassenhaus lemma are true only in some cases. The following theorem and proposition summarize the situation.

Theorem 0.4.8 (Theorem 3.3.4 page 91). *Let $\mathbb{L} \leq \mathbb{H}$ and $\mathbb{M}$ be closed quantum subgroups of a locally compact quantum group $\mathbb{G}$ such that $\mathbb{L}$ normalizes $\mathbb{M}$. Then, we have*

$$\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L}. \tag{0.4.2}$$

if either

(a) $\mathbb{L}$ is compact, or

(b) $\mathbb{H}$ is open in $\mathbb{G}$.

Proposition 0.4.9 (Proposition 3.3.9 page 93). *Let $\mathbb{A}' \trianglelefteq \mathbb{A}$ and $\mathbb{B}' \trianglelefteq \mathbb{B}$ be either*

(a) compact or

(b) open

quantum subgroups of a locally compact quantum group $\mathbb{G}$. Then, we have an isomorphism

$$\frac{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B}')} \cong \frac{\mathbb{B}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{B}' \vee (\mathbb{A}' \wedge \mathbb{B})}.$$

0.5 The Schreier refinement theorem and the Jordan-Hölder theorem for quantum groups

In this section, we present the quantum analogue of the Schreier refinement theorem and the Jordan-Hölder theorem for compact and discrete quantum groups (see for example [81, Chapter 5, Theorem 5.11] and [81, Chapter 5, Theorem 5.12]) for the classical analogue for ordinary discrete groups). For this purpose, we first defined the quantum analogue of the notion of subnormal series and composition series.

Definition 0.5.1 (Definition 2.3.1 page 81)**.** Let $\mathbb{G}$ be either a compact or (algebraic) discrete quantum group. A finite system

$$\mathbb{G} = \mathbb{G}_0 \geq \mathbb{G}_1 \geq \mathbb{G}_2 \geq \mathbb{G}_3 \geq \cdots \geq \mathbb{G}_k = 1 \tag{0.5.1}$$

of closed quantum subgroups of $\mathbb{G}$ is called a *subnormal series* of $\mathbb{G}$ if every subgroup $\mathbb{G}_i$ is a proper normal closed quantum subgroup of $\mathbb{G}_{i-1}$, $i \in \{1, 2, \cdots, k\}$. In particular, $\mathbb{G}_1$ is a normal closed quantum subgroup of $\mathbb{G}$, $\mathbb{G}_2$ is a normal closed quantum subgroup of $\mathbb{G}_1$, but not necessarily of $\mathbb{G}$, and so on.

A subnormal series is *normal* if each $\mathbb{G}_i$ is normal in the ambient group $\mathbb{G}$.

The corresponding subquotient quantum groups

$$\mathbb{G}_1 \backslash \mathbb{G}, \ \mathbb{G}_2 \backslash \mathbb{G}_1, \ \cdots, \ \mathbb{G}_k \backslash \mathbb{G}_{k-1}$$

of $\mathbb{G}$ are the *factors* of the (sub)normal series (0.5.1).

The integer k is the *length* of the series (0.5.1).

Definition 0.5.2. A subnormal series

$$\mathbb{G} = \mathbb{H}_0 \geq \mathbb{H}_1 \geq \mathbb{H}_2 \geq \mathbb{H}_3 \geq \cdots \geq \mathbb{H}_l = 1 \tag{0.5.2}$$

is called a *refinement* of the subnormal series (0.5.1) if every quantum subgroup $\mathbb{G}_i$ of (0.5.1) coincides with one of the quantum subgroups $\mathbb{H}_j$, i.e. if every quantum subgroup that occurs in (0.5.1) also occurs in (0.5.2).

In particular, every subnormal series is a refinement of itself. The lengths of the normal series (0.5.1) and its refinement (0.5.2) of course satisfy the inequality $k \leq l$.

Two subnormal series of a compact quantum groups are called *equivalent* if their lengths are equal and their constituent subquotients are isomorphic up to permutation.

We are now ready to state the quantum analogue of the Schreier refinement theorem and the Jordan-Hölder theorem, and their proofs follow respectively from the Zassenhaus lemma and Schreier refinement theorem.

Theorem 0.5.3 (Theorem 2.3.3 page 82)**.** *Any two subnormal series of a compact / discrete quantum group $\mathbb{G}$ have equivalent refinements.*

The same holds for any two normal series of a linearly reductive quantum group.

Definition 0.5.4 (Definition 2.4.1 page 82)**.** A subnormal series (0.5.1) is a *composition series* of $\mathbb{G}$ if $\mathbb{G}_i$ is a proper maximal normal closed quantum subgroup of $\mathbb{G}_{i-1}$ for $1 \leq i \leq k$.

Theorem 0.5.5 (Theorem 2.4.3 page 82)**.** *Any two composition series of a compact or discrete quantum group $\mathbb{G}$ are equivalent.*

For locally compact quantum groups, the quantum analog of the Schreier refinement theorem and the Jordan-Hölder theorem are also true in some cases. Let us first note that the Definition 0.5.1, Definition 0.5.2 and Definition 0.5.4 remain valid for locally compact quantum groups.

The following two theorems summarize the situation.

We write $\{\mathbb{G}_\ell\}_{\ell \geq 0}$ for the generic (sub)normal series

$$\mathbb{G} = \mathbb{G}_0 \geq \mathbb{G}_1 \geq \mathbb{G}_2 \geq \mathbb{G}_3 \geq \cdots \geq \mathbb{G}_k = 1. \tag{0.5.3}$$

of closed quantum subgroups of a locally compact quantum group $\mathbb{G}$.

Theorem 0.5.6 (Theorem 3.4.1 page 94)**.** *Let $\mathbb{G}$ be a locally compact quantum group. Then, any two subnormal series $\{\mathbb{G}_\ell\}$ and $\{\mathbb{G}'_t\}$ of $\mathbb{G}$ admit equivalent refinements, provided*

$$\mathbb{G}_\ell,\ \ell \geq 1 \quad \textit{and} \quad \mathbb{G}'_t,\ t \geq 1$$

are all

(a) compact or

(b) open.

Theorem 0.5.7 (Theorem 3.4.2 page 94)**.** *Let $\mathbb{G}$ be a locally compact quantum group. Then, all composition series of $\mathbb{G}$ consisting of all*

(a) compact or all

(b) open

quantum subgroups are equivalent.

The compact versions of Theorem 0.5.6 and Theorem 0.5.7 refer to subnormal series (0.5.3) in which all $\mathbb{G}_\ell$, $\ell \geq 1$ are compact, but $\mathbb{G} = \mathbb{G}_0$ need not be so. Let us note that this is equivalent to the subquotient $\mathbb{G}/\mathbb{G}_1$ being non-compact. Indeed, we have

Proposition 0.5.8 (Proposition 3.4.3 page 95)**.** *A locally compact quantum group $\mathbb{G}$ is compact if and only if it admits a subnormal series (3.4.1) with compact quotients $\mathbb{G}_i/\mathbb{G}_{i+1}$.*

0.6 Characterization of ergodic actions and idempotents states of the compact quantum group $O_{-1}(2)$

An idempotent measure on a classical Hausdorff compact group is the Haar measure of a closed subgroup. Pal showed in [78] that an analogue result is no longer possible in the quantum case by giving the example of an idempotent measure on the Kac-Paljutkin compact quantum group of dimension 8 that does not result from the Haar measure of a quantum subgroup. In this section, we first define a compact quantum group before we announce the result of Franz-Skalski[39], who first established a characterization of the idempotent states of a compact quantum group. This result was generalized for locally compact quantum groups by Salmi-Skalski[85].

In this section we will give a complete list of the ergodic actions and embedabble ergodic actions of the compact quantum group $O_{-1}(2)$ and characterize the idempotent states of the latter relying on the result of Franz- Skalski [39, Theorem 4.1].

First, let us note that listing the ergodic actions of the compact quantum group $O_{-1}(2)$ is equivalent to listing the ergodic actions of the classical compact group $O(2)$.

Indeed, a fundamental result of Banica-Bichon-Collins, more precisely [6, Theorem 4.3], stipulates that the category of corepresentations of the compact quantum group $O_{-1}(n)$ is tensor equivalent to the category of representation of the classical compact group $O(n)$. From this we deduce that the compact quantum groups $O_{-1}(2)$ and $O(2)$ (which is also a compact quantum group with $q = 1$) are monoidally equivalent.

Another fundamental result of De Rijt-Vander Vennet, more precisely [30, Theorem 7.3], specifies that the ergodic actions of two monoidally equivalent compact quantum groups are in bijective correspondence. We deduce from this that the ergodic actions of the compact quantum group $O_{-1}(2)$ are in bijective correspondence with the ergodic actions of the classical compact group $O(2)$. The first difficulty was to list the ergodic actions of the classical compact group $O(2)$.

Let us first remark that if G is a classical compact group, H a closed subgroup, and $H \overset{\alpha}{\curvearrowright} N$ an ergodic action of H on a von Neumann algebra N, we have the induced ergodic action of G on

$$\begin{aligned} \mathrm{Ind}_H^G(N) &= \{f \in L^\infty(G,N) \mid \forall g \in G, h \in H, \alpha_h(f(gh)) = f(g)\} \\ &\subseteq L^\infty(G)\bar{\otimes}N, \end{aligned}$$

given by

$$(\alpha_g(f))(g') = f(g^{-1}g').$$

In particular, if π is an irreducible projective representation of H on a Hilbert space V_π, we have an ergodic action of G on the von Neumann algebra $\mathrm{Ind}_H^G(B(V_\pi))$ where $B(V_\pi)$ is equipped with the ergodic H-action

$$\alpha_h(x) = \pi(h)x\pi(h)^*.$$

For example, when π is the trivial representation (or a one-dimensional representation), we have $\mathrm{Ind}_H^G(\mathbb{C}) = L^\infty(G/H)$ with the left translation action.

In the following, we will identify $O(2) \cong C_2 \ltimes T$, with $T = \{z \in \mathbb{C} \mid |z| = 1\}$ the circle group, and with the cyclic group $C_2 = \{1, \sigma\}$ acting on T by $\sigma(z) = \bar{z}$.

The following result is a private communication from Kenny De Commer.

Theorem 0.6.1 (Theorem 4.1.2 page 98)**.** *Let $O(2) \underset{\alpha}{\curvearrowright} M$ be an ergodic action. Then $M \cong \mathrm{Ind}_H^{O(2)}(B(V_\pi))$ for some closed subgroup $H \subseteq O(2)$ and π an irreducible representation of H.*

After having determined the isomorphisms between the induced actions, we have obtained the complete list of the non-equivalent ergodic actions of the classical compact group $O(2)$.

The irreducible (projective) representations of $O(2)$ give either the trivial action $\beta_0^{(\infty)}$ on $\mathbb{C}$ (for the characters) or the ergodic action $\beta_{l/2}^{(\infty)} = \alpha$ on $M_2(\mathbb{C})$ by

$$\alpha_z \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z^l b \\ z^{-l} c & d \end{pmatrix}, \quad \alpha_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

for $l \in \mathbb{N} \setminus \{0\}$ (with the even l coming from representations and the odd l coming from non-trivial projective representations).

The irreducible representations of T all give the same action $\alpha = \alpha^{(\infty)}$ on $\mathbb{C}^2 = L^\infty(O(2)/T)$, namely

$$\alpha_z((x,y)) = (x,y), \quad \alpha_\sigma(x,y) = (y,x).$$

The irreducible representations of C_k for some fixed k all give the same action $\alpha = \alpha^{(k)}$ on $L^\infty(O(2)/C_k) = L^\infty(T/C_k) \oplus L^\infty(T/C_k)$, namely

$$\alpha_z(f,g) = (f_z, g_z), \quad \alpha_\sigma(f,g) = (g,f),$$

where f_z denotes the z-translate of f.

Finally, for the dihedral group D_k we have the action $\alpha = \beta_0^{(k)}$ coming from the characters of D_k, giving the action on $L^\infty(O(2)/D_k)$, or for the non-character (projective) representations the actions $\beta_{l/2}^{(k)}$ where $0 < l < k$ a natural number and $\beta_{l/2}^{(k)}$ the induced action coming from the (projective) D_k-representation

$$\alpha_z \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z^l b \\ z^{-l} c & d \end{pmatrix}, \quad \alpha_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Hence, we obtain:

Proposition 0.6.2 (Proposition 4.1.3 page 100)**.** *The full list of non-equivalent ergodic actions of $O(2)$:*

$$\{\beta_{l/2}^{(k)}, \beta_{l'/2}^{(\infty)}, \alpha^{(\infty)}, \alpha^{(k')} \mid k, k' \in \mathbb{N}_0, l' \in \mathbb{N}, 0 \leq l < k\}.$$

We want to deduce the list of the embeddable ergodic actions of $O_{-1}(2)$. To do this, we proceed as follows.

First, note that by definition an embeddable ergodic action is a comodule $*$-algebra of the CQG algebra $\mathcal{A}_{-1}$ associated to $O_{-1}(2)$ which embeds into $\mathcal{A}_{-1}$ as such (i.e. by an embedding that preserves all of the structure: comodule, algebra, etc.).

Since the twisting equivalence $\lambda\triangleright$ (see the notation from Section 4.4) that implements Theorem 1.4.19 also implements an equivalence between the categories of coideal $*$-algebras over $\mathcal{A}_{-1}$ and the untwisted version $\mathcal{A}$ (algebra of representative functions on the classical group $O(2)$), it will be sufficient to identify the ergodic $O(2)$-action $\mathcal{B}$ in the list of Proposition 0.6.2 for which

$$\lambda \triangleright \mathcal{B} \cong \mathcal{A}_{-1}$$

as $\mathcal{A}_{-1}$ comodule $*$-algebras, and to then also identify the members of that list that embed into $\mathcal{B}$.

We will see that there is only one candidate for $\mathcal{B}$ (namely $\beta_{1/2}^{(2)}$) using the Peter-Weyl theorem to determine the representation type of the ergodic actions identified in Proposition 0.6.2 (where by representation type we mean the multiplicities of the various irreducible $O(2)$-representations). Indeed, this is the substance of the following result.

Proposition 0.6.3 (Proposition 4.2.1 page 100)**.** *The only comodule algebras among those in Proposition 0.6.2 that are isomorphic to $\mathcal{A}$ as $O(2)$-representations are $\alpha^{(1)} \cong \mathcal{A}$ itself and $\beta_{1/2}^{(2)}$.*

We then obtained as a consequence the complete list of the embeddable ergodic coactions of the compact quantum group $O_{-1}(2)$.

Corollary 0.6.4 (Corollary 4.2.2 page 101)**.** *The twisting equivalence $\lambda\triangleright$ induces a bijection between*

$$\{\alpha^{(k)},\ \beta_{l/2}^{(k)} \mid k = \infty \text{ or even },\ l = 0 \text{ or odd }\}$$

from Proposition 0.6.2 and the isomorphism classes of embeddable ergodic actions of $O_{-1}(2)$.

The complete list of non-equivalent embeddable ergodic coactions of $O_{-1}(2)$ is then:

$$\{\alpha^{(k)},\ \beta_{l/2}^{(k)} \mid k = \infty \text{ or even },\ l = 0 \text{ or odd }\}$$

To finish this section, we asked ourselves the following question motivated by Theorem 1.4.19 page 71:

Question 0.6.1 (Question 1.4.1 page 71). *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent compact quantum groups. Is there a bijective correspondence between their embeddable ergodic actions?*

Remark 0.6.5. *An affirmative answer to the previous question would extend the fundamental result* [30, Theorem 7.3] *of De Rijt-Vander Vennet to embeddable ergodic actions. But the answer is negative in general (See Question 0.6.2 below) except in the case we have an additional condition: See Proposition 4.4.5 page 106).*

Specifically, let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent compact quantum groups. Suppose furthermore that the monoidal equivalence is implemented by twisting by a cocycle

$$\lambda : C(\mathbb{G}_1)^{\otimes 2} \to \mathbb{C}.$$

Then, according to the proof of Theorem 1.4.19, we have a twisting operation $\lambda\triangleright$ that implements an equivalence between the category $\mathcal{E}rg(\mathbb{G}_1)$ of ergodic $\mathbb{G}_1$-actions and the analogous category $\mathcal{E}rg(\mathbb{G}_2)$. In this setting, the formulation of Question 0.6.1 that we answer here is

Question 0.6.2. *Does*

$$\lambda\triangleright : \mathcal{E}rg(\mathbb{G}_1) \to \mathcal{E}rg(\mathbb{G}_2)$$

restrict to an equivalence between subcategories of embeddable ergodic coactions?

Unfortunately, the answer to Question 0.6.2 is negative in general, as shown in the counter-example below.

Take the compact quantum group $\mathbb{G}_1$ to be the order-64 Heisenberg group

$$H_4 \;:=\; \langle \varepsilon_1, \varepsilon_2, \delta \mid \varepsilon_1^4 = \varepsilon_2^4 = \delta^4 = 1,\ \delta\varepsilon_1 = \varepsilon_1\delta,\ \varepsilon_2\delta = \delta\varepsilon_2,\ \varepsilon_1\varepsilon_2 = \delta\varepsilon_2\varepsilon_1\rangle.$$

Its deformation $\mathbb{G}_2$ (with underlying function algebra $H = C(H_4)$) will be the one used in [16], described in some detail in Section 6 of [16]. Although we will not write the cocycle

$$\lambda : C(H_4)^{\otimes} \to \mathbb{C}$$

explicitly, recall that it is obtained as the composition

$$C(H_4)^{\otimes 2} \longrightarrow C(\Gamma)^{\otimes 2} \longrightarrow \mathbb{C}, \qquad \lambda : C(H_4)^{\otimes 2} \to \mathbb{C}$$

where the left hand arrow is restriction along the inclusion

$$(\mathbb{Z}/2)^2 \cong \Gamma := \langle \varepsilon_1^2, \varepsilon_2^2\rangle \subset H_4$$

and the right hand arrow is some cocycle inducing the non-zero cohomology class in

$$H^2(\widehat{\Gamma}, \mathbb{C}^\times) \cong \mathbb{Z}/2.$$

It will be more convenient to dualize and work with the group algebra $\mathbb{C}H_4$ and its deformed version H^*, which coincides with $\mathbb{C}H_4$ as an algebra but has twisted comultiplication.

We proved that:

Proposition 0.6.6 (Corollary 4.4.3 page 106)**.** *There does not exist an inclusion, dimension, and G-preserving bijection between the coideal subalgebras of $\mathbb{C}H_4$ and those of H^*.*

As one final simple observation, we remark that if $\mathbb{G}_1$ happens to have trivial 2-cohomology set $H^2(\widehat{\mathbb{G}_1}, \mathbb{S}^1)$, the answer to Question 0.6.2 is affirmative:

Proposition 0.6.7 (Proposition 4.4.5 page 106)**.** *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent compact quantum groups, and assume $H^2(\widehat{\mathbb{G}_2}, \mathbb{S}^1)$ is trivial. Then,*

$$\lambda\triangleright : \mathcal{E}rg(\mathbb{G}_1) \to \mathcal{E}rg(\mathbb{G}_2)$$

restricts to an equivalence between subcategories of embeddable ergodic actions.

Idempotent states of the compact quantum group $O_{-1}(2)$

Having succeeded to make the list of the embeddable ergodic actions of $O_{-1}(2)$ and since by definition, an embeddable ergodic action is identified with a coideal subalgebra and since according to the fundamental result [39, Theorem 4.1] by Franz-Skalski, there is a bijective correspondence between the expected coideal subalgebras and the idempotent states of a compact quantum group; Our project has met a final concern: *How to express by a concrete formula an idempotent state from an action of a compact quantum group?*

In fact, although we characterized them by listing embeddable ergodic action using [39], we still have to list explicitly the idempotent states of $O_{-1}(2)$ from the list of its embeddable ergodic actions.

Considering that:

1. Let us recall that if one takes the comultiplication itself as the action (i.e the quantum group itself is taken as the coideal subalgebra), it is not immediate to produce the corresponding Haar state from the multiplication.

2. Also if we consider the subalgebra $\mathbb{C}$, then we see that we can not produce the Haar state from the action which is simply $\mathbb{C} \to \mathbb{C} \otimes \mathbb{C}, 1 \mapsto 1 \otimes 1$.

The problem is that starting from the proof of [39, Theorem 4.1], there are several elements that we have abstractly but non-explicitly, and they depend on each other. Indeed, to obtain the idempotent state of the coideal subalgebra, one needs the conditional expectation E; And to get E, we need unfortunately the state in the proof of [39, Theorem 4.1]. It is therefore circular and does not lead to something explicit (just to the theorem of the paper of Franz-Skalski, i.e the fact that there is this bijection). We believe that if there is a chance to obtain a concrete formula of the idempotent state from an embeddable ergodic action, it will be by expressing the conditional expectation E only as a function of the embeddable ergodic action α by not involving the state itself, then compose E with the counit ε.

The following question summarizes our point of view.

Question 0.6.3 (Question 4.3.1 page 104)**.** *Is it possible to obtain such a concrete formula in general for a compact quantum group at least in the Kac case?*

Chapter 1

Preliminaries

This chapter collects the necessary preliminaries for the whole thesis. The first section is a brief introduction to locally compact quantum group (in the von Neumann algebra setting). In the second section we present some preliminaries on linearly reductive compact quantum groups, including some basic notions and properties, as well as some typical constructions such as the free products and the Drinfeld-Jimbo deformation.

1.1 Preliminaries for locally compact quantum groups

The theory of locally compact quantum groups is formulated in terms of operator algebras. Operator algebra theory is divided into two parts. In order to explain this division let us fix a Hilbert space H. The set of all bounded operators acting on H forms a normed $*$-algebra which we denote by $B(\mathsf{H})$. This algebra, except the norm topology carries a host of locally convex topologies: strong, σ-strong, weak, σ-weak and others. Although the aforementioned distinction does not depend on the choice of the topology listed above, we choose the σ-weak topology on $B(\mathsf{H})$ for its description. In this paper the scalar product $(\cdot|\cdot)$ on a Hilbert space will be linear in the second variable.

Definition 1.1.1. Let I be a directed set and H a Hilbert space. Let $(T_i)_{i\in I}$ be a net of bounded operators acting on H and let $T \in B(\mathsf{H})$. We say that $(T_i)_{i\in I}$ σ-weakly converges to $T \in B(\mathsf{H})$ if for all sequences $(\xi_n)_{n\in\mathbb{N}}, (\zeta_n)_{n\in\mathbb{N}} \in \mathsf{H}$ satisfying $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty, \sum_{n=1}^{\infty} \|\zeta_n\|^2 < \infty$ we have

$$\lim_i \sum_{n=1}^{\infty} (\xi_n|T_i\zeta_n) = \sum_{n=1}^{\infty} (\xi_n|T\zeta_n).$$

We say that $(T_i)_{i\in I}$ σ-$*$ strongly converges to T if

$$\lim_i \sum_{n=1}^{\infty} \|(T - T_i)\zeta_n\|^2 = 0,$$

$$\lim_i \sum_{n=1}^{\infty} \|(T^* - T_i^*)\zeta_n\|^2 = 0.$$

Definition 1.1.2. Let H be a Hilbert space.

(i) A $*$-subalgebra A of $B(\mathsf{H})$ which is closed in the norm topology is called a concrete C*-algebra.

(ii) A unital $*$-subalgebra N of $B(\mathsf{H})$ which is closed in the σ-weak topology is called von Neumann algebra.

Usually we shall skip the term concrete and say that $\mathsf{A} \subset B(\mathsf{H})$ is a C*-algebra.

Let $X \subset B(\mathsf{H})$ be a non-empty subset. The commutant X' of X is defined as

$$X' = \{y \in B(\mathsf{H}) : xy = yx \text{ for all } x \in X\}.$$

We shall write $X'' = (X')'$. The famous bicommutant theorem implies that a *-subalgebra $\mathsf{A} \subset B(\mathsf{H})$ is a von Neumann algebra if and only if $\mathsf{A} = (\mathsf{A}')'$.

Let Y be a topological vector space and X a non-empty subset of Y. The closure of the linear span of X will be denoted X^{cls}. If X is a C*-algebra then the norm closure of the linear span of X will also be denoted by $\mathsf{X}^{\|\cdot\|-\text{cls}}$. If X is a von Neumann algebra then the σ-weak closure of the linear span of X will be denoted by $\mathsf{X}^{\sigma-\text{cls}}$.

Given a pair of C*-algebras $\mathsf{A}_1 \subset B(\mathsf{H}_1)$ and $\mathsf{A}_2 \subset B(\mathsf{H}_2)$, the (spatial) tensor product $\mathsf{A}_1 \otimes \mathsf{A}_2 \subset B(\mathsf{H}_1 \otimes \mathsf{H}_2)$ is defined as

$$\mathsf{A}_1 \otimes \mathsf{A}_2 = \{x \otimes y : x \in \mathsf{A}_1, y \in \mathsf{A}_2\}^{\|\cdot\|-\text{cls}}.$$

Similarly, given a pair of von Neumann algebras $\mathsf{N}_1 \subset B(\mathsf{H}_1)$ and $\mathsf{N}_2 \subset B(\mathsf{H}_2)$, we define

$$\mathsf{N}_1 \bar{\otimes} \mathsf{N}_2 = \{x \otimes y : x \in \mathsf{N}_1, y \in \mathsf{N}_2\}^{\sigma-\text{cls}}.$$

The Banach dual of the Banach space $(B(\mathsf{H}), \|\cdot\|)$ will be denoted by $B(\mathsf{H})^*$. For $\zeta, \xi \in \mathsf{H}$ we define a bounded functional $\omega_{\zeta,\xi} \in B(\mathsf{H})^*$: $\omega_{\zeta,\xi}(T) = (\zeta|T\xi)$ for all $T \in B(\mathsf{H})$. Let us consider a subset X of $B(\mathsf{H})^*$:

$$X = \{\omega_{\zeta,\xi} : \zeta, \xi \in \mathsf{H}\}.$$

We shall denote $B(\mathsf{H})_* = X^{\|\cdot\|-\text{cls}}$. We say that $\omega \in B(\mathsf{H})_*$ is a normal functional on $B(\mathsf{H})$. We have $(B(\mathsf{H})_*)^* = B(\mathsf{H})$ and the σ-weak topology coincide with the weak $*$-topology on $B(\mathsf{H})$.

There is an abstract version of a (concrete) concept of a C*-algebra and a von Neumann algebra formulated in Definition 1.1.2.

Definition 1.1.3. Let A be a Banach $*$-algebra. We say that A is a C*-algebra if the C*-identity $\|a^*a\| = \|a\|^2$ holds for all $a \in \mathsf{A}$. Let N be a C*-algebra. We say that N is a W^*-algebra if N admits a predual Banach space.

Every C*-algebra can be identified with a concrete C*-algebra. A C*-algebra N can be identified with a von Neumann algebra if and only if N is a W^*-algebra. The predual space of a W^*-algebra N is uniquely determined by N and it will be denoted by N_*

In this paper we shall always consider concrete C*-algebras which are non-degenerate.

Definition 1.1.4. Let $\mathsf{A} \subset B(\mathsf{H})$ be a concrete C*-algebra. We say that A is non-degenerate if $\bigcap_{a\in\mathsf{A}} \ker a = \{0\}$.

Let $\mathsf{A} \subset B(\mathsf{H})$ be a non-degenerate C*-algebra. The C*-algebra

$$\mathrm{M}(\mathsf{A}) = \{x \in B(\mathsf{H}) : xa, ax \in \mathsf{A}\}$$

is called a *multiplier* C**-algebra* of A.

It can be proved that the multiplier C*-algebra $\mathrm{M}(\mathsf{A})$ of a concrete C*-algebra $\mathsf{A} \subset B(\mathsf{H})$ does not depend on the embedding $\mathsf{A} \subset B(\mathsf{H})$. To be more precise if K is a Hilbert space and $\pi : \mathsf{A} \to B(\mathsf{K})$ is an injective $*$-homomorphism then $\pi(\mathsf{A})$ is a C*-subalgebra of $B(\mathsf{K})$ and $\mathrm{M}(\mathsf{A})$ and $\mathrm{M}(\pi(\mathsf{A}))$ are isomorphic (as C*-algebras).

Let B be a C*-algebra and C a C*-subalgebra of $\mathrm{M}(\mathsf{B})$. The set

$$\{cb : c \in \mathsf{C}, b \in \mathsf{B}\}^{\|\cdot\|-\mathrm{cls}}$$

will be denoted CB. Let $\pi : \mathsf{A} \to \mathrm{M}(\mathsf{B})$ be a $*$-homomorphism. We say that π is non-degenerate if $\pi(\mathsf{A})\mathsf{B} = \mathsf{B}$. The set of non-degenerate $*$-homomorphisms from A to $\mathrm{M}(\mathsf{B})$ will be denoted by $\mathrm{Mor}(\mathsf{A}, \mathsf{B})$. It can be checked that there exists a unique $*$-homomorphism $\overline{\pi} : \mathrm{M}(\mathsf{A}) \to \mathrm{M}(\mathsf{B})$ such that for all $x \in \mathrm{M}(\mathsf{A})$ and $a \in \mathsf{A}$ we have $\pi(xa) = \overline{\pi}(x)\pi(a)$. In particular $\overline{\pi}$ extends π and in what follows this extension will be denoted by π. Note that for $\pi \in \mathrm{Mor}(\mathsf{A}, \mathsf{B})$ and $\rho \in \mathrm{Mor}(\mathsf{B}, \mathsf{C})$ we can form $\rho \circ \pi \in \mathrm{Mor}(\mathsf{A}, \mathsf{C})$. This composition gives rise to the category of C*-algebras with $\mathrm{Mor}(\mathsf{A}, \mathsf{B})$ being morphisms.

Let N and M be von Neumann algebras. A unital $*$-homomorphism $\pi : \mathsf{N} \to \mathsf{M}$ is said to be normal if it is continuous in the σ-weak topologies. The set of positive elements of N will be denoted by N^+.

Definition 1.1.5. Let M be a von Neumann algebra. A weight on M is a function $\psi : \mathsf{M}^+ \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $\psi(0) = 0$, $\psi(x + y) = \psi(x) + \psi(y)$ and $\psi(tx) = t\psi(x)$ for all $t \in \mathbb{R}_{\geq 0}$ and $x, y \in \mathsf{M}^+$. We say that ψ is normal if it is lower semi-continuous in the σ-weak topology on M^+. We say that ψ is semifinite if the set

$$\{x \in \mathsf{M}^+ : \psi(x) < \infty\}$$

is σ-weakly dense in M^+. We say that that ψ is faithful if $\psi(x) = 0 \implies x = 0$. A normal semifinite faithful weight will be called an n.s.f. weight.

Let ψ be an n.s.f. weight on M. Then we define the following sets:

- $\mathcal{M}_\psi^+ = \{x \in \mathsf{M}^+ : \psi(x) < \infty\}$,
- $\mathcal{N}_\psi = \{x \in \mathsf{M} : \psi(x^*x) < \infty\}$,
- $\mathcal{M}_\psi = \mathrm{Lin}\mathcal{M}_\psi^+$.

Let us note that $\mathcal{N}_\psi$ forms a left ideal in N. It can be checked that $\mathcal{M}_\psi = \mathrm{Lin}\{x^*y : x, y \in \mathcal{N}_\psi\}$ and ψ yields a linear map $\psi : \mathcal{M}_\psi \to \mathbb{C}$.

The GNS-construction based on ψ is a triple $(\mathsf{H}_\psi, \pi_\psi, \eta)$ where H_ψ is a Hilbert space $\pi_\psi : \mathsf{N} \to B(\mathsf{H}_\psi)$ is a normal $*$-homomorphism and $\eta : \mathcal{N}_\psi \to \mathsf{H}_\psi$ is a σ-$*$ strongly closed linear map such that

- $(\eta(x)|\eta(y)) = \psi(x^*y)$ for all $x, y \in \mathcal{N}_\psi$,
- $\eta(xy) = \pi_\psi(x)\eta(y)$ for all $x \in \mathsf{N}$ and $y \in \mathcal{N}_\psi$.

A GNS construction for ψ always exists and is essentially unique.

For the theory of locally compact quantum groups we refer to [63, 65].

Definition 1.1.6. A von Neumann algebraic locally compact quantum group is a quadruple $\mathbb{G} = (\mathsf{M}, \Delta_\mathbb{G}, \varphi_\mathbb{G}, \psi_\mathbb{G})$, where M is a von Neumann algebra, $\Delta : \mathsf{M} \to \mathsf{M} \bar{\otimes} \mathsf{M}$ is a normal injective $*$-homomorphism satisfying

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

and $\varphi_{\mathbb{G}}$ and $\psi_{\mathbb{G}}$ are, respectively, normal semifinite faithful left and right invariant weights on M, i.e.

$$\psi_{\mathbb{G}}((\mathrm{id}\otimes\omega)(\Delta(x)) = \psi_{\mathbb{G}}(x)$$
$$\varphi_{\mathbb{G}}((\omega\otimes\mathrm{id})(\Delta(x)) = \varphi_{\mathbb{G}}(x)$$

for all $x \in \mathsf{M}^+$ and $\omega \in \mathsf{M}_*^+$.

Let $\mathbb{G}$ be a locally compact quantum group. We shall use a notation $\mathsf{M} = \mathrm{L}^\infty(\mathbb{G})$, $\Delta = \Delta_{\mathbb{G}}$. The GNS Hilbert space of the right Haar weight $\psi_{\mathbb{G}}$ will be denoted by $\mathrm{L}^2(\mathbb{G})$ and the corresponding GNS map will be denoted by $\eta_{\mathbb{G}}$. $\mathbb{G}$ is assigned with the *antipode*, the *scaling group* and the *unitary antipode* which are denoted by S, $(\tau_t)_{t\in\mathbb{R}}$ and R. A fundamental role in the theory of locally compact quantum groups is played by the multiplicative unitary $\mathrm{W}^{\mathbb{G}} \in B(\mathrm{L}^2(\mathbb{G})\otimes\mathrm{L}^2(\mathbb{G}))$, which is a unique unitary operator such that

$$\mathrm{W}^{\mathbb{G}}(\eta_{\mathbb{G}}(x)\otimes\eta_{\mathbb{G}}(y)) = (\eta_{\mathbb{G}}\otimes\eta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x)(\mathbb{1}\otimes y))$$

for all $x, y \in D(\eta_{\mathbb{G}})$; $\mathrm{W}^{\mathbb{G}}$ satisfies the pentagonal equation $\mathrm{W}^{\mathbb{G}}_{12}\mathrm{W}^{\mathbb{G}}_{13}\mathrm{W}^{\mathbb{G}}_{23} = \mathrm{W}^{\mathbb{G}}_{23}\mathrm{W}^{\mathbb{G}}_{12}$. Note that we use the leg numbering notation, e.g. $\mathrm{W}^{\mathbb{G}}_{12} = \mathrm{W}^{\mathbb{G}}\otimes\mathbb{1} \in B(\mathrm{L}^2(\mathbb{G})\otimes\mathrm{L}^2(\mathbb{G})\otimes\mathrm{L}^2(\mathbb{G}))$. Using $\mathrm{W}^{\mathbb{G}}$ one can recover $\mathrm{L}^\infty(\mathbb{G})$ and $\Delta_{\mathbb{G}}$

$$\begin{aligned}\mathrm{L}^\infty(\mathbb{G}) =& \big\{(\omega\otimes\mathrm{id})(\mathrm{W}^{\mathbb{G}})\,\big|\,\omega\in B(\mathrm{L}^2(\mathbb{G}))_*\big\}^{\sigma-\mathrm{cls}},\\ \Delta_{\mathbb{G}}(x) =& \mathrm{W}^{\mathbb{G}}(x\otimes\mathbb{1})\mathrm{W}^{\mathbb{G}^*}.\end{aligned}$$

A locally compact quantum group $\mathbb{G}$ admits a C*-version, which can also be recovered from $\mathrm{W}^{\mathbb{G}}$. For example the C*-algebra assigned to $\mathbb{G}$ denoted by $\mathrm{C}_0(\mathbb{G})$ is given by

$$\mathrm{C}_0(\mathbb{G}) = \big\{(\omega\otimes\mathrm{id})(\mathrm{W}^{\mathbb{G}})\,\big|\,\omega\in B(\mathrm{L}^2(\mathbb{G}))_*\big\}^{\|\cdot\|-\mathrm{cls}}.$$

We say that $\mathbb{G}$ is a compact quantum group if $\mathbb{1}\in\mathrm{C}_0(\mathbb{G})$. A locally compact quantum group admits a dual object $\widehat{\mathbb{G}} = (\mathrm{L}^\infty(\widehat{\mathbb{G}}), \Delta_{\widehat{\mathbb{G}}}, \varphi_{\widehat{\mathbb{G}}}, \psi_{\widehat{\mathbb{G}}})$. For the detailed description of the Haar weights $\varphi_{\widehat{\mathbb{G}}}, \psi_{\widehat{\mathbb{G}}}$ we refer to [63]; let us only mention that we have $\mathrm{L}^2(\widehat{\mathbb{G}}) = \mathrm{L}^2(\mathbb{G})$. The multiplicative unitary assigned to $\widehat{\mathbb{G}}$ is given by $\mathrm{W}^{\widehat{\mathbb{G}}} = \boldsymbol{\sigma}(\mathrm{W}^{\mathbb{G}})^*$, where $\boldsymbol{\sigma}$ denotes the flipping morphism $\boldsymbol{\sigma}(a\otimes b) = b\otimes a$. In particular we have

$$\begin{aligned}\mathrm{L}^\infty(\widehat{\mathbb{G}}) &= \big\{(\omega\otimes\mathrm{id})(\mathrm{W}^{\widehat{\mathbb{G}}})\,\big|\,\omega\in B(\mathrm{L}^2(\mathbb{G}))_*\big\}^{\sigma-\mathrm{cls}},\\ \Delta_{\widehat{\mathbb{G}}}(x) &= \mathrm{W}^{\widehat{\mathbb{G}}}(x\otimes\mathbb{1})\mathrm{W}^{\widehat{\mathbb{G}}^*}.\end{aligned}$$

Moreover

$$\mathrm{C}_0(\widehat{\mathbb{G}}) = \big\{(\omega\otimes\mathrm{id})(\mathrm{W}^{\widehat{\mathbb{G}}})\,\big|\,\omega\in B(\mathrm{L}^2(\mathbb{G}))_*\big\}^{\|\cdot\|-\mathrm{cls}}$$

and we have $\mathrm{W}^{\mathbb{G}}\in\mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{G}})\otimes\mathrm{C}_0(\mathbb{G}))$.

Definition 1.1.7. Let $\mathbb{G}$ be a locally compact quantum group. The opposite locally compact quantum group $\mathbb{G}^{\mathrm{op}}$ is defined as $(\mathrm{L}^\infty(\mathbb{G}), \Delta^{\mathrm{op}}_{\mathbb{G}}, \psi_{\mathbb{G}}, \varphi_{\mathbb{G}})$ where $\Delta^{\mathrm{op}}_{\mathbb{G}} = \boldsymbol{\sigma}\circ\Delta_{\mathbb{G}}$. We say that $\mathbb{G}$ is abelian if $\Delta_{\mathbb{G}} = \Delta_{\mathbb{G}^{\mathrm{op}}}$; in other words $\mathbb{G}$ is abelian if and only if $\widehat{\mathbb{G}}$ is classical, i.e. $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is commutative.

Definition 1.1.8. Let $\mathbb{G}$ be a locally compact quantum group, N a von Neumann algebra and $\alpha : \mathsf{N}\to\mathrm{L}^\infty(\mathbb{G})\bar{\otimes}\mathsf{N}$ a normal, unital injective $*$-homomorphism. We say that α is a left action of $\mathbb{G}$ on N if

$$(\Delta_{\mathbb{G}}\otimes\mathrm{id})\circ\alpha = (\mathrm{id}\otimes\alpha)\circ\alpha.$$

We say that the action α is *integrable* if the set

$$\{x \in \mathsf{N}^+ : (\psi_{\mathbb{G}} \otimes \mathrm{id})(\alpha(x)) \in \mathsf{N}^+\}$$

is σ-weakly dense in N^+.

For an action α of $\mathbb{G}$ on N we have (see [59, Corollary 2.6])

$$\mathsf{N} = \{(\omega \otimes \mathrm{id})(\alpha(x)) : \omega \in \mathrm{L}^\infty(\mathbb{G})_*, x \in \mathsf{N}\}^{\sigma-\mathrm{cls}}. \tag{1.1.1}$$

We also have a right counterpart of the concept of an action and the integrability condition.

In the sequel we shall often use the right adjoint action $\beta : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ of $\mathbb{G}$ on $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ where

$$\beta(x) = \mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*} \tag{1.1.2}$$

for all $x \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$.

A von Neumann subalgebra $\mathsf{N} \subset \mathrm{L}^\infty(\mathbb{G})$ is called

- *Left coideal* if $\Delta_{\mathbb{G}}(\mathsf{N}) \subset \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathsf{N}$;
- *Invariant subalgebra* if $\Delta_{\mathbb{G}}(\mathsf{N}) \subset \mathsf{N} \bar{\otimes} \mathsf{N}$;
- *Baaj-Vaes subalgebra* if N is an invariant subalgebra of $\mathrm{L}^\infty(\mathbb{G})$ which is preserved by unitary antipode R and the scaling group $(\tau_t)_{t\in\mathbb{R}}$ of $\mathbb{G}$;
- *Normal* if $\mathrm{W}^{\mathbb{G}}(\mathbb{1} \otimes \mathsf{N})\mathrm{W}^{\mathbb{G}*} \subset \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathsf{N}$;
- *Integrable* if the set of integrable elements with respect to the right Haar weight $\psi_{\mathbb{G}}$ is dense in N^+; in other words, the restriction of $\psi_{\mathbb{G}}$ to N is semifinite.

In the sequel a von Neumann subalgebra of $\mathrm{L}^\infty(\mathbb{G})$ which is a left coideal will be called a $\mathbb{G}$-coideal or simply a coideal. Note that $\Delta_{\mathbb{G}}|_{\mathsf{N}}$ is an action of $\mathbb{G}$ on N. In particular (see (1.1.1))

$$\mathsf{N} = \{(\omega \otimes \mathrm{id})(\Delta_{\mathbb{G}}(x)) : \omega \in \mathrm{L}^\infty(\mathbb{G})_*, x \in \mathsf{N}\}^{\sigma-\mathrm{cls}}. \tag{1.1.3}$$

Let $\mathsf{N} \subset \mathrm{L}^\infty(\mathbb{G})$ be a Baaj-Vaes subalgebra. The restriction of $\Delta_{\mathbb{G}}$ to N will be denoted by $\Delta_{\mathsf{N}} : \mathsf{N} \to \mathsf{N} \bar{\otimes} \mathsf{N}$. We shall often use the so called Baaj-Vaes theorem [98, Proposition 10.5], which states that $(\mathsf{N}, \Delta|_{\mathsf{N}})$ admits a structure of a locally compact quantum group. To be more precise there exists a pair of n.s.f. weights $\varphi_{\mathsf{N}}, \psi_{\mathsf{N}}$ on N such that $(\mathsf{N}, \Delta_{\mathbb{G}}|_{\mathsf{N}}, \varphi_{\mathsf{N}}, \psi_{\mathsf{N}})$ is a locally compact quantum group.

Definition 1.1.9. Let $\mathbb{G}$ be a locally compact quantum group. The set of $\mathbb{G}$-coideals will be denoted by $\mathcal{CI}(\mathbb{G})$. We equip $\mathcal{CI}(\mathbb{G})$ with the poset structure: for $\mathsf{N}, \mathsf{M} \in \mathcal{CI}(\mathbb{G})$ we write $\mathsf{N} \leq \mathsf{M}$ if $\mathsf{N} \subset \mathsf{M}$. The poset $(\mathcal{CI}(\mathbb{G}), \leq)$ admits two operations $\wedge, \vee$

- $\mathsf{N} \wedge \mathsf{M} = \mathsf{N} \cap \mathsf{M}$,
- $\mathsf{N} \vee \mathsf{M} = \{xy : x \in \mathsf{N}, y \in \mathsf{M}\}''$.

$(\mathcal{CI}(\mathbb{G}), \leq, \wedge, \vee)$ forms a lattice which will be called a *lattice of coideals* of $\mathbb{G}$.

The subset of $\mathcal{CI}(\mathbb{G})$ of normal $\mathbb{G}$-coideals will be denoted $\mathcal{NCI}(\mathbb{G})$. The subset of $\mathcal{CI}(\mathbb{G})$ of Baaj-Vaes subalgebras of $\mathrm{L}^\infty(\mathbb{G})$ will be denoted $\mathcal{BV}(\mathbb{G})$.

It is easy to check that $\mathcal{NCI}(\mathbb{G})$ and $\mathcal{BV}(\mathbb{G})$ form sublattices of $\mathcal{CI}(\mathbb{G})$. Similarly $\mathcal{NCI}(\mathbb{G}) \cap \mathcal{BV}(\mathbb{G})$ forms a sublattice of $\mathcal{CI}(\mathbb{G})$.

Remark 1.1.10. *Using* [55, Theorem 3.9] *we get a bijective map* $\mathcal{CI}(\mathbb{G}) \ni \mathsf{N} \mapsto \tilde{\mathsf{N}} \in \mathcal{CI}(\widehat{\mathbb{G}})$ *where*

$$\tilde{\mathsf{N}} = \mathsf{N}' \cap \mathrm{L}^\infty(\widehat{\mathbb{G}}).$$

The coideal $\tilde{\mathsf{N}}$ *is said to be a codual of* N *and the map* $\mathsf{N} \mapsto \tilde{\mathsf{N}}$ *is denoted by cd. Note that* $cd : \mathcal{CI}(\mathbb{G}) \to \mathcal{CI}(\widehat{\mathbb{G}})$ *is an anti-isomorphism of lattices:*

$$\begin{aligned} cd(\mathsf{N}) \leq cd(\mathsf{M}) \ &\textit{iff}\ \mathsf{M} \leq \mathsf{N}, \\ cd(\mathsf{N} \wedge \mathsf{M}) &= cd(\mathsf{N}) \vee cd(\mathsf{M}), \\ cd(\mathsf{N} \vee \mathsf{M}) &= cd(\mathsf{N}) \wedge cd(\mathsf{M}). \end{aligned}$$

Moreover $cd^2 = \mathrm{id}$ *(note that the coduality* $\mathcal{CI}(\widehat{\mathbb{G}}) \to \mathcal{CI}(\mathbb{G})$*) is also denoted by cd).*

1.2 Preliminaries for linearly reductive quantum groups

In Chapter 2 we work with Hopf algebras over an algebraically closed field k, regarded as the function algebras of the quantum groups in question. For this reason, we typically speak of either the quantum group $\mathbb{G}$ or the Hopf algebra $\mathcal{O}(\mathbb{G})$ associated to it. $\mathbb{G}$ is then a non-commutative analogue of a linear algebraic group [9]. Unless specified otherwise, antipodes are assumed to be bijective. For general background on coalgebras, bialgebras or Hopf algebras (which we recall somewhat briefly and selectively) the reader may consult e.g. [1, 27, 80, 92]; the various papers we cite are also good sources on specific points that arise in the course of the discussion below.

We use Sweedler notation for the comultiplication of a Hopf algebra (or more generally coalgebra) H, writing

$$\Delta : x \mapsto x_1 \otimes x_2$$

for the comultiplication $\Delta : H \to H \otimes H$. The reader should note that the symbol $\otimes$ has double meaning in this thesis - one in the context of C*-algebras and other in the context of algebras. Counits and antipodes are denoted by ε and S. Finally, for a linear subspace $V \subseteq H$ of H, we denote

$$V^- := \ker(\varepsilon|_V).$$

We denote categories of left / right modules over an algebra A by ${}_A\mathcal{M}$ and $\mathcal{M}_A$ respectively. Similarly, the categories of left / right C-comodules for a coalgebra C are denoted by ${}^C\mathcal{M}$ and $\mathcal{M}^C$ respectively. Following standard terminology (see e.g. [73, Definition 1.4]), the quantum group $\mathbb{G}$ is *linearly reductive* when $\mathcal{O}(\mathbb{G})$ is cosemisimple, i.e. its category $\mathcal{M}^{\mathcal{O}(\mathbb{G})}$ is semisimple.

Every coalgebra is the union of its finite-dimensional subcoalgebras (the so-called fundamental theorem of coalgebras; e.g. [92, Theorem 2.2.1]), and cosemisimple coalgebras are direct sums of their *simple* subcoalgebras, i.e. those that have no proper non-zero subcoalgebras [92, §14]. This latter decomposition is dual to the usual decomposition of semisimple algebras as (finite) products of simple algebras. In fact, simple coalgebras are dual to simple algebras, and hence, since we are working over an algebraically closed field, all simple subcoalgebras are of the form M_n^* (duals of the matrix algebras $M_n = M_n(k)$).

We will also deal with *discrete* quantum groups in a slightly more general setting than in Chapter 3. Definition 1.2.1, summarizing our conventions, will be sufficient for our purposes.

Definition 1.2.1. The category $\mathcal{QG}$ of *(linear algebraic) quantum groups* over a fixed field k is the category opposite to that of Hopf algebras over k with bijective antipode.

The category $\mathcal{RQG}$ of *linearly reductive quantum groups* over a fixed field k is the full subcategory of $\mathcal{QG}$ consisting of cosemisimple Hopf algebras.

The category $\mathcal{DQG}$ of *algebraic discrete quantum groups* over a fixed field k is the opposite category $\mathcal{RQG}^{op}$.

We often drop the adjective 'algebraic' below.

Remark 1.2.2. *In other words, we regard discrete quantum groups as dual to linearly reductive groups. This mimics the usual machinery in the locally compact case, except that we allow here arbitrary algebraically closed fields of arbitrary characteristic, and there are no $*$ structures.*

In the spirit of Definition 1.2.1, we regard the underlying Hopf algebra $\mathcal{O}(\widehat{\mathbb{G}})$ of a linearly reductive quantum group $\widehat{\mathbb{G}}$ as the group algebra of its discrete Pontryagin dual $\mathbb{G}$ of $\widehat{\mathbb{G}}$ and (working over the algebraically closed field k) use the notation

$$k\mathbb{G} = \mathcal{O}(\widehat{\mathbb{G}})$$

when we want to emphasize this point of view.

One particular class of cosemisimple Hopf algebras are the *CQG algebras* of [32], which in the context of Section 1.1 are dense complex Hopf $*$-subalgebras of $L^\infty(\mathbb{G})$ for some *compact* quantum group $\mathbb{G}$. Some of the results of Chapter 2 only apply to CQG algebras.

Definition 1.2.3. A *quantum subgroup* of the linearly reductive quantum group $\mathbb{G}$ is a quotient Hopf algebra $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{K})$.

Let $\mathbb{G}$ be a discrete quantum group. A *quantum subgroup* of $\mathbb{G}$ is a Hopf subalgebra $A \subseteq k\mathbb{G}$.

Remark 1.2.4. *Note that quantum subgroups of a discrete quantum group are automatically discrete, because cosemisimplicity is preserved by passing to Hopf subalgebras. Thus denoting $A = k\mathbb{K}$ in the second part of Definition 1.2.3 we see that a quantum subgroup $\mathbb{K}$ of a discrete quantum group $\mathbb{G}$ is a Hopf subalgebra $k\mathbb{K} \subseteq k\mathbb{G}$. A quantum subgroups of linearly reductive quantum groups need not be reductive, however: consider the classical situation whereby the function algebra of $GL_2(\mathbb{C})$ surjects onto that of the subgroup of upper triangular invertible matrices.*

Quantum subgroups of a given quantum group form a lattice both in $\mathcal{QG}$ and in $\mathcal{DQG}$.

Definition 1.2.5. Let $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H}_i)$, $i = 1, 2$ be two quantum subgroups of $\mathbb{G} \in \mathcal{QG}$. Then, the *intersection* $\mathbb{H}_1 \wedge \mathbb{H}_2$ whose underlying Hopf algebra $\mathcal{O}(\mathbb{H}_1 \wedge \mathbb{H}_2)$ is defined as the quotient of $\mathcal{O}(\mathbb{G})$ by the smallest Hopf ideal I invariant under the inverse S^{-1} of the antipode of $\mathbb{G}$, and which contains both ideals

$$\ker\left(\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H}_i)\right),\ i = 1, 2. \tag{1.2.1}$$

Similarly, the subgroup $\mathbb{H}_1 \vee \mathbb{H}_2$ *generated by* $\mathbb{H}_i$ is defined as the object in $\mathcal{QG}$ whose underlying Hopf algebra is the quotient of $\mathcal{O}(\mathbb{G})$ by the largest Hopf ideal invariant under S^{-1} contained in both (1.2.1).

Now let $k\mathbb{H}_i \subseteq k\mathbb{G}$ be two quantum subgroups of a discrete quantum group $\mathbb{G}$. Then, the *intersection* $\mathbb{H}_1 \wedge \mathbb{H}_2$ is the discrete quantum group whose underlying group algebra $k(\mathbb{H}_1 \wedge \mathbb{H}_2)$ is the intersection of $k\mathbb{H}_i$ in $k\mathbb{G}$.

Similarly, the subgroup *generated by* $\mathbb{H}_i$ is defined as the discrete quantum group whose underlying Hopf algebra is the Hopf subalgebra of $k\mathbb{G}$ generated as an algebra by $k\mathbb{H}_i$.

We can verify that in both cases the operations $\wedge$ and $\vee$ are well defined and turn the sets of quantum subgroups into lattices.

Remark 1.2.6. *Classically, the intersection $\mathbb{H} \wedge \mathbb{K}$ can be defined as the pullback*

$$\begin{array}{ccccc} & & \mathbb{H} & & \\ & \nearrow & & \searrow & \\ \mathbb{H} \wedge \mathbb{K} & & & & \mathbb{G} \\ & \searrow & & \nearrow & \\ & & \mathbb{K} & & \end{array}$$

in whatever category of groups is convenient (linear algebraic, etc.). The analogue in the $\mathcal{QG}$ case of Definition 1.2.5 is the observation that we have a pushout

$$\begin{array}{ccccc} & & \mathcal{O}(\mathbb{H}) & & \\ & \swarrow & & \nwarrow & \\ \mathcal{O}(\mathbb{H} \wedge \mathbb{K}) & & & & \mathcal{O}(\mathbb{G}) \\ & \nwarrow & & \swarrow & \\ & & \mathcal{O}(\mathbb{K}) & & \end{array}$$

in the category of algebras, or equivalently, that of Hopf algebras (or Hopf algebras with bijective antipode). In other words, the left hand corner is universal among quotients that make the diagram commutative.

We will make frequent (mostly implicit) use of an algebraic version of the correspondence cd from Section 1.1 throughout Chapter 2. We elaborate on the construction here.

First, for any Hopf algebra H, define $\mathcal{CI}(H)$ as the set of *right* coideal subalgebras $A \subseteq H$, i.e. those subalgebras for which

$$\Delta(A) \subseteq A \otimes H.$$

(in opposition to Section 1.1, we use right rather than left coideals in order to preserve compatibility with much of the literature on Hopf algebras accessible through our references).

Now, for each $A \subseteq H$ in $\mathcal{CI}(H)$ denote

$$\mathrm{cd}(A) = H/HA^{-}.$$

This is a left module quotient coalgebra of H in the sense of Definition 1.2.7 (e.g. [94, Proposition 1]), which justifies denoting the set of such module quotients by $\mathcal{MQ}(H)$.

Definition 1.2.7. A (left) *module coalgebra* over a Hopf algebra H is a coalgebra C equipped with an H-module action

$$H \otimes C \to C$$

that is a coalgebra map.

A (left) *module quotient coalgebra* is a module coalgebra $H \otimes C \to C$ as before equipped with a surjection $H \to C$ of module coalgebras, i.e. a surjection that is both a coalgebra morphism and a morphism of left H-modules.

On the other hand, given $\pi : H \to C$ in $\mathcal{MQ}(H)$, define $\mathrm{cd}(\pi)$ (or usually $\mathrm{cd}(C)$ by a slight abuse of notation) to be

$$\mathrm{cd}(\pi) := \{x \in H \mid \pi(x_1) \otimes x_2 = \pi(\mathbb{1}) \otimes x\}.$$

It can be shown to be an object in $\mathcal{CI}(H)$ ([94, Proposition 1]).

Note that we are using the symbol cd for two different maps, relating $\mathcal{CI}$ and $\mathcal{MQ}$ in two opposite directions. They are not, in general, mutual inverses; that requires additional technical conditions, as we now recall.

Definition 1.2.8. Let $\iota : A \to H$ be an algebra map. H is left (resp. right) *faithfully flat* if the functor

$$- \otimes_A H : \mathcal{M}_A \to \mathcal{M}_H$$

resp.

$$H \otimes_A - : {}_A\mathcal{M} \to {}_H\mathcal{M}$$

preserves morphism injectivity.

Dually, let $\pi : H \to C$ be a coalgebra map. Then, H is left (resp. right) *faithfully coflat* if the functor

$$-\square_C H : \mathcal{M}^C \to \mathcal{M}^H$$

resp.

$$H\square_C - : {}^C\mathcal{M} \to {}^H\mathcal{M}$$

preserves morphism surjectivity.

The notion of tensor coproduct $\square$, dual to that of tensor product, (see e.g. [94, §1]) can be briefly described as follows:

Given a right C-comodule

$$\rho_V : V \to V \otimes C$$

and a left C-comodule

$$\rho_W : W \to C \otimes W,$$

$V\square_C W$ is the subspace of $V \otimes W$ on which the two arrows

$$V \otimes W \underset{\mathrm{id}_V \otimes \rho_W}{\overset{\rho_V \otimes \mathrm{id}_W}{\rightrightarrows}} V \otimes C \otimes W$$

agree.

Then, part of the content of [94, Theorems 1] is that

$$\mathrm{cd}^2 : \mathcal{CI}(H) \to \mathcal{CI}(H)$$

restricts to the identity to those $A \in \mathcal{CI}(H)$ for which H is left A-faithfully flat.

Similarly, [94, Theorem 2] says (among other things) that

$$\mathrm{cd}^2 : \mathcal{MQ}(H) \to \mathcal{MQ}(H)$$

restricts to the identity on those $\pi : H \to C$ over which H is right faithfully coflat.

We will be concerned almost exclusively with situations where either $\iota : A \to H$ or $\pi : H \to C$ is a Hopf algebra map. For that reason, we make the following simple observation:

Lemma 1.2.9. *Let $\pi : H \to C$ be a quotient Hopf algebra, and set $A = cd(\pi)$. Then, A is invariant under the right adjoint action of H on itself defined by*

$$a \triangleleft x = S(x_1)ax_2.$$

One consequence of Lemma 1.2.9 that we will use later is

Corollary 1.2.10. *Let $A, B \in \mathcal{CI}(H)$ and suppose $A = cd(\pi)$ for some $\pi : H \to C$ in $\mathcal{MQ}(C)$. Then, the linear span*

$$BA = \mathrm{span}\{ba \mid a \in A,\ b \in B\}$$

is a coideal subalgebra.

Proof. Since it is clear that the space in question is a coideal, it suffices to show that it is a subalgebra. Specifically, we have to prove that for any $a \in A$ and $b \in B$, the product ab belongs to BA. This follows from the identity

$$ab = b_1(S(b_2)ab_3),$$

together with the fact that according to Lemma 1.2.9 the parenthetic factors on the right hand side belong to A. □

Let us also record the dual version of Lemma 1.2.9 (the proof is entirely analogous; we once more do not include it):

Lemma 1.2.11. *Let $\iota : A \to H$ be an inclusion of Hopf algebras, and set $\pi : H \to C$ to be $cd(A)$. Then, the left adjoint coaction of H on itself defined by*

$$x \mapsto x_1 S(x_3) \otimes x_2$$

descends to a coaction of H on C through the quotient $\pi : H \to C$.

We will often encounter the situation when *both* $\iota : A \to H$ and $\pi : H \to C$ are morphisms in $\mathcal{QG}$. Given the results recalled briefly above on the importance of (co)flatness, we fix our terminology as follows.

Definition 1.2.12. An *exact sequence* of quantum groups is a diagram

$$k \to \mathcal{O}(\mathbb{H}) \to \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{K}) \to k \tag{1.2.2}$$

in $\mathcal{QG}$ where the second arrow is an inclusion, the third arrow is a surjection, cd interchanges these two arrows, and moreover $\mathcal{O}(\mathbb{G})$ is (co)flat over $\mathcal{O}(\mathbb{H})$ (respectively $\mathcal{O}(\mathbb{K})$). In this case we denote $\mathbb{H} = \mathbb{G}/\mathbb{K}$.

The quantum subgroup $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{K})$ of $\mathbb{G}$ is *normal* if it fits into an exact sequence (1.2.2).

The discrete quantum subgroup $k\mathbb{K} \subseteq k\mathbb{G}$ of $k\mathbb{G} \in \mathcal{DQG}$ is *normal* if the inclusion in question is the second arrow in an exact sequence (1.2.2).

Remark 1.2.13. *Cf.* [3]*, where the definition of an exact sequence is the same, minus the (co)flatness conditions.*

1.3 Lattice of closed quantum subgroups: basic facts

Definition 1.3.1. Let $\mathbb{G}$ be a locally compact quantum group. The lattice $\mathcal{BV}(\widehat{\mathbb{G}})$ will be denoted $\mathcal{QS}(\mathbb{G})$ and called a *lattice of quantum subgroups* of $\mathbb{G}$.

The lattice $\mathcal{NCI}(\widehat{\mathbb{G}}) \cap \mathcal{BV}(\widehat{\mathbb{G}})$ will be denoted $\mathcal{NQS}(\mathbb{G})$ and called a *lattice of normal quantum subgroups* of $\mathbb{G}$.

Let $\mathsf{N} \in \mathcal{QS}(\mathbb{G})$. Using Baaj-Vaes theorem, we conclude the existence of a locally compact quantum group $\mathbb{H}$ such that $\mathsf{N} = \mathrm{L}^\infty(\widehat{\mathbb{H}})$. Thus when convenient we will write $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$. Similarly for $\mathbb{H}_1, \mathbb{H}_2 \in \mathcal{QS}(\mathbb{G})$ we write $\mathbb{H}_1 \wedge \mathbb{H}_2, \mathbb{H}_1 \vee \mathbb{H}_2$.

Remark 1.3.2. *Let $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$ and let $cd : \mathcal{CI}(\mathbb{G}) \to \mathcal{CI}(\widehat{\mathbb{G}})$ be the coduality (see Remark 1.1.10). Then $cd(\mathrm{L}^\infty(\widehat{\mathbb{H}})) \in \mathcal{CI}(\mathbb{G})$ is denoted by $\mathrm{L}^\infty(\mathbb{G}/\mathbb{H})$. It can be checked that (see e.g.* [56]*)*

- *$cd(\mathcal{QS}(\mathbb{G})) \subset \mathcal{NCI}(\mathbb{G})$,*
- *$cd(\mathcal{NQS}(\mathbb{G})) = \mathcal{NQS}(\widehat{\mathbb{G}})$.*

If $\mathbb{H} \in \mathcal{NQS}(\mathbb{G})$ then the normal quantum subgroup $cd(\mathbb{H}) \in \mathcal{NQS}(\widehat{\mathbb{G}})$ is denoted by $\widehat{\mathbb{G}/\mathbb{H}}$. For the concept of short exact sequence of locally compact quantum groups we refer to [99, Definition 3.2.]*. Up to natural isomorphisms all examples are of the form*

$$\bullet \to \mathbb{H} \to \mathbb{G} \to \mathbb{G}/\mathbb{H} \to \bullet \tag{1.3.1}$$

where $\bullet$ denotes a trivial group. Since $cd^2 = \mathrm{id}$ we also have the dual exact sequence

$$\bullet \to \widehat{\mathbb{G}/\mathbb{H}} \to \widehat{\mathbb{G}} \to \widehat{\mathbb{H}} \to \bullet. \tag{1.3.2}$$

Let $\mathbb{G}$ be a locally compact quantum group. As formulated in Definition 1.3.1, a closed quantum subgroup of $\mathbb{G}$ corresponds to a Baaj-Vaes subalgebra of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. In particular a locally compact quantum group $\mathbb{G}$ can be assigned with a quantum subgroup $\mathcal{Z}(\mathbb{G}) \leq \mathbb{G}$ which is called a center of $\mathbb{G}$: by definition $\mathrm{L}^\infty(\widehat{\mathcal{Z}(\mathbb{G})})$ is the largest Baaj-Vaes subalgebra contained in the center of the von Neumann algebra $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. In particular $\mathcal{Z}(\mathbb{G})$ is a normal quantum subgroup of $\mathbb{G}$ and one can form the quotient group $\mathbb{G}/\mathcal{Z}(\mathbb{G})$. For the detailed description of the corresponding exact sequence

$$\bullet \to \mathcal{Z}(\mathbb{G}) \to \mathbb{G} \to \mathbb{G}/\mathcal{Z}(\mathbb{G}) \to \bullet$$

see [54]. In what follows we shall describe the quantum analog of the quotient of $\mathbb{G}$ by its commutator subgroup.

Proposition 1.3.3. *Let $\mathbb{G}$ be a locally compact quantum group and let us consider*

$$\mathsf{M} = \{x \in \mathrm{L}^\infty(\mathbb{G}) : (\mathrm{id} \otimes \Delta^{op}_{\mathbb{G}})(\Delta_{\mathbb{G}}(x)) = (\mathrm{id} \otimes \Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x))\}. \tag{1.3.3}$$

Then M is a Baaj-Vaes subalgebra of $\mathrm{L}^\infty(\mathbb{G})$. The quantum group $\mathbb{H}$ such that $\mathsf{M} = \mathrm{L}^\infty(\mathbb{H})$ is abelian. Let N be another Baaj-Vaes subalgebra and $\mathbb{L}$ be the locally compact quantum group assigned to N. If $\mathbb{L}$ is abelian then $\mathsf{N} \subset \mathsf{M}$.

Proof. Clearly M is a von Neumann subalgebra of $\mathrm{L}^\infty(\mathbb{G})$. We shall first show that $\Delta_{\mathbb{G}}(\mathsf{M}) \subset \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathsf{M}$. Let $x \in \mathsf{M}$. Then

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_{\mathbb{G}})(\mathrm{id} \otimes \Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x)) &= (\Delta_{\mathbb{G}} \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x)) \\ &= (\Delta_{\mathbb{G}} \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{\mathrm{op}}_{\mathbb{G}})(\Delta_{\mathbb{G}}(x)) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \Delta^{\mathrm{op}}_{\mathbb{G}})(\mathrm{id} \otimes \Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x))\end{aligned}$$

and we get $\Delta_{\mathbb{G}}(x) \in \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathsf{M}$.

Using (1.1.3) and Definition 1.3.3 we see that $\Delta_{\mathbb{G}}|_{\mathsf{M}} = \Delta^{\mathrm{op}}_{\mathbb{G}}|_{\mathsf{M}}$. In particular $\Delta_{\mathbb{G}}(\mathsf{M}) \subset \mathsf{M} \bar{\otimes} \mathsf{M}$.

The $\tau^{\mathbb{G}}_t$-invariance of M follows easily from the relation $\Delta_{\mathbb{G}} \circ \tau^{\mathbb{G}}_t = (\tau^{\mathbb{G}}_t \otimes \tau^{\mathbb{G}}_t) \circ \Delta_{\mathbb{G}}$. Since $\Delta_{\mathbb{G}} \circ R^{\mathbb{G}} = (R^{\mathbb{G}} \otimes R^{\mathbb{G}}) \circ \Delta^{\mathrm{op}}_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}|_{\mathsf{M}} = \Delta^{\mathrm{op}}_{\mathbb{G}}|_{\mathsf{M}}$ we get $R^{\mathbb{G}}(\mathsf{M}) \subset \mathsf{M}$. Summarizing M forms a Baaj-Vaes subalgebra.

If $\mathsf{N} \subset \mathrm{L}^\infty(\mathbb{G})$ is a Baaj-Vaes subalgebra such that $\Delta_{\mathbb{G}}|_{\mathsf{N}} = \Delta^{\mathrm{op}}_{\mathbb{G}}|_{\mathsf{N}}$ then it is clear that for all $x \in \mathsf{N}$ the condition $(\mathrm{id} \otimes \Delta^{\mathrm{op}}_{\mathbb{G}})(\Delta_{\mathbb{G}}(x)) = (\mathrm{id} \otimes \Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x))$ holds, i.e. $\mathsf{N} \subset \mathsf{M}$. □

Remark 1.3.4. *In other words, $\mathsf{M} \in \mathrm{L}^\infty(\mathbb{G})$ introduced in Theorem 1.3.3 is the largest cocommutative Baaj-Vaes subalgebra. This means that it corresponds to the largest classical closed quantum subgroup of $\widehat{\mathbb{G}}$.*

Example 1.3.5. Let $\mathbb{G}$ be a classical locally compact group. Adopting the notation of Proposition 1.3.3 we see that $f \in \mathsf{M}$ if for all (up to measure zero subset) $(p,q,r) \in \mathbb{G}^3$ we have $f(pqr) = f(prq)$. This condition is equivalent with $f(pqrq^{-1}r^{-1}) = f(p)$. Thus $f \in \mathsf{M}$ if and only if f is constant on the cosets of the commutator subgroup $\mathbb{N} \subset \mathbb{G}$ where $\mathbb{N}$ is defined as the smallest closed subgroup of $\mathbb{G}$ containing $\{qrq^{-1}r^{-1} : q, r \in \mathbb{G}\}$. In conclusion, we have $\mathsf{M} = \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$.

Remark 1.3.6. *Now, let $\mathbb{G}$ be a locally compact quantum group and $\mathsf{M} \subset \mathrm{L}^\infty(\mathbb{G})$ the Baaj-Vaes algebra described in Proposition 1.3.3. In general a normal quantum subgroup $\mathbb{N} \trianglelefteq \mathbb{G}$ such that $\mathsf{M} = \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ does not exist. Actually such $\mathbb{N}$ exists if and only if M, viewed as a coideal $\mathrm{L}^\infty(\mathbb{G})$, is normal; the normality of M in turn is equivalent with the equality*

$$\mathrm{W}^{\mathbb{G}}_{13}\mathrm{W}^{\mathbb{G}}_{14}(\mathrm{id}\otimes\Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x))_{234}\mathrm{W}^{\mathbb{G}\,*}_{14}\mathrm{W}^{\mathbb{G}\,*}_{13} = \mathrm{W}^{\mathbb{G}}_{14}\mathrm{W}^{\mathbb{G}}_{13}(\mathrm{id}\otimes\Delta_{\mathbb{G}})(\Delta_{\mathbb{G}}(x))_{234}\mathrm{W}^{\mathbb{G}\,*}_{13}\mathrm{W}^{\mathbb{G}\,*}_{14} \quad (1.3.4)$$

being satisfied for all $x \in \mathsf{M}$. Example 1.3.7 below shows that (1.3.4) does not always hold; when it does, we call $\mathbb{N} \trianglelefteq \mathbb{G}$ as above the commutator subgroup of $\mathbb{G}$.

Example 1.3.7. As indicated in Remark 1.3.6 above, the largest cocommutative Baaj-Vaes subalgebra of $L^\infty(\mathbb{G})$ is not, in general, of the form $L^\infty(\mathbb{G}/\mathbb{N})$ for a normal closed quantum subgroup $\mathbb{N} \trianglelefteq \mathbb{G}$. To see this, note that upon dualizing, the claim takes the form, that there exist locally compact quantum groups with the property that the largest classical closed quantum subgroup is not normal (see Remark 1.3.4).

For examples of this latter phenomenon, consider one of the free unitary groups U_n^+ for some $n \geq 2$ (these are the quantum groups whose underlying CQG algebras $A_u(n)$ are defined in [106] as being freely generated by $n \times n$ unitary matrix of generators u_{ij} such that (u_{ij}^*) is also unitary).

It's largest classical quantum subgroup is the ordinary unitary group U_n obtained as the object dual to the largest commutative CQG quotient algebra of $A_u(n)$, whereas it is known [18, Corollary 12] that proper normal quantum subgroups of U_n^+ are contained in the common center $\mathbb{T}$ of $U_n < U_n^+$.

Let us move on to the discussion of *morphisms* of locally compact quantum groups. This requires the universal C*-version of a given locally compact quantum group $\mathbb{G}$ (see e.g. [65]). The universal version $\mathrm{C}_0^u(\mathbb{G})$ of $\mathrm{C}_0(\mathbb{G})$ is equipped with a comultiplication $\Delta^u_{\mathbb{G}} \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0^u(\mathbb{G}) \otimes \mathrm{C}_0^u(\mathbb{G}))$. The multiplicative unitary $\mathrm{W}^{\mathbb{G}} \in \mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{G}))$ admits the universal lift $\mathbb{W}^{\mathbb{G}} \in \mathrm{M}(\mathrm{C}_0^u(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\mathbb{G}))$. The reducing morphisms for $\mathbb{G}$ and $\widehat{\mathbb{G}}$ will be denoted by

$$\Lambda_{\mathbb{G}} \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0(\mathbb{G})),$$
$$\Lambda_{\widehat{\mathbb{G}}} \in \mathrm{Mor}(\mathrm{C}_0^u(\widehat{\mathbb{G}}), \mathrm{C}_0(\widehat{\mathbb{G}}))$$

respectively. Then

$$(\Lambda_{\widehat{\mathbb{G}}} \otimes \Lambda_{\mathbb{G}})(\mathbb{W}^{\mathbb{G}}) = \mathrm{W}^{\mathbb{G}}$$

We shall also use the half-lifted versions of $\mathrm{W}^{\mathbb{G}}$

$$\mathbb{W}^{\mathbb{G}} = (\mathrm{id} \otimes \Lambda_{\mathbb{G}})(\mathbb{W}^{\mathbb{G}}) \in \mathrm{M}(\mathrm{C}_0^u(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{G})),$$
$$\mathcal{W}^{\mathbb{G}} = (\Lambda_{\widehat{\mathbb{G}}} \otimes \mathrm{id})(\mathbb{W}^{\mathbb{G}}) \in \mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\mathbb{G}))$$

which satisfy the appropriate versions of the pentagonal equation

$$\mathrm{W}^{\mathbb{G}}_{12}\mathrm{W}^{\mathbb{G}}_{13}\mathrm{W}^{\mathbb{G}}_{23} = \mathrm{W}^{\mathbb{G}}_{23}\mathrm{W}^{\mathbb{G}}_{12},$$

$$\mathrm{W}^{\mathbb{G}}_{12}\mathbb{W}^{\mathbb{G}}_{13}\mathbb{W}^{\mathbb{G}}_{23} = \mathbb{W}^{\mathbb{G}}_{23}\mathrm{W}^{\mathbb{G}}_{12}.$$

The half-lifted versions of the comultiplications will be denoted by $\Delta_r^{r,u} \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{G}), \mathrm{C}_0(\mathbb{G})\otimes \mathrm{C}_0^u(\mathbb{G}))$ and $\widehat{\Delta}_r^{r,u} \in \mathrm{Mor}(\mathrm{C}_0(\widehat{\mathbb{G}}), \mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\widehat{\mathbb{G}}))$, e.g.

$$\Delta_r^{r,u}(x) = \mathbb{W}^{\mathbb{G}}(x \otimes \mathrm{id})\mathbb{W}^{\mathbb{G}*}, \qquad x \in \mathrm{C}_0(\mathbb{G}).$$

We have

$$\begin{aligned}(\Lambda_{\mathbb{G}} \otimes \mathrm{id}) \circ \Delta^u_{\mathbb{G}} &= \Delta_r^{r,u} \circ \Lambda_{\mathbb{G}},\\ (\Lambda_{\widehat{\mathbb{G}}} \otimes \mathrm{id}) \circ \Delta^u_{\widehat{\mathbb{G}}} &= \widehat{\Delta}_r^{r,u} \circ \Lambda_{\widehat{\mathbb{G}}}.\end{aligned} \tag{1.3.5}$$

Given two locally compact quantum groups $\mathbb{G}$ and $\mathbb{H}$, a morphism $\Pi : \mathbb{H} \to \mathbb{G}$ (see e.g. [71]) is represented by a C*-morphism $\pi \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0^u(\mathbb{H}))$ intertwining the respective coproducts:

$$(\pi \otimes \pi) \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{H}} \circ \pi.$$

It can be equivalently described via:

- a *bicharacter* from $\mathbb{H}$ to $\mathbb{G}$, i.e. a unitary $V \in \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ such that

  $$(\Delta_{\widehat{\mathbb{G}}} \otimes \mathrm{id}_{\mathrm{C}_0(\mathbb{H})})(V) = V_{23}V_{13},$$

  $$(\mathrm{id}_{\mathrm{C}_0(\widehat{\mathbb{G}})} \otimes \Delta_{\mathbb{H}})(V) = V_{12}V_{13}.$$

  In fact $V \in \mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{H}))$ and $V = (\mathrm{id} \otimes \Lambda^{\mathbb{H}} \circ \pi)(\mathbb{W}^{\mathbb{G}})$. We shall also use $\mathbb{V} = (\mathrm{id} \otimes \pi)(\mathbb{W}^{\mathbb{G}}) \in \mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\mathbb{H}))$.

- a *right quantum group homomorphism* i.e. an action $\alpha : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ of $\mathbb{H}$ on $\mathrm{L}^\infty(\mathbb{G})$ satisfying

  $$(\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha) \circ \Delta_{\mathbb{G}}$$

  In fact $\alpha(x) = V(x \otimes \mathbb{1})V^*$. We shall also use the obvious left version of the concept of a right quantum group homomorphism, which is refereed to as a *left quantum group homomorphism*.

Let $\Pi : \mathbb{H} \to \mathbb{G}$. The right quantum group homomorphism assigned to Π will be denoted α_Π or $\alpha_{\mathbb{H}\to\mathbb{G}}$ when convenient.

Example 1.3.8. Let $\mathbb{G}$ be a locally compact quantum group and $\mathbb{H} \le \mathbb{G}$. Since $\mathrm{L}^\infty(\widehat{\mathbb{H}})$ is a Baaj-Vaes subalgebra of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$, the multiplicative unitary $\mathrm{W}^{\mathbb{H}} \in \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ can be viewed as an element $V \in \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$. Since V is a bicharacter from $\mathbb{H}$ to $\mathbb{G}$, $\mathbb{H} \le \mathbb{G}$ yields a morphism from $\mathbb{H}$ to $\mathbb{G}$.

Let $\mathbb{G}$ and $\mathbb{H}$ be locally compact quantum groups and $\Pi : \mathbb{H} \to \mathbb{G}$ a morphism. We say that Π identifies $\mathbb{H}$ with a closed quantum subgroup of $\mathbb{G}$ if there exists a normal injective $*$-homomorphism $\gamma : \mathrm{L}^\infty(\widehat{\mathbb{H}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}})$ such that $V = (\gamma \otimes \mathrm{id})(\mathrm{W}^{\mathbb{H}})$ (see [28]).

Clearly, a closed quantum subgroup $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$ is normal if and only if

$$\beta(\mathrm{L}^\infty(\widehat{\mathbb{H}})) \subset \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$$

where β is the adjoint action (1.1.2). Let us also note the following result whose classical version is well known.

Proposition 1.3.9. *Let $\mathbb{G}$ be a locally compact quantum group and $\mathbb{N} \trianglelefteq \mathbb{G}$ an abelian normal quantum subgroup of $\mathbb{G}$. Then for every $x \in \mathrm{L}^\infty(\widehat{\mathbb{N}})$ we have*

$$\mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*} \in \mathrm{L}^\infty(\widehat{\mathbb{N}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$$

In particular the restriction of the adjoint action β to $\mathrm{L}^\infty(\widehat{\mathbb{N}})$ gives rise to the action of $\mathbb{G}/\mathbb{N}$ on $\mathrm{L}^\infty(\widehat{\mathbb{N}})$. Conversely if the adjoint action restricted to $\mathrm{L}^\infty(\widehat{\mathbb{N}})$ gives rise to the action of $\mathbb{G}/\mathbb{N}$ then $\mathbb{N}$ is abelian.

Proof. Let $\alpha : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{N})$ be the right quantum group homomorphism assigned to $\mathbb{N} \trianglelefteq \mathbb{G}$. Since $(\mathrm{id} \otimes \alpha)(\mathrm{W}^{\mathbb{G}}) = \mathrm{W}^{\mathbb{G}}_{12}\mathrm{W}^{\mathbb{N}}_{13}$ and $\mathbb{N}$ is abelian (i.e. $\mathrm{L}^\infty(\widehat{\mathbb{N}})$ is commutative), we conclude that

$$(\mathrm{id} \otimes \alpha)(\mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*}) = \mathrm{W}^{\mathbb{G}}_{12}\mathrm{W}^{\mathbb{N}}_{13}(x \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^{\mathbb{N}*}_{13}\mathrm{W}^{\mathbb{G}*}_{12} = (\mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*}) \otimes \mathbb{1}$$

for all $x \in \mathrm{L}^\infty(\widehat{\mathbb{N}})$. Thus $\mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*} \in \mathrm{L}^\infty(\widehat{\mathbb{N}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$.

Conversely, the condition

$$\mathrm{W}^{\mathbb{G}}_{12}\mathrm{W}^{\mathbb{N}}_{13}(x \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^{\mathbb{N}*}_{13}\mathrm{W}^{\mathbb{G}*}_{12} = (\mathrm{W}^{\mathbb{G}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{G}*}) \otimes \mathbb{1}$$

holds if and only if $\mathrm{W}^{\mathbb{N}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{N}*} = (x \otimes \mathbb{1})$ for all $x \in \mathrm{L}^\infty(\widehat{\mathbb{N}})$, which is equivalent to $\mathbb{N}$ being abelian. ☐

Let $\Pi : \mathbb{H} \to \mathbb{G}$ be a morphism of locally compact quantum groups. It turns out that Π cannot (in general) be assigned with a quantum analog of the kernel subgroup $\ker \Pi \leq \mathbb{H}$ (the case $\mathbb{H} = \mathbb{G}$ and Π being a projection $\Pi^2 = \Pi$ was thoroughly studied in [56]). In particular Π cannot be assigned with the exact sequence

$$\bullet \to \ker \Pi \to \mathbb{H} \to \mathbb{H}/\ker \Pi \to \bullet. \tag{1.3.6}$$

As noted in [53], the quantum analog of $\mathbb{H}/\ker \Pi$ can always be constructed. In what follows we shall provide a number of descriptions of $\mathbb{H}/\ker \Pi$ and formulate the condition which yields the existence of $\ker \Pi$ entering the exact sequence (1.3.6).

The von Neumann algebra $\mathrm{L}^\infty(\mathbb{H}/\ker \Pi)$ is defined as (see [53, Definition 4.4]

$$\mathrm{L}^\infty(\mathbb{H}/\ker \Pi) = \{(\omega \otimes \mathrm{id})(V) : \omega \in \mathrm{L}^\infty(\widehat{\mathbb{G}})_*\}^{\sigma-\mathrm{cls}}. \tag{1.3.7}$$

To be more precise the right hand side of (1.3.7) forms a Baaj-Vaes subalgebra of $\mathrm{L}^\infty(\mathbb{H})$, thus yields a locally compact quantum group which we denote $\mathbb{H}/\ker \Pi$. Since $V = (\mathrm{id} \otimes \Lambda^{\mathbb{H}} \circ \pi)(\mathcal{W}^{\mathbb{G}})$ the following holds

$$\mathrm{L}^\infty(\mathbb{H}/\ker \Pi) = \{\Lambda_{\mathbb{H}}(\pi(x)) : x \in \mathrm{C}_0^u(\mathbb{G})\}^{\sigma-\mathrm{cls}} \tag{1.3.8}$$

which is the second description of $\mathrm{L}^\infty(\mathbb{H}/\ker \Pi)$. The third description is the subject of [53, Theorem 4.7]:

$$\mathrm{L}^\infty(\mathbb{H}/\ker \Pi) = \{(\omega \otimes \mathrm{id})(\alpha(x)) : \omega \in \mathrm{L}^\infty(\mathbb{G})_*, x \in \mathrm{L}^\infty(\mathbb{G})\}''. \tag{1.3.9}$$

In what follows we shall give a simple proof of a slightly stronger version of (1.3.9).

Lemma 1.3.10. *Given a morphism $\Pi : \mathbb{H} \to \mathbb{G}$ we have*

$$\mathrm{L}^\infty(\mathbb{H}/\ker \Pi) = \{(\omega \otimes \mathrm{id})(\alpha(x)) : \omega \in \mathrm{L}^\infty(\mathbb{G})_*, x \in \mathrm{L}^\infty(\mathbb{G})\}^{\sigma-cls}$$

Proof. The bicharacter equation for V yields

$$V_{23}\mathrm{W}^{\mathbb{G}}_{12}V^*_{23} = \mathrm{W}^{\mathbb{G}}_{12}V_{13}.$$

In particular, since

$$\mathrm{L}^\infty(\mathbb{G}) = \{(\mu\otimes\mathrm{id})(\mathrm{W}^{\mathbb{G}}) : \mu\in\mathrm{L}^\infty(\widehat{\mathbb{G}})_*\}^{\sigma-\mathrm{cls}}$$

we have

$$\begin{aligned}\{(\omega\otimes\mathrm{id})(\alpha(x)) : \omega\in\mathrm{L}^\infty(\mathbb{G})_*, x\in\mathrm{L}^\infty(\mathbb{G})\}^{\sigma-\mathrm{cls}}\\ &= \{(\mu\otimes\omega\otimes\mathrm{id})(V_{23}\mathrm{W}^{\mathbb{G}}_{12}V^*_{23}) : \mu\in\mathrm{L}^\infty(\widehat{\mathbb{G}})_*, \omega\in\mathrm{L}^\infty(\mathbb{G})_*\}^{\sigma-\mathrm{cls}}\\ &= \{(\mu\otimes\omega\otimes\mathrm{id})(\mathrm{W}^{\mathbb{G}}_{12}V_{13}) : \mu\in\mathrm{L}^\infty(\widehat{\mathbb{G}})_*, \omega\in\mathrm{L}^\infty(\mathbb{G})_*\}^{\sigma-\mathrm{cls}}\\ &= \{(\mu\otimes\omega\otimes\mathrm{id})(V_{13}) : \mu\in\mathrm{L}^\infty(\widehat{\mathbb{G}})_*, \omega\in\mathrm{L}^\infty(\mathbb{G})_*\}^{\sigma-\mathrm{cls}}\\ &= \{(\omega\otimes\mathrm{id})(V) : \omega\in\mathrm{L}^\infty(\mathbb{G})_*,\}^{\sigma-\mathrm{cls}} = \mathrm{L}^\infty(\mathbb{H}/\ker\Pi)\end{aligned}$$

where in third equality $\sigma-\mathrm{cls}$ in the subscript and unitarity of $\mathrm{W}^{\mathbb{G}}$ enabled us to absorb $\mathrm{W}^{\mathbb{G}}$ into the functional $\mu\otimes\omega$ without changing the resulting set. □

Let $\Pi:\mathbb{H}\to\mathbb{G}$. Then the embedding $\mathrm{L}^\infty(\mathbb{H}/\ker\Pi)\subset\mathrm{L}^\infty(\mathbb{H})$ can be interpreted as $\widehat{\mathbb{H}/\ker\Pi}\leq\widehat{\mathbb{H}}$. In particular a short exact sequence (1.3.2) starting with

$$\bullet\to\widehat{\mathbb{H}/\ker\Pi}\to\widehat{\mathbb{H}}$$

exists if and only if $\widehat{\mathbb{H}/\ker\Pi}\in\mathcal{NQS}(\widehat{\mathbb{H}})$. In this case defining $\mathrm{L}^\infty(\widehat{\ker\Pi}) = \mathrm{cd}(\mathrm{L}^\infty(\mathbb{H}/\ker\Pi))\in\mathcal{NQS}(\mathbb{H})$ we get a short exact sequence of locally compact quantum groups (1.3.6).

A morphism $\Pi:\mathbb{H}\to\mathbb{G}$ is assigned with the dual morphism $\widehat{\Pi}:\widehat{\mathbb{G}}\to\widehat{\mathbb{H}}$ which in terms of bicharacter is given by $\widehat{V}=\sigma(V)^*$. The locally compact quantum group $\widehat{\mathbb{G}}/\ker\widehat{\Pi}$ will be denoted by $\overline{\mathrm{im}\Pi}$ (see [53, Definition 4.3]). In particular, using (1.3.7) we can see that $V\in\mathrm{L}^\infty(\widehat{\mathbb{G}})\bar{\otimes}\mathrm{L}^\infty(\mathbb{H})$ is actually an element of $\mathrm{L}^\infty(\widehat{\overline{\mathrm{im}\Pi}})\bar{\otimes}\mathrm{L}^\infty(\mathbb{H}/\ker\Pi)$. Using the inclusions

$$\mathrm{L}^\infty(\widehat{\overline{\mathrm{im}\Pi}})\bar{\otimes}\mathrm{L}^\infty(\mathbb{H}/\ker\Pi)\subset\mathrm{L}^\infty(\widehat{\mathbb{G}})\bar{\otimes}\mathrm{L}^\infty(\mathbb{H}/\ker\Pi)$$
$$\mathrm{L}^\infty(\widehat{\overline{\mathrm{im}\Pi}})\bar{\otimes}\mathrm{L}^\infty(\mathbb{H}/\ker\Pi)\subset\mathrm{L}^\infty(\widehat{\overline{\mathrm{im}\Pi}})\bar{\otimes}\mathrm{L}^\infty(\mathbb{H})$$

we see that a morphism $\Pi_{\mathbb{H}\to\mathbb{G}}:\mathbb{H}\to\mathbb{G}$ induces three morphisms $\Pi_{\mathbb{H}/\ker\Pi\to\overline{\mathrm{im}\Pi}}$, $\Pi_{\mathbb{H}/\ker\Pi\to\mathbb{G}}$ and $\Pi_{\mathbb{H}\to\overline{\mathrm{im}\Pi}}$. Using [53, Theorem 6.2, Corollary 6.5] we shall now formulate the First Isomorphism Theorem for locally compact quantum groups.

Theorem 1.3.11. *Let $\mathbb{H}$ and $\mathbb{G}$ be locally compact quantum groups, $\Pi:\mathbb{H}\to\mathbb{G}$ a morphism and let $\Pi_{\mathbb{H}/\ker\Pi\to\overline{im\Pi}}$, $\Pi_{\mathbb{H}/\ker\Pi\to\mathbb{G}}$, $\widehat{\Pi}_{\widehat{\overline{im\Pi}}\to\widehat{\mathbb{H}}}$ be the morphisms induced by Π as described above. Then the following conditions are equivalent:*

(i) $\Pi_{\mathbb{H}/\ker\Pi\to\overline{im\Pi}}$ is an isomorphism;

(ii) the action $\alpha:\mathrm{L}^\infty(\overline{im\Pi})\to\mathrm{L}^\infty(\overline{im\Pi})\bar{\otimes}\mathrm{L}^\infty(\mathbb{H}/\ker\Pi)$ is integrable;

(iii) $\Pi_{\mathbb{H}/\ker\Pi\to\mathbb{G}}$ identifies $\mathbb{H}/\ker\Pi$ with a closed quantum subgroup of $\mathbb{G}$;

(iv) $\widehat{\Pi}_{\widehat{\overline{im\Pi}}\to\widehat{\mathbb{H}}}$ identifies $\widehat{\overline{im\Pi}}$ with a closed quantum subgroup of $\widehat{\mathbb{H}}$.

Remark 1.3.12. *Let $\Pi : \mathbb{H} \to \mathbb{G}$ be a morphism of locally compact quantum groups. Clearly $\overline{\mathrm{im}\Pi}$ is abelian if and only if $\mathbb{H}/\ker\Pi$ is abelian. Let $\mathsf{M} \subset \mathrm{L}^\infty(\mathbb{G})$ be the Baaj-Vaes algebra described in Proposition 1.3.3. Then $\mathbb{H}/\ker\Pi$ is abelian if and only if $\mathrm{L}^\infty(\mathbb{H}/\ker\Pi) \subset \mathsf{M}$. For further discussion let us suppose that $\ker\Pi \trianglelefteq \mathbb{G}$ exists (see the paragraph containing (1.3.6)) and there exists a $\mathbb{N} \trianglelefteq \mathbb{G}$ such that $\mathsf{M} = \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$. Then $\mathrm{L}^\infty(\mathbb{H}/\ker\Pi) \leq \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ if and only if $\mathbb{N} \leq \ker\Pi$. Thus in the discussed case we get a quantum analog of the well known classical fact: the closed image of $\Pi : \mathbb{H} \to \mathbb{G}$ is abelian if and only if the kernel $\ker\Pi \trianglelefteq \mathbb{H}$ contains the commutator subgroup $\mathbb{N} \trianglelefteq \mathbb{H}$.*

The next lemma will be needed further.

Lemma 1.3.13. *Let $\Pi : \mathbb{H} \to \mathbb{G}$. Then*

$$cd(\mathrm{L}^\infty(\mathbb{H}/\ker\Pi)) = \{y \in \mathrm{L}^\infty(\widehat{\mathbb{H}}) : \widehat{\alpha}(y) = y \otimes \mathbb{1}\}$$

Proof. Let $V \in \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ be the bicharacter corresponding to Π. The right quantum group homomorphism $\widehat{\alpha} : \mathrm{L}^\infty(\widehat{\mathbb{H}}) \to \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{G}})$ corresponding to $\widehat{\Pi}$ is given by

$$\widehat{\alpha}(y) = \boldsymbol{\sigma}(V^*(\mathbb{1} \otimes y)V)$$

for all $y \in \mathrm{L}^\infty(\widehat{\mathbb{H}})$. In particular $\widehat{\alpha}(y) = y \otimes \mathbb{1}$ if and only if

$$(\mathbb{1} \otimes y)V = V(\mathbb{1} \otimes y)$$

We conclude using (1.3.7). □

Example 1.3.14. Let us consider $\mathbb{H} \in \mathcal{NQS}(\mathbb{G})$ and the exact sequence

$$\bullet \to \mathbb{H} \to \mathbb{G} \to \mathbb{G}/\mathbb{H} \to \bullet.$$

Let us denote the morphism $\mathbb{G} \to \mathbb{G}/\mathbb{H}$ by Π. Since $\mathrm{L}^\infty(\mathbb{G}/\mathbb{H})$ is defined as a Baaj-Vaes subalgebra of $\mathrm{L}^\infty(\mathbb{G})$, the dual morphism $\widehat{\Pi} : \widehat{\mathbb{G}/\mathbb{H}} \to \widehat{\mathbb{G}}$ identifies $\widehat{\mathbb{G}/\mathbb{H}}$ with a closed quantum subgroup of $\widehat{\mathbb{G}}$. In particular $\overline{\mathrm{im}\Pi} = \mathbb{G}/\mathbb{H}$ and $\mathbb{G}/\ker\Pi = \mathbb{G}/\mathbb{H}$.

Let $\widehat{\alpha} : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{G}/\mathbb{H}})$ be the right quantum group homomorphism assigned to $\widehat{\Pi} : \widehat{\mathbb{G}/\mathbb{H}} \to \widehat{\mathbb{G}}$. Since $\widehat{\Pi}$ identifies $\widehat{\mathbb{G}/\mathbb{H}}$ with a closed quantum subgroup of $\widehat{\mathbb{G}}$ we get

$$\mathrm{L}^\infty(\widehat{\mathbb{G}/\mathbb{H}}) = \{(\omega \otimes \mathrm{id})(\widehat{\alpha}(a)) : \omega \in \mathrm{L}^\infty(\widehat{\mathbb{G}})_*, a \in \mathrm{L}^\infty(\widehat{\mathbb{G}})\}^{\sigma-\mathrm{cls}}. \tag{1.3.10}$$

Finally, using Lemma 1.3.13 we see that

$$\mathrm{L}^\infty(\widehat{\mathbb{H}}) = \{y \in \mathrm{L}^\infty(\widehat{\mathbb{G}}) : \widehat{\alpha}(y) = y \otimes \mathbb{1}\}. \tag{1.3.11}$$

Lemma 1.3.15. *Let $\mathbb{G}$ and $\mathbb{K}$ be locally compact quantum groups $\Pi_1 : \mathbb{G} \to \mathbb{K}$ a homomorphism and $\widehat{\alpha}_1 : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{K}})$ the corresponding right quantum group homomorphism. Let $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$, $\Pi_2 : \mathbb{H} \to \mathbb{K}$ the restriction of Π_1 to $\mathbb{H} \leq \mathbb{G}$ and $\widehat{\alpha}_2 : \mathrm{L}^\infty(\widehat{\mathbb{H}}) \to \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{K}})$ the right quantum group homomorphism corresponding to $\widehat{\Pi}_2$. Then $\widehat{\alpha}_2 = \widehat{\alpha}_1|_{\mathrm{L}^\infty(\widehat{\mathbb{H}})}$.*

Proof. Let $V \in \mathrm{L}^\infty(\widehat{\mathbb{K}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ and $U \in \mathrm{L}^\infty(\widehat{\mathbb{K}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ be the bicharacters corresponding to Π_1 and Π_2 respectively. Let $\mathbb{V} \in \mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{K}}) \otimes \mathrm{C}_0^u(\mathbb{G}))$ be the universal lift of V:

$$V_{12}^* \mathbb{W}_{23}^{\mathbb{G}} V_{12} = \mathbb{V}_{13} \mathbb{W}_{23}^{\mathbb{G}}. \tag{1.3.12}$$

Applying quantum group morphism $\pi \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0^u(\mathbb{H}))$ (corresponding to $\mathbb{H} \leq \mathbb{G}$) to the third leg of (1.3.12) and reducing the result we get

$$V_{12}^* \mathrm{W}_{23}^{\mathbb{H}} V_{12} = U_{13} \mathrm{W}_{23}^{\mathbb{H}} = U_{12}^* \mathrm{W}_{23}^{\mathbb{H}} U_{12} \tag{1.3.13}$$

(note that we use the embedding $\mathrm{L}^\infty(\widehat{\mathbb{H}}) \subset \mathrm{L}^\infty(\widehat{\mathbb{G}})$ on the left side of (1.3.13)). We conclude by recalling that $\widehat{\alpha}_1$ is implemented by $\widehat{V}$ and $\widehat{\alpha}_2$ is implemented by $\widehat{U}$. □

We shall also need the following

Lemma 1.3.16. *Let $\mathbb{G}$ be a locally compact quantum group, $\mathsf{N} \in \mathcal{NCI}(\mathbb{G})$ and $\mathsf{M} \in \mathcal{BV}(\mathbb{G})$. Let $\mathbb{H}$ be a locally compact quantum group such that $\mathsf{M} = \mathrm{L}^\infty(\mathbb{H})$. Then*

$$\mathrm{W}^{\mathbb{H}}(\mathbb{1} \otimes \mathsf{N})\mathrm{W}^{\mathbb{H}*} \subset \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathsf{N}. \tag{1.3.14}$$

In particular $\mathsf{N} \wedge \mathrm{L}^\infty(\mathbb{H}) \in \mathcal{NCI}(\mathbb{H})$. Moreover

$$\mathsf{N} \vee \mathrm{L}^\infty(\mathbb{H}) = \{xy : x \in \mathsf{N}, y \in \mathsf{M}\}^{\sigma-cls}. \tag{1.3.15}$$

Proof. As explained in Example 1.3.8 the embedding $\mathrm{L}^\infty(\mathbb{H}) \subset \mathrm{L}^\infty(\mathbb{G})$ corresponds to a morphism $\Pi : \mathbb{G} \to \mathbb{H}$. Let $\widehat{\Pi} : \widehat{\mathbb{H}} \to \widehat{\mathbb{G}}$ be the dual morphism and $\widehat{\alpha} : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{H}})$ the corresponding right quantum group homomorphism. Applying $(\widehat{\alpha} \otimes \mathrm{id})$ to the normality condition

$$\mathrm{W}^{\mathbb{G}}(\mathbb{1} \otimes \mathsf{N})\mathrm{W}^{\mathbb{G}*} \subset \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathsf{N}$$

and using

$$(\widehat{\alpha} \otimes \mathrm{id})(\mathrm{W}^{\mathbb{G}}) = \mathrm{W}_{23}^{\mathbb{H}} \mathrm{W}_{13}^{\mathbb{G}}$$

we get

$$\mathrm{W}_{23}^{\mathbb{H}} \mathrm{W}_{13}^{\mathbb{G}}(\mathbb{1} \otimes \mathbb{1} \otimes \mathsf{N})\mathrm{W}_{13}^{\mathbb{G}*} \mathrm{W}_{23}^{\mathbb{H}*} \in \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathsf{N}. \tag{1.3.16}$$

Using (1.1.1) in the context of the $\widehat{\mathbb{G}}$-action

$$\alpha : \mathsf{N} \ni x \to \mathrm{W}^{\mathbb{G}}(\mathbb{1} \otimes x)\mathrm{W}^{\mathbb{G}*}$$

we get

$$\mathsf{N} = \{(\omega \otimes \mathrm{id})(\mathrm{W}^{\mathbb{G}}(\mathbb{1} \otimes x)\mathrm{W}^{\mathbb{G}*}) : \omega \in \mathrm{L}^\infty(\widehat{\mathbb{G}})_*, x \in \mathsf{N}\}^{\sigma-\mathrm{cls}}.$$

Thus (1.3.16) implies that

$$\mathrm{W}^{\mathbb{H}}(\mathbb{1} \otimes \mathsf{N})\mathrm{W}^{\mathbb{H}*} \subset \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathsf{N}.$$

Let us fix $x \in \mathsf{N}$ and $y \in \mathrm{L}^\infty(\mathbb{H})$ of the form $y = (\omega_{p,q} \otimes \mathrm{id})(\mathrm{W}^{\mathbb{H}*})$ where $p, q \in \mathrm{L}^2(\mathbb{H})$. In order to check that (1.3.15) holds it suffices to check that

$$xy \in \{\mathrm{L}^\infty(\mathbb{H})\mathsf{N}\}^{\sigma-\mathrm{cls}}. \tag{1.3.17}$$

Indeed the latter enables us to conclude that the right hand side of (1.3.15) forms a von Neumann algebra and this suffices since $\mathsf{N} \vee \mathrm{L}^\infty(\mathbb{H})$ is the von Neumann algebra generated by N and $\mathrm{L}^\infty(\mathbb{H})$. In the following computation we fix an orthonormal basis $(e_i)_{i \in I}$ of $\mathrm{L}^2(\mathbb{H})$

$$\begin{aligned} xy &= x(\omega_{p,q} \otimes \mathrm{id})(\mathrm{W}^{\mathbb{H}*}) \\ &= (\omega_{p,q} \otimes \mathrm{id})(\mathrm{W}^{\mathbb{H}*}(\mathrm{W}^{\mathbb{H}}(\mathbb{1} \otimes x)\mathrm{W}^{\mathbb{H}*})) \\ &= \textstyle\sum_{i \in I}(\omega_{p,e_i} \otimes \mathrm{id})(\mathrm{W}^{\mathbb{H}*})(\omega_{e_i,q} \otimes \mathrm{id})(\mathrm{W}^{\mathbb{H}}(\mathbb{1} \otimes x)\mathrm{W}^{\mathbb{H}*}). \end{aligned}$$

This computation together with (1.3.14) shows that (1.3.17) indeed holds which ends the proof. □

Definition 1.3.17. Let $\mathbb{G}$ be a locally compact quantum group and $\mathbb{H},\mathbb{M} \in \mathcal{QS}(\mathbb{G})$. We say that $\mathbb{H}$ is normalized by $\mathbb{M}$ if $\mathbb{H} \in \mathcal{NQS}(\mathbb{H} \vee \mathbb{M})$.

Let $\mathbb{H}$ and $\mathbb{M}$ be as in Definition 1.3.17. Using Lemma 1.3.16 we get

$$\mathrm{W}^{\widehat{\mathbb{M}}}(\mathbb{1} \otimes \mathrm{L}^\infty(\widehat{\mathbb{H}}))\mathrm{W}^{\widehat{\mathbb{M}}*} \subset \mathrm{L}^\infty(\mathbb{M}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{H}}). \tag{1.3.18}$$

More generally the following holds.

Lemma 1.3.18. *Let $\mathbb{G}$ be a locally compact quantum group, $\mathbb{H},\mathbb{M} \in \mathcal{QS}(\mathbb{G})$ and suppose that $\mathbb{H}$ is normalized by $\mathbb{M}$. Let $\mathbb{L} \leq \mathbb{M}$. Then $\mathbb{H}$ is normalized by $\mathbb{L}$. Moreover*

$$\mathrm{L}^\infty(\widehat{\mathbb{H} \vee \mathbb{L}}) = \{xy : x \in \mathrm{L}^\infty(\widehat{\mathbb{L}}), y \in \mathrm{L}^\infty(\widehat{\mathbb{H}})\}^{\sigma-cls}$$

and $\mathbb{H} \wedge \mathbb{L} \in \mathcal{NQS}(\mathbb{L})$. In particular if $\mathbb{H} \in \mathcal{NQS}(\mathbb{G})$ then $\mathbb{H}$ is normalized by every $\mathbb{L} \in \mathcal{QS}(\mathbb{G})$ and $\mathbb{H} \wedge \mathbb{L} \in \mathcal{NQS}(\mathbb{L})$.

Proposition 1.3.19. *Let $\mathbb{G}$ be a locally compact quantum group and $\mathbb{H},\mathbb{M} \in \mathcal{QS}(\mathbb{G})$ and suppose that $\mathbb{H}$ is normalized by $\mathbb{M}$. Then $\alpha_{\mathbb{M}\to\mathbb{G}}(\mathrm{L}^\infty(\mathbb{G}/\mathbb{H})) \subset \mathrm{L}^\infty(\mathbb{G}/\mathbb{H}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{M})$.*

Proof. Let us first recall that

$$\alpha_{\mathbb{M}\to\mathbb{G}}(x) = \mathrm{W}^{\mathbb{M}}(x \otimes \mathbb{1})\mathrm{W}^{\mathbb{M}*}$$

Let us fix $x \in \mathrm{L}^\infty(\mathbb{G}/\mathbb{H})$, i.e. $x \in \mathrm{L}^\infty(\mathbb{G})$ and

$$\mathrm{W}^{\mathbb{H}}(x \otimes \mathbb{1}) = (x \otimes \mathbb{1})\mathrm{W}^{\mathbb{H}}.$$

We have to prove that

$$\mathrm{W}^{\mathbb{H}}_{13}\mathrm{W}^{\mathbb{M}}_{12}(x \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^{\mathbb{M}*}_{12} = \mathrm{W}^{\mathbb{M}}_{12}(x \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^{\mathbb{M}*}_{12}\mathrm{W}^{\mathbb{H}}_{13}. \tag{1.3.19}$$

Using (1.3.18) we see that

$$\mathrm{W}^{\mathbb{M}*}_{12}\mathrm{W}^{\mathbb{H}}_{13}\mathrm{W}^{\mathbb{M}}_{12} \subset \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{M}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H}) \tag{1.3.20}$$

and we compute

$$\begin{aligned}\mathrm{W}^{\mathbb{H}}_{13}\mathrm{W}^{\mathbb{M}}_{12}(x \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^{\mathbb{M}*}_{12} &= \mathrm{W}^{\mathbb{M}}_{12}\mathrm{W}^{\mathbb{M}*}_{12}\mathrm{W}^{\mathbb{H}}_{13}\mathrm{W}^{\mathbb{M}}_{12}(x \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^{\mathbb{M}*}_{12}\\ &= \mathrm{W}^{\mathbb{M}}_{12}(x \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^{\mathbb{M}*}_{12}\mathrm{W}^{\mathbb{H}}_{13}\mathrm{W}^{\mathbb{M}}_{12}\mathrm{W}^{\mathbb{M}*}_{12}\\ &= \mathrm{W}^{\mathbb{M}}_{12}(x \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^{\mathbb{M}*}_{12}\mathrm{W}^{\mathbb{H}}_{13}\end{aligned}$$

where in the second equality we use (1.3.20) and the fact that $x \in \mathrm{L}^\infty(\widehat{\mathbb{H}})'$ □

Remark 1.3.20. *Note that in the proof of Proposition 1.3.19 we needed somewhat less than Definition 1.3.17: it is enough to have*

$$\mathrm{W}^{\widehat{\mathbb{M}}}(\mathbb{1} \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{H}}))\mathrm{W}^{\widehat{\mathbb{M}}*} \subset \mathrm{L}^\infty(\mathbb{M}) \otimes \mathrm{L}^\infty(\widehat{\mathbb{H}}),$$

which is more akin to the classical notion of one group normalizing another.

The following simple observation regarding a universal property of quotient quantum groups will come in handy repeatedly in Chapter 3.

Lemma 1.3.21. *Let $\Pi : \mathbb{G} \to \mathbb{P}$ be a morphism of quantum groups, and $\mathbb{N} \in \mathcal{NQS}(\mathbb{G})$. Then, Π factors as*

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{\quad\Pi\quad} & \mathbb{P} \\ & \searrow \quad \mathbb{G}/\mathbb{N} \quad \nearrow & \end{array}$$

if and only if the composition $\mathbb{N} \to \mathbb{G} \to \mathbb{P}$ is trivial (i.e the image of the composition is the trivial group).

Proof. The direct implication is clear. Conversely, suppose the composition $\mathbb{N} \to \mathbb{G} \to \mathbb{P}$ is trivial. We will apply Lemma 1.3.15 to $\Pi_1 = \Pi$ and $\mathbb{H} = \mathbb{N} \in \mathcal{QS}(\mathbb{G})$. The right quantum group homomorphism $\widehat{\alpha} : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{P}})$ assigned to $\widehat{\Pi}$ is given by

$$\widehat{\alpha}(x) = \boldsymbol{\sigma}(V^*)(x \otimes 1)\boldsymbol{\sigma}(V^*),$$

where $x \in \mathrm{L}^\infty(\widehat{\mathbb{G}})$ and $V \in \mathrm{L}^\infty(\widehat{\mathbb{P}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ is the bicharacter assigned Π. Using Lemma 1.3.15 we conclude that V is contained in $\mathrm{L}^\infty(\widehat{\mathbb{P}}) \bar{\otimes} (\mathrm{L}^\infty(\widehat{\mathbb{N}})' \cap \mathrm{L}^\infty(\mathbb{G}))$, and hence must be contained in $\mathrm{L}^\infty(\widehat{\mathbb{P}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$. In particular V viewed as an element $\mathrm{L}^\infty(\widehat{\mathbb{P}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ defines a morphism $\mathbb{G}/\mathbb{N} \to \mathbb{P}$. Running through the way in which bicharacters, regarded as morphisms, compose in the category of locally compact quantum groups, this means precisely that Π factors through $\mathbb{G}/\mathbb{N}$. □

Remark 1.3.22. *From the perspective of the category of locally compact quantum groups, Lemma 1.3.21 simply says that $\mathbb{G} \to \mathbb{G}/\mathbb{N}$ is the coequalizer of the inclusion $\mathbb{N} \to \mathbb{G}$ and the trivial map $\mathbb{N} \to \mathbb{1} \to \mathbb{G}$.*

Moreover, by the self-duality of the category of locally compact quantum groups, we can conclude that the inclusion $\mathbb{N} \to \mathbb{G}$ of a normal subgroup is the equalizer of the arrows $\mathbb{G} \to \mathbb{G}/\mathbb{N}$ and $\mathbb{G} \to \mathbb{1} \to \mathbb{G}/\mathbb{N}$.

In fact, we can improve on Lemma 1.3.21 somewhat. For future reference, we record the result in Lemma 1.3.23 below. Before its formulation let us consider an action $\alpha : \mathsf{N} \to \mathsf{N} \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ of a locally compact quantum group $\mathbb{G}$ on a von Neumann algebra N. Then given a left quantum group homomorphism $\gamma : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{H}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ there exists a unique action $\beta : \mathsf{N} \to \mathsf{N} \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ such that

$$(\mathrm{id} \otimes \gamma) \circ \alpha = (\beta \otimes \mathrm{id}) \circ \alpha.$$

In particular, given $\mathbb{H} \leq \mathbb{G}$ we get β which we denote by $\alpha|_{\mathbb{H}} : \mathsf{N} \to \mathsf{N} \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ and we say that $\alpha|_{\mathbb{H}}$ is the restriction of α to $\mathbb{H}$. The details yielding the existence of β are left to the reader.

Lemma 1.3.23. *Let $\alpha : \mathsf{N} \to \mathsf{N} \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ and $\mathbb{N} \trianglelefteq \mathbb{G}$ a closed normal subgroup. Then, α factors as*

$$\begin{array}{ccc} \mathsf{N} & \xrightarrow{\quad\alpha\quad} & \mathsf{N} \bar{\otimes} \mathrm{L}^\infty(\mathbb{G}) \\ & \searrow \quad \mathsf{N} \bar{\otimes} \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) \quad \nearrow & \end{array}$$

through an action by $\mathbb{G}/\mathbb{N}$ on N if and only if $\mathbb{N}$ acts trivially on N.

Proof. Once again, one implication is trivial, so we prove the other one; that is, we assume that the restriction $\alpha|_{\mathbb{N}}$ of α to $\mathbb{N}$ is trivial. Let $\gamma : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{N}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G})$ be the left quantum group homomorphism assigned to $\mathbb{N} \trianglelefteq \mathbb{G}$. Using the identity $(\alpha|_{\mathbb{N}} \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \gamma) \circ \alpha$ and the trviality of $\alpha|_{\mathbb{N}}$ we conclude that $\alpha(\mathsf{N}) \subset \mathsf{N} \bar{\otimes} \mathrm{L}^\infty(\mathbb{N}\backslash\mathbb{G})$ where

$$\mathrm{L}^\infty(\mathbb{N}\backslash\mathbb{G}) = \{x \in \mathrm{L}^\infty(\mathbb{G}) : \gamma(x) = \mathbb{1} \otimes x\}.$$

We conclude by noting that normality of $\mathbb{N}$ yields $\mathrm{L}^\infty(\mathbb{N}\backslash\mathbb{G}) = \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ (see e.g. [51, Eq. (1.4)]).

□

Remark 1.3.24. *In phrasing of [53, §4.3], Lemma 1.3.23 says that the quotient of $\mathbb{G}$ by the kernel of α factors through $\mathbb{G}/\mathbb{N}$ if and only if $\alpha|_{\mathbb{N}}$ is trivial.*

In addition to Lemma 1.3.21, Proposition 1.3.9 above is also a consequence of Lemma 1.3.23; in effect, the intuitive content of that proposition is that since the abelian normal subgroup $\mathbb{N} \trianglelefteq \mathbb{G}$ acts trivially on itself by conjugation, the conjugation action of $\mathbb{G}$ on $\mathbb{N}$ descends to a $(\mathbb{G}/\mathbb{N})$-action.

Now, we shall discuss open quantum subgroups. Let us begin with [51, Definition 2.2].

Definition 1.3.25. Let $\mathbb{H}$ and $\mathbb{G}$ be locally compact quantum groups and $\pi : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{H})$ a normal surjective $*$-homomorphism. We say that π identifies $\mathbb{H}$ with an open quantum subgroup of $\mathbb{G}$ if $\Delta_{\mathbb{H}} \circ \pi = (\pi \otimes \pi) \circ \Delta_{\mathbb{G}}$.

Let $\mathbb{H}$ be a locally compact quantum group which is identified with an open quantum subgroup of $\mathbb{G}$ via $\pi : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{H})$. The central support of π (i.e. the smallest projection of $P \in \mathrm{L}^\infty(\mathbb{G})$ such that $\pi(P) = \mathbb{1}$) will be denoted by $\mathbb{1}_{\mathbb{H}}$ and it will be refereed to as a group-like projection assigned to π. The morphism π defines a morphism $\Pi : \mathbb{H} \to \mathbb{G}$ which in terms of the bicharacter is given by $V = (\mathrm{id} \otimes \pi)(\mathrm{W}^{\mathbb{G}}) \in \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$. Using [51, Theorem 3.6] we see that Π identifies $\mathbb{H}$ with a closed quantum subgroup of $\mathbb{G}$ as described in Example 1.3.8.

Let $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$. Then as proved in [53, Corollary 3.4], $\mathbb{H}$ can be identified with an open quantum subgroup of $\mathbb{G}$, (we shall say shortly that $\mathbb{H}$ is open in $\mathbb{G}$) if and only if the Haar weight $\psi^{\widehat{\mathbb{G}}}$ restricts to the Haar weight on $\mathrm{L}^\infty(\widehat{\mathbb{H}})$. In other words if and only if the restriction of $\psi^{\widehat{\mathbb{G}}}$ to $\mathrm{L}^\infty(\widehat{\mathbb{H}}) \subset \mathrm{L}^\infty(\widehat{\mathbb{G}})$ is semifinite. This in turn is equivalent with the existence of a conditional expectation $T : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}})$ onto $\mathrm{L}^\infty(\widehat{\mathbb{H}})$ satisfying (see [52, Theorem 6.1]

$$(T \otimes \mathrm{id}) \circ \Delta_{\widehat{\mathbb{G}}} = \Delta_{\widehat{\mathbb{G}}} \circ T = (\mathrm{id} \otimes T) \circ \Delta_{\widehat{\mathbb{G}}}. \tag{1.3.21}$$

The subset of $\mathcal{QS}(\mathbb{G})$ that consists of open quantum subgroups of $\mathbb{G}$ will be denoted by $\mathcal{OQS}(\mathbb{G})$. In what follows we shall investigate the structure of $\mathcal{OQS}(\mathbb{G})$ showing in particular that it forms a lattice.

Proposition 1.3.26. *Let $\mathbb{G}$ be a locally compact quantum group, $\mathbb{H} \in \mathcal{OQS}(\mathbb{G})$ and $\mathbb{M} \in \mathcal{QS}(\mathbb{G})$. Then $\mathbb{H} \wedge \mathbb{M} \in \mathcal{OQS}(\mathbb{M})$.*

Moreover, if

$$T : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{H}}) \text{ and } T' : \mathrm{L}^\infty(\widehat{\mathbb{M}}) \to \mathrm{L}^\infty(\widehat{\mathbb{H} \wedge \mathbb{M}})$$

are the expectations associated to the respective Haar weights, we have

$$T' = T|_{\mathrm{L}^\infty(\widehat{\mathbb{M}})}.$$

Proof. Let $T : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{H}})$ be the conditional expectation assigned to $\mathbb{H} \in \mathcal{OQS}(\mathbb{G})$ (see (1.3.21)). Consider the restriction of T to $\mathrm{L}^\infty(\widehat{\mathbb{M}}) \leq \mathrm{L}^\infty(\widehat{\mathbb{G}})$. The equality (see (1.1.3))

$$\mathrm{L}^\infty(\widehat{\mathbb{M}}) = \{(\omega \otimes \mathrm{id})(\Delta_{\widehat{\mathbb{G}}}(x)) : \omega \in \mathrm{L}^\infty(\widehat{\mathbb{G}})_*, x \in \mathrm{L}^\infty(\widehat{\mathbb{M}})\}^{\sigma-\mathrm{cls}}$$

together with (see (1.3.21))

$$(\mathrm{id} \otimes T) \circ \Delta_{\widehat{\mathbb{G}}} = (T \otimes \mathrm{id}) \circ \Delta_{\widehat{\mathbb{G}}}$$

imply that $T(\mathrm{L}^\infty(\widehat{\mathbb{M}})) \subset \mathrm{L}^\infty(\widehat{\mathbb{M}})$. Thus we get an inclusion $T(\mathrm{L}^\infty(\widehat{\mathbb{M}})) \subset \mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge \mathrm{L}^\infty(\widehat{\mathbb{M}})$. On the other hand, this restriction is the identity on

$$\mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge \mathrm{L}^\infty(\widehat{\mathbb{M}}) = \mathrm{L}^\infty(\widehat{\mathbb{H} \wedge \mathbb{M}}).$$

In conclusion, $T|_{\mathrm{L}^\infty(\widehat{\mathbb{M}})}$ is an expectation onto $L^\infty(\widehat{\mathbb{H} \wedge \mathbb{M}})$ which clearly satisfies (1.3.21). Both claims of the conclusion now follow from [52, Theorem 6.1]. □

Let us also note in passing that in the particular case when $\mathbb{H} \le \mathbb{M}$ we obtain:

Corollary 1.3.27. *If $\mathbb{H} \le \mathbb{M} \le \mathbb{G}$ is a sequence of closed quantum group embeddings with $\mathbb{H}$ open in $\mathbb{G}$ then $\mathbb{H} \le \mathbb{M}$ is also open.*

In fact, we can improve on Corollary 1.3.27 as in the next result, proving a strong transitivity of openness.

Proposition 1.3.28. *Let $\mathbb{H} \le \mathbb{M} \le \mathbb{G}$ be a chain of closed embeddings of locally compact quantum groups. Then, $\mathbb{H}$ is open in $\mathbb{G}$ if and only if*

$$\mathbb{H} \le \mathbb{M} \quad \textit{and} \quad \mathbb{M} \le \mathbb{G}$$

are both open.

Proof. In order to see the leftward implication '⇐' we use [53, Corollary 3.4].

Let us check the rightward implication '⇒'. We have already seen in Corollary 1.3.27 that $\mathbb{H} \le \mathbb{M}$ is open. On the other hand, [53, Theorem 3.3] shows that $\mathbb{M} \le \mathbb{G}$ is open if and only if there is some non-zero element of

$$\mathrm{L}^\infty(\widehat{\mathbb{M}}) \le \mathrm{L}^\infty(\widehat{\mathbb{G}})$$

that is square-integrable with respect to the (either left or right) Haar weight of $\widehat{\mathbb{G}}$. An application of the same result in the opposite direction shows that there are such elements in

$$\mathrm{L}^\infty(\widehat{\mathbb{H}}) \le \mathrm{L}^\infty(\widehat{\mathbb{M}}) \le \mathrm{L}^\infty(\widehat{\mathbb{G}}).$$

This concludes the proof. □

Using Proposition 1.3.26 and Proposition 1.3.28 we get:

Corollary 1.3.29. *The set $\mathcal{OQS}(\mathbb{G})$ forms a sublattice of $\mathcal{QS}(\mathbb{G})$.*

Remark 1.3.30. *Let $\mathbb{H} \le \mathbb{G}$ be an open quantum subgroup and $\omega \in \mathrm{C}_0^u(\widehat{\mathbb{G}})^*$ the idempotent state assigned to $\mathbb{H} \le \mathbb{G}$ as described in* [52, Remark 6.3]*. The group-like projection $\mathbb{1}_{\mathbb{H}} \in \mathrm{L}^\infty(\mathbb{G})$ assigned to $\mathbb{H} \le \mathbb{G}$ is given by*

$$\mathbb{1}_{\mathbb{H}} = (\omega \otimes \mathrm{id})(\mathrm{W}^{\mathbb{G}}).$$

Let us denote $\mathbb{1}^u_{\mathbb{H}} = (\omega \otimes \mathrm{id})(\mathbb{W}) \in \mathrm{M}(\mathrm{C}_0^u(\mathbb{G}))$; applying the reducing morphism $\Lambda_{\mathbb{G}} \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0(\mathbb{G}))$ we get $\Lambda_{\mathbb{G}}(\mathbb{1}^u_{\mathbb{H}}) = \mathbb{1}_{\mathbb{H}}$. Denoting by $\pi^u \in \mathrm{M}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0^u(\mathbb{H}))$ the morphism assigned to $\mathbb{H} \le \mathbb{G}$ we see that

$$\begin{aligned}(\pi^u \otimes \mathrm{id})(\Delta^u(\mathbb{1}^u_{\mathbb{H}})) &= (\pi^u \otimes \mathrm{id})(\Delta^u((\omega \otimes \mathrm{id})(\mathbb{W}))) \\ &= (\omega \otimes \mathrm{id} \otimes \mathrm{id})((\mathrm{id} \otimes \pi^u)(\mathbb{W}_{12})\mathbb{W}_{13}) \qquad (1.3.22) \\ &= \mathbb{1} \otimes \mathbb{1}^u_{\mathbb{H}}\end{aligned}$$

where in the third equality we used the identity $\omega(x\hat{\pi}^u(y)) = \omega(x)\hat{\varepsilon}(y))$ *which holds for all* $x \in \mathrm{C}_0^u(\widehat{\mathbb{G}})$ *and* $y \in \mathrm{C}_0^u(\widehat{\mathbb{H}})$*. The latter can be easily concluded from the fact that the image of the conditional expectation* $T^u = (\mathrm{id} \otimes \omega) \circ \widehat{\Delta}^u$ *is equal to* $\hat{\pi}^u(\mathrm{C}_0^u(\widehat{\mathbb{H}}))$*.*

Similarly we can prove that $(\mathrm{id} \otimes \pi^u)(\Delta^u(\mathbb{1}_{\mathbb{H}}^u)) = \mathbb{1}_{\mathbb{H}}^u \otimes \mathbb{1}$*. In particular using (1.3.22) and (1.3.5) we get*

$$\begin{aligned}(\mathbb{1}_{\mathbb{H}} \otimes \mathbb{1})\Delta_r^{r,u}(\mathbb{1}_{\mathbb{H}}) &= \mathbb{1}_{\mathbb{H}} \otimes \mathbb{1}_{\mathbb{H}}^u,\\ (\mathbb{1} \otimes \mathbb{1}_{\mathbb{H}})\Delta_r^{u,r}(\mathbb{1}_{\mathbb{H}}) &= \mathbb{1}_{\mathbb{H}}^u \otimes \mathbb{1}_{\mathbb{H}}.\end{aligned} \tag{1.3.23}$$

Now let $\mathbb{M} \leq \mathbb{G}$ *be a closed quantum subgroup and* $\rho \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0(\mathbb{M}))$ *the corresponding morphism. The group-like projection assigned to the open containment* $\mathbb{H} \wedge \mathbb{M} \leq \mathbb{M}$ *(see Proposition 1.3.26) will be denoted by* $\mathbb{1}_{\mathbb{H}\wedge\mathbb{M}} \in \mathrm{L}^\infty(\mathbb{M})$*. Using Proposition 1.3.26 we get*

$$\mathbb{1}_{\mathbb{H}\wedge\mathbb{M}} = \rho(\mathbb{1}_{\mathbb{H}}^u) \tag{1.3.24}$$

which follows from the computation

$$(\mathbb{1} \otimes \mathbb{1}_{\mathbb{H}\wedge\mathbb{M}})\mathrm{W}^{\mathbb{M}} = (T' \otimes \mathrm{id})(\mathrm{W}^{\mathbb{M}}) = (T \otimes \rho)(\mathbb{W}^{\mathbb{G}}) = (\mathbb{1} \otimes \rho(\mathbb{1}_{\mathbb{H}}^u))\mathrm{W}^{\mathbb{M}}$$

(in this computation we use the embedding $\mathrm{L}^\infty(\widehat{\mathbb{M}}) \subset \mathrm{L}^\infty(\widehat{\mathbb{G}})$ *and the notation of the proof of Proposition 1.3.26). Denoting the action*

$$\alpha \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{G}), \mathrm{C}_0(\mathbb{G}) \otimes \mathrm{C}_0(\mathbb{M}))$$

describing the embedding $\mathbb{M} \leq \mathbb{G}$ *we get*

$$(\mathbb{1}_{\mathbb{H}} \otimes \mathbb{1})\alpha(\mathbb{1}_{\mathbb{H}}) = \mathbb{1}_{\mathbb{H}} \otimes \mathbb{1}_{\mathbb{H}\wedge\mathbb{M}}$$

Indeed, this follows from (1.3.23), (1.3.24) and the the identity $\alpha = (\mathrm{id}\otimes\rho)\circ\Delta_r^{r,u}$*. Similarly we get*

$$(\mathbb{1} \otimes \mathbb{1}_{\mathbb{H}})\alpha(\mathbb{1}_{\mathbb{H}}) = \mathbb{1}_{\mathbb{H}\wedge\mathbb{M}} \otimes \mathbb{1}_{\mathbb{H}}.$$

1.3.1 Well positioned quantum subgroups

For subgroups $\mathbb{H} \leq \mathbb{G}$, we will be working with the quantum homogeneous spaces (see Remark 1.3.2)

$$A_{\mathbb{H}} = \mathrm{L}^\infty(\mathbb{G}/\mathbb{H}) = \mathrm{cd}(\mathrm{L}^\infty(\widehat{\mathbb{H}})) \subseteq \mathrm{L}^\infty(\mathbb{G}).$$

Definition 1.3.31. Let $\mathbb{H}$ and $\mathbb{M}$ be two closed quantum subgroups of a locally compact quantum group $\mathbb{G}$. We say that $\mathbb{H}$ and $\mathbb{M}$ are *(relatively) well positioned* if we have the equality

$$A_{\mathbb{H}} \vee A_{\mathbb{M}} = \{A_{\mathbb{H}}A_{\mathbb{M}}\}^{\sigma-\mathrm{cls}} \tag{1.3.25}$$

(or equivalently its analogue with $\mathbb{H}$ and $\mathbb{M}$ reversed).

As we will see in Theorem 3.3.4, the well positioning property is relevant to the modular law for quantum subgroups of a locally compact quantum group. Here, we discuss sufficient conditions that ensure well positioning. Let us also note that in the algebraic context the counterpart of well positioning always holds as noted in Corollary 1.2.10.

Proposition 1.3.32. *The closed quantum subgroups* $\mathbb{H}, \mathbb{M} \leq \mathbb{G}$ *are relatively well positioned if*

(a) $\mathbb{G}$ *is classical;*

(b) one of $\mathbb{H}$ and $\mathbb{M}$ is compact;

(c) one of $\mathbb{H}$ and $\mathbb{M}$ is normal;

(d) $\mathbb{G}$ is dual-classical.

Proof. We prove the different points separately, as they require different techniques.

(a) This is immediate: $\mathrm{L}^\infty(\mathbb{G})$ is then commutative, and hence it does not matter in which order we multiply elements of $A_\mathbb{H}$ and $A_\mathbb{M}$.

(b) The condition is symmetric, so let us assume that $\mathbb{H}$ is compact and show that $A_\mathbb{M} A_\mathbb{H}$ is linearly dense in $A_\mathbb{H} \vee A_\mathbb{M}$. We will adapt the proof of Lemma 1.3.16.

We write $\mathrm{W} = \mathrm{W}^\mathbb{G}$ and φ for the Haar state on the compact quantum group $\mathbb{H}$. We further denote by α the canonical coaction

$$\mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$$

and by $V \in \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ the bicharacter corresponding to α.

Note that the elements of the form

$$x = (\mathrm{id} \otimes \varphi)\alpha((\omega_{p,q} \otimes \mathrm{id})\mathrm{W}) = (\omega_{p,q} \otimes \mathrm{id} \otimes \varphi)(\mathrm{W}_{12} V_{13}) \tag{1.3.26}$$

span a dense subset of $A_\mathbb{H}$ for $p, q \in \mathrm{L}^2(\mathbb{G})$. Hence, it suffices to show that an element obtained by multiplying x as in (1.3.26) and an arbitrary $a \in A_\mathbb{M}$ belongs to the closure of linear span on $A_\mathbb{M} A_\mathbb{H}$.

With this purpose in mind, we first use

$$(\mathrm{id} \otimes \alpha)\mathrm{W} = \mathrm{W}_{12} V_{13}$$

again to write

$$xa = (\omega_{p,q} \otimes \mathrm{id} \otimes \varphi)(\mathrm{W}_{12} V_{13}(1 \otimes a \otimes 1)) = (\omega_{p,q} \otimes \mathrm{id} \otimes \varphi)(\mathrm{W}_{12}(1 \otimes a \otimes 1)\mathrm{W}_{12}^* \mathrm{W}_{12} V_{13})).$$

Using

$$\omega_{pq}(\bullet -) = \sum_i \omega_{p,e_i}(\bullet)\omega_{e_i,q}(-),$$

the expression turns into

$$\sum_i (\omega_{p,e_i} \otimes \mathrm{id})(\mathrm{W}_{12}(1 \otimes a)\mathrm{W}_{12}^*) \cdot (\omega_{e_i,q} \otimes \mathrm{id} \otimes \varphi)(\mathrm{W}_{12} V_{13}).$$

Now, the left hand side of the '$\cdot$' symbol belongs to $A_\mathbb{M}$ by the normality condition

$$\mathrm{W}(1 \otimes A_\mathbb{M})\mathrm{W}^* \subseteq \mathrm{L}^\infty(\widehat{\mathbb{G}}) \otimes A_\mathbb{M},$$

whereas the right hand side is of the same form as (1.3.26) and hence belongs to $A_\mathbb{H}$.

(c) Once again the condition is symmetric, so for the sake of making a choice we assume $\mathbb{H}$ is normal. But then $A_\mathbb{H}$ is a Baaj-Vaes subalgebra of $\mathrm{L}^\infty(\mathbb{G})$, and hence the desired result follows from an application of Lemma 1.3.16 (in the form of Lemma 1.3.15) to $\mathsf{N} = A_\mathbb{M}$ and $\mathsf{M} = A_\mathbb{H}$.

(d) Since $\mathbb{G}$ is abelian, quantum subgroups of $\mathbb{G}$ are normal and part (c) applies. □

Remark 1.3.33. *Let $\mathbb{G}$ be a locally compact quantum group and $\mathsf{N} \subset B(\mathrm{L}^2(\mathbb{G}))$ a von Neumann algebra such that $\mathrm{W}(\mathbb{1} \otimes \mathsf{N})\mathrm{W}^* \subset \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathsf{N}$. Let M be a von Neumann subalgebra of $\mathrm{L}^\infty(\mathbb{G})$ equipped with a conditional expectation $E : \mathrm{L}^\infty(\mathbb{G}) \to \mathsf{M}$. Let $\mathsf{N} \vee \mathsf{M}$ be the von Neumann algebra generated by N and M. Using the method of the proof of point (b) of Proposition 1.3.32 we get*

$$\mathsf{N} \vee \mathsf{M} = \{\mathsf{N}\mathsf{M}\}^{\sigma-cls}.$$

1.4 Preliminaries on ergodic actions and idempotents states of a compact quantum group

In this section, we note $C_0(\mathbb{X})$ a C^*-algebra and $\mathbb{X}$ its associated *locally compact quantum space*. The notion of compact quantum groups has been introduced by Woronowicz. Here we adopt his definition in [118]. The symbol $\otimes$ denotes the spatial tensor product of C^*-algebras.

Definition 1.4.1. A *compact quantum group* is a pair (A, Δ) where A is a unital C^*-algebra and $\Delta : A \to A \otimes A$ is a unital $*$-homomorphism which is coassociative:

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta$$

and A satisfies the quantum cancellation properties:

$$\overline{Lin}\,((1 \otimes A) \circ \Delta(A)) = \overline{Lin}\,((A \otimes 1) \circ \Delta(A)) = A \otimes A$$

One of the most important features of compact quantum groups is the existence of a unique Haar state h i.e a unique state $h \in A^*$ such that:

$$(h \otimes id_A) \circ \Delta(a) = (id_A \otimes h) \circ \Delta(a) = h(a)1 \qquad \forall a \in A.$$

A compact quantum groups is said of *Kac type* if h is tracial i.e $h(ab) = h(ba) \qquad \forall a, b \in A$.

Remark 1.4.2. *All coidalgebras of a compact quantum group of Kac type are expected i.e there exists a conditional expectation (see Definition 1.4.8).*

A compact quantum group $\mathbb{G} = (A, \Delta)$ is co-amenable if A^* is unital, this equivalent to the existence of a non-zero multiplicative functional on A.

Example 1.4.3. $O_{-1}(2)$ (see Definition 1.4.16) is a coamenable compact quantum group of Kac type.

The following definition generalising the notion of an idempotent probability measure on a compact group:

Definition 1.4.4. A state φ on a compact quantum group A is said to be an idempotent state if

$$(\varphi \otimes \varphi) \circ \Delta = \varphi.$$

By [39, Theorem 4.1], we have:

Theorem 1.4.5. *Let (A, Δ) be a coamenable compact quantum group. Then there exists an order-preserving bijection between the expected right coidalgebras in (A, Δ) and the idempotent states on (A, Δ).*

Remark 1.4.6. *The bijection of Theorem 1.5 extends to non-coamenable and non-unimodular locally compact quantum group (See* [84, Theorem 1]*). In the general case, we must just be careful and consider the idempotent states in the universal version $C_u(\mathbb{G})$ or in $Pol(\mathbb{G})$ (See* [38, Theorem 2.3]*). On the other hand, the coideals are in the reduced version $C_r(\mathbb{G})$ and there is a one to one correspondence between idempotent states and integrable coideals preserved by the scaling group (See* [52, Corollary 4.3]*).*

Lets now define the notion of right action, natural conditional expectation, ergodic action, ergodic action of quotient type and embeddable ergodic action in compact quantum group setting. We adopt here the same definition as K. De Commer in [25].

Definition 1.4.7. A (continuous) *right action* $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ consists of

- a compact quantum group $\mathbb{G}$,
- a locally compact quantum space $\mathbb{X}$, and
- a $*$-homomorphism, called *right coaction*,

$$\alpha : C_0(\mathbb{X}) \to C_0(\mathbb{X}) \otimes C(\mathbb{G})$$

  such that:

  - the *coaction property* holds,

$$(\alpha \otimes Id_{\mathbb{G}}) \circ \alpha = (id_{\mathbb{X}} \otimes \Delta) \circ \alpha,$$

    and
  - the following density condition, called *Podleś condition*, holds,

$$\alpha\left(C_0(\mathbb{X})\right)\left(1_{\mathbb{X}} \otimes C(\mathbb{G})\right) = C_0(\mathbb{X}) \otimes C(\mathbb{G}).$$

Definition 1.4.8. Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ and $\mathbb{Y} = \mathbb{X}/\mathbb{G}$. The *natural conditional expectation* onto $C_0(\mathbb{Y})$ is the map

$$E_{\mathbb{Y}} : \begin{cases} C_0(\mathbb{X}) & \to C_0(\mathbb{X}) \\ a & \mapsto (id_{\mathbb{X}} \otimes \varphi_{\mathbb{G}})\,\alpha(a) \end{cases} \quad \text{with} \quad \varphi_{\mathbb{G}} \quad \text{the Haar state of the compact quantum group} \quad \mathbb{G}.$$

Definition 1.4.9. An action $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$ is called *ergodic* (or *homogeneous*) if

$$C(\mathbb{X}/\mathbb{G}) = \{a \in C_0(\mathbb{X}) | \quad \alpha(a) = a \otimes 1_{\mathbb{G}}\}\mathbb{C}1_{\mathbb{X}}.$$

Remark 1.4.10. *Since $\mathbb{Y}$ reduces to a point in the case of homogeneous actions, we obtain in particular that the conditional expectation $E_{\mathbb{Y}}$ becomes a state on $C(\mathbb{X})$.*

Definition 1.4.11. Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. One calls α of *quotient type* if there exists a compact quantum subgroup $\mathbb{H} \subset \mathbb{G}$ with corresponding quotient map $\pi_{\mathbb{H}} : C(\mathbb{G}) \to C(\mathbb{H})$ and a $*$-isomorphism

$$\theta : C(\mathbb{X}) \to C(\mathbb{H} \setminus \mathbb{G}) = \{g \in C(\mathbb{G}) \mid \quad (\pi_{\mathbb{H}} \otimes id_{\mathbb{G}})\,\Delta(g) = 1_{\mathbb{H}} \otimes g\}$$

such that

$$(\theta \otimes id_{\mathbb{G}}) \circ \alpha = \Delta \circ \theta.$$

Definition 1.4.12. Let $\mathbb{X} \overset{\alpha}{\curvearrowleft} \mathbb{G}$. One calls α *embeddable* if there exists a faithful $*$-homomorphism

$$\theta : C(\mathbb{X}) \hookrightarrow C(\mathbb{G})$$

such that

$$(\theta \otimes id_{\mathbb{G}}) \circ \alpha = \Delta \circ \theta.$$

Remark 1.4.13. *When seen as subalgebras of $C(\mathbb{G})$, embeddable actions are also referred to as (right) coideal C^*-subalgebras or coidalgebras.*

By Theorem 1.4.5 and Remark 1.4.2, we deduce that:

Lemma 1.4.14. *Let (A, Δ) be a coamenable compact quantum group of Kac type. Then there exists an order-preserving bijection between the embeddable ergodic action of (A, Δ) and the idempotent states on (A, Δ).*

Lets $\mathbb{G}$ be a group and $\mathbb{K}$ a field.

Definition 1.4.15. A *projective representation* of $\mathbb{G}$ over $\mathbb{K}$ is a map $\pi : \mathbb{G} \to GL(n, \mathbb{K})$ such that:

$$\pi(x)\pi(y) = \lambda(x, y)\pi(xy) \quad \forall x, y \in \mathbb{G} \tag{1.4.1}$$

where λ is called the associated multiplier (or a factor or a 2-cocycle).

In order to study the ergodic actions of $O_{-1}(2)$ (see Chapter 4 Section 4.1), let's first recall the definition of $O_{-1}(2)$.

Definition 1.4.16. The compact quantum group $O_{-1}(2)$ is defined as the compact matrix quantum group with fundamental copresentation $y = (y_{jk})_{1 \leq j,k \leq 2}$ and the relations

1. y is orthogonal, i.e. the generators y_{jk} are self-adjoint and satisfy the unitarity relations $y_{1j}y_{1k} + y_{2j}y_{2k} = \delta_{jk} = y_{j1}y_{k1} + y_{j2}y_{k2}$ for $j, k = 1, 2$;
2. $y_{jk}y_{j\ell} = -y_{j\ell}y_{jk}$ and $y_{kj}y_{\ell j} = -y_{\ell j}y_{kj}$ for $k \neq \ell$;
3. $y_{jk}y_{\ell m} = y_{\ell m}y_{jk}$ for $j \neq \ell$ and $k \neq m$.

The coproduct, counit and antipode of $O_{-1}(2)$ are given by:

$$\Delta(y_{jk}) = \sum_i y_{ji} \otimes y_{ik}, \qquad \varepsilon(y_{jk}) = \delta_{jk}, \qquad S(y_{jk}) = y_{kj}.$$

The following notion of monoidal equivalence was introduced in [8]. We adopt here the definition of [30].

Definition 1.4.17. Two compact quantum groups $\mathbb{G}_1 = (A_1, \Delta_1)$ and $\mathbb{G}_2 = (A_2, \Delta_2)$ are said to be *monoidally equivalent* if there exists a bijection $\psi : Irred(\mathbb{G}_1) \to Irred(\mathbb{G}_2)$ satisfying $\psi(\varepsilon) = \varepsilon$, together with linear isomorphisms

$$\psi : Mor(x_1 \otimes \cdots \otimes x_r, y_1 \otimes \cdots \otimes y_k) \to Mor(\psi(x_1) \otimes \cdots \otimes \psi(x_r), \psi(y_1) \otimes \cdots \otimes \psi(y_k))$$

satisfying the following conditions:

$$\psi(1) = 1, \quad \psi(S^*) = (\psi(S))^*, \quad \psi(ST) = \psi(S)\psi(T), \quad \psi(S \otimes T) = \psi(S) \otimes \psi(T) \quad \forall S \subset A_1 \text{ and } T \subset A_2.$$

whenever the formulas make sense. Such a collection of maps ψ is called a *monoidal equivalence* between $\mathbb{G}_1$ and $\mathbb{G}_2$.

Remark 1.4.18. *Concrete examples of monoidally equivalent compact quantum groups are given in section 4 of* [30].

Lets recall now the principal result of [30]. In fact, in their paper, the autors proved that there is bijective correspondence between ergodic actions of two monoidally equivalent compact quantum group. Let's recall now [30, Theorem 7.3].

Theorem 1.4.19. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent compact quantum group. Then there is a natural bijective correspondence between their ergodic actions.*

We finish this subsection by stating a question motivated by Theorem 1.4.19, and fundamentally related to Section 4.2 and Section 4.3.

Question 1.4.1. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent compact quantum group. Do we have a bijective correspondence between their embeddable ergodic actions?*

The answer of this question will be the aim of Chapter 4 Section 4.4.

Chapter 2

Lattices of quantum subgroups of a linearly reductive quantum group

In this chapter we tackle some analogues of the group isomorphism theorems in the setting of (mostly linearly reductive) linear algebraic quantum groups.

2.1 The second isomorphism theorem

We will prove a version of the second isomorphism theorem [81, Theorem 2.26] for function algebras of linearly reductive quantum groups, i.e. cosemisimple Hopf algebras (see Definition 1.2.1). The general setup is as follows.

Recall from Section 1.2 that unless specified otherwise, we work over an algebraically closed field k of arbitrary characteristic. $\mathcal{O}(\mathbb{G})$ denotes a cosemisimple Hopf algebra, to be thought of as the algebra of regular functions on a quantum group $\mathbb{G}$. We fit the latter into an exact sequence

$$1 \to \mathbb{K} \to \mathbb{G} \to \mathbb{G}/\mathbb{K} \to 1$$

in the sense that we have an exact sequence

$$k \to \mathcal{O}(\mathbb{G}/\mathbb{K}) \to \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{K}) \to k$$

as in Definition 1.2.12. Note that $\mathcal{O}(\mathbb{G}/\mathbb{K})$ is automatically cosemisimple (being a Hopf subalgebra of a cosemisimple Hopf algebra), and hence ([20, Theorem 2.1]) the inclusion $\mathcal{O}(\mathbb{G}/\mathbb{K}) \to \mathcal{O}(\mathbb{G})$ is automatically faithfully flat both on the left and the right. It then follows [20, Theorem 2.5] that $\mathcal{O}(\mathbb{K})$ is itself cosemisimple.

Assume now that we have another linearly reductive quantum subgroup $\mathbb{H} \leq \mathbb{G}$, i.e. a quotient cosemisimple Hopf algebra $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$. We will now examine the issue of whether or not the intersection $\mathbb{H} \wedge \mathbb{K}$ from Definition 1.2.5 is linearly reductive.

First, define the Hopf subalgebra $\bullet$ of $\mathcal{O}(\mathbb{H})$ so as to make the following diagram commute.

$$\begin{array}{ccccc} & \hookrightarrow & \mathcal{O}(\mathbb{G}) & \twoheadrightarrow & \\ \mathcal{O}(\mathbb{G}/\mathbb{K}) & & & & \mathcal{O}(\mathbb{H}) \\ & \twoheadrightarrow & \bullet & \hookrightarrow & \end{array} \tag{2.1.1}$$

$\bullet$ is then a Hopf subalgebra of $\mathcal{O}(\mathbb{H})$, and hence automatically cosemisimple. It is also invariant under the adjoint actions of $\mathcal{O}(\mathbb{H})$ on itself, since $\mathcal{O}(\mathbb{G}/\mathbb{K})$ is ad-invariant in $\mathcal{O}(\mathbb{G})$. This means that $\bullet$ is of the form

$$\mathcal{O}(\mathbb{H}/\mathbb{N}) \subseteq \mathcal{O}(\mathbb{H})$$

for some normal linearly reductive quantum subgroup $\mathcal{O}(\mathbb{H}) \to \mathcal{O}(\mathbb{N})$ of $\mathbb{H}$. By construction, we have a morphism

$$\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{K}.$$

Our goal is to argue that we have

$$\mathbb{H} \wedge \mathbb{K} = \mathbb{N},$$

for the intersection operation $\wedge$ as in Definition 1.2.5. According to Remark 1.2.6, this is achieved by the following result.

Proposition 2.1.1. *In the setting above, the diagram*

$$\begin{array}{ccccc} & & \mathcal{O}(\mathbb{H}) & & \\ & \swarrow & & \nwarrow & \\ \mathcal{O}(\mathbb{N}) & & & & \mathcal{O}(\mathbb{G}) \\ & \nwarrow & & \swarrow & \\ & & \mathcal{O}(\mathbb{K}) & & \end{array} \tag{2.1.2}$$

is a pushout in the category of algebras, or equivalently, that of bialgebras, or Hopf algebras, or Hopf algebras with bijective antipode.

Proof. For the fact that the forgetful functor from Hopf algebras to bialgebras or algebras is a left adjoint and hence preserves colimits (such as pushouts) we refer to [79]. Hence, we will focus on showing that the diagram is a pushout of algebras.

The exactness of the sequence

$$k \to \mathcal{O}(\mathbb{G}/\mathbb{K}) \to \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{K}) \to k$$

implies that the kernel of the surjection $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{K})$ is the ideal

$$\mathcal{O}(\mathbb{G})\mathcal{O}(\mathbb{G}/\mathbb{K})^- = \mathcal{O}(\mathbb{G}/\mathbb{K})^-\mathcal{O}(\mathbb{G}).$$

But this means that the pushout of the two right hand arrows of (2.1.2) is the quotient of $\mathcal{O}(\mathbb{H})$ by the ideal generated by the kernel of the counit of

$$\mathcal{O}(\mathbb{H}/\mathbb{N}) := \mathrm{Im}(\mathcal{O}(\mathbb{G}/\mathbb{K}) \to \mathcal{O}(\mathbb{H})).$$

Finally, the general theory of exact sequences of Hopf algebras as covered in [3] and recalled in Section 1.2 above says that this is precisely right hand quotient in the sequence

$$k \to \mathcal{O}(\mathbb{H}/\mathbb{N}) \to \mathcal{O}(\mathbb{H}) \to \Box \to k,$$

which is by definition our $\mathcal{O}(\mathbb{H}) \to \mathcal{O}(\mathbb{N})$. $\Box$

Remark 2.1.2. *The substance of Proposition 2.1.1 is that the algebra colimit in question is automatically cosemisimple as a Hopf algebra. This is analogous to the classical fact that a normal subgroup of a linearly reductive linear algebraic group is automatically linearly reductive, as follows easily, for instance, from the classification of linearly reductive groups* [75].

Finally, suppose $\mathbb{H}$ and $\mathbb{K}$ generate $\mathbb{G}$ in the following representation-theoretic sense: a linear map $f: V \to W$ between comodules $V, W \in \mathcal{M}^{\mathcal{O}(\mathbb{G})}$ is a $\mathbb{G}$-intertwiner if and only if it is both an $\mathbb{H}$- and a $\mathbb{K}$-intertwiner (see e.g. [11] for the identical notion of topological generation for compact quantum groups, or [21], where the same property is phrased in terms of the injectivity of the map from $\mathcal{O}(\mathbb{G})$ into the product $\mathcal{O}(\mathbb{K}) \times \mathcal{O}(\mathbb{H})$ in the category of Hopf algebras).

Remark 2.1.3. *One can show that the condition above is equivalent to $\mathbb{H} \vee \mathbb{K} = \mathbb{G}$, for the operation '$\vee$' from Definition 1.2.5.*

With all of this in place, we have

Theorem 2.1.4. *Let $\mathbb{H} \leq \mathbb{G}$ and $\mathbb{K} \trianglelefteq \mathbb{G}$ be linearly reductive quantum subgroups of a linearly reductive quantum group. If $\mathbb{H}$ and $\mathbb{K}$ generate $\mathbb{G}$, then the canonical morphism $\mathbb{H}/\mathbb{H} \wedge \mathbb{K} \to \mathbb{G}/\mathbb{K}$ is an isomorphism.*

Proof. By construction,

$$\mathcal{O}(\mathbb{G}/\mathbb{K}) \to \mathcal{O}(\mathbb{H}/\mathbb{H} \wedge \mathbb{K})$$

is onto. In order to prove injectivity and complete the proof, it suffices to show that the functor from $(\mathbb{G}/\mathbb{K})$-representations to $\mathbb{H}$-representations induced by the the upper composition in (2.1.1) is full, i.e. it induces a bijection between sets of morphisms. The fact that this condition is equivalent to the bijectivity of a map of coalgebras that is known to be onto follows e.g. from [86, Lemmas 2.2.12, 2.2.13].

Let V and W be finite-dimensional $(\mathbb{G}/\mathbb{K})$-representations, and $f: V \to W$ an $\mathbb{H}$-intertwiner between them. Since the category $\mathcal{M}^{\mathcal{O}(\mathbb{G}/\mathbb{K})}$ of comodules over $\mathcal{O}(\mathbb{G}/\mathbb{K})$ is the full subcategory of $\mathcal{M}^{\mathbb{G}}$ consisting of objects that break up as copies of the trivial representation when restricted to $\mathbb{K}$, f is also a $\mathbb{K}$-intertwiner. But then, by the hypothesis that $\mathbb{H}$ and $\mathbb{K}$ generate $\mathbb{G}$, f is a $\mathbb{G}$- and hence a $(\mathbb{G}/\mathbb{K})$-intertwiner. This completes the proof. □

Remark 2.1.5. *When working over $\mathbb{C}$ and all Hopf algebras in sight are CQG, Theorem 2.1.4 gives an alternate proof of the case of Proposition 3.1.1 when all quantum groups are compact.*

We also have a version of Theorem 2.1.4 taking place in $\mathcal{DQG}$.

Theorem 2.1.6. *If the subgroup $\mathbb{H}$ and the normal subgroup $\mathbb{K}$ generate the discrete quantum group $\mathbb{G}$, then the canonical morphism $\mathbb{H}/\mathbb{H} \wedge \mathbb{K} \to \mathbb{G}/\mathbb{K}$ is an isomorphism.*

Proof. The hypothesis that $\mathbb{H} \vee \mathbb{K} = \mathbb{G}$ means, in the context of algebraic discrete quantum groups, that we have

$$k\mathbb{G} = k\mathbb{H}k\mathbb{K},$$

and the surjectivity of the canonical map

$$k(\mathbb{H}/\mathbb{H} \wedge \mathbb{K}) = k\mathbb{H}/(k\mathbb{H}(k\mathbb{H} \cap k\mathbb{K})^-) \to k\mathbb{G}/k\mathbb{G}k\mathbb{K}^- = k(\mathbb{G}/\mathbb{K})$$

follows from this.

As for injectivity, it amounts to showing that those simple $k\mathbb{H}$-comodules that become trivial (i.e. break up as direct sums of copies of the trivial comodule) over $k(\mathbb{G}/\mathbb{K})$ are precisely those corresponding to subcoalgebras of $k\mathbb{H} \cap k\mathbb{K}$; this is immediate, using the fact that a $k\mathbb{G}$-comodule becomes trivial over $k(\mathbb{G}/\mathbb{K})$ if and only if it is a $k\mathbb{K}$-comodule. □

Note incidentally that a trivial version of the first isomorphism theorem is implicit in the proof of Theorem 2.1.4. For a morphism $\Pi : \mathbb{H} \to \mathbb{G}$ of locally compact quantum groups, $\mathbb{H}/\ker\Pi$ is essentially the smallest "quotient LCQG" $\mathbb{H} \to ?$ for which Π factors as

$$\begin{tikzcd} \mathbb{H} \arrow[rr, "\Pi"] \arrow[dr, two heads] & & \mathbb{G} \\ & ? \arrow[ur] & \end{tikzcd} \tag{2.1.3}$$

(see e.g. Definition 1.3.8) Similarly, $\overline{\mathrm{im}\Pi}$ is the smallest $? \leq \mathbb{G}$ such that Π factors similarly to (2.1.3) as

$$\begin{tikzcd} \mathbb{H} \arrow[rr, "\Pi"] \arrow[dr] & & \mathbb{G} \\ & ? \arrow[ur, hook] & \end{tikzcd} \tag{2.1.4}$$

In the algebraic case, the image of a Hopf algebra morphism $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$ clearly has both factorization universality properties, and hence by default the two concepts coincide. For this reason, we do not state a First Isomorphism Theorem in the present section.

2.2 The modular law and Zassenhaus lemma

Throughout this section $\mathbb{G}$ denotes a linearly reductive quantum group. We will be interested in studying its poset of subgroups $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$.

First, recall the *modular law* for subgroups of a discrete group G: whenever M, N and H are subgroups of G with $N \leq H$ we have

$$H \cap MN = (H \cap M)N,$$

where the juxtaposition AB of subgroups $A, B \leq G$ means the set

$$\{ab \mid a \in A,\ b \in B\}.$$

We will be interested in cases where the set products in question are actually subgroups. To this end, we first prove

Proposition 2.2.1. *Let $\mathbb{N} \leq \mathbb{H}$ and $\mathbb{M}$ be normal subgroups of the linearly reductive quantum group $\mathbb{G}$. Then, we have*

$$\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{N}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{N}. \tag{2.2.1}$$

Proof. We will dualize the picture, and study quotient Hopf algebras $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\bullet)$ corresponding to normal quantum subgroups from the perspective of the corresponding Hopf subalgebras $A_\bullet = \mathcal{O}(\bullet\backslash\mathbb{G}) \subseteq \mathcal{O}(\mathbb{G})$.

This dualization procedure reverses the lattice operations on quotient Hopf algebras and Hopf subalgebras. For this reason, the Hopf subalgebra

$$A = A_{\mathbb{H}\wedge(\mathbb{M}\vee\mathbb{N})}$$

corresponding to the left hand side of (2.2.1) is equal to

$$A_{\mathbb{H}}(A_{\mathbb{M}} \wedge A_{\mathbb{N}}). \tag{2.2.2}$$

Similarly, the Hopf subalgebra

$$B = A_{(\mathbb{H}\wedge\mathbb{M})\vee\mathbb{N}}$$

corresponding to the right hand side is

$$(A_{\mathbb{H}}A_{\mathbb{M}}) \wedge A_{\mathbb{N}}. \tag{2.2.3}$$

Now note that (2.2.2) is the sum of those simple subcoalgebras of $\mathcal{O}(\mathbb{G})$ whose simple comodules

$$V \le W \otimes X,$$

where W is a simple $A_{\mathbb{H}}$-comodule and X is a simple comodule over both $A_{\mathbb{M}}$ and $A_{\mathbb{N}}$ (see the discussion on cosemisimple coalgebras in Section 1.2).

On the other hand, the simple comodules of (2.2.3) are characterized by the fact that they are $A_{\mathbb{N}}$-comodules, and also embed into tensor products of the form $W \otimes X$, for simple comodules

$$W \in \mathcal{M}^{A_{\mathbb{H}}},\ X \in \mathcal{M}^{A_{\mathbb{M}}}. \tag{2.2.4}$$

Clearly, the latter property for a simple comodule $V \in \mathcal{M}^{\mathcal{O}(\mathbb{G})}$ is weaker than the former, and hence $A \le B$.

On the other hand, suppose the simple comodule $V \in \mathcal{M}^{A_{\mathbb{N}}}$ embeds into $W \otimes X$ with W and X as in (2.2.4) (and hence $V \in \mathcal{M}^{B}$). Then we have a non-zero morphism

$$V \to W \otimes X,$$

which by duality gives a non-zero morphism

$$W^* \otimes V \to X$$

(automatically an epimorphism, since X is assumed to be simple). But since

$$V \in \mathcal{M}^{A_{\mathbb{N}}},\ W \in \mathcal{M}^{A_{\mathbb{H}}} \subseteq \mathcal{M}^{A_{\mathbb{N}}},$$

we get $X \in \mathcal{M}^{A_{\mathbb{N}}}$, and hence X is actually a comodule over

$$A_{\mathbb{M}} \wedge A_{\mathbb{N}}.$$

This means that V is actually an A-comodule, and the proof is complete. □

Remark 2.2.2. *Alternatively, we can restate Proposition 2.2.1 as saying that the normal quantum subgroups of a linearly reductive quantum group form a modular lattice.*

We can prove somewhat more when $\mathbb{G}$ is a compact quantum group. As noted above, the identity

$$H \cap MN = (H \cap M)N$$

holds for *all* subgroups $N \le H \le G$ and $M \le G$. Our version (Proposition 2.2.4 below) will still not be as general as this, but we will impose just enough restrictions to ensure that classically, the product sets MN and $(H \cap M)N$ are actually subgroups. To this end, we need

Definition 2.2.3. A quantum subgroup $\mathbb{L} \le \mathbb{G}$ *normalizes* another $\mathbb{M} \le \mathbb{G}$ if the latter is normal in the quantum subgroup $\mathbb{M} \vee \mathbb{L}$.

We are now ready to state

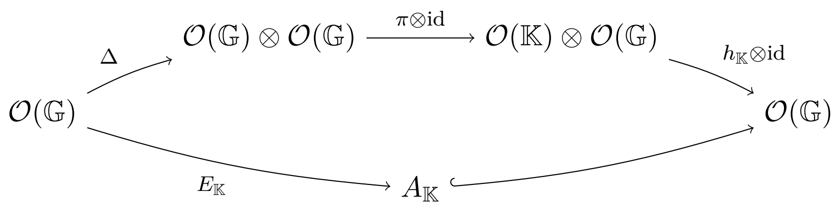
Proposition 2.2.4. *Let $\mathbb{G}$ be a compact quantum group, with quantum subgroups $\mathbb{L} \le \mathbb{H} \le \mathbb{G}$ and $\mathbb{M} \le \mathbb{G}$ such that $\mathbb{L}$ normalizes $\mathbb{M}$. Then, the equality $\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L}$ holds.*

Proof. As in the proof of Proposition 2.2.1, the goal is to show that we have

$$A_{\mathbb{H}}(A_{\mathbb{M}} \wedge A_{\mathbb{L}}) = A_{\mathbb{H}}A_{\mathbb{M}} \wedge A_{\mathbb{L}}, \tag{2.2.5}$$

or rather that the right hand side is contained in the left hand side (the opposite inclusion being immediate). Note that we have used Lemma 1.2.9 and Corollary 1.2.10 implicitly in order to conclude that the subspace products in (2.2.5) are both coideal subalgebras.

For any quantum subgroup $\pi : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{K})$ we have an expectation $E_{\mathbb{K}} = \mathcal{O}(\mathbb{G}) \to A_{\mathbb{K}}$ defined as



It is automatically an $A_{\mathbb{K}}$-bimodule map, and intertwines $h_{\mathbb{G}}$ and its restriction to $A_{\mathbb{K}}$.

Now consider the expectation $E_{\mathbb{L}} : \mathcal{O}(\mathbb{G}) \to A_{\mathbb{L}}$. Applied to an element x in the right hand side of (2.2.5), it fixes x that element (because $E_{\mathbb{L}}$ acts as the identity on $A_{\mathbb{L}}$). On the other hand, writing $x = x_{\mathbb{H}}x_{\mathbb{M}}$ for

$$x_{\mathbb{H}} \in \mathbb{H},\ \ x_{\mathbb{M}} \in \mathbb{M},$$

we have

$$E_{\mathbb{L}}(x) = x_{\mathbb{H}}E_{\mathbb{L}}(x_{\mathbb{M}})$$

because $x_{\mathbb{H}} \in A_{\mathbb{H}} \le A_{\mathbb{L}}$ and $E_{\mathbb{L}}$ is the identity on $A_{\mathbb{L}}$. In conclusion, we would be done if we could show that $E_{\mathbb{L}}(x_{\mathbb{M}}) \in A_{\mathbb{M}} \wedge A_{\mathbb{L}}$. This is taken care of by Lemma 2.2.5 below. □

For the next result we will use the same notation as in the proof of Proposition 2.2.4 for coideal subalgebras $A_{\mathbb{L}} = \mathcal{O}(\mathbb{L}\backslash\mathbb{G})$, expectations $E_{\mathbb{L}} : \mathcal{O}(\mathbb{G}) \to A_{\mathbb{L}}$, etc. We will also denote by $\pi_{\mathbb{L}}$ the surjection $\mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{L})$ onto the function algebra of a quantum subgroup.

Lemma 2.2.5. *Let $\mathbb{M}$ and $\mathbb{L}$ be quantum subgroups of a compact quantum group $\mathbb{G}$ such that $\mathbb{L}$ normalizes $\mathbb{M}$. Then, we have*

$$E_{\mathbb{L}}(A_{\mathbb{M}}) \subseteq A_{\mathbb{M}} \wedge A_{\mathbb{L}}.$$

Proof. Since the range of $E_{\mathbb{L}}$ is $A_{\mathbb{L}}$, we are trying to show that $A_{\mathbb{M}}$ is invariant under $E_{\mathbb{L}}$. To this end, let $f \in A_{\mathbb{M}}$ be an arbitrary element. This means by definition that

$$\pi_{\mathbb{M}}(f_1) \otimes f_2 = 1 \otimes f. \tag{2.2.6}$$

Also by definition, the expression for the expectation is

$$E_{\mathbb{L}}(f) = h_{\mathbb{L}}(\pi_{\mathbb{L}}(f_1))f_2,$$

and hence our goal is to prove that we have

$$h_{\mathbb{L}}(\pi_{\mathbb{L}}(f_1))\pi_{\mathbb{M}}(f_2) \otimes f_3 = 1 \otimes h_{\mathbb{L}}(\pi_{\mathbb{L}}(f_1))f_2.$$

More generally, we will show that in fact we have

$$\pi_{\mathbb{L}}(f_1) \otimes \pi_{\mathbb{M}}(f_2) \otimes f_3 = \pi_{\mathbb{L}}(f_1) \otimes 1 \otimes f_2. \tag{2.2.7}$$

Moreover, by substituting $\mathbb{L} \vee \mathbb{M}$ for $\mathbb{L}$, we may as well assume that $\mathbb{M} \trianglelefteq \mathbb{L}$.

Using the defining property of the antipode, the left hand side of (2.2.7) equals

$$\pi_{\mathbb{L}}(f_1 S(f_3) f_4) \otimes \pi_{\mathbb{M}}(f_2) \otimes f_5. \tag{2.2.8}$$

The normality assumption $\mathbb{M} \trianglelefteq \mathbb{L}$ implies by Lemma 1.2.11 that the surjection $\mathcal{O}(\mathbb{L}) \to \mathcal{O}(\mathbb{M})$ is one of left $\mathcal{O}(\mathbb{L})$-comodules under the left adjoint coaction

$$x \mapsto x_1 S(x_3) \otimes x_2.$$

For this reason, (2.2.8) is the result of first applying the left adjoint $\mathcal{O}(\mathbb{L})$-coaction to the left hand leg of

$$\pi_{\mathbb{M}}(f_1) \otimes f_2,$$

and then subjecting the result to the operation

$$\bullet \otimes \bullet \otimes \square \mapsto \bullet(\pi_{\mathbb{L}} \square_1) \otimes \bullet \otimes \square_2. \tag{2.2.9}$$

The conclusion now follows from (2.2.6), which ensures that the input of (2.2.9) is $1 \otimes 1 \otimes f$. $\square$

Remark 2.2.6. *The proof of Lemma 2.2.5 is a quantum version of the following classical argument that will be much more transparent:*

Given a continuous function f on $\mathbb{G}$, the expression for its expectation $E_{\mathbb{L}} : L^\infty(\mathbb{G}) \to A_{\mathbb{L}}$ is

$$(E_{\mathbb{L}} f)(\mathbb{L}_g) = \int_{\mathbb{L}} f(l_g)\ \mathrm{d}l.$$

We want to argue that if f is invariant under left translation by $\mathbb{M}$, then so is $E_{\mathbb{L}} f$. In order to see this, let $m \in \mathbb{M}$. We then have

$$(E_{\mathbb{L}} f)(m-) = \int_{\mathbb{L}} f(lm-)\ \mathrm{d}l = \int_{\mathbb{L}} f(lml^{-1} \cdot l-)\ \mathrm{d}l,$$

which, because $lml^{-1} \in \mathbb{M}$ and f is $\mathbb{M}$-invariant, equals

$$\int_{\mathbb{L}} f(l-)\ \mathrm{d}l = E_{\mathbb{L}} f.$$

Remark 2.2.7. *We note that Proposition 2.2.4 would be problematic in the more general setting of linearly reductive quantum groups (which is why we only have Proposition 2.2.1 in the latter case).*

The reason is that even classically, intersections of linearly reductive subgroups (such as $(\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L}$) need not be linearly reductive again, as Example 2.2.8 below shows.

Example 2.2.8. Let

$$\mathbb{G} = SL_3 = SL_3(\mathbb{C}).$$

Using the correspondence between complex Lie subalgebras of $\mathfrak{g} = \mathfrak{sl}_3$ and complex linear algebraic subgroups of $\mathbb{G}$ ([72, discussion preceding 3.42 and Theorem 4.22]) and the fact that this correspondence is compatible with intersections ([72, Proposition 3.19] or [9, 6.12]), it suffices to exhibit two semisimple Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$ whose intersection is not semisimple. We take the span of

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2.10}$$

for $\mathfrak{a}$. The three displayed elements are a so-called $\mathfrak{sl}_2$ *triple*; this means that their identification with

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

respectively implements an isomorphism $\mathfrak{a} \cong \mathfrak{sl}_2$.

Similarly, we take the conjugate of $\mathfrak{a}$ by

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2.11}$$

for $\mathfrak{b}$.

Since (2.2.11) commutes with the leftmost element e of (2.2.10) and conjugates the semisimple element h outside of $\mathfrak{a}$ (as can easily be seen), the intersection $\mathfrak{a} \cap \mathfrak{b}$ is the one-dimensional span of e, and hence at the level of groups the corresponding intersection is a copy of the (non-linearly reductive) additive algebraic group $\mathbb{G}_a$ over $\mathbb{C}$ (i.e. just $\mathbb{C}$ with its usual additive group structure).

There is also a dual version to Proposition 2.2.4, dealing with discrete quantum groups in the sense of Definition 1.2.1.

Quantum subgroups $\mathbb{M} \le \mathbb{G}$ then correspond to Hopf subalgebras

$$k\mathbb{M} \subseteq k\mathbb{G}.$$

Definition 2.2.9. Let $\mathbb{G}$ be an algebraic discrete quantum group. A quantum subgroup $\mathbb{L}$ *normalizes* another $\mathbb{M} \le \mathbb{G}$ if the group algebra $k\mathbb{M}$ of the latter is invariant under the adjoint action of $k\mathbb{L}$ on $k\mathbb{G}$.

Proposition 2.2.10. *Let $\mathbb{G}$ be an algebraic discrete quantum group, with quantum subgroups $\mathbb{L} \le \mathbb{H} \le \mathbb{G}$ and $\mathbb{M} \le \mathbb{G}$ such that $\mathbb{L}$ normalizes $\mathbb{M}$. Then, the equality (2.2.1) holds.*

Proof. The proof is essentially the same as that of Proposition 2.2.1, once we substitute $k\mathbb{G}$ for $\mathcal{O}(\mathbb{G})$ in that result, and similarly substitute the Hopf subalgebras

$$k\mathbb{H},\ k\mathbb{L},\ k\mathbb{M}\ \subseteq\ k\mathbb{G}$$

of $k\mathbb{G}$ for

$$A_{\mathbb{L}},\ A_{\mathbb{H}},\ A_{\mathbb{M}}\ \subseteq\ \mathcal{O}(\mathbb{G})$$

respectively. □

Propositions 2.2.4, 2.2.10 will allow us to prove the following version of the Zassenhaus (or butterfly) lemma ([62, Vol. 1, p. 77] or [81, Chapter 5, Lemma 5.10]) for compact and discrete quantum groups.

Proposition 2.2.11. *Let $\mathbb{A}' \trianglelefteq \mathbb{A}$ and $\mathbb{B}' \trianglelefteq \mathbb{B}$ be quantum subgroups of either a compact or an algebraic discrete quantum group $\mathbb{G}$. Then, we have an isomorphism*

$$\frac{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B}')} \cong \frac{\mathbb{B}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{B}' \vee (\mathbb{A}' \wedge \mathbb{B})}.$$

The analogous statement holds for linearly reductive $\mathbb{G}$ provided $\mathbb{A}$, $\mathbb{A}'$, etc. are all normal in $\mathbb{G}$.

Proof. We focus first on the compact / discrete case, following the usual strategy (as in [62, Vol. 1, p. 77] or the proof of [81, Chapter 5, Lemma 5.10], for instance) of proving that we have isomorphisms

$$\frac{\mathbb{A}'\vee(\mathbb{A}\wedge\mathbb{B})}{\mathbb{A}'\vee(\mathbb{A}\wedge\mathbb{B}')} \xrightarrow{\cong} \frac{\mathbb{A}\wedge\mathbb{B}}{(\mathbb{A}'\wedge\mathbb{B})\vee(\mathbb{A}\wedge\mathbb{B}')} \xleftarrow{\cong} \frac{\mathbb{B}'\vee(\mathbb{A}\wedge\mathbb{B})}{\mathbb{B}'\vee(\mathbb{A}'\wedge\mathbb{B})}$$

By symmetry, it suffices to focus on the left hand side of this diagram. The required isomorphism will follow from the compact / discrete quantum version of the second isomorphism theorem (Theorems 2.1.4 and 2.1.6) applied to

$$\mathbb{H} = \mathbb{A} \wedge \mathbb{B} \text{ and } \mathbb{K} = \mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B}')$$

once we prove that we have

$$(\mathbb{A} \wedge \mathbb{B}) \wedge (\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B}')) = (\mathbb{A}' \wedge \mathbb{B}) \vee (\mathbb{A} \wedge \mathbb{B}').$$

In turn, this follows from Proposition 2.2.4 or Proposition 2.2.10 applied to $\mathbb{H} = \mathbb{A} \wedge \mathbb{B}$, $\mathbb{L} = \mathbb{A} \wedge \mathbb{B}'$ and $\mathbb{M} = \mathbb{A}'$.

As for the last claim regarding the linearly reductive case, its proof is virtually identical, using Proposition 2.2.1 instead of Proposition 2.2.4. □

2.3 The Schreier refinement theorem

In this section we prove an analogue of the Schreier refinement theorem for compact and discrete quantum groups (see e.g. [81, Chapter 5, Theorem 5.11] for the classical analogue for ordinary discrete groups). To this aim, we need to define a quantum analogue of the notion of (sub)normal series.

Definition 2.3.1. Let $\mathbb{G}$ be either a compact or (algebraic) discrete quantum group. A finite system

$$\mathbb{G} = \mathbb{G}_0 \geq \mathbb{G}_1 \geq \mathbb{G}_2 \geq \mathbb{G}_3 \geq \cdots \geq \mathbb{G}_k = 1 \tag{2.3.1}$$

of closed quantum subgroups of $\mathbb{G}$ is called a *subnormal series* of $\mathbb{G}$ if every subgroup $\mathbb{G}_i$ is a proper normal closed quantum subgroup of $\mathbb{G}_{i-1}$, $i \in \{1, 2, \cdots, k\}$. In particular, $\mathbb{G}_1$ is a normal closed quantum subgroup of $\mathbb{G}$, $\mathbb{G}_2$ is a normal closed quantum subgroup of $\mathbb{G}_1$, but not necessarily of $\mathbb{G}$, and so on.

A subnormal series is *normal* if each $\mathbb{G}_i$ is normal in the ambient group $\mathbb{G}$.

The corresponding subquotient quantum groups

$$\mathbb{G}_1\backslash\mathbb{G},\ \mathbb{G}_2\backslash\mathbb{G}_1,\ \cdots,\ \mathbb{G}_k\backslash\mathbb{G}_{k-1}$$

of $\mathbb{G}$ are the *factors* of the (sub)normal series (2.3.1).

The integer k is the *length* of the series (2.3.1).

Definition 2.3.2. A subnormal series

$$\mathbb{G} = \mathbb{H}_0 \geq \mathbb{H}_1 \geq \mathbb{H}_2 \geq \mathbb{H}_3 \geq \cdots \geq \mathbb{H}_l = 1 \tag{2.3.2}$$

is called a *refinement* of the subnormal series (2.3.1) if every quantum subgroup $\mathbb{G}_i$ of (2.3.1) coincides with one of the quantum subgroups $\mathbb{H}_j$, i.e. if every quantum subgroup that occurs in (2.3.1) also occurs in (2.3.2).

In particular, every normal series is a refinement of itself. The lengths of the normal series (2.3.1) and its refinement (2.3.2) of course satisfy the inequality $k \leq l$.

Two subnormal series of a compact quantum groups are called *equivalent* if their lengths are equal and their constituent subquotients are isomorphic up to permutation.

We are now ready for the following analogue of Schreier's refinement theorem. As we will see, its proof, given the Zassenhaus lemma (Proposition 2.2.11) is virtually automatic.

Theorem 2.3.3. *Any two subnormal series of a compact / discrete quantum group* $\mathbb{G}$ *have equivalent refinements.*

The same holds for any two normal series of a linearly reductive quantum group.

Proof. We focus first on the claim relating to compact and algebraic discrete quantum groups.

Let (2.3.1) and (2.3.2) be two normal series of a compact quantum group $\mathbb{G}$, and set

$$\mathbb{G}_{ij} = \mathbb{G}_i \vee (\mathbb{G}_{i-1} \wedge \mathbb{H}_j),\ \mathbb{H}_{ij} = \mathbb{H}_j \vee (\mathbb{H}_{j-1} \wedge \mathbb{G}_i).$$

For $i \in \{1, 2, \cdots, k\}$ and $j \in \{1, 2, \cdots, l\}$ we obtain two new refinements of (2.3.1) and (2.3.2) respectively:

$$\mathbb{G}_{i-1} = \mathbb{G}_{i0} > \mathbb{G}_{i(j-1)} > \mathbb{G}_{ij} > \mathbb{G}_{ii} = \mathbb{G}_i; \tag{2.3.3}$$

$$\mathbb{H}_{j-1} = \mathbb{H}_{0j} > \mathbb{H}_{(i-1)j} > \mathbb{H}_{ij} > \mathbb{G}_{kj} = \mathbb{H}_j. \tag{2.3.4}$$

By Proposition 2.2.11, $\mathbb{G}_{ij}$ is a normal closed quantum subgroup of $\mathbb{G}_{i(j-1)}$ and $\mathbb{H}_{ij}$ is normal in $\mathbb{H}_{(i-1)j}$, and moreover

$$\mathbb{G}_{ij} \backslash \mathbb{G}_{i(j-1)} \cong \mathbb{H}_{ij} \backslash \mathbb{H}_{(i-1)j}. \tag{2.3.5}$$

The refinements induced by (2.3.3) and (2.3.4) have the same length, and (2.3.5) says that they are equivalent.

The proof of the second claim follows similarly, using the corresponding second half of Proposition 2.2.11. □

Remark 2.3.4. *This is simply an adaptation to the quantum setting of the usual proof of the Schreier refinement theorem (see e.g. the proof of* [81, Chapter 5, Theorem 5.11]*). As mentioned above, once we have the Zassenhaus lemma the standard argument goes through mechanically. The same goes for the Jordan-Hölder theorem below.*

2.4 The Jordan-Hölder theorem

In this section we prove analogues of the Jordan-Hölder theorem for compact and discrete quantum groups (and a weaker form of it in the linearly reductive case). We begin with the following definition.

Definition 2.4.1. A subnormal series (2.3.1) is a *composition series* of $\mathbb{G}$ if $\mathbb{G}_i$ is a proper maximal normal closed quantum subgroup of $\mathbb{G}_{i-1}$ for $1 \le i \le k$.

Remark 2.4.2. *In other words, a composition series is a subnormal series that cannot be refined further.*

The main result of this section is

Theorem 2.4.3. *Any two composition series of a compact or discrete quantum group* $\mathbb{G}$ *are equivalent.*

Proof. This is immediate from Theorem 2.3.3 together with the observation (made in Remark 2.4.2) that composition series cannot be refined strictly: two composition series have equivalent refinements, and hence they must already be equivalent. □

Analogously, making use of the second half of Theorem 2.3.3, we have

Proposition 2.4.4. *Any two normal series of a linearly reductive quantum group which are maximal with respect to refinement are equivalent.*

Chapter 3

Isomorphism theorems, modular law: the locally compact case

3.1 The second isomorphism theorem

We shall first consider the setting of the second isomorphism theorem for ordinary discrete groups, transported to the present framework: $\mathbb{G}$ is a locally compact quantum group, $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$ and $\mathbb{N} \in \mathcal{NQS}(\mathbb{G})$.

In order to make sense of the statement of Proposition 3.1.1 below, note first that according to Lemma 1.3.21, $\mathbb{H} \to \mathbb{G} \to \mathbb{G}/\mathbb{N}$ factors through a morphism

$$\mathbb{H}/\mathbb{H} \wedge \mathbb{N} \to \mathbb{G}/\mathbb{N}$$

inducing an action of $\mathbb{H}/\mathbb{H} \wedge \mathbb{N}$ on $\mathbb{G}/\mathbb{N}$.

Proposition 3.1.1. *Let $\mathbb{G}$ be a locally compact quantum group, $\mathbb{H} \in \mathcal{QS}(\mathbb{G})$ and $\mathbb{N} \in \mathcal{NQS}(\mathbb{G})$. Let us denote by $\Pi : \mathbb{H} \to \mathbb{G}/\mathbb{N}$ the induced morphism. Then $\mathbb{H} \wedge \mathbb{N} = \ker \Pi$. If moreover $\mathbb{G} = \mathbb{H} \vee \mathbb{N}$ then $\overline{im \Pi} = \mathbb{G}/\mathbb{N}$.*

Proof. Let us consider homomorphisms $\Pi_1 : \mathbb{G} \to \mathbb{G}/\mathbb{N}$ and $\Pi : \mathbb{H} \to \mathbb{G} \to \mathbb{G}/\mathbb{N}$. The right quantum group homomorphism assigned to $\widehat{\Pi}_1$ and $\widehat{\Pi}$ will be denoted by $\widehat{\alpha}_1 : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{G}/\mathbb{N}})$ and $\widehat{\alpha} : \mathrm{L}^\infty(\widehat{\mathbb{H}}) \to \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{G}/\mathbb{N}})$ respectively. Viewing $\mathrm{L}^\infty(\widehat{\mathbb{H}})$ as a subalgebra of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ we have $\widehat{\alpha} : \mathrm{L}^\infty(\widehat{\mathbb{H}}) \to \mathrm{L}^\infty(\widehat{\mathbb{H}}) \bar{\otimes} \mathrm{L}^\infty(\widehat{\mathbb{G}/\mathbb{N}})$ where $\widehat{\alpha} = \widehat{\alpha}_1|_{\mathrm{L}^\infty(\widehat{\mathbb{H}})}$ (see Lemma 1.3.15). Let $x \in \mathrm{cd}(\mathrm{L}^\infty(\mathbb{H}/\ker \Pi))$. Then $x \in \mathrm{L}^\infty(\widehat{\mathbb{H}})$ and $\widehat{\alpha}_1(x) = x \otimes \mathbb{1}$. By Lemma 1.3.13 we have $x \in \mathrm{L}^\infty(\widehat{\mathbb{N}}) \wedge \mathrm{L}^\infty(\widehat{\mathbb{H}})$. Thus $\mathrm{L}^\infty(\mathbb{H}/\ker \Pi) = \mathrm{L}^\infty(\mathbb{H}/\mathbb{H} \wedge \mathbb{N})$, i.e. $\ker \Pi = \mathbb{H} \wedge \mathbb{N}$.

Since $\mathbb{G} = \mathbb{H} \vee \mathbb{N}$ and $\mathbb{N} \in \mathcal{NQS}(\mathbb{G})$ we conclude from Lemma 1.3.18 that

$$\mathrm{L}^\infty(\widehat{\mathbb{G}}) = \{\mathrm{L}^\infty(\widehat{\mathbb{N}})\,\mathrm{L}^\infty(\widehat{\mathbb{H}})\}^{\sigma-\mathrm{cls}}. \tag{3.1.1}$$

Using (1.3.11) we can see that for all $\omega \in \mathrm{L}^\infty(\widehat{\mathbb{G}})_*$, $x \in \mathrm{L}^\infty(\widehat{\mathbb{H}})$ and $y \in \mathrm{L}^\infty(\widehat{\mathbb{N}})$ we have

$$(\omega \otimes \mathrm{id})(\widehat{\alpha}_1(xy)) = (y \cdot \omega \otimes \mathrm{id})(\widehat{\alpha}(x)). \tag{3.1.2}$$

Using (1.3.10), (3.1.1), (3.1.2) we get

$$\mathrm{L}^\infty(\widehat{\mathbb{G}/\mathbb{N}}) = \{(\omega \otimes \mathrm{id})(\widehat{\alpha}(a)) : \omega \in \mathrm{L}^\infty(\widehat{\mathbb{H}})_*, a \in \mathrm{L}^\infty(\widehat{\mathbb{H}})\}^{\sigma-\mathrm{cls}}$$

i.e. the closure of the image of Π is also $\mathbb{G}/\mathbb{N}$. ☐

Using Theorem 1.3.11 and Proposition 3.1.1 we get

Corollary 3.1.2. *The homomorphism* $\mathbb{H}/\mathbb{H}\wedge\mathbb{N}\to\mathbb{G}/\mathbb{N}$ *is an isomorphism if and only if the corresponding action of* $\mathbb{H}/\mathbb{H}\wedge\mathbb{N}$ *on* $\mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ *is integrable.*

Remark 3.1.3. *The assumptions of Corollary 3.1.2 hold trivially when* $\mathbb{H}$ *is compact.*

Assume on the other hand that $\mathbb{N}$ *is compact. Then, since the action of* $\mathbb{H}$ *on* $\mathrm{L}^\infty(\mathbb{G})$ *is integrable and we have a conditional expectation onto* $\mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$*, the integrability assumption also holds in this case.*

Finally, note that every morphism Π *of discrete quantum groups automatically identifies* $\mathbb{H}/ker\Pi$ *with* $\overline{im\Pi}$*. In conclusion, Corollary 3.1.2 also goes through when all quantum groups in sight are discrete.*

Let us also note that equivalent statements in Corollary 3.1.2 fails (and hence so does the second isomorphism theorem) in general even classically, for locally compact abelian groups, as the following example shows.

Example 3.1.4. Consider the group $\mathbb{G}=\mathbb{T}^2\times\mathbb{R}$, and the subgroups

$$\mathbb{H}=\{(e^{it\theta},1,t)\mid t\in\mathbb{R}\}\text{ and }\mathbb{N}=\{(1,e^{is\phi},s)\mid s\in\mathbb{R}\}$$

for real numbers θ and ϕ that are incommensurable (i.e. linearly independent over $\mathbb{Q}$). Then, the subgroup

$$\{(e^{it\theta},e^{-it\phi},0)\}\subset\mathbb{T}^2\times\{0\}$$

of $\mathbb{HN}$ is dense $\mathbb{T}^2\times\{0\}$ and hence the closure $\mathbb{H}\vee\mathbb{N}$ of $\mathbb{HN}$ contains $\mathbb{T}^2\times\{0\}$. But the product of this latter group with $\mathbb{H}$ is clearly all of $\mathbb{G}$, and we have $\mathbb{H}\vee\mathbb{N}=\mathbb{G}$.

Now, $\mathbb{H}/\mathbb{H}\wedge\mathbb{N}$ is a one-dimensional Lie group whereas $\mathbb{G}/\mathbb{N}$ is a two-dimensional one, and hence the conditions of Corollary 3.1.2 cannot possibly hold.

The fundamental characteristic of Example 3.1.4 is that the naive product $\mathbb{HN}$ is not closed in $\mathbb{G}$, and hence $\mathbb{H}\vee\mathbb{N}$ is "larger than expected". Indeed, classically, it is this failure of $\mathbb{HN}$ to be closed that prevents the conditions of Corollary 3.1.2 from holding. This is summarized in the following result.

Proposition 3.1.5. *Let* $\mathbb{G}$ *be a classical locally compact group, and* $\mathbb{H}\le\mathbb{G}$ *and* $\mathbb{N}\trianglelefteq\mathbb{G}$ *closed subgroups.*

Then, $\mathbb{H}/\mathbb{H}\wedge\mathbb{N}$ *acts integrably on* $\mathbb{G}/\mathbb{N}$ *if and only if for every* $(\mathbb{H}\wedge\mathbb{N})$*-invariant closed subset* $\mathbb{F}$ *of* $\mathbb{H}$ *the product* $\mathbb{FN}$ *is closed.*

Proof. According to Corollary 3.1.2, the integrability of the action in the statement is equivalent to the canonical map

$$\mathbb{H}/(\mathbb{H}\wedge\mathbb{N})\to(\mathbb{H}\vee\mathbb{N})/\mathbb{N}\tag{3.1.3}$$

being an isomorphism. We will use this equivalence throughout the proof, implicitly or explicitly.

($\Leftarrow$) Suppose $\mathbb{FN}$ is closed in $\mathbb{G}$ for every closed $\mathbb{F}\subseteq\mathbb{H}$. Applying this to $\mathbb{F}=\mathbb{H}$ first, we have $\mathbb{H}\vee\mathbb{N}=\mathbb{HN}$ and hence the canonical one-to-one morphism (3.1.3) is also onto.

Now note that the composition

$$\mathbb{H}\to\mathbb{G}\to\mathbb{G}/\mathbb{N}$$

realizes $\mathbb{H}/(\mathbb{H} \wedge \mathbb{N})$ as a closed subgroup of the right hand side. Indeed, it induces an embedding of the former group into $\mathbb{G}/\mathbb{N}$, and the condition on $\mathbb{F}\mathbb{N}$ being closed means precisely that the induced one-to-one map

$$\mathbb{H}/(\mathbb{H} \wedge \mathbb{N}) \to \mathbb{G}/\mathbb{N}$$

is closed.

All in all, (3.1.3) is a bijective inclusion of one closed subgroup of $\mathbb{G}/\mathbb{N}$, namely $\mathbb{H}/(\mathbb{H} \wedge \mathbb{N})$, into another, i.e. $\mathbb{H}\mathbb{N}/\mathbb{N}$. It is then an isomorphism, and the conclusion follows from Corollary 3.1.2.

($\Rightarrow$) Conversely, suppose the action in question is integrable, and hence by Corollary 3.1.2 the morphism (3.1.3) is bijective. The diagram

$$\begin{array}{ccccc} & & \mathbb{H} \vee \mathbb{N} & & \\ & \nearrow & & \searrow & \\ \mathbb{H} & & & & (\mathbb{H} \vee \mathbb{N})/\mathbb{N} \\ & \searrow & & \nearrow_{\cong} & \\ & & \mathbb{H}/(\mathbb{H} \wedge \mathbb{N}) & & \end{array}$$

shows that $\mathbb{H} \vee \mathbb{N}$ is generated as a (plain, not topological) group by $\mathbb{H}$ and $\mathbb{N}$. Since $\mathbb{N}$ is normal, this in turn implies $\mathbb{H} \vee \mathbb{N} = \mathbb{H}\mathbb{N}$, so that the latter product must be closed.

Moreover, the fact that (3.1.3) is a homeomorphism implies that it is in particular closed. This means that the image of every closed subset $\mathbb{F} \subseteq \mathbb{H}$ as in the statement is closed in $\mathbb{G}/\mathbb{N}$, and hence its preimage $\mathbb{F}\mathbb{N}$ through the quotient map $\mathbb{G} \to \mathbb{G}/\mathbb{N}$ is closed. $\square$

Although quite explicit, the closure condition in Proposition 3.1.5 might be somewhat inconvenient to check. In view of this, one might wonder whether the seemingly weaker condition that $\mathbb{H}\mathbb{N}$ be closed in $\mathbb{G}$ is sufficient. Example 3.1.6 shows that this is not the case, even in the case of classical *abelian* locally compact groups.

Before spelling out the example, let us clarify what it is meant to do. Placing ourselves entirely within the context of locally compact abelian groups, consider for simplicity the case when $\mathbb{H} \wedge \mathbb{N}$ is trivial. Moreover, we may further assume harmlessly that the subgroup $\mathbb{H}\mathbb{N} \le \mathbb{G}$ (which is supposed to be closed anyway) is all of $\mathbb{G}$.

All in all, we will have an algebraic decomposition

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{N}. \tag{3.1.4}$$

Then, the condition from Proposition 3.1.5 and its symmetric counterpart (i.e. with the roles of $\mathbb{H}$ and $\mathbb{N}$ interchanged) jointly mean precisely that the decomposition (3.1.4) is one of *topological* abelian groups as well as abstract ones.

In conclusion, in order to show that the closedness of $\mathbb{H}\mathbb{N}$ does not entail the second isomorphism theorem, it suffices to exhibit a locally compact abelian group $\mathbb{G}$ which decomposes as (3.1.4) abstractly for closed subgroups $\mathbb{H}$ and $\mathbb{N}$, but not topologically. Example 3.1.6 achieves this by choosing $\mathbb{H}$ and $\mathbb{N}$ to be discrete, whereas $\mathbb{G}$ is not.

Example 3.1.6. We take $\mathbb{G}$ to be the direct product between a copy of the compact additive group $\mathbb{Z}_p$ of p-adic integers for some odd prime number p, and a discrete copy Γ of the self-same group $\mathbb{Z}_p$ (in other words, Γ is $\mathbb{Z}_p$ as an abstract group, but with discrete topology).

Now, in $\mathbb{G} = \mathbb{Z}_p \times \Gamma$ we have a diagonal subgroup

$$\mathbb{H} = \{(g, g) \mid g \in \mathbb{Z}_p\}$$

as well as an anti-diagonal one,

$$\mathbb{N} = \{(g, -g) \mid g \in \mathbb{Z}_p\}.$$

We have $\mathbb{H} \wedge \mathbb{N} = \{0\}$ because $\mathbb{Z}_p$ is torsion-free, and also $\mathbb{H} + \mathbb{N} = \mathbb{G}$ because $\mathbb{Z}_p$ is divisible by 2. Moreover, $\mathbb{H}$ and $\mathbb{N}$ are easily seen to both be closed in $\mathbb{G}$ and discrete. By construction, though, $\mathbb{G}$ is not. The preceding discussion explains why this will do.

3.2 The third isomorphism theorem

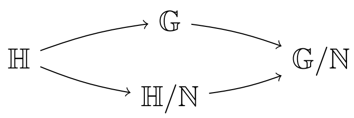
Recall ([34, §3.3, Theorem 19]) that this states that given normal subgroups N and H of G with $N \leq H$, we have $H/N \trianglelefteq G/N$ and moreover

$$(G/N)/(H/N) \cong G/H.$$

Consider now the typical setup for the third isomorphism theorem: a locally compact quantum group $\mathbb{G}$, and normal closed quantum subgroups $\mathbb{N} \leq \mathbb{H}$ of $\mathbb{G}$. Then, because the composition

$$\mathbb{N} \to \mathbb{H} \to \mathbb{G} \to \mathbb{G}/\mathbb{N}$$

is trivial, Lemma 1.3.21 ensures that we have a factorization



of the top composition $\mathbb{H} \to \mathbb{G} \to \mathbb{G}/\mathbb{N}$. We will now examine bottom right morphism $\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$.

In general, we say that a morphism $\Pi : \mathbb{P} \to \mathbb{Q}$ of locally compact quantum groups *has trivial kernel* if the quotient quantum group

$$\mathbb{P} \to \mathbb{P}/\mathrm{ker}\Pi$$

of [53, Definition 4.4] is an isomorphism. Let us recal that $\Pi : \mathbb{P} \to \mathbb{Q}$ induces a morphism $\Pi_1 : \mathbb{P}/\ker \Pi \to \mathbb{Q}$ which has trivial kernel.

Lemma 3.2.1. *The canonical morphism $\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$ has trivial kernel.*

Proof. Let us consider the morphism

$$\eta : \mathbb{H} \to \mathbb{G} \to \mathbb{G}/\mathbb{N}.$$

Using Proposition 3.1.1 we see that $\ker \eta = \mathbb{H} \wedge \mathbb{N} = \mathbb{N}$. In particular the kernel of the induced morphism $\mathbb{H}/\ker \eta \to \mathbb{G}/\mathbb{N}$ is trivial. □

Lemma 3.2.2. *The closed image of the canonical map $\mathbb{G}/\mathbb{N} \to \mathbb{G}/\mathbb{H}$ is full.*

Proof. As noted in Example 1.3.14 the closed image of $\Pi : \mathbb{G} \to \mathbb{G}/\mathbb{H}$ is full. Since $\mathbb{N} \subset \mathbb{H} = \ker \Pi$ the induced morphism $\mathbb{G}/\mathbb{N} \to \mathbb{G}/\mathbb{H}$ exists and its closed image coincides with the one of $\Pi : \mathbb{G} \to \mathbb{G}/\mathbb{H}$ thus it is also full. □

Let us gather up all of the ingredients we have so far in the form of Lemma 3.2.1 and Lemma 3.2.2 into a weak version of the third isomorphism theorem (to be improved on later):

Proposition 3.2.3. *Given normal closed quantum subgroups $\mathbb{N} \leq \mathbb{H}$ of a locally compact quantum group $\mathbb{G}$, the canonical morphism $\Pi_1 : \mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$ has trivial kernel and its closed image is precisely the kernel of $\Pi_2 : \mathbb{G}/\mathbb{N} \to \mathbb{G}/\mathbb{H}$.*

Proof. Let $\Pi : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ be the morphism which is dual to the embedding $\mathbb{H} \leq \mathbb{G}$. Let us recall that $\mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ being a Baaj-Vaes subalgebra of $\mathrm{L}^\infty(\mathbb{G})$ can be interpreted as $\widehat{\mathbb{G}/\mathbb{N}} \leq \widehat{\mathbb{G}}$. Using Lemma 1.3.15 to $\Pi : \widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$ and $\widehat{\mathbb{G}/\mathbb{N}} \leq \widehat{\mathbb{G}}$ we conclude that the right quantum group homomorphism $\alpha_{\mathbb{H}\to\mathbb{G}/\mathbb{N}} : \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) \to \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ is the restriction of the right quantum group homomorphism $\alpha_{\mathbb{H}\to\mathbb{G}} : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ to $\mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) \subset \mathrm{L}^\infty(\mathbb{G})$. Using Lemma 3.2.1 we conclude that the kernel of the morphism $\mathbb{H} \to \mathbb{G}/\mathbb{N}$ is equal $\mathbb{N}$ and using Lemma 1.3.10 we get $\alpha_{\mathbb{H}\to\mathbb{G}/\mathbb{N}}(\mathrm{L}^\infty(\mathbb{G}/\mathbb{N})) \subset \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H}/\mathbb{N})$. Summarizing the restriction of $\alpha_{\mathbb{H}\to\mathbb{G}} : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H})$ to $\mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ induces right quantum group homomorphism

$$\alpha_{\mathbb{H}/\mathbb{N}\to\mathbb{G}/\mathbb{N}} : \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) \to \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{H}/\mathbb{N}).$$

In particular

$$\mathrm{L}^\infty(\mathbb{G}/\mathbb{H}) = \{x \in \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) : \alpha_{\mathbb{H}/\mathbb{N}\to\mathbb{G}/\mathbb{N}}(x) = x \otimes \mathbb{1}\}. \tag{3.2.1}$$

Recalling that $\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$ is denoted by Π_1 let us consider $\overline{\mathrm{im}\Pi_1} \leq \mathbb{G}/\mathbb{N}$. Equation (3.2.1) then shows that

$$\mathrm{L}^\infty((\mathbb{G}/\mathbb{N})/\overline{\mathrm{im}\Pi_1}) = \mathrm{cd}(\mathrm{L}^\infty(\widehat{\overline{\mathrm{im}\Pi_1}})) = \mathrm{L}^\infty(\mathbb{G}/\mathbb{H}). \tag{3.2.2}$$

On the other hand noting that $\Pi_2 : \mathbb{G}/\mathbb{N} \to \mathbb{G}/\mathbb{H}$ is represented by the bicharacter $\mathrm{W}^{\mathbb{G}/\mathbb{H}} \in \mathrm{L}^\infty(\widehat{\mathbb{G}/\mathbb{H}}) \bar{\otimes} \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$ (where we used that $\mathrm{L}^\infty(\mathbb{G}/\mathbb{H}) \subset \mathrm{L}^\infty(\mathbb{G}/\mathbb{N})$) we get

$$\mathrm{L}^\infty((\mathbb{G}/\mathbb{N})/\ker\Pi_2) = \{(\omega \otimes \mathrm{id})(\mathrm{W}^{\mathbb{G}/\mathbb{H}}) : \omega \in \mathrm{L}^\infty(\widehat{\mathbb{G}/\mathbb{H}})_*\}^{\sigma-\mathrm{cls}} = \mathrm{L}^\infty(\mathbb{G}/\mathbb{H})$$

which together with (3.2.2) shows that $\ker\Pi_2 = \overline{\mathrm{im}\Pi_1}$.

□

In order to have a full analogue of [34, §3.3, Theorem 19], we would further want to know that the canonical morphism $\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$ identifies the former with a closed quantum subgroup of the latter. Moreover, in view of Proposition 3.2.3 and Theorem 1.3.11, this amounts to showing that the action of $\mathbb{H}/\mathbb{N}$ on $\mathbb{G}/\mathbb{N}$ is integrable.

To this end, we will first need the following Weyl-integral-formula-type result.

Proposition 3.2.4. *Given a normal closed quantum subgroup $\mathbb{N} \trianglelefteq \mathbb{G}$ a left-invariant Haar weight $\varphi_{\mathbb{G}}$ can be expressed as*

$$\varphi_{\mathbb{G}/\mathbb{N}} \circ T,$$

where

$$\mathrm{L}^\infty(\mathbb{G}) \longrightarrow \mathrm{L}^\infty(\mathbb{G}) \otimes \mathrm{L}^\infty(\mathbb{N}) \xrightarrow{\mathrm{id}\otimes\varphi_{\mathbb{N}}} \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}), \qquad T : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}/\mathbb{N}) \tag{3.2.3}$$

is a faithful semifinite normal operator-valued weight.

Proof. The fact that the composition (3.2.3) is a faithful normal operator-valued weight (in the sense of [93, Definition IX.4.12]) into the right hand side (one needs to check that it lands in the algebra of $\mathbb{N}$-invariants of $L^\infty(\mathbb{G})$) is essentially [97, Proposition 1.3].

The integrability [53, Theorem 6.2] of the action of the closed subgroup $\mathbb{N}$ on $\mathbb{G}$ means by definition that T is semifinite, and hence pre-composing with T turns semifinite weights on $L^\infty(\mathbb{G}/\mathbb{N})$ into semifinite weights on $L^\infty(\mathbb{G})$ (see also e.g. [64, Definition 8.1]). Finally, the requisite invariance property of $\varphi_{\mathbb{G}/\mathbb{N}}\circ T$ is a routine computation, using the invariance properties of $\varphi_{\mathbb{G}/\mathbb{N}}$ and T. ☐

Given a morphism $\Pi : \mathbb{P} \to \mathbb{Q}$ of locally compact quantum groups, we will denote by $T_{\mathbb{P}\to\mathbb{Q}}$ the operator-valued weight

$$L^\infty(\mathbb{Q}) \longrightarrow L^\infty(\mathbb{Q}) \mathbin{\bar{\otimes}} L^\infty(\mathbb{P}) \xrightarrow{\mathrm{id}\otimes\varphi_{\mathbb{P}}} L^\infty(\mathbb{Q}).$$

Let us note that in general $T_{\mathbb{P}\to\mathbb{Q}}$ is not semifinite.

Finally, Proposition 3.2.4 will help in proving the missing integrability ingredient we remarked on above:

Proposition 3.2.5. *Given closed normal subgroups $\mathbb{N} \le \mathbb{H}$ of a locally compact quantum group $\mathbb{G}$, the canonical action of $\mathbb{H}/\mathbb{N}$ on $\mathbb{G}/\mathbb{N}$ is integrable.*

Proof. We have to show that the operator-valued weight $T_{\mathbb{H}/\mathbb{N}\to\mathbb{G}/\mathbb{N}}$ defined as

$$L^\infty(\mathbb{G}/\mathbb{N}) \longrightarrow L^\infty(\mathbb{G}/\mathbb{N}) \mathbin{\bar{\otimes}} L^\infty(\mathbb{H}/\mathbb{N}) \xrightarrow{\mathrm{id}\,\bar{\otimes}\,\varphi_{\mathbb{H}/\mathbb{N}}} L^\infty(\mathbb{G}/\mathbb{N})$$

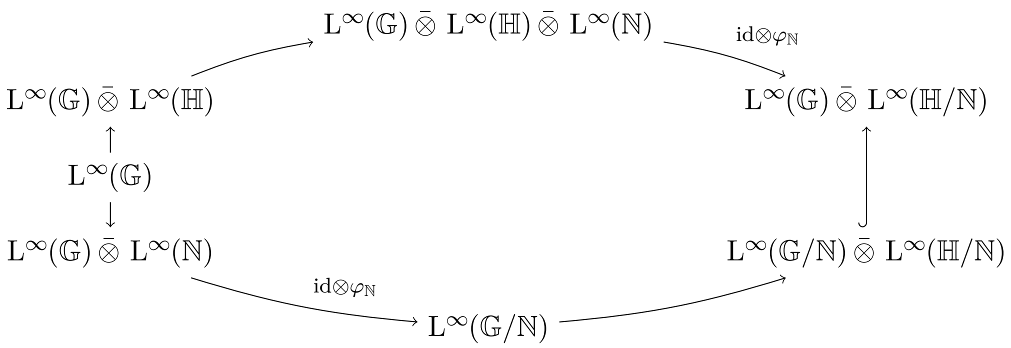
is semifinite, or equivalently, that there is at least one element of $L^\infty(\mathbb{G}/\mathbb{N})$ that is integrable with respect to the $(\mathbb{H}/\mathbb{N})$-action ([64, Proposition 6.2]).

We have already observed via [53, Theorem 6.2] that actions of closed quantum subgroups are integrable, and hence $T_{\mathbb{N}\to\mathbb{G}}$ is semifinite. Similarly, $T_{\mathbb{H}\to\mathbb{G}}$ is semifinite. We will argue that for any $x \in L^\infty(\mathbb{G})^+$ that is $\mathbb{H}$-integrable, its image

$$T_{\mathbb{N}\to\mathbb{G}}(x) \in L^\infty(\mathbb{G}/\mathbb{N})$$

is $(\mathbb{H}/\mathbb{N})$-integrable; as observed, this is sufficient to finish the proof of the proposition.

First, consider the following diagram of operator-valued weights and von Neumann algebra homomorphisms, where commutativity is immediate from the definitions:



(3.2.4)

Now further glue the commutative square

$$\begin{array}{ccc} L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{H}/\mathbb{N}) & \xrightarrow{\mathrm{id}\otimes\varphi_{\mathbb{H}/\mathbb{N}}} & L^\infty(\mathbb{G}) \\ \uparrow & & \uparrow \\ L^\infty(\mathbb{G}/\mathbb{N}) \otimes L^\infty(\mathbb{H}/\mathbb{N}) & \xrightarrow{\mathrm{id}\otimes\varphi_{\mathbb{H}/\mathbb{N}}} & L^\infty(\mathbb{G}/N) \end{array} \tag{3.2.5}$$

to the right hand side of (3.2.4).

Using the Weyl integration formula (Proposition 3.2.4) for the normal subgroup $\mathbb{N} \trianglelefteq \mathbb{H}$, we can see that the composition of the top half of (3.2.4) with the top horizontal arrow of (3.2.5) yields precisely the semifinite operator-valued weight $T_{\mathbb{H}\to\mathbb{G}}$. The commutativity of the compound diagram obtained by gluing (**??**) then proves our assertion that the image through $T_{\mathbb{N}\to\mathbb{G}}$ of an $\mathbb{H}$-integrable element of $L^\infty(\mathbb{G})$ is $(\mathbb{H}/\mathbb{N})$-integrable, thus completing the proof. □

In summary, we obtain

Theorem 3.2.6. *Let $\mathbb{N} \le \mathbb{H} \trianglelefteq \mathbb{G}$ be inclusions of closed locally compact quantum subgroups, and assume furthermore that $\mathbb{N}$ is normal in $\mathbb{G}$. Then, we have*

$$\mathbb{H}/\mathbb{N} \trianglelefteq \mathbb{G}/\mathbb{N} \quad \text{and} \quad (\mathbb{G}/\mathbb{N})/(\mathbb{H}/\mathbb{N}) \cong \mathbb{G}/\mathbb{H}.$$

Proof. As noted above, Theorem 1.3.11 and Proposition 3.2.3 reduce the problem to showing that the action of $\mathbb{H}/\mathbb{N}$ on $\mathbb{G}/\mathbb{N}$ resulting from the canonical morphism $\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$ is integrable. This is exactly what Proposition 3.2.5 says. □

In fact, some of the above results generalize somewhat so as to allow us to recover standard results on topological groups in the locally compact quantum setting. For instance, the conclusion that

$$\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$$

is a closed embedding does not actually require the normality of $\mathbb{H}$, and hence Theorem 3.2.6 extends to this general setup.

Theorem 3.2.7. *Let $\mathbb{N} \le \mathbb{H} \le \mathbb{G}$ be closed embeddings of locally compact quantum groups, with $\mathbb{N}$ normal in $\mathbb{G}$. Then the canonical morphism*

$$\mathbb{H}/\mathbb{N} \to \mathbb{G}/\mathbb{N}$$

is a closed embedding, and

$$L^\infty((\mathbb{G}/\mathbb{N})/(\mathbb{H}/\mathbb{N})) = L^\infty(\mathbb{G}/\mathbb{H})$$

Let us now briefly go back to the setup of Proposition 3.1.1: $\mathbb{H}$ and $\mathbb{N}$ are closed quantum subgroups of $\mathbb{G}$, with $\mathbb{N}$ normal. Then, by Lemma 1.3.21, the composition $\mathbb{H} \to \mathbb{G} \to \mathbb{G}/\mathbb{N}$ always factors as

$$\begin{array}{ccccc} & & \mathbb{G} & & \\ & \nearrow & & \searrow & \\ \mathbb{H} & & & & \mathbb{G}/\mathbb{N} \\ & \searrow & & \nearrow & \\ & & \mathbb{H}/\mathbb{H}\wedge\mathbb{N} & & \end{array}$$

Moreover, Theorem 3.2.7 ensures that we can regard $\mathbb{H} \vee \mathbb{N}/\mathbb{N}$ as a closed subgroup of $\mathbb{G}/\mathbb{N}$, and an examination of the proof of Proposition 3.1.1 shows that we actually have the following amplification of Proposition 3.1.1.

Theorem 3.2.8. *Let $\mathbb{H} \leq \mathbb{G}$ and $\mathbb{N} \trianglelefteq \mathbb{G}$ be closed quantum subgroups of a locally compact quantum group. Then, the canonical morphism*

$$\mathbb{H}/\mathbb{H} \wedge \mathbb{N} \to \mathbb{G}/\mathbb{N}$$

has trivial kernel and closed image $\mathbb{H} \vee \mathbb{N}/\mathbb{N} \leq \mathbb{G}/\mathbb{N}$.

3.3 The modular law and Zassenhauss lemma

We now proceed to address an analogue of the Zassenhauss lemma for locally compact quantum groups. First, recall the classical (non-topological) statement, for instance as in [62, Vol. 1, p. 77].

Proposition 3.3.1. *Let $A' \trianglelefteq A$ and $B' \trianglelefteq B$ be subgroups of a group G. Then, we have a canonical isomorphism*

$$\frac{A'(A \cap B)}{A'(A \cap B')} \cong \frac{B'(A \cap B)}{B'(A' \cap B)}.$$

Remark 3.3.2. *The statement implicitly includes the claims that the group products appearing in the formula (such as $A'(A \cap B)$) are indeed subgroups of G, and the groups appearing as denominators are normal in those appearing as numerators.*

Recall ([62, Vol. 1, p. 77]) that the proof typically proceeds through the second isomorphism theorem for groups (which Proposition 3.1.1 replicates) by using it to implement connecting isomorphisms

$$\frac{A'(A\cap B)}{A'(A\cap B')} \xrightarrow{\cong} \frac{A\cap B}{(A'\cap B)(A\cap B')} \xleftarrow{\cong} \frac{B'(A\cap B)}{B'(A'\cap B)}$$

We will adopt a similar approach here, but we need some preparatory remarks.

First, note that it is implicit in the proof sketch we have just recalled that under the assumptions of the Zassenhaus lemma we have e.g.

$$(A \cap B) \cap (B'(A' \cap B)) = (A \cap B')(A' \cap B).$$

Given that $A' \cap B$ is a normal subgroup of $A \cap B$ and normalizes B', this follows from the modularity law for subgroups which we will use in the following form:

$$L \leq H \leq G, \quad M \leq G \quad \text{and} \quad L \text{ normalizes } M \quad \Rightarrow \quad H \cap ML = (H \cap M)L.$$

Theorem 3.3.4 is an analogue of modularity in the locally compact quantum setting. Let us first prove an easy inclusion.

Lemma 3.3.3. *Let $\mathbb{G}$ be a locally compact quantum groups $\mathbb{M}, \mathbb{H} \leq \mathbb{G}$, and $\mathbb{L} \leq \mathbb{H}$. Then*

$$(\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L} \leq \mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}).$$

Proof. Let us note that $\mathbb{H} \wedge \mathbb{M} \leq \mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L})$. Moreover, by assumption $\mathbb{L} \leq \mathbb{H}$ and clearly $\mathbb{L} \leq \mathbb{M} \vee \mathbb{L}$ thus $\mathbb{L} \leq \mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L})$. This altogether shows that $(\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L} \leq \mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L})$. □

We now turn to sufficient conditions for an inclusion reversal in Lemma 3.3.3. The material surrounding Definition 1.3.31 above will be needed here.

Theorem 3.3.4. *Let $\mathbb{L} \le \mathbb{H}$ and $\mathbb{M}$ be closed quantum subgroups of a locally compact quantum group $\mathbb{G}$ such that $\mathbb{L}$ normalizes $\mathbb{M}$. Then, we have*

$$\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = (\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L}. \tag{3.3.1}$$

if either

(a) $\mathbb{L}$ *is compact, or*

(b) $\mathbb{H}$ *is open in* $\mathbb{G}$.

Proof. We address the two versions of the result separately.

Proof of part (a). Here we rephrase the desired conclusion in terms of the quantum homogeneous spaces $A_\bullet$ for $\bullet = \mathbb{H}$, $\mathbb{M}$, etc (see notation in §1.3.1). Since cd is an anti-isomorphism of lattices the sought-after conclusion is

$$A_{\mathbb{H}} \vee (A_{\mathbb{M}} \wedge A_{\mathbb{L}}) = (A_{\mathbb{H}} \vee A_{\mathbb{M}}) \wedge A_{\mathbb{L}}. \tag{3.3.2}$$

Using lemma 3.3.3 we see that the right hand side contains the left hand side. We hence focus on proving the opposite inclusion.

Let us first consider the case when $\mathbb{H}$ and $\mathbb{M}$ are relatively well positioned in the sense of Definition 1.3.31. Now, as in the proof of Proposition 3.2.5, consider the operator-valued weights

$$T_{\bullet \to \mathbb{G}} : \mathrm{L}^\infty(\mathbb{G}) \to A_\bullet.$$

Since $\mathbb{L}$ is assumed to be compact, $T = T_{\mathbb{L} \to \mathbb{G}}$ actually restricts to the identity on $A_{\mathbb{L}}$, and hence also on the right hand side of (3.3.2).

On the other hand, in order to study the result of applying T to the right hand side algebra of (3.3.2), it suffices by (1.3.25) to look at products

$$x = x_{\mathbb{H}} x_{\mathbb{M}},\ x_{\mathbb{H}} \in A_{\mathbb{H}},\ x_{\mathbb{M}} \in A_{\mathbb{M}}$$

When applied to the latter, due to the preservation by T of bimodule structures over $A_{\mathbb{H}} \subseteq A_{\mathbb{L}}$, T produces the element

$$x_{\mathbb{H}} T(x_{\mathbb{M}}).$$

We would be finished if we could show that $T(x_{\mathbb{M}}) \in A_{\mathbb{M}} \wedge A_{\mathbb{L}}$; this is what Lemma 3.3.6 below does.

In order to drop the well positioning assumption let us consider $\mathbb{D} = \mathbb{H} \wedge (\mathbb{L} \vee \mathbb{M})$. The containment $\mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge (\mathrm{L}^\infty(\widehat{\mathbb{M}}) \vee \mathrm{L}^\infty(\widehat{\mathbb{L}})) \subset (\mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge \mathrm{L}^\infty(\widehat{\mathbb{M}})) \vee \mathrm{L}^\infty(\widehat{\mathbb{L}})$, which is effectively proved above under the well positioning assumption of $\mathbb{H}$ and $\mathbb{L}$, is equivalent with the following containment

$$\mathrm{L}^\infty(\widehat{\mathbb{D}}) \wedge (\mathrm{L}^\infty(\widehat{\mathbb{M}}) \vee \mathrm{L}^\infty(\widehat{\mathbb{L}})) \subset (\mathrm{L}^\infty(\widehat{\mathbb{D}}) \wedge \mathrm{L}^\infty(\widehat{\mathbb{M}})) \vee \mathrm{L}^\infty(\widehat{\mathbb{L}}). \tag{3.3.3}$$

Since $\mathbb{D}, \mathbb{L}, \mathbb{M} \le \mathbb{L} \vee \mathbb{M}$, proving (3.3.3) we can substitute $\mathbb{M} \vee \mathbb{L}$ for $\mathbb{G}$. After this substitution the normalization assumption of $\mathbb{M}$ by $\mathbb{L}$ gets replaced by the normality of $\mathbb{M}$ in $\mathbb{G}$. Using Proposition 1.3.32 we see that $\mathbb{D}$ and $\mathbb{M}$ are well positioned and by the first part of the proof (3.3.3) holds, thus we are done.

Proof of part (b). Here, we translate the claim into an equivalent statement for the underlying von Neumann algebras of the dual groups $\widehat{\mathbb{G}}$, $\widehat{\mathbb{H}}$, etc.

Since $\mathbb{L}$ normalizes $\mathbb{M}$ we can use Lemma 1.3.18 and get

$$\mathrm{L}^\infty(\widehat{\mathbb{M}\vee\mathbb{L}}) = \{\mathrm{L}^\infty(\widehat{\mathbb{M}})\,\mathrm{L}^\infty(\widehat{\mathbb{L}})\}^{\sigma-\mathrm{cls}}.$$

In conclusion, the von Neumann subalgebra of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ corresponding to the left hand side of (3.3.1) is

$$\mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge \{\mathrm{L}^\infty(\widehat{\mathbb{M}})\,\mathrm{L}^\infty(\widehat{\mathbb{L}})\}^{\sigma-\mathrm{cls}}.$$

Similarly, the right hand side corresponds to

$$\{(\mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge \mathrm{L}^\infty(\widehat{\mathbb{M}}))\,\mathrm{L}^\infty(\widehat{\mathbb{L}})\}^{\sigma-\mathrm{cls}},$$

and we seek to prove

$$\mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge \{\mathrm{L}^\infty(\widehat{\mathbb{M}})\,\mathrm{L}^\infty(\widehat{\mathbb{L}})\}^{\sigma-\mathrm{cls}} = \{(\mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge \mathrm{L}^\infty(\widehat{\mathbb{M}}))\,\mathrm{L}^\infty(\widehat{\mathbb{L}})\}^{\sigma-\mathrm{cls}}. \tag{3.3.4}$$

As in the first part, the inclusion of the right hand side in the left hand side is Lemma 3.3.3, and we only need to prove '$\subseteq$'.

We will use essentially the same strategy as in the proof of part (1), substituting for $T_{\mathbb{L}\to\mathbb{G}} : \mathrm{L}^\infty(\mathbb{G}) \to A_{\mathbb{L}}$ from that other proof the expectation

$$T : \mathrm{L}^\infty(\widehat{\mathbb{G}}) \to \mathrm{L}^\infty(\widehat{\mathbb{H}})$$

corresponding to the compatible Haar weights on the two von Neumann algebras (this is where the openness of $\mathbb{H}$ is essential; see e.g. [51, Theorem 7.5]).

As before, applying T to the left hand side of (3.3.4) on the one hand acts as the identity, and on the other produces from a product

$$x = x_{\mathbb{M}}x_{\mathbb{L}},\ \ x_{\mathbb{M}} \in \mathrm{L}^\infty(\widehat{\mathbb{M}}),\ \ x_{\mathbb{L}} \in \mathrm{L}^\infty(\widehat{\mathbb{L}})$$

the element

$$T(x_{\mathbb{M}})x_{\mathbb{L}}$$

due to the $\mathrm{L}^\infty(\widehat{\mathbb{L}})$-bimodule map property of T. The conclusion that $x = T(x)$ belongs to the right hand side of (3.3.4) now follows from the fact that

$$T(x_{\mathbb{M}}) \in \mathrm{L}^\infty(\widehat{\mathbb{H}}) \wedge \mathrm{L}^\infty(\widehat{\mathbb{M}}),$$

which in turn relies on Proposition 1.3.26. □

Remark 3.3.5. *The reader should note that since $\mathbb{L} = \mathbb{H} \wedge \mathbb{L}$, the modular law is really a form of the distributive law $\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = (\mathbb{H} \wedge \mathbb{M}) \vee (\mathbb{H} \wedge \mathbb{L})$. The latter, however, is false in general.*

Lemma 3.3.6. *Let $\mathbb{G}$ be a locally compact quantum group, $\mathbb{L} \leq \mathbb{G}$ a compact quantum subgroup, and $\mathbb{M} \leq \mathbb{G}$ a closed quantum subgroup normalized by $\mathbb{L}$. Then, the expectation*

$$T : \mathrm{L}^\infty(\mathbb{G}) \to A_{\mathbb{L}}$$

leaves $A_{\mathbb{M}}$ invariant.

Proof. Indeed, the normalization condition ensures that the right action of $\mathbb{L}$ on $\mathbb{G}$ descends to an action on the quantum homogeneous space $\mathbb{G}/\mathbb{M}$ via the commutative diagram (see Lemma 1.3.19)

$$\begin{array}{ccc} \mathrm{L}^\infty(\mathbb{G}) & \longrightarrow & \mathrm{L}^\infty(\mathbb{G}) \mathbin{\bar{\otimes}} \mathrm{L}^\infty(\mathbb{L}) \\ \uparrow & & \uparrow \\ A_{\mathbb{M}} & \longrightarrow & A_{\mathbb{M}} \mathbin{\bar{\otimes}} \mathrm{L}^\infty(\mathbb{L}) \end{array}$$

The conclusion now follows from the definition of the expectation T as the coaction

$$\mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \mathbin{\bar{\otimes}} \mathrm{L}^\infty(\mathbb{L})$$

followed by an application of the Haar state $\psi_{\mathbb{L}}$ to the right hand tensorand. □

Remark 3.3.7. *We note that an appropriately rephrased version of Lemma 3.3.6 holds under the weaker requirement that $\mathbb{L}/\mathbb{L}\wedge\mathbb{M}$ acts integrably on $\mathbb{G}/\mathbb{M}$. T would then restrict to a semifinite operator-valued weight*

$$A_{\mathbb{M}} \to A_{\mathbb{L}} \wedge A_{\mathbb{M}}.$$

We do, however, need compactness in the proof of Theorem 3.3.4 above, where the operator-valued weight T had to be an expectation and hence fix $A_{\mathbb{L}}$ pointwise.

Note that Theorem 3.3.4 does not hold in full generality, even for classical locally compact abelian groups. In order to see this, we can modify Example 3.1.4 as follows.

Example 3.3.8. Our ambient group $\mathbb{G} = \mathbb{T}^4 \times \mathbb{R}$ is written as in Example 3.1.4, multiplicatively in the first four variables and additively in the last.

We then take

$$\mathbb{M} = \{(e^{ix\theta_1}, \cdots, e^{ix\theta_4}, x) \mid x \in \mathbb{R}\}$$

and

$$\mathbb{L} = \{(e^{is\phi}, 1, 1, 1, s) \mid s \in \mathbb{R}\}.$$

with ϕ and θ_i linearly independent over $\mathbb{Q}$. Finally, let

$$\mathbb{H} = \{(e^{is\phi}, e^{it\phi}, 1, 1, s+t) \mid s, t \in \mathbb{R}\}.$$

$\mathbb{H}$ is easily seen to be a two-dimensional closed Lie subgroup of $\mathbb{G}$ that contains $\mathbb{L}$ and intersects $\mathbb{M}$ trivially. Since $\mathbb{M}\mathbb{L}$ is dense in $\mathbb{G}$, we have $\mathbb{H} \wedge (\mathbb{M} \vee \mathbb{L}) = \mathbb{H}$ on the left hand side of (3.3.1). On the other hand, the right hand side $(\mathbb{H} \wedge \mathbb{M}) \vee \mathbb{L}$ is $\mathbb{L}$.

As in Chapter 2 above, we have the following consequence of modularity (i.e. of Theorem 3.3.4).

Proposition 3.3.9. *Let $\mathbb{A}' \trianglelefteq \mathbb{A}$ and $\mathbb{B}' \trianglelefteq \mathbb{B}$ be either*

(a) compact or

(b) open

quantum subgroups of a locally compact quantum group $\mathbb{G}$. Then, we have an isomorphism

$$\frac{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{A}' \vee (\mathbb{A} \wedge \mathbb{B}')} \cong \frac{\mathbb{B}' \vee (\mathbb{A} \wedge \mathbb{B})}{\mathbb{B}' \vee (\mathbb{A}' \wedge \mathbb{B})}.$$

Proof. This follows from Theorem 3.3.4 in much the same way in which Proposition 2.2.11 follows from Proposition 2.2.4 and Proposition 2.2.10, by applying the earlier result to $\mathbb{H} = \mathbb{A} \wedge \mathbb{B}$, $\mathbb{L} = \mathbb{A} \wedge \mathbb{B}'$ and $\mathbb{M} = \mathbb{A}'$.

Everything goes through as before, modulo the observation that in the open case we need Proposition 1.3.26 in order to conclude that $\mathbb{A} \wedge \mathbb{B}$ is open in the open subgroup $\mathbb{B}$, and hence is also open in $\mathbb{G}$ by Proposition 1.3.28. □

Remark 3.3.10. *In case (a) of Proposition 3.3.9 it is enough that $\mathbb{L}$ (and hence say $\mathbb{B}'$) be compact.*

3.4 Schreier and Jordan-Hölder-type results

We devote this section to certain partial analogues of Theorems 2.3.3, 2.4.3 and 2.4.4 in the setting of locally compact quantum groups.

In this context, the relevant notions of (sub)normal series and refinements thereof make sense virtually verbatim, so we point to Definitions **??** for a reminder.

We write $\{\mathbb{G}_\ell\}_{\ell\ge 0}$ for the generic (sub)normal series

$$\mathbb{G} = \mathbb{G}_0 \ge \mathbb{G}_1 \ge \mathbb{G}_2 \ge \mathbb{G}_3 \ge \cdots \ge \mathbb{G}_k = 1. \tag{3.4.1}$$

of closed quantum subgroups of a locally compact quantum group $\mathbb{G}$.

Theorem 3.4.1. *Let $\mathbb{G}$ be a locally compact quantum group. Then, any two subnormal series $\{\mathbb{G}_\ell\}$ and $\mathbb{G}'_t$ of $\mathbb{G}$ admit equivalent refinements, provided*

$$\mathbb{G}_\ell,\ \ell \ge 1 \quad \textit{and} \quad \mathbb{G}'_t,\ t \ge 1$$

are

(a) compact or

(b) open.

Proof. One can simply imitate the proof of Theorem 2.3.3, making use of parts (a) and (b) of Proposition 3.3.9 respectively for the two parts of the present result. □

As for an analogue of Theorem 2.4.3 and Proposition 2.4.4, we have

Theorem 3.4.2. *Let $\mathbb{G}$ be a locally compact quantum group. Then, all composition series of $\mathbb{G}$ consisting of*

(a) compact or

(b) open

quantum subgroups are equivalent.

Proof. Just as the proof of Theorem 2.4.3, this follows mechanically once we have Theorem 3.4.1 above. □

The compact versions of Theorems 3.4.1 and 3.4.2 refer to subnormal series (3.4.1) in which all $\mathbb{G}_\ell$, $\ell \ge 1$ are compact, but $\mathbb{G} = \mathbb{G}_0$ need not be so. Let us note that this is equivalent to the subquotient $\mathbb{G}/\mathbb{G}_1$ being non-compact. Indeed, we have

Proposition 3.4.3. *A locally compact quantum group* $\mathbb{G}$ *is compact if and only if it admits a subnormal series* (3.4.1) *with compact quotients* $\mathbb{G}_i/\mathbb{G}_{i+1}$.

Proof. The direct implication '⇒' is immediate by considering the trivial length-zero series, so we focus on the opposite implication.

By induction, it suffices to show that if $\mathbb{L} \trianglelefteq \mathbb{G}$ is compact along with $\mathbb{G}/\mathbb{L}$, then so is $\mathbb{G}$. This in turn follows from the fact that by the Weyl integration formula proven above (Proposition 3.2.4) the Haar weight of $\mathbb{G}$ is a state. □

Chapter 4

Classification results for the compact quantum group $O_{-1}(2)$

In this chapter, we will give first the list of ergodic actions of $O_{-1}(2)$ and we will apply the results of Chapter 1 Section 1.4 to obtain the list of embeddable ergodic actions and the list of idempotent states of $O_{-1}(2)$.

4.1 Ergodic actions of $O_{-1}(2)$

In this section, we will give the complete list of ergodic coactions of $O_{-1}(2)$. Let's recall [6, Theorem 4.3]:

Theorem 4.1.1. *The category of corepresentations of $C(O_n^{-1})$ is tensor equivalent to the category of representations of O_n.*

By Theorem 4.1.1, the compact quantum groups $O_{-1}(2)$ and $O(2)$ are monoidally equivalent and by Theorem 1.4.19, the ergodic actions of $O(2)$ are in bijective correspondence with the ergodic actions of $O_{-1}(2)$. Then it is sufficient to classify the ergodic actions of $O(2)$. Recall that if G is a compact group, H a closed subgroup, and $H \overset{\alpha}{\curvearrowright} N$ an ergodic action of H on a von Neumann algebra N, we have the induced ergodic action of G on

$$\begin{aligned} \mathrm{Ind}_H^G(N) &= \{f \in L^\infty(G, N) \mid \forall g \in G, h \in H, \alpha_h(f(gh)) = f(g)\} \\ &\subseteq L^\infty(G)\bar{\otimes}N, \end{aligned}$$

given by

$$(\alpha_g(f))(g') = f(g^{-1}g').$$

In particular, if π is an irreducible projective representation of H on a Hilbert space V_π, we have an ergodic action of G on the von Neumann algebra $\mathrm{Ind}_H^G(B(V_\pi))$ where $B(V_\pi)$ is equipped with the ergodic H-action

$$\alpha_h(x) = \pi(h)x\pi(h)^*.$$

For example, when π is the trivial representation (or a one-dimensional representation), we have $\mathrm{Ind}_H^G(\mathbb{C}) = L^\infty(G/H)$ with the left translation action.

In the following, we will identify $O(2) \cong \mathbb{Z}_2 \ltimes \mathbb{T}$, with $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ the circle group, and with the cyclic group $\mathbb{Z}_2 = \{1, \sigma\}$ acting on $\mathbb{T}$ by $\sigma(z) = \bar{z}$.

Theorem 4.1.2. *Let $O(2) \underset{\alpha}{\curvearrowright} M$ be an ergodic action. Then $M \cong \mathrm{Ind}_H^{O(2)}(B(V_\pi))$ for some closed subgroup $H \subseteq O(2)$ and π an irreducible representation of H.*

Proof. For $n \in \mathbb{Z}$, we write M_n for the n-th spectral subspace of M with respect to the action of $\mathbb{T}$, i.e. $\alpha_z(x) = z^n x$ for $x \in M_n$. In particular, M_0 is the space of $\mathbb{T}$-invariant elements. As M_0 is stabilised by σ, and as $\mathbb{Z}_2$ must act ergodically on M_0 since $O(2)$ acts ergodically on M, we must have either $M_0 \cong \mathbb{C}$ with σ acting trivially or $M_0 \cong \mathbb{C} \oplus \mathbb{C}$ with σ interchanging the factors.

In the first case, $\mathbb{T}$ already acts ergodically on M_0. If then for some $n \in \mathbb{Z}$ we have $M_n \neq \{0\}$, it follows that for a non-zero $x \in M_n$ we have $x^*x \in M_0$, and hence $x^*x = \lambda \in \mathbb{C}\backslash\{0\}$. Similarly, $xx^* = \lambda' \in \mathbb{C}\backslash\{0\}$. We must hence have $\lambda = \lambda'$ and x a scalar multiple of a unitary. If y is another non-zero element in M_n, we must have also x^*y a scalar multiple of the unit, hence $y \in \mathbb{C}x$. It follows that all spectral subspaces are one-dimensional. Moreover, if $u \in M_n$ and $v \in M_m$ are non-zero, then $uv \in M_{m+n}$ non-zero. Hence either M_n is zero for all $n \neq 0$, and $M = \mathbb{C}$, or there exists a least $n \in \mathbb{N} \setminus \{0\}$ such that $M_n \neq 0$. In the latter case, any non-zero spectral subspace is of the form $M_{kn} = \mathbb{C}u^k$ for $k \in \mathbb{Z}$ and u a fixed unitary element in M_n. In particular, it follows that M is commutative, and hence of the form $L^\infty(O(2)/H)$ for H a closed subgroup of $O(2)$.

In the second case, let p be a non-trivial projection in M_0. Then also the central support $z(p)$ of p in M must be a projection in M_0. Hence there are two cases: $z(p) = p$ or $z(p) = 1$. In the first case, p is central in M, and hence $M = pM \oplus (1-p)M$ with the action of $\mathbb{T}$ on both pM and $(1-p)M$ ergodic. By the first part of the proof, it follows that then M is commutative, and hence of the form $L^\infty(O(2)/H)$ for H a closed subgroup of $O(2)$. In the second situation, it follows that MpM is weakly dense in M. We must have then as well that $M(1-p)M$ is weakly dense in M, and hence we can decompose M as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = pMp$, $B = pM(1-p)$, $C = (1-p)Mp$ and $D = (1-p)M(1-p)$. In particular, A, D are von Neumann algebras with ergodic actions of $\mathbb{T}$, and hence commutative. Writing $\alpha_\sigma = \sigma$, then as $\sigma(p) = 1-p$, it follows that $\sigma(A) = D$ and $\sigma(B) = C$, and we can write

$$M = \begin{pmatrix} A & B \\ \sigma(B) & \sigma(A) \end{pmatrix}.$$

It follows also that $B_0 = \{0\}$, as otherwise $\begin{pmatrix} 0 & x \\ \sigma(x) & 0 \end{pmatrix}$ would be a non-zero, non-scalar $O(2)$-invariant element in M for any non-zero $x \in B_0$, contradicting ergodicity. On the other hand, not all B_n can be zero, as B is non-zero (since p is not central). Now as before, we can show that each non-zero B_n must be one-dimensional, and spanned by an element u such that $u^*u = 1-p$ and $uu^* = p$. Fixing a non-zero B_l and such a u, we further have that $B = uu^*B \subseteq u\sigma(A) \subseteq B$, and hence $B = u\sigma(A)$. Similarly, $B = Au$.

If now $A = A_0 = \mathbb{C}p$, it follows that also $\sigma(A) = \sigma(A)_0 = \mathbb{C}(1-p)$, and B and $\sigma(B)$ are one-dimensional, i.e. $M = M_2(\mathbb{C})$. But an ergodic action $O(2) \curvearrowright M_2(\mathbb{C})$ determines an irreducible projective representation of $O(2)$ on $\mathbb{C}^2$, hence $M = B(\mathbb{C}^2) = \mathrm{Ind}_{O(2)}^{O(2)}(B(\mathbb{C}^2))$.

In case where A is not one-dimensional, we must have that $A \cong L^\infty(SO(2)/H)$ for some finite cyclic group $H = \mathbb{Z}_k \subseteq SO(2)$. As A then contains non-zero spectral subspaces

For a projection $p \in M$ the smallest projection q in the center $\mathcal{Z}(M)$ of M such that $p \leq q$ is called the central support and is denoted by $z(p)$.

precisely at the values $k\mathbb{Z}$, it follows that B contains non-zero spectral subspaces at the values in $l+k\mathbb{Z}$. In particular, as B_0 is zero, we must have l not a multiple of k, and we can assume $0<l<k$.

Take v a unitary in A_k. It follows that the direct sum M of all $O(2)$-spectral subspaces has a basis consisting of partial isometries of either the form $v_+^{(r)} = \begin{pmatrix} v^r & 0 \\ 0 & 0 \end{pmatrix}$, $v_-^{(r)} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma(v)^r \end{pmatrix}$, $u_+^{(r)} = \begin{pmatrix} 0 & uv^r \\ 0 & 0 \end{pmatrix}$ and $u_-^{(r)} = \begin{pmatrix} 0 & 0 \\ \sigma(u)\sigma(v)^r & 0 \end{pmatrix}$, where $\sigma(v_\pm^{(r)}) = v_\mp^{(r)}$ and $\sigma(u_\pm^{(r)}) = u_\mp^{(r)}$, and where $\mathbb{T}$ acts by

$$\alpha_z(v_\pm^{(r)}) = z^{\pm kr} v_\pm^{(r)}, \quad \alpha_z(u_\pm^{(r)}) = z^{\pm(l+rk)} u_\pm^{(r)}.$$

But let $\mathbb{D}_k = \mathbb{Z}_2 \ltimes \mathbb{Z}_k$, acting ergodically on $B(\mathbb{C}^2)$ by $\alpha_z \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z^l b \\ z^{-l}c & d \end{pmatrix}$ and $\alpha_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$. Then it is easy to see that $\mathrm{Ind}_{\mathbb{D}_k}^{O(2)}(B(\mathbb{C}^2))$ is spanned by the partial isometries

$$V_+^{(r)}(z) = \begin{pmatrix} z^{-kr} & 0 \\ 0 & 0 \end{pmatrix}, \quad V_+^{(r)}(\sigma z) = \begin{pmatrix} 0 & 0 \\ 0 & z^{-kr} \end{pmatrix},$$

$V_-^{(r)}(g) = V_+^{(r)}(\sigma g)$ and

$$U_+^{(r)}(z) = \begin{pmatrix} 0 & z^{-l-kr} \\ 0 & 0 \end{pmatrix}, \quad U_+^{(r)}(\sigma z) = \begin{pmatrix} 0 & 0 \\ z^{-l-kr} & 0 \end{pmatrix},$$

$U_-^{(r)}(g) = U_+^{(r)}(\sigma g)$. We then have a unique $O(2)$-equivariant *-isomorphism $\mathrm{Ind}_{\mathbb{D}_k}^{O(2)}(B(\mathbb{C}^2)) \to M$ such that $V_\pm^{(r)} \mapsto v_\pm^{(r)}$ and $U_\pm^{(r)} \mapsto u_\pm^{(r)}$.

□

It remains to determine isomorphisms between the induced actions.

The irreducible (projective) representations of $O(2)$ give either the trivial action $\beta_0^{(\infty)}$ on $\mathbb{C}$ (for the characters) or the ergodic action $\beta_{l/2}^{(\infty)} = \alpha$ on $M_2(\mathbb{C})$ by

$$\alpha_z \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z^l b \\ z^{-l}c & d \end{pmatrix}, \quad \alpha_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

for $l \in \mathbb{N}\setminus\{0\}$ (with the even l coming from representations and the odd l coming from non-trivial projective representations).

The irreducible representations of $\mathbb{T}$ all give the same action $\alpha = \alpha^{(\infty)}$ on $\mathbb{C}^2 = L^\infty(O(2)/T)$, namely

$$\alpha_z((x,y)) = (x,y), \quad \alpha_\sigma(x,y) = (y,x).$$

The irreducible representations of $\mathbb{Z}_k$ for some fixed k all give the same action $\alpha = \alpha^{(k)}$ on $L^\infty(O(2)/\mathbb{Z}_k) = L^\infty(T/\mathbb{Z}_k) \oplus L^\infty(T/\mathbb{Z}_k)$, namely

$$\alpha_z(f,g) = (f_z, g_z), \quad \alpha_\sigma(f,g) = (g,f),$$

where f_z denotes the z-translate of f.

Finally, for the $\mathbb{D}_k$ we have the action $\alpha = \beta_0^{(k)}$ coming from the characters of $\mathbb{D}_k$, giving the action on $L^\infty(O(2)/\mathbb{D}_k)$, or for the non-character (projective) representations the actions $\beta_{l/2}^{(k)}$ where $0 < l < k$ a natural number and $\beta_{l/2}^{(k)}$ the induced action coming from the (projective) $\mathbb{D}_k$-representation

$$\alpha_z \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z^l b \\ z^{-l} c & d \end{pmatrix}, \quad \alpha_\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

Hence, we obtain:

Proposition 4.1.3. *The full list of non-equivalent ergodic actions of $O(2)$ is:*

$$\{\beta_{l/2}^{(k)}, \beta_{l'/2}^{(\infty)}, \alpha^{(\infty)}, \alpha^{(k')} \mid k, k' \in \mathbb{N}_0, l' \in \mathbb{N}, 0 \leq l < k\}.$$

Question 4.1.4. *Does monoidal equivalence allow us to determine the ergodic action $\mathbb{X}_\sigma \overset{\widehat{\alpha}}{\curvearrowleft} O_{-1}(2)$ corresponding to each family of ergodic action $\mathbb{X} \overset{\alpha}{\curvearrowleft} O(2)$?*

Consider an action $\alpha : \mathbb{G} \to Aut(M) \cong \widehat{\alpha} : \begin{cases} G \times M & \to M \\ (g, m) & \mapsto \alpha_g(m) \end{cases}$ of a group $\mathbb{G}$ on a von Neumann algebra M. This induces a map $\widehat{\alpha} : \begin{cases} M & \to M \otimes L^\infty(G) \cong L^\infty(G, M) \\ m & \mapsto (g \mapsto \alpha_g(m)) \end{cases}$ such that: $\widehat{\alpha}(m)(g) \in M$. In fact, it easy to verify that $\widehat{\alpha}$ is an homomorphism and verifies $(\widehat{\alpha} \otimes id) \circ \widehat{\alpha} = (id \otimes \Delta) \circ \widehat{\alpha}$. That is the map $\widehat{\alpha}$ is a coaction on a compact quantum group.

4.2 Embeddable ergodic actions of $O_{-1}(2)$

In this section we determine the embeddable ergodic actions on $O_{-1}(2)$, based on those of $O(2)$ classified above in Proposition 4.1.3. The plan for achieving this is as follows.

First, note that by definition an embeddable ergodic action is by definition a comodule $*$-algebra of the CQG algebra $\mathcal{A}_{-1}$ associated to $O_{-1}(2)$ which embeds into $\mathcal{A}_{-1}$ as such (i.e. by an embedding that preserves all of the structure: comodule, algebra, etc.).

Since the twisting equivalence $\lambda\triangleright$ (see the notation from Section 4.4) that implements Theorem 1.4.19 also implements an equivalence between the categories of coideal $*$-algebras over $\mathcal{A}_{-1}$ and the untwisted version $\mathcal{A}$ (algebra of representative functions on the classical group $O(2)$), it will be sufficient to identify the ergodic $O(2)$-action $\mathcal{B}$ in the list of Proposition 4.1.3 for which

$$\lambda \triangleright \mathcal{B} \cong \mathcal{A}_{-1}$$

as $\mathcal{A}_{-1}$ comodule $*$-algebras, and to then also identify the members of that list that embed into $\mathcal{B}$.

We will see that there is only one candidate for $\mathcal{B}$ (namely $\beta_{1/2}^{(2)}$) using the Peter-Weyl theorem to determine the representation type of the ergodic actions identified in Proposition 4.1.3 (where by representation type we mean the multiplicities of the various irreducible $O(2)$-representations). Indeed, this is the substance of the following result.

Proposition 4.2.1. *The only comodule algebras among those in Proposition 4.1.3 that are isomorphic to $\mathcal{A}$ as $O(2)$-representations are $\alpha^{(1)} \cong \mathcal{A}$ itself and $\beta_{1/2}^{(2)}$.*

Proof. The (∞)-superscript $O(2)$-representations are finite-dimensional, so we can discount them for the purposes of this proposition.

For the other members of the list, we will use the Frobenius reciprocity formula

$$\hom_{O(2)}(V, \mathrm{Ind}_H^{O(2)} W) \cong \hom_H(V, W) \tag{4.2.1}$$

for $V \in \mathrm{Rep}_{O(2)}$ and $W \in \mathrm{Rep}_H$ in order to compute the multiplicities of various irreducible $O(2)$-representations.

For each $k \geq 1$ we have a 2-dimensional $O(2)$-representation V_k whose restriction to $\mathbb{T}$, upon identifying the Pontryagin dual

$$\widehat{T} \cong \mathbb{Z},$$

splits as $k \oplus (-k)$.

Now, for $k \geq 2$, $\alpha^{(k)}$ is induced from the non-trivial cyclic group $\mathbb{Z}_k \subset \mathbb{T}$. Taking $H = \mathbb{Z}_k$, W to be trivial, and $V = V_1$ in (4.2.1), the right hand side vanishes and hence so must the left hand side. This means that V_1 is not a summand of $\alpha^{(k)}$, $k \geq 2$, and hence these list members can also be dropped as candidates for an isomorphism to $\mathcal{A}$ as $\mathcal{A}$-comodules.

Next we look at the representations $\beta_0^{(k)}$ for all $k \geq 1$ induced from the trivial representation of the order-$2k$ dihedral groups $\mathbb{D}_k \subset O(2)$. In these cases, (4.2.1) with $H = \mathbb{D}_k$, W trivial and V being the non-trivial character of $O(2)$ annihilates the right hand side, and hence the left hand side too. In conclusion, the non-trivial character of $O(2)$ does not appear in $\beta_0^{(k)}$; this disqualifies these representations.

Finally, we consider $\beta_{\ell/2}^{(k)}$ for $\ell > 0$ and $k \geq 2$. Here, we apply (4.2.1) with $H = \mathbb{D}_k$, W the representation of $\mathbb{D}_k$ on M_2 described in the discussion preceding Proposition 4.1.3, and $V = V_1$. There are now a few possibilities:

(a) If $\ell > 1$ then the right hand side of (4.2.1) is zero, so these cases can be discarded;

(b) If $\ell = 1$ and $k \geq 3$ then the right hand side of (4.2.1) is one-dimensional, because the restriction of V_1 to $\mathbb{D}_k$ is irreducible. In conclusion V_1 appears in $\beta_{\ell/2}^{(k)}$ with multiplicity one, but it appears in $\mathcal{A}$ with multiplicity two(by Peter-Weyl, since it is a two-dimensional irreducible representation). Once more, these cases do not qualify for the purposes of the proposition;

(c) Finally, $\ell = 1$ and $k = 2$ is left, in which case one easily checks that the multiplicities match as expected. Indeed, $\mathbb{D}_k$ is then the Klein group $(\mathbb{Z}/2)^2$, and its 4-dimensional representation W that is induced up to $O(2)$ to produce $\beta_{1/2}^{(2)}$ breaks up as a sum of all of its characters.

It follows from the previous paragraph that if the irreducible $O(2)$-representation V is one-dimensional then the right hand side of (4.2.1) is also one-dimensional, whereas if V is two-dimensional then its restriction to D_2 breaks up as a sum of two distinct characters, and hence the right hand side of (4.2.1) is two-dimensional.

This finishes the proof of the proposition. □

We can now record the consequence alluded to above.

Corollary 4.2.2. *The twisting equivalence $\lambda\triangleright$ induces a bijection between*

$$\{\alpha^{(k)},\ \beta_{l/2}^{(k)} \mid k = \infty \text{ or even },\ l = 0 \text{ or odd }\}$$

from Proposition 4.1.3 and the isomorphism classes of embeddable ergodic actions of $O_{-1}(2)$.

Proof. The function algebra of $O_{-1}(2)$ can be obtained from that of $O(2)$ by twisting the multiplication both on the right and the left, by the cocycle λ and its convolution inverse λ^{-1}. Since $\lambda\triangleright$ by definition twists by λ on the right, the $\mathcal{A}$-comodule algebra $\mathcal{B}$ from the introductory remarks to Section 4.2 is a twist of $\mathcal{A}$ on the left and hence cannot be abelian, and yet must have the same representation type as $\mathcal{A}$ as a right $\mathcal{A}$-comodule. It must thus be $\beta^{(2)}_{1/2}$ by Proposition 4.2.1.

In summary, the desired conclusion will follow once we show that the ergodic $O(2)$-actions listed in the statement are precisely those that embed into $\beta^{(2)}_{1/2}$.

Throughout the proof, we denote by W the D_2-representation on M_2 that gives rise to $\beta^{(2)}_{1/2}$ by induction to $O(2)$. We examine the representations listed in Proposition 4.1.3 systematically.

Type-α actions.

$\alpha^{(\infty)}$ is two-dimensional and clearly embeds into $\beta^{(2)}_{1/2}$, as the diagonal subalgebra of the realization of W as 2×2 matrices.

As for $\alpha^{(k)}$ for positive integers k, consider first the case when k is odd. If we had an embedding

$$\alpha^{(k)} \subseteq \beta^{(2)}_{1/2},$$

then the Frobenius adjunction (4.2.1) would turn it into a map

$$\mathrm{Res}^{O(2)}_{H}\alpha^{(k)} \to W \tag{4.2.2}$$

of algebras in Rep_{D_2}. The condition that k be odd then ensures that this map is surjective, since in that case all four characters of D_2 appear in the restriction of $\alpha^{(k)}$. Since however the left hand side of (4.2.2) is commutative while the right hand side is not, we obtain a contradiction.

For even k on the other hand, we can embed $\alpha^{(k)}$ into $\beta^{(2)}_{1/2}$ by inducing in stages. First, embed

$$\mathrm{Ind}^{\mathbb{D}_k}_{\mathbb{Z}_k}\mathbb{C} \subseteq \mathrm{Ind}^{\mathbb{D}_k}_{D_2} W$$

as the diagonal subalgebra of the 2×2 matrix realization of W, and then induce the entire embedding further to $O(2)$.

Type-β actions, $l = 0$.

$\beta^{(\infty)}_0$ is simply the trivial representation and hence is embeddable into $\beta^{(2)}_{1/2}$. We note also that $\beta^{(k)}_0$ for odd k can be eliminated in exactly the same way we did $\alpha^{(k)}$ above.

For even k $\beta^{(k)}_0$ is again embeddable into $\beta^{(2)}_{1/2}$ by the case of even $\alpha^{(k)}$, since we have

$$\beta^{(k)}_0 \subset \alpha^{(k)}.$$

Type-β actions, $l > 0$.

Consider the case of $\beta^{(k)}_{l/2}$ (including $k = \infty$) for even positive l. Here we have an embedding

$$\beta^{(\infty)}_{l/2} \subseteq \beta^{(k)}_{l/2} \tag{4.2.3}$$

of algebras in $\mathrm{Rep}_{O(2)}$, and hence an embedding of the right hand side into $\beta^{(2)}_{1/2}$ would imply the existence of a morphism of the left hand side into W in the category Rep_{D_2}.

This is impossible: both the left hand side of (4.2.3) and W are 2×2 matrix algebras and hence the morphism would have to be one-to-one, but the evenness of l ensures that when restricted to D_2 the left hand side of (4.2.3) has a two-dimensional space of invariants.

When k is positive and odd, then for every l we have an even l' such that

$$l' \equiv l \ (\text{mod} k).$$

We have an embedding

$$\beta^{(\infty)}_{l'/2} \subseteq \beta^{(k)}_{l/2}$$

of algebras in $\text{Rep}_{O(2)}$ and we can repeat the argument above to conclude that $\beta^{(k)}_{l/2}$ is not embeddable into $\beta^{(2)}_{1/2}$.

For even k (including by abuse the case $k = \infty$ with $\mathbb{D}_k = O(2)$) and positive odd l the restriction of $\beta^{(\infty)}_{l/2}$ to $\mathbb{D}_k$ embeds into

$$\text{Ind}^{\mathbb{D}_k}_{D_2} W,$$

and hence $\beta^{(k)}_{l/2}$ is embeddable into $\beta^{(2)}_{1/2}$, as desired.

This concludes the last case and the proof of the result. □

4.3 Idempotent states of $O_{-1}(2)$

In this section, we will deduce from Section 4.2, the list of idempotent states of $O_{-1}(2)$. In fact, by Example 1.4.3 and Lemma 1.4.14, there exists then an order-preserving bijection between the embeddable ergodic action of $O_{-1}(2)$ and the idempotent states on $O_{-1}(2)$. Having succeeded to make the list of the embeddable ergodic actions of $O_{-1}(2)$ and since by definition, an embeddable ergodic action is identified with a coideal subalgebra and since according to the fundamental result [39, Theorem 4.1] by Franz-Skalski, there is a bijective correspondence between the expected coideal subalgebras and the idempotent states of a compact quantum group; Our project has met a final concern: *How to express by a concrete formula an idempotent state from an action of a compact quantum group* In fact, although we characterized them by listing embeddable ergodic action using [39], we still have to list explicitly the idempotent states of $O_{-1}(2)$ from the list of its embeddable ergodic actions.

Considering that:

1. Let us recall that if one takes the comultiplication itself as the action (i.e the quantum group itself is taken as the coideal subalgebra), it is not immediate to produce the corresponding Haar state from the multiplication.

2. Also if we consider the subalgebra $\mathbb{C}$, then we see that we can not produce the Haar state from the action which is simply $\mathbb{C} \to \mathbb{C} \otimes \mathbb{C}, 1 \mapsto 1 \otimes 1$.

The problem is that starting from the proof of [39, Theorem 4.1], there are several elements that we have abstractly but non-explicitly, and they depend on each other. Indeed, to obtain the idempotent state of the coideal subalgebra, one needs the conditional expectation E; And to get E, we need the state. We believe that if there is a chance to obtain a concrete formula of the idempotent state from an embeddable ergodic action, it will be by expressing the conditional expectation E as a function of the embeddable ergodic action α only by not involving the state itself, then compose E with the counit ε. This question summarizes our point of view.

Question 4.3.1. *Is it possible to obtain such a concrete formula in general for a compact quantum group at least in the Kac case?*

4.4 A counterexample

In the section, we will show that in general, one can not extend the principal result of [30] for embeddable ergodic actions, answering Question 1.4.1 (or rather to the more precise version of it we give below) in the negative.

Specifically, let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent compact quantum groups. Suppose furthermore that the monoidal equivalence is implemented by twisting by a cocycle

$$\lambda : C(\mathbb{G}_1)^{\otimes 2} \to \mathbb{C}.$$

Then, according to the proof of Theorem 1.4.19, we have a twisting operation $\lambda \triangleright$ that implements an equivalence between the category $\mathcal{E}rg(\mathbb{G}_1)$ of ergodic $\mathbb{G}_1$-actions and the analogous category $\mathcal{E}rg(\mathbb{G}_2)$. In this setting, the formulation of Question 1.4.1 that we answer here is

Question 4.4.1. *Does*

$$\lambda \triangleright : \mathcal{E}rg(\mathbb{G}_1) \to \mathcal{E}rg(\mathbb{G}_2)$$

restrict to an equivalence between subcategories of embeddable ergodic coactions?

To describe the counterexample (and explain why it is one), we start out by taking $\mathbb{G}_1$ to be the order-64 Heisenberg group

$$H_4 \ := \ \langle \varepsilon_1, \varepsilon_2, \delta \mid \varepsilon_1^4 = \varepsilon_2^4 = \delta^4 = 1,\ \delta\varepsilon_1 = \varepsilon_1\delta,\ \varepsilon_2\delta = \delta\varepsilon_2,\ \varepsilon_1\varepsilon_2 = \delta\varepsilon_2\varepsilon_1 \rangle.$$

Its deformation $\mathbb{G}_2$ (with underlying function algebra $H = C(H_4)$) will be the one used in [16], described in some detail in Section 6 of that paper. Although we will not write the cocycle

$$\lambda : C(H_4)^{\otimes} \to \mathbb{C}$$

explicitly, recall that it is obtained as the composition

$$\begin{array}{ccccc} & & \lambda & & \\ C(H_4)^{\otimes 2} & & \longrightarrow & & \mathbb{C} \\ & \searrow & & \nearrow & \\ & & C(\Gamma)^{\otimes 2} & & \end{array}$$

where the left hand arrow is restriction along the inclusion

$$(\mathbb{Z}/2)^2 \cong \Gamma := \langle \varepsilon_1^2, \varepsilon_2^2 \rangle \subset H_4$$

and the right hand arrow is some cocycle inducing the non-zero cohomology class in

$$H^2(\widehat{\Gamma}, \mathbb{C}^\times) \cong \mathbb{Z}/2.$$

We refer to [16, Section 6] for a deduction of the needed structural results on H, which we will now recollect. It will be more convenient to dualize and work with the group algebra $\mathbb{C}H_4$ and its deformed version H^*, which coincides with $\mathbb{C}H_4$ as an algebra but has twisted comultiplication.

First, the order-16 subgroup

$$(\mathbb{Z}/2)^2 \times (\mathbb{Z}/4) \cong G := \langle \varepsilon_1^2 \rangle \times \langle \varepsilon_2^2 \rangle \times \langle \delta \rangle$$

of H_4 survives the deformation, in the sense that the new comultiplication on $\mathbb{C}H_4$ preserves the inclusion $\mathbb{C}G \subset \mathbb{C}H_4$, and hence the left hand side is a Hopf subalgebra of H^*. The elements of G are precisely the grouplikes of H^*.

The balance of $64-16=48$ dimensions of H^* consists of twelve 2×2 matrix coalgebras, corresponding to twelve simple 2-dimensional comodules. Setting $\eta=\delta^2$, these coalgebras are the images of

$$D_i = \mathrm{span}\{\varepsilon_i^{\pm 1}, \varepsilon_i^{\pm 1}\eta\}, i=1,2$$

and

$$D_3 = \mathrm{span}\{\varepsilon_1\varepsilon_2,\ \varepsilon_1^{-1}\varepsilon_2^{-1},\ \varepsilon_1\varepsilon_2\eta,\ \varepsilon_1^{-1}\varepsilon_2^{-1}\eta\}$$

through the action of G by left multiplication; each D_j, $j=1,2,3$ has a size-4 orbit under this action.

Now, using the correspondence between coideals of a quantum group and those of its dual (e.g. [55]), right coideal $*$-subalgebras of $C(H_4)$ and H (which embeddable ergodic coactions map isomorphically onto) correspond in an order-reversing fashion to the coideal $*$-subalgebras of $\mathbb{C}H_4$ and H^*.

Moreover, if the coideal $*$-subalgebra A of H corresponds to $\mathrm{cd}(A)\in H^*$, then we have

$$\dim(A)\cdot\dim(\mathrm{cd}(A)) = \dim(H) = 64.$$

This follows for instance from the fact that every coideal subalgebra $A\subset H$ we are considering is a finite-dimensional C^*-algebra and hence Frobenius, which implies [70] that H is free as a right A-module, say $H\cong A^{\oplus m}$. But then the dual

$$\mathrm{cd}(A)^* = H/H\ker(\varepsilon|_A) \cong \mathbb{C}^{\oplus m}$$

as vector spaces, giving the desired conclusion (since $m=\frac{\dim(H)}{\dim(A)}$).

All in all, if the answer to Question 4.4.1 were affirmative, then we would have a dimension and inclusion-preserving bijection between the coideal $*$-subalgebras of $\mathbb{C}H_4$ and those of H^* (note that the preceding discussion was needed in order to pass from coideal subalgebras of $C(H_4)$ and H to those of their duals). Moreover, because $G\subset H_4$ is untwisted, this bijection would preserve G along with its subgroups. This is precisely what we will show is not the case. Specifically, we have

Proposition 4.4.2. *The only 4-dimensional coideal $*$-subalgebra of H^* that does not intersect $\mathbb{C}G\subset H^*$ and which contain $\mathbb{C}\langle\varepsilon_1^2\rangle$ is the group algebra of $\langle\varepsilon_1^2\rangle\times\langle\varepsilon_2^2\rangle$.*

Proof. Let $A\subset H^*$ be a four-dimensional coideal subalgebra as in the statement, different from

$$\mathbb{C}\langle\varepsilon_i^2\rangle. \tag{4.4.1}$$

We have already noted that G accounts for all of the grouplikes of H^*; since the group (4.4.1) is the only order-4 subgroup of G that does not intersect the center $Z(G)=Z(H_4)=\langle\delta\rangle$, it follows that as a right H^*-comodule, A must break up as

$$\mathbb{C}\oplus\mathbb{C}\varepsilon_1^2\oplus V$$

for some two-dimensional irreducible comodule V. Since moreover A is a subalgebra, V must be invariant under tensoring with $\mathbb{C}\varepsilon_1^2$. The description of the coalgebra structure of H^* given above then implies that we have

$$V\subset D_1,\ \delta D_1,\ \varepsilon_2^2 D_1,\ \text{or } \delta\varepsilon_2^2 D_1.$$

Now, it is shown in [16, Lemma 6.2] that

$$\begin{cases} a := \frac{1}{2}(\varepsilon_1 + \varepsilon_1\eta) & b := \frac{1}{2}(\varepsilon_1 - \varepsilon_1\eta) \\ c := \frac{1}{2}(\varepsilon_1^{-1} - \varepsilon_1^{-1}\eta) & d := \frac{1}{2}(\varepsilon_1^{-1} + \varepsilon_1^{-1}\eta) \end{cases}$$

are matrix counits for D_1, and $\varepsilon_1^2 V = V$ then implies that we have

$$V = \mathrm{span}\{a + c, b + d\},\ V = \mathrm{span}\{a - c, d - b\},$$

or their images under multiplication by δ, ε_2^2 or $\delta\varepsilon_2^2$. But it is easily checked that in none of these cases is $\mathbb{C} \oplus \mathbb{C}\varepsilon_1^2 \oplus V$ a subalgebra of H^*. For instance, when $V \subset D_1$, the decomposition of $(a+c)^2$ in one case and $(a-c)^2$ in the other with respect to the H_4 basis contains $\varepsilon_1^2\eta$; analogous arguments work in the other three cases. □

Consequently,

Corollary 4.4.3. *There does not exist an inclusion, dimension, and G-preserving bijection between the coideal subalgebras of $\mathbb{C}H_4$ and those of H^*.*

Proof. This follows from Proposition 4.4.2 and the fact that $\mathbb{C}H_4$ has an additional coideal subalgebra as in the statement of that result, namely the group algebra of the subgroup

$$\langle \varepsilon_1 \rangle \cong \mathbb{Z}/4$$

of H_4. □

Note however that the answer to Question 4.4.1 is affirmative for those compact quantum groups $\mathbb{G} = \mathbb{G}_1$ whose only ergodic comodule $*$-algebra admitting all irreducible comodules of $\mathcal{O}(\mathbb{G})$ with full multiplicity is $\mathcal{O}(\mathbb{G})$ itself.

The latter comodule algebras (usually referred to as *full multiplicity* ergodic actions) are classified by the cohomology set $H^2(\widehat{\mathbb{G}}, \mathbb{S}^1)$ of the discrete quantum dual of $\mathbb{G}$ described for instance in [77, §2.4]. This is an extension to the quantum realm of the classification carried out in [108] in terms of cocycles.

Remark 4.4.4. *Note that when $\mathbb{G} = \widehat{\Gamma}$ for an ordinary discrete group Γ, the set $H^2(\widehat{\mathbb{G}}, \mathbb{S}^1)$ is simply the standard 2-cohomology group $H^2(\Gamma, \mathbb{S}^1)$ of Γ valued in the trivial Γ-module $\mathbb{S}^1$.*

As one final simple observation, we remark that if $\mathbb{G}_1$ happens to have trivial 2-cohomology set $H^2(\widehat{\mathbb{G}_1}, \mathbb{S}^1)$, the answer to Question 4.4.1 is affirmative:

Proposition 4.4.5. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two monoidally equivalent compact quantum groups, and assume $H^2(\widehat{\mathbb{G}_2}, \mathbb{S}^1)$ is trivial. Then,*

$$\lambda\triangleright : \mathcal{E}rg(\mathbb{G}_1) \to \mathcal{E}rg(\mathbb{G}_2)$$

restricts to an equivalence between subcategories of embeddable ergodic actions.

Proof. The functor $\lambda\triangleright$ implements a bijection between the isomorphism classes of the ergodic coideal algebras of $\mathcal{O}(\mathbb{G}_1)$ embeddable into $\mathcal{O}(\mathbb{G}_1)$ as coideal algebras and the ergodic coideal $*$-algebras of $\mathbb{G}_2$ embeddable into $\lambda\triangleright\mathcal{O}(\mathbb{G}_1)$. The latter is a full-multiplicity ergodic action of $\mathbb{G}_2$, and hence by assumption it must be (isomorphic to) $\mathcal{O}(\mathbb{G}_2)$ itself. □

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